A Trivialist's Travails[1]

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It's always a pleasure to read a small book with big ambitions. Agustín Rayo's manifesto, *The Construction of Logical Space*,[2] is an outstanding example. In only a couple of hundred pages, Rayo presents a novel metaphysics *and* epistemology for mathematics *and* modality, a new defence of a Stalnakerian theory of the propositional attitudes, and some philosophy of language and reflections on philosophical methodology.

In his introduction, Rayo states that he wants to 'explain how mathematical knowledge is possible' [pg. *ix*]. In this critical notice, I will focus on Rayo's epistemology of mathematics. There are two discussions of this in the book. The first is part of his discussion of 'trivialist Platonism' in the first four chapters; the second is in chapter 8, where he introduces a position that I'll call 'postulationism'.[3] I'll discuss trivialist Platonism in §1 and §2. In §3 I'll explain Rayo's postulationism. I'll raise a problem for postulationism in §4, to which I'll offer a solution in §5.

1. Background: 'just is'-statements

1.1 Introducing the 'just is' operator

Let's start by taking a look at Rayo's 'just is' operator. He introduced the operator in earlier work (Rayo [2009]) but his new book contains a much more thorough discussion of its uses. Here's one of his examples, slightly adapted [pg. 3]:

> To be composed of water just is to be composed of H_2O.

This can be paraphrased, 'there is no difference between being composed of water and being composed of H_2O', or 'the property of being composed of water is identical with the property of being composed of H_2O'.[4] 'Just is' can also be used to identify relations with various arities:

> For *x* to be taller than *y* just is for *y* to be shorter than *x*.
> For *x* to be the child of *y* and *z* just is for *y* and *z* to be the parents of *x*.

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[2] Rayo [2013]; throughout the rest of this paper, whenever I refer to Rayo's work it is this book that I have in mind, unless I specify otherwise.

[3] This is not Rayo's term. I've taken the word from Potter [2004, pg. 10].

[4] Rayo would add that he accepts this latter paraphrase only on the condition that the property-talk is 'understood in a suitably deflationary way' [pg. 68].

The following exemplifies a rather different use of the expression 'just is':

> For it to be the case that Chicago is in Illinois just is for it to be the case that Illinois contains Chicago.

In this case, it's tempting to say that the 'just is' operator is being used to assert an identity between propositions. But since questions about the nature of propositions are so difficult and highly contested, it is understandable that Rayo is cautious about this way of putting it [pg. 66]. Instead, he prefers to paraphrase this 'just is'-statement like so [pg. 52]:

> 'Chicago is in Illinois' has the same truth-condition as 'Illinois contains Chicago'.

As we will see in §2, one of Rayo's claims is that true statements in pure mathematics have 'trivial truth-conditions', in the sense that 'nothing is required of the world in order for the truth-conditions of a mathematical truth to be satisfied' [pg. 98]. Rayo uses his 'just is' operator to clarify this assertion: Rayo's claim is that if φ is a true purely mathematical sentence, the truth-condition of φ is the same as the truth-condition of '∀x x=x'; that is, the following is true:

> ⌜ For it to be the case that φ just is for it to be the case that ∀x x=x. ⌝

Of course, the choice of '∀x x=x' in this definition is rather arbitrary; one could also use (for example) any truth of the form ⌜ φ→φ ⌝.[5]

1.2 'Just is'-statements and necessity

According to Rayo, any set *S* of 'just is'-statements defines a 'conception of logical space'. Another set of statements *T* describes a possible scenario, relative to this conception, just in case *T* is consistent with *S*. A statement φ is necessary, relative to this conception, just in case *S* entails φ. For example, if you accept that to be composed of H_2O just is to be composed of water, you thereby adopt a conception of logical space relative to which the sentence 'This raindrop is composed of water, but not H_2O' does not describe a possible scenario.

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[5] Rayo's definition of 'trivial' is on pg. 53. I've changed the definition slightly, but it's easy to see that my definition is equivalent to Rayo's.

Let's look at a mathematical example. Consider:[6]

STRONG NUMBERS
For it to be the case $\#_x\, \varphi = n$ just is for it to be the case that $\exists!_n\, x\, \varphi$.

WEAK NUMBERS
$\#_x\, \varphi = n \leftrightarrow \exists!_n\, x\, \varphi$

Here, the operator $\#_x$ … x … means *the number of things x such that … x …* . For example, '$\#_x$ Dog(x)' refers to the number of dogs, while '$\#_x$ (cat(x)∧black(x))' refers to the number of black cats. ⌜ $\exists!_n\, x\, \varphi$ ⌝ means *there exist exactly n things x such that φ,* where this is defined recursively in the usual way. Now if one accepts STRONG NUMBERS, one thereby adopts a conception of modal space according to which WEAK NUMBERS is necessarily true.

1.3 The epistemology of 'just is'

Rayo argues that when deciding which 'just is'-statements to accept, we should use a certain sort of cost-benefit analysis. His example is this statement of the kinetic theory of heat:

For a gas to be hot just is for it to have high mean kinetic energy.[7]

The *benefit* of accepting this 'just is'-statement is that one thereby avoids having to answer certain questions, which might otherwise be problematic:

> [Suppose you accept this 'just is'-statement. Then] you should think there is no need to answer the following question. 'I can see that the gas is hot. But why does it also have high mean kinetic energy?' You should think, in particular, that the question rests on a false presupposition. It presupposes that there is a *gap* between the gas's being hot and its having high mean kinetic energy – a gap that should be plugged with a bit of theory. But to accept the 'just is'-statement is to think that the gap is illusory. There is no need to explain how the gas's being hot might be correlated with its having high mean kinetic energy because there is no difference between the two… [Pg. 18]

[6] Strictly speaking, WEAK NUMBERS and STRONG NUMBERS are schemas; for convenience, I will talk as though they are statements. What I call 'STRONG NUMBERS', Rayo calls simply 'NUMBERS'.
[7] As Rayo points out [pg. 18, fn. 9] this statement of the kinetic theory is 'baldly inaccurate'. This doesn't matter: it's just an example.

More generally, Rayo claims that anyone who accepts a 'just is'-statement is consequently exempted from having to explain the corresponding universally quantified biconditional. He calls this 'why closure' [section 2.2.4].

But accepting a 'just is'-statement comes with a cost too. By accepting the above 'just is'-statement, one adopts a conception of modal space relative to which there are no possible scenarios at which a gas is hot but does not have high mean kinetic energy, and no possible scenarios at which a gas has high mean kinetic energy but is not hot. Rayo comments, 'having extra scenarios to work with can … prove advantageous, since it makes room for additional theoretical positions, some of which could deliver fruitful theorizing' [pg. 19]. In order to make our final decision about whether to accept this 'just is'-statement, we should weigh the cost and the benefit against each other:

> There is no quick-and-easy criterion for determining whether the extra theoretical space is fruitful enough to justify paying the price of having to answer a new range of potentially problematic questions. The only reasonable way to proceed is to roll up one's sleeves and do some metaphysics. [Pg. 19]

We'll see this style of cost-benefit analysis at work in §2, when we'll look at Rayo's defence of STRONG NUMBERS.

1.4 Objectivism and subjectivism

Rayo comments:

> [T]he decision to adopt a particular conception should be guided by its ability to combine with the rest of one's theorizing to deliver a fruitful tool for scientific or philosophical inquiry. But fruitfulness is a goal-relative notion: a theoretical apparatus that constitutes a fruitful way of pursuing one set of goals may not be a fruitful way of pursuing another. So one might end up with a situation in which one has grounds for accepting a particular conception of logical space relative to one set of goals, and a different conception relative to another. [Pg. 57]

Let's suppose that conception C_1 is optimal relative to one set of goals that you have, and conception C_2 is optimal relative to some other set of goals that you have. What should your attitude be towards these two conceptions?

Here, we need to distinguish the 'objectivist' from the 'subjectivist' versions of Rayo's account of necessity and possibility.[8] According to the objectivist, there is one privileged, 'objectively correct' conception of logical space. So at most one of C_1 and C_2 can be 'objectively correct', even if you are unable to choose between them. The subjectivist on the other hand rejects the claim that there is an 'objectively correct' conception; for the subjectivist, asking whether it is C_1 or C_2 which is 'objectively correct' is like asking whether it is chess or draughts which has the 'objectively correct' rules [pg. 58].

In earlier work, Rayo committed himself to the subjectivist position (Rayo [2009, pg. 255]), but in the current book he is more cautious. He challenges the objectivist with the question, 'What does it mean to say that a conception of logical space is objectively correct?' 'The most straightforward answer,' he goes on, 'would be to say that for a conception of logical space to be objectively correct is for the 'just is'-statements it is based on to be objectively true' [pg. 57]. But Rayo argues that it is difficult to understand what it means to say that a 'just is'-statement is objectively true. So Rayo's current position is this. He *prefers* subjectivism, because he doesn't know how to make sense of the idea that one conception of logical space is 'objectively correct'. However, he leaves open the possibility that someone in the future will find a way of making sense of this idea, thereby vindicating objectivism.

So far, what I have said has been purely expository. However, to finish the section I will make a point of my own. I will argue that, however difficult it may be to make sense of the idea that some 'just is'-statements are objectively true, even a subjectivist should admit that some 'just is'-statements are objectively false. I'll start by introducing an ugly technical term (my own, not Rayo's): Given a 'just is'-statement, its 'demodalisation' is the result of replacing the 'just is' operator with a universally quantified biconditional. So for example, the demodalisation of (a) is (a*):

(a) To be composed of H_2O just is to be composed of water.
(a*) Everything that is composed of water is also composed of H_2O and *vice versa.*

I presume that every 'just is'-statement entails its demodalisation; e.g. (a) entails (a*). Now imagine someone who accepts the following 'just is'-statement:

(b) To be a bird just is to be winged creature.

[8] Rayo uses the term 'objectivist' on pg. 57. He does *not* use the term 'subjectivist', but it's the obvious term to use.

The demodalisation of (b) is:

(b*) Every bird is a winged creature, and every winged creature is a bird.

Now (b*) is false, and presumably objectively false.[9] Since (b) entails (b*), (b) must be objectively false too.

This point will be important later, when we turn to Rayo's views about mathematics. Recall these two statements:

STRONG NUMBERS
For it to be the case $\#_x \varphi = n$ just is for it to be the case that $\exists!_n x\, \varphi$.

WEAK NUMBERS
$\#_x \varphi = n \leftrightarrow \exists!_n x\, \varphi$

As I will explain in §2, Rayo accepts STRONG NUMBERS. So he puts himself at odds with nominalists – those who think that numbers don't exist. Nominalists will say that WEAK NUMBERS is false (objectively false, if you like) from which it follows that STRONG NUMBERS is (objectively) false too. If Rayo's defence of STRONG NUMBERS is to be successful, he must establish that this nominalist position is mistaken.[10]

2. Trivialist Platonism

2.1 Introducing trivialist Platonism

Chapters 3 and 4 of Rayo's book are largely devoted to the defence of what he calls 'trivialist Platonism'. Rayo's 'Platonism' is 'the claim that mathematical objects exist' [pg. *viii*]. 'Trivialism' is rather harder to define. Rayo himself defines trivialism as the 'the view that the truths of pure mathematics have trivial truth-conditions, and the falsities of pure mathematics have trivial falsity-conditions' [pg. 74] (see §1.1 of this paper for an explanation of the term 'trivial').

[9] The adverb 'objectively' is far from the clearest term in the philosophers' lexicon, but I take it that on any reasonable construal of this term, (b*) is 'objectively' false.
[10] For an interesting discussion of the role of subjectivism in Rayo's epistemology of mathematics, see Burgess [forthcoming].

However, Rayo makes it clear that being a trivialist *also* involves accepting STRONG NUMBERS [pg. 35]:

> STRONG NUMBERS
> For it to be the case $\#_x\, \varphi = n$ just is for it to be the case that $\exists!_n\, x\, \varphi$.

For our purposes, it will suffice to define 'trivialist' in a rather open ended way by saying that the trivialist is committed to the view that a range of mathematical sentences have trivial truth-conditions, including the truths of pure mathematics, and WEAK NUMBERS.[11]

Rayo begins his discussion of trivialist Platonism with a cost-benefit analysis, designed to show that we should accept that WEAK NUMBERS and the truths of pure mathematics have trivial truth-conditions. I will consider this cost-benefit analysis in detail in §2.2 and §2.3. He then deals with some other issues:

- It follows from STRONG NUMBERS that, for example, the truth-condition of '$\#_x$ Planet(x) = 8' is the same as the truth-condition of '$\exists!_8$ x Planet(x)'. But what should the trivialist say about the truth conditions of more complicated statements about numbers, such as this?

  $$2 + \#_x \text{WelshCity}(x) = \#_x \text{Planet}(x)$$

  In order to answer this question, in section 3.3 Rayo develops a 'compositional specification of truth-conditions for arithmetical sentences that assigns to each sentence in the language of arithmetic the truth-conditions that a trivialist thinks it should have' [pg. 76].
- In section 3.4, Rayo extends his compositional semantics to cover set-theoretic vocabulary.
- Finally, in chapter four, Rayo offers an account of 'cognitive accomplishment in logic and mathematics'. Rayo adopts a Stalnakerian[12] account of the propositional attitudes, according to which the objects of belief are sets of possible worlds. On this view, anyone who believes the proposition that 7+5=12 also believes every theorem of set theory. It would seem that anyone who maintains this position will have a hard time explaining what mathematical learning consists in. Rayo's goal in chapter 4 is to deal with this problem.

My goal in the rest of §2 is to criticise Rayo's cost-benefit analysis, which he uses to defend his trivialism.

[11] Thanks are due to an anonymous referee at *Philosophia Mathematica* for pointing out that Rayo's definition of 'trivialist' doesn't quite capture his intention.
[12] See Stalnaker [1984].

2.2 Rayo's cost-benefit analysis

As I said, the trivialist thinks that a number of mathematical statements (including WEAK NUMBERS and all purely mathematical truths) have trivial truth-conditions. When Rayo defends trivialism using cost-benefit analysis, he uses WEAK NUMBERS as an example. Since I intend to criticise the cost-benefit analysis in some detail, I'll quote the relevant passage in full:

> [I]t is natural to think that the costs of accepting [STRONG NUMBERS] are far outweighed by the benefits. For by accepting [STRONG NUMBERS], one eliminates the need to answer questions such as the following:
>
> > I can see that there are no dinosaurs. What I want to know is whether it is also true that the number of the dinosaurs is Zero. And I would like to understand how one could ever be justified in taking a stand on the issue, given that we have no causal access to the purported realm of abstract objects.
>
> There is no need to explain how the non-existence of dinosaurs might be correlated with dinosaurs' having Zero as a number because there is *no difference* between the two: for the number of the dinosaurs to be Zero just is for there to be no dinosaurs.
>
> It is true that there is also a cost. By accepting [STRONG NUMBERS] one loses access to a certain amount of theoretical space, since one is no longer in a position to work with scenarios in which there are no numbers. But it seems to me that this is not much of a price to pay, since the availability of such scenarios is not very likely to lead to fruitful theorizing. The upshot is that there is significant theoretical pressure to accept [STRONG NUMBERS] – at least provided that it can be used to construct a viable philosophy of mathematics. [Pg. 74] [13]

Let's unpack this argument, which is stated rather quickly. Rayo identifies two benefits and one cost of accepting STRONG NUMBERS.

The first benefit of accepting STRONG NUMBERS is epistemological: by accepting STRONG NUMBERS, the trivialist can deal with the question of 'how one could ever be justified in taking a stand on the issue [of whether $\#_x$ Dinosaur(x) = 0].'

[13] There is an almost identical discussion on pg. 22.

The second benefit is that for someone who accepts STRONG NUMBERS, '[t]here is no need to explain how the non-existence of dinosaurs might be correlated with dinosaurs' having Zero as a number'. I take it that the correlation in question here is something like this:

> *Dinosaur Correlation*
> For any time *t*, the number of dinosaurs at *t* is Zero just in case there are no dinosaurs at *t*.

This is an instance of 'why closure' (see §1.2 of this paper).

Rayo also mentions a cost associated with accepting STRONG NUMBERS. The cost is that 'one loses access to a certain amount of theoretical space, since one is no longer in a position to work with scenarios in which there are no numbers'. Rayo comments that this cost is small, since 'the availability of such scenarios is not very likely to lead to fruitful theorizing'.

2.3 A critique of Rayo's cost-benefit analysis

Before getting into the details, it will be helpful to establish some names for the alternative positions. I'll use the term 'nominalism' for the view that there are no mathematical objects, and so no numbers. According to the nominalist, WEAK NUMBERS is false. Platonists on the other hand believe that numbers exist. I'll distinguish two varieties of Platonist. *Trivialist* Platonists accept STRONG NUMBERS; *non-trivialist* Platonists accept WEAK NUMBERS but not STRONG NUMBERS.[14] The non-trivialist Platonist agrees that WEAK NUMBERS is true, but adopts a conception of logical space relative to which there are some possible scenarios at which WEAK NUMBERS is false – perhaps these are scenarios at which there are no numbers.

Now let's take a closer look at Rayo's cost-benefit analysis. My conclusion will be that while Rayo's analysis may indeed show that trivialist Platonism is preferable to non-trivialist Platonism, it does not show that Platonism itself is true.

The first of the two benefits that Rayo claims for trivialist Platonism is epistemological.

Suppose that a sentence *S* is true, but not trivially true. That is, suppose that *S* is true, but there is a possible scenario at which *S* is false. According to Rayo's epistemology, to be justified in accepting *S*, one must *rule out* those possible scenarios at which *S* is false [pg. 36-8]. One is justified for example in

[14] This categorization is not exhaustive: there are Platonist positions according to which even WEAK NUMBERS is false. We need not consider such positions, however.

accepting 'John is in Paris' only once one has ruled out scenarios in which he is in Rome, scenarios in which he is in Omaha, and so on. Now suppose that one adopts a conception of logical space according to which WEAK NUMBERS is non-trivially true. Then to know that this sentence is true one has to rule out possible scenarios at which it is false. But since we have 'no causal access to the purported realm of abstract objects' [pg. 22] it is hard to see how we could ever rule out such scenarios. So non-trivialist Platonism is an unstable position: if the non-trivialist Platonist is right that WEAK NUMBERS is not trivial, she cannot justify her claim that it is true. This is Rayo's version of the 'access' or 'Benacerraf problem'.[15] Rayo argues that the trivialist Platonist doesn't face this problem. For her, there are *no* possible scenarios at which the WEAK NUMBERS is false, and so knowing that this sentence is true doesn't involve ruling out any scenarios ([Pg. 98]; See also [pg. 22] and [pg. 74]).

I am inclined to agree with Rayo that trivialist Platonism is preferable to non-trivialist Platonism for this epistemological reason. At least, I am happy to concede the point. However, I don't think that there is an argument here for preferring Platonism to nominalism. To deal with an epistemological objection to Platonism is not to provide a positive argument for the Platonist position. Someone who was initially agnostic between the trivialist Platonist, non-trivialist Platonist, and nominalist positions might be convinced by this epistemological argument that she should take non-trivialist Platonism off the table. But she has been given no reason to reject nominalism.[16]

Now let's consider the second putative benefit of trivialist Platonism: Rayo's claim is that it is an advantage of the trivialist Platonist position that (because of 'why closure') she doesn't have to explain the following correlation:

> *Dinosaur Correlation*
> For any time *t*, the number of dinosaurs at *t* is zero just in case there are no dinosaurs at *t*.

Once again, it seems to me that however well this works as a defence of the trivialist version of Platonism over the non-trivialist version, this point provides no justification for Platonism itself. It is perhaps a negative feature of non-trivialist Platonism that its proponent is stuck with the difficult task of explaining *Dinosaur Correlation.* However, nominalism does not share this negative feature: the nominalist, after all, doesn't accept *Dinosaur Correlation* and so is under no pressure to offer an explanation for it. So the nominalist's position is not threatened by this point.

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[15] See Benacerraf [1973].

[16] We could distinguish trivialist from non-trivialist versions of nominalism: the trivialist thinks that it is necessary that there are no numbers; the non-trivialist thinks that it is contingently true that there are no numbers. A variant on Rayo's epistemological argument against non-trivialist Platonism could be used to attack non-trivialist nominalism. But it seems that trivialist nominalism (like trivialist Platonism) survives such arguments.

Finally, let's look at Rayo's assessment of the cost of trivialist Platonism. Rayo claims that the disadvantage of the trivialist position is that by accepting it 'one loses access to … scenarios in which there are no numbers'; he comments, 'this is not much of a price to pay, since the availability of such scenarios is not very likely to lead to fruitful theorizing' [pg. 74]. A nominalist might reasonably accuse Rayo of begging the question at this point. According to the nominalist, the *actual scenario* is one at which there are no numbers, and so to 'lose access' to such scenarios would be a *huge* theoretical cost.

My conclusion, to repeat, is that while Rayo's cost-benefit analysis may establish that the trivialist version of Platonism is preferable to the non-trivialist version, it does not establish Platonism itself. On its own, this point is hardly devastating – for Rayo can appeal to his postulationism in defence of Platonism. But as we shall see, postulationism has its own problems.

To finish, it's worth pointing out a further limitation of Rayo's cost-benefit analysis: Rayo has not yet provided us with a complete account of what justifies us in accepting particular purely mathematical statements – hence, he has not yet completely explained 'how mathematical knowledge is possible' [pg. *ix*]. Imagine, to pick an example more or less at random, someone who is unsure about whether the axiom of choice is true. Rayo's cost-benefit analysis plausibly establishes that the axiom is trivially true if it is true at all, but that doesn't tell us whether it is true. For the trivialist, if the axiom is true, then in order to establish this fact one does not need to 'go to the world to check whether any requirements have been met' [pg. 98]; on the contrary, it suffices to show that the axiom has trivial truth-conditions. But how is one to do *that?* Rayo is aware of this issue:

> It is important to keep in mind that getting clear about the truth-conditions of a given mathematical sentence can be highly non-trivial, so determining whether [a given purely mathematical sentence] is true is not, in general, a trivial affair – more on this later. [Pg. 98]

The 'later' that Rayo mentions here must be his discussion of postulationism. So let's take a look at that.

3. Rayo's postulationism

The final chapter of *The Construction of Logical Space* is titled, 'Introducing Mathematical Vocabulary'. Rayo begins by stating the goals of the chapter, and these stated goals are modest. He tells the reader that '[a] familiar way of introducing mathematical vocabulary is by linguistic stipulation'; he promises an account of how such stipulations work, including a 'sufficient condition for successful stipulation' [pg. 180]. However, later on it becomes clear that Rayo's goals in the chapter are at least partly epistemological.[17] So I will suppose that this chapter forms part of Rayo's attempt to 'explain how mathematical knowledge is possible' [pg. *ix*].

I'll explain some of the details presently; for now, here's the gist. One can learn mathematical truths by deducing them from stipulative definitions: both explicit definitions, and implicit definitions (which are more usually called 'axioms'). For example, one might learn number theory by stipulating that the Peano axioms are implicit definitions of 'number', 'plus', 'times', 'successor' and 'zero', and then deducing theorems from these axioms, introducing additional terms by explicit definition as necessary.

Two comments. First, Rayo does not make the mistaken claim that this is how actual mathematical research advances.[18] I suppose that Rayo's claim is, more modestly, that *in principle* one could acquire mathematical knowledge in this way.

Second, in the past philosophers who have defended the view that mathematical theorems are entailed by definitions have also defended the claim that these theorems are true *a priori.* In earlier work, Rayo himself made this claim (Rayo [2008]). In the current book, however, Rayo is more cautious.[19] He does not commit himself to the view that purely mathematical claims are *a priori;* at the same time, he does not at any point suggest that empirical data might be needed when using the method described. So we should keep in mind the question, 'Can one, using Rayo's method, achieve *a priori* knowledge in pure mathematics?'

Now, the idea that one might learn mathematical truths by deducing them from explicit and implicit definitions is hardly original to Rayo. It has been defended, for example, by Reichenbach and (arguably) by Carnap.[20] What's novel in Rayo's book is a defence of this position against a certain now standard objection, which I will presently explain. It's interesting to note that in earlier work, Rayo himself used a version of this objection when criticising a position rather like postulationism.[21] Let's take a look at the objection.

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[17] See in particular pg. 185, where Rayo indicates that he's working on an 'account of mathematical knowledge'.

[18] The claim would be mistaken because there have been many cases in the history of mathematics in which theorems were established *before* adequate definitions of the relevant terms were given. For example, much was known about continuity before Bolzano gave the first rigorous definition of 'continuous' in 1817. As Lakatos put it, we should reject the claim that 'the logic of discovery is deduction' [Lakatos 1976, pg. 143].

[19] See in particular pg. 185, where Rayo is conspicuously non-committal on this question.

[20] Reichenbach [1924]; Carnap [1934/1937].

[21] See Rayo [2003], which is a critique of Crispin Wright's neofregeanism (for which see Hale and Wright [2001]). See section 3.2 of Rayo's book for his current take on neofregeanism.

Suppose that someone introduces the name 'Goliath' stipulatively using this sentence as a definition:

> Goliath is a horse at least 10cm taller than any other horse.

It is obvious that this definition will succeed (that is, become true) only if there exists a horse at least 10cm taller than any other. If such a horse does not exist, the definition will 'fail': the term 'Goliath' will be an empty name, and the definition will be either false or truth-value-less. We could say that the 'success condition' of the definition is that there exists a horse at least 10cm taller than any other. More generally, the success condition of a definition is the condition that needs to be met in order for the definition to succeed.

It seems that in order to know that one's definition is true, one must know *independently* that its success condition is met. To see what I mean by 'independently', imagine meeting someone who claims to know that the definition of 'Goliath' is true; you ask her if she knows that its success condition is met, and she says that she does; she tells you that the success condition of the definition is that there exists a horse at least 10cm taller than any other, and it is analytic that this condition is met, because this follows from the definition of 'Goliath'. This is surely absurd. One can't measure horses *a priori*.

Now let's apply this in the case of mathematics. Let's imagine an agent – 'Clare', say – who sets out to learn some mathematics using Rayo's method. We'll suppose that Clare defines some primitive number-theoretic vocabulary using some axioms as implicit definitions, perhaps the Peano axioms.[22] What is the success condition of Clare's definition?

Here's a natural line of thought. A definition will succeed only if there exist suitable referents for any newly introduced singular terms – for example, the success of the definition of 'Goliath' was contingent on the existence of a suitably tall horse to act as referent for the name. We can suppose that Clare's new singular terms include:

> $0, 0', 0'', 0''', 0'''', \ldots$

Now it's not quite clear what some things would have to be like in order to be 'suitable' referents for these terms, but it does seem that no two of these singular terms should be assigned the *same* referent. And so it seems that the definition can succeed only if there exist infinitely many things. And so, apparently, the success condition of Clare's definition is at least that there exist infinitely many things.

[22] For simplicity, I assume that she uses a multi-sorted language, with a new style of variables and new quantifiers to range over the natural numbers. Also, I suppose that her axioms contain no vocabulary other than logical vocabulary and the new number-theoretic vocabulary, and that only the new number-theoretic quantifiers are used in the theory.

So in order to know that her definition has succeeded, Clare must establish *independently* that there exist infinitely many things. In order to know *a priori* that the definition has succeeded, Clare would have to establish independently and *a priori* that there exist infinitely many things. And it is not clear how Clare could do this.

Now it might be replied that Clare can establish independently and *a priori* that infinitely many things exist, because this is an implication of set theory (or some other mathematical theory), which is *a priori*. This is not a very helpful response, for it leaves us with the task of explaining how knowledge (*a priori* or otherwise) in this other branch of mathematics is possible.[23] It would not do to reply that knowledge in this other branch of mathematics can be achieved by deducing theorems from implicit and explicit definitions – that would be the first step in an unending regress.

Alternatively, it might be replied that Clare can defend the claim that infinitely many things exist by showing empirically that there exist infinitely many *physical* things – say, by showing that there exist infinitely many spacetime-points. I will return to this idea in §5, but *prima facie* this approach is rather unattractive. First, if Rayo takes this line he will be left without a defence of the claim that pure mathematical knowledge is *a priori.* Second, it's really not clear that we are justified in believing that there are infinitely many physical things: so in taking this line, Rayo would make the success of his epistemology of mathematics dependent on a difficult empirical claim, which would require its own defence.

This is the 'infinity problem'. Let's take a look at Rayo's response.

According to Rayo, the infinity problem results from a faulty understanding of how definitions work. For Rayo, the function of a definition is *not* to assign referents to the newly introduced terms; rather, the function of a definition is to assign truth conditions (thought of as sets of possible worlds) to whole sentences containing the new terms. A definition will succeed, Rayo claims, provided that a suitable assignment of truth conditions to the newly introduced sentences exists.

So Rayo must address the question of what it takes for an assignment of truth conditions to be 'suitable'. Let's assume that L is the set of sentences in Clare's 'old' language, sentences that do not contain the new terms, and that L^+ is the set of sentences in Clare's new, extended language. We can assume that there already exists an assignment J of truth conditions to the sentences in L; we want to know what a 'suitable' assignment J^+ of truth conditions to the sentences in L^+ would have to look like. Rayo supposes that any assignment which meets four conditions will be suitable.

[23] In addition, if Clare already knows set theory, she can introduce her number-theoretic vocabulary just using explicit definition. In this case, implicit definition is otiose.

Here's the first condition:[24]

(C1) If δ is one of the new definitions, $J^+(\delta)$ is the set of all possible worlds.

Now this might not be an appropriate condition for all definitions. For example, if one defines 'Jack the Ripper' using the sentence, 'Jack the Ripper is the man who murdered five women in Whitechapel in 1888', one would presumably not intend one's definition to be *necessarily* true. But in our case, I think condition (C1) is appropriate. We can assume that Clare intends her axioms to be true at all possible worlds. The second condition is also straightforward:[25]

(C2) For any φ in L, $J(\varphi)=J^+(\varphi)$.

This condition is motivated by the thought that, in introducing her new terms, Clare does not intend to change the meanings of any of her existing sentences. She is *extending* her language, without changing those parts of the language which already exist. Next:

(C3) J^+ may be a partial function, but it must assign a truth condition to all sentences that Clare 'wishes to make available for use.'[26]

For example, Clare may have no use for the sentence '2=Julius Caesar',[27] in which case J^+ need not assign any truth condition to this peculiar sentence. However, any sentence that Clare intends to use should have a truth condition. Finally:[28]

(C4) J^+ should 'respect logical consequence' in the following sense. If Γ is a subset of L^+, and J^+ is defined at each element of Γ, and possible world $w \in J^+(\gamma)$ for each $\gamma \in \Gamma$, and if φ is a logical consequence of Γ, then J^+ is defined at φ and $w \in J^+(\varphi)$.

It is very plausible that these four conditions are *necessary* for the success of Clare's definition. What is less clear is whether they are jointly *sufficient*. I'll return to this question later.

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24 This is condition (a) on [pg. 183].

25 Rayo does not state this condition explicitly, but it is clearly assumed throughout. This is especially clear in the parenthetical remark at the end of the antepenultimate paragraph of section 8.2.3.

26 This is condition 2 from [pg. 181].

27 This is an allusion to Frege [1884/1974: pg. 68].

28 This is condition 3, on [pg. 181]. (I've adapted Rayo's condition slightly, but as far as I can tell the change makes no relevant difference).

Rayo shows that an interpretation meeting these four conditions will exist provided that Clare's axioms are 'internally coherent' [pg. 183].[29] So Rayo concludes that the success condition of Clare's definition is no stronger than the claim that her axioms are internally coherent. Plausibly, this is something that Clare can confirm *a priori*, and so the infinity problem is apparently solved.

The resulting position is reminiscent of Hilbert:

> if the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist. This is for me the criterion of truth and existence.[30]

So according to Rayo, when Clare stipulates that her axioms are to be necessary truths, her stipulation *will* succeed provided that the axioms are internally coherent. Clare can know that her axioms are true provided she can establish that they are coherent. And she can know *a priori* that her axioms are true provided that she is able to establish *a priori* that they are coherent. She does not, however, need any independent way of showing that there exist enough objects to serve as referents for her newly introduced terms.

One feature of this position deserves particular emphasis. Rayo thinks that in order to come to know that her axioms are true, Clare must *first* show that they are coherent. Rayo might, on the contrary, have adopted the position that as long as her axioms are *in fact* coherent, she will be justified in thinking that they are true without having first to *check* that they are coherent, at least in the absence of a defeater. According to this latter position, her axioms are 'innocent until proven guilty', as it were. I call this 'the externalist approach'. I think that Rayo was right to reject the externalist approach, and I'd like to finish the section with an (all too brief!) explanation of this point.[31]

Yitang Zhang recently proved that there exist infinitely many pairs of prime numbers that differ by no more than seventy million[32] – a major step forward in the search for a proof of the twin primes hypothesis. Now according to the externalist approach, mathematicians could have established the truth of this statement decades ago, by just including it as one of the axioms of number theory. This is surely wrong: mathematical knowledge is not so easily acquired. So it seems that Rayo is correct to insist that one must establish that one's axioms are coherent before one knows them to be true.[33]

[29] I'll sometimes omit 'internally' for stylistic reasons.
[30] This is from a letter to Frege, 29th December 1899. See McGuinness and Kaal [1980: pg. 39].
[31] Rayo himself skillfully criticizes the externalist approach (not under this name) in Rayo [2003].
[32] See Zhang (Forthcoming). Actually, Zhang's theorem is rather stronger than the result stated.
[33] See Ebert and Shapiro [2009] for a similar argument in a difference context. Also see Boghossian [2003].

4. But what is coherence?

As we've seen, the following thesis is crucial to Rayo's epistemology:

> *The Rayovian Thesis*
> As long as Clare's axioms are coherent, there will exist a 'suitable' interpretation for her newly extended language.

There are several different ways of understanding the term 'coherent'.[34] We will see that on one natural interpretation on the term, *The Rayovian Thesis* is just false. There are other interpretations of the term on which *The Rayovian Thesis* seems to be true, but these interpretations raise other problems for Rayo's view, as we will see.

I will proceed by looking a number of different interpretations of the term 'coherent' in turn.

Option One: Coherence as proof-theoretic consistency

First, let's consider the suggestion that some sentences are coherent just in case they are 'consistent' in the sense of proof theory: a set of sentences Γ is 'consistent' (in this proof-theoretic sense) just in case it is not the case that there exists a proof of $\bot$ from premises drawn from Γ.[35] Will this interpretation of 'coherent' work out for Rayo?

It won't: it turns out that the Rayovian thesis is false on this interpretation of 'coherent'. Let $\alpha_1,\ldots,\alpha_N \vdash_T \beta$ mean 'β is provable from premises $\alpha_1,\ldots, \alpha_N$ in formal system *T*' and suppose that the property of *being a proof in T* is effectively decidable. Then assuming that the second-order Peano axioms ('PA^2') is consistent in T, there will exist a formula $\varphi(x)$ in the language of second order arithmetic in which only x occurs free, such that:

(i) $PA^2 \vdash_T \varphi(\underline{n})$, for each numeral $\underline{n}$.

(ii) $PA^2 \nvdash_T \forall x\, \varphi(x)$.

Now consider the set $PA^* = PA^2 \cup \{\neg\forall x\varphi(x)\}$. By (ii), $PA^* \nvdash_T \bot$, and so PA* is consistent in *T*, and hence 'coherent' in the current sense. Nevertheless, PA* doesn't have a suitable interpretation. For suppose

[34] There are also several ways of interpreting the term 'logical consequence' in condition (C4). It is clear from the argument on pg. 185 that Rayo's intention is that a set of sentences is coherent just in case $\bot$ is not a logical consequence of the set. I will make this assumption in what follows.

[35] Strictly speaking, of course, there are many different proof-theoretic notions of consistency, corresponding to different formal deductive systems.

that J^{+} meets conditions (C1)-(C4). Then by (C1), $J^{+}(\neg\forall x\varphi(x))$ is the set of all possible worlds, so by (C4), $J^{+}(\forall x\varphi(x))=\emptyset$. But by (i), and conditions (C1) and (C4), $J^{+}(\varphi(\underline{n}))$ is the set of all possible worlds, for each numeral $\underline{n}$. But this violates what ought to be a fifth constraint on suitability:

$$J^{+}(\ulcorner \forall x\, \varphi(x) \urcorner) = J^{+}(\varphi(0)) \cap J^{+}(\varphi(0')) \cap J^{+}(\varphi(0'')) \cap J^{+}(\varphi(0''')) \cap \ldots$$

J^{+} fails to respect the intended range of the number-theoretic quantifiers. So it is false that whenever a set of axioms is consistent in *T*, the set has a suitable interpretation.

Option Two: Coherence as consistency in the informal sense

Mathematicians frequently ask questions of the form 'Is such-and-such provable?' or 'Is such-and-such provable without such-and-such assumption?' without having any particular formal system of proof in mind. When a mathematician does this, it would be gratuitous to assume that her question must be understood as making some tacit reference to a particular formal system. So there's a notion of provability—let's call it 'informal provability'—that isn't captured by the proof-theoretic definitions. Now there may be some formal system *T* such that provability in *T* coincides with informal provability, but if so this is an important and interesting feature of system *T,* not a triviality. Having introduced the notion of informal provability, we can say that a set of sentences Γ is consistent in the informal sense just in case a contradiction is not provable, in the informal sense, from Γ. I will now consider the suggestion that Rayo's 'coherence' is consistency in the informal sense.

Consider again *The Rayovian Thesis*:

> *The Rayovian Thesis*
> As long as Clare's axioms are coherent, there will exist a 'suitable' interpretation for her newly extended language.

I showed a moment ago that this thesis is false if 'coherent' is understood as meaning 'consistent in *T*' for some formal system *T* that meets certain reasonable conditions. The same argument establishes that the thesis is false if 'coherent' means 'consistent in the informal sense' and informal provability *coincides* with provability in some such formal system. So in order to maintain *The Rayovian Thesis* on this interpretation, Rayo will have to commit himself to the view that informal provability outstrips provability in any formal system. This is by no means absurd: but it is a substantial theoretical commitment, which would require defence.

Option Three: Coherence as Model-Theoretic Satisfiability

A set of sentences Γ is 'satisfiable' in the model-theoretic sense just in case there is some model $\mathfrak{M}$ (for the relevant language) such that every element of Γ is true at $\mathfrak{M}$. Perhaps Rayo's 'internally coherent' means *satisfiable*?

On this interpretation of 'coherent', the Rayovian thesis seems very plausible. But this interpretation of Rayo's position gives rise to another problem. Consider Clare again – and suppose that her axioms are the Peano axioms. Rayo's position, as I've said, is that Clare will only come to know that her axioms are true if she first establishes that they internally coherent. On the current interpretation, this means that in order for Clare to know that the Peano axioms are true she must first establish that they have a model. But the claim that the Peano axioms have a model is already a substantial mathematical claim, and we're left with the task of explaining how Clare could establish it. It won't do to say that Clare could derive this model-theoretic claim from implicit and explicit definitions – that would be the first step of an unending regress.

The problem is particularly acute because showing that a model exists for the Peano axioms would involve showing that there exist infinitely many things – so this approach just reintroduces the infinity problem in a new form.

Rayo might reply:

> My goal in chapter 8 was not to describe a method for achieving mathematical knowledge *ab initio:* more modestly, I was attempting to describe a method for learning one mathematical theory, given pre-existing knowledge of some other mathematical theory or theories.

If this is the extent of Rayo's ambition in the chapter, then I have no objection to the position he describes. However, on this reading, Rayo fails 'to explain how mathematical knowledge is possible' [pg *ix*] which is one of the stated goals of the book. To achieve this goal, it does not suffice to explain how one can extend a pre-existing body of mathematical knowledge. An analogy might help. Suppose someone offers the following as an explanation of the possibility of mathematical knowledge:

> One can obtain mathematical knowledge by deduction: one simply deduces new results from statements already known.

While it is true that one can extend one's mathematical knowledge by deduction, this is hardly a satisfactory explanation of 'how mathematical knowledge is possible'.

Option Four: Coherence as Informal Satisfiability

Suppose you are asked to establish that the following theory is coherent:

$\forall x\, Rxx \qquad \forall x \forall y \forall z[(Rxy \wedge Ryz) \rightarrow Rxz]$

You might respond by offering an 'interpretation' on which both sentences in the theory are true. For example, you might point out that both sentences are true relative to an interpretation on which the quantifiers range over people, and 'R' means *is at least as old as*. Notice that in giving this interpretation you do not commit yourself to the existence of a *set* of people, or indeed any set. So this interpretation is not a model in the normal sense. The term 'interpretation' does not have a formal definition, but informally the idea is this: to specify an interpretation for a language, you have to identify some objects for the quantifiers to range over, you have to choose referents for the singular terms, and you have to say which objects satisfy the various predicates in the language. We can say that some sentences are 'informally satisfiable' if there is an interpretation on which all of the sentences are true.[36]

Now let's consider a version of Rayo's position on which 'internal coherence' is informal satisfiability. On this version of the view, in order for Clare to establish that her definition has succeeded, she would have first to establish that there is an interpretation on which her axioms are all true. It's not at all clear how Clare could do this. Suppose, for example, that her axioms are the Peano axioms. Then specifying an interpretation for the axioms would involve identifying *infinitely many* objects – to serve as referents for the singular terms, to form a domain of quantification, and to form extensions for the predicates. So, on this view, if Clare is to establish that her axioms are true she must first show that there exist infinitely many things. And if she is to establish *a priori* that her axioms are true, she must first show *a priori* that there exist infinitely many things. This introduces a new version of the infinity problem.

If Clare already has sufficient mathematical knowledge, she may be able to use this knowledge to establish that her new axioms, the Peano axioms, are informally satisfiable. For example, if she already knows ZFC set theory she can interpret the Peano axioms in the universe of sets. But then again we are stuck with the question of how Clare could establish this pre-existing body of mathematical knowledge. Once again, we are threatened with a regress.

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[36] Informal satisfiability is a close relative of Kriesel's 'informal validity': see Kreisel [1972] and Smith [2011]. $S_1,\ldots,S_n$ are informally satisfiable just in case $\neg(S_1 \wedge \ldots \wedge S_n)$ is not informally valid.

Alternatively, it might be said that Clare can establish that her axioms are informally satisfiable by interpreting them in some well establish physical theory. For example, she could 'identify' the natural numbers with an ω-sequence of spacetime points. I will return to this idea in §5, but *prima facie* this proposal is unsatisfactory. It's not clear that we are justified in believing that there exist infinitely many physical things, so if Rayo takes this approach he makes his epistemology of mathematics dependent on a difficult empirical claim, which would require independent defence. Second, even if the approach is workable in the case of Peano arithmetic, it seems unlikely to work for richer theories, such as second-order Zermelo-Fraenkel set theory. It also worth noting that if Rayo takes this line, he will be left without a defence of the claim that mathematical knowledge is *a priori*.

Option Five: Primitivism about logical necessity

Hartry Field[37] has recently been defending the view that the concept of logical necessity (represented with a box '☐') should be used as a primitive: that is, we should continue to use the term, but we should not attempt to define it. Now given this primitive concept, we can introduce a concept of coherence (at least for finite sets of sentences) using the following schematic definition:

$\{S_1, \ldots, S_n\}$ is coherent iff $\neg\Box\neg(S_1 \wedge \ldots \wedge S_n)$

I don't have much to say about this proposal, except that if Rayo takes this approach he is left with two *substantial* tasks to carry out. First, he would have to defend the Rayovian thesis understood in this new way. We've seen that this thesis is false on one way of understanding the term 'coherent'; it's not obvious that the thesis is true if 'coherent' is defined as suggested above using a primitive notion of logical necessity. Second, Rayo would have to explain how Clare could establish that (say) the Peano axioms are 'coherent' in this sense, without drawing upon some pre-existing body of mathematical knowledge. I do not claim that these tasks are impossible – perhaps Rayo could make this view work – but I do claim that further theorizing is required.

5. Towards a solution to the coherence problem.

I've now explained my criticisms of Rayo's account as it stands. I'd like to finish, very briefly, by recommending a way forward for Rayo.

[37] See for example Field [2008: pp. 47-8]. See Smith [2011] for criticisms of Field's argument.

Rayo's thinks that the following method is knowledge-producing:

> *Method 1:* Suppose that some set of mathematical terms are entirely defined by definitions $D_1,\ldots,D_n$, and that no other non-logical terms occur in $D_1,\ldots,D_n$. Then if you can show that $D_1,\ldots,D_n$ are coherent, you may infer that they are all true.[38]

In the last section, I discussed various interpretations of 'coherent'. I now suggest that Rayo identify coherence with informal satisfiability (that's 'option four'). The problem with Rayo's method, on this interpretation, is that it is not clear how someone not already equipped with a good deal of mathematics could establish that her definitions are coherent.

I suggest that Rayo should respond to this difficulty by specifying further methods of acquiring logico-mathematical knowledge. Each such method would no doubt require considerable discussion, but for now I will do no more than provide a rough-and-ready list of suggestions.

Given the informal semantic account of 'coherence', one can establish that a set of sentences is consistent by defining an interpretation relative to which they are all true; hence:

> *Method 2:* Suppose that *S* is a set of sentences, and that *T* is a theory you know to be true. Then if you can interpret S in *T*, you may infer that *S* is coherent.

The term 'interpret' here is meant to be understood as it is used in proof-theory. For example, someone who already knew the axioms of ZF to be true might establish that the Peano axioms are coherent by 'interpreting' those axioms in ZF (by identifying the natural numbers with, say, the finite von Neumann ordinals). Another example: using *Method 2* one might (or might not!) be able to establish that the axioms of Peano arithmetic are coherent by interpreting them within an empirically well-confirmed theory of space-time, by identifying the natural numbers with an ω-sequence of space-time points. Next:

> *Method 3:* Suppose that *S* is a set of sentences. Then if you can deduce $\bot$ from *S*, you may conclude that *S* is incoherent.

For example, Russell showed that the axioms presented in Frege's *Grundgesetze* are incoherent by deriving $\bot$ from them. Next:

> *Method 4:* Suppose that you try, and repeatedly fail, to deduce $\bot$ from a set *S*. Then with repeated attempts you accumulate evidence that *S* is in fact coherent.

In using both *Method 3* and *Method 4*, one is in effect using consistency in the informal sense as a guide to informal satisfiability. Next, a Quinean proposal:

> *Method 5:* If a mathematical theory *T* plays an indispensable role in an empirically well-confirmed scientific theory, one has good reason for believing that *T* is true.[39]

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[38] In order to deal with cases of mathematical theories which entail that there exist only a limited number of things, we should add here as an extra condition that the quantifiers in $D_1,\ldots,D_n$ are appropriately restricted. For example, the quantifiers in one's theory of the natural numbers might be restricted so that they range over only the numbers.

[39] See Colyvan [2001] for an interpretation and defence of Quine's views about indispensability. See also Putnam [1972] for another version.

The next method is inspired by the work of Charles Parsons:[40]

> *Method 6:* One can establish truths about *types* of strings of symbols by experimenting (both on paper and in imagination) with *token* strings of symbols. One can then establish the coherence of Peano arithmetic by interpreting it within one's theory of string-types – for example by 'identifying' the natural numbers with the following sequence of strings:
>
> 1, 11, 111, 1111, 11111, …

I have one more method to add:

> *Method 7:* Changes to one's overall system of logico-mathematical beliefs can be justified on grounds of simplicity, elegance etc..[41]

As I say, each of these methods require a great deal of discussion. But my very tentative suggestion is that these methods together could be used to achieve a good deal of mathematical knowledge. This is, at least, a position which Rayo might like to explore. A couple of consequences of the view require mention. First, notice that on this view even rather basic mathematical knowledge might turn out not to be *a priori*. Second, notice that on this view there is no priority of logical knowledge to mathematical knowledge.

6. Summary of Conclusions

- In his discussions of 'trivialist Platonism', Rayo provides a response to the 'Benacerraf problem' or 'access problem' for Platonists. However, he does not in these discussions provide any convincing positive argument for Platonism.
- Rayo provides a convincing defence of the claim that this method of belief-formation produces knowledge:

> Suppose that some set of mathematical terms are entirely defined by definitions $D_1,\ldots,D_n$, and that no other non-logical terms occur in $D_1,\ldots,D_n$. Then if you can show that $D_1,\ldots,D_n$ are coherent, you may infer that they are all true.

- However, it is by no means clear how someone who doesn't already have a good deal of mathematical knowledge could establish that (say) the axioms of Peano arithmetic are coherent.
- I suggest that Rayo should respond to the problem just described by specifying *other* methods for acquiring logico-mathematical knowledge.

[40] See in particular Parsons [1979] and Parsons [2009]. Thanks are due to Sofia Ortiz and Michael Friedman for suggesting that this method be included.

[41] A great deal of Penelope Maddy's work could be read as a demonstration of the power of 'Method 7'. See for example her famous papers Maddy [1988a] and [1988b].

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