Foundation of a Rigorous Implication

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This paper presents an English translation of the work titled "Begründung einer strengen Implikation" [1] by the German logician and mathematician Wilhelm Ackermann (1896–1962), first published in *The Journal of Symbolic Logic* (Vol. 21, No. 2, pp. 113–128) in June 1956.

The phrase "rigorous implication" translates Ackermann's German phrase *strenge Implikation*. Belnap and Anderson [3] adopted this translation based on a suggestion from Robert Feys. Ackermann claimed that his system formalizes the properties of a robust and natural form of implication, avoiding implicational paradoxes, and considered it as a "narrower" relation than strict implication. The choice of the German word "strenge" and its English translation, "rigorous", as proposed by Feys, is grounded in this rationale.

The German term "Begründung" encompasses both "justification" and "foundation". In this translation, the preference for "foundation" is motivated by its ability to convey a more comprehensive concept. While "justification" may emphasize providing reasons or rationale for a rigorous implication, "foundation" not only includes justification but also extends to encompass the systematic construction of the logical framework. Essentially, "foundation" provides a more precise and perspicuous idea of the technical development within Ackermann's paper.

In this translation, I have preserved Ackermann's original notation. The following table provides a summary:

Symbol	Meaning
$A, B, C, \ldots$	propositional formulas
$\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$	compound formulas
$\&, \vee, \overline{\cdot}$	conjunction, disjunction, negation
$\supset$	material implication
$\rightarrow$	rigorous implication
⅄	the 'absurd', falsity constant
N, M, U	necessary, possible, impossible
$\forall, \exists$	universal and existential quantifiers

The numbering of theorems, derivations, and proofs is in line with Ackermann's. Any additional information introduced for clarity and readability is enclosed in square brackets [...]. Finally, no changes or additions have been made to the references originally selected by Ackermann. The pages of the original German article are also indicated in square brackets, highlighted in red.

An overview of Ackermann's paper

§1. Introduction: Ackermann presents the motivation for introducing his rigorous implication (*strenge Implikation*), specifically the need to establish a conditional that exhibits a "logical connection" (*logischer Zusammenhang*) between the antecedent and the consequent, in order to avoid the paradoxes of both the material conditional and Lewis' strict implication (cf. also [2]).

§2. A system for the [classical] propositional calculus: In the second section of his paper, Ackermann presents two systems for classical propositional logic. First, he introduces the system Σ, which consists of a set of "basic (or inferential) schemas" (*Grundschemata* or *Ableitungsschemata*) and "inferential rules" (*Ableitungsregeln*) designed to construct specific derivations. Second, he provides an axiomatic formulation, denoted as Π, consisting of what Ackermann terms *Grundformeln* ("basic formulas"). The motivation for introducing these two formulations – particularly the structure of Σ – is to facilitate the transformation of this system into one that omits $\supset$ but incorporates $\rightarrow$.

§3. The RI-calculus: In the third section, the reader is introduced to the **RI** logic, specifically to the two calculi featuring rigorous implication. Analogous to the systems for classical logic, Ackermann presents and examines the calculus Σ' (comprising both basic schemas and inferential rules) and the axiomatic formulation Π' for the **RI** logic.

§4. Equivalence of Systems Σ' and Π' of §3: In the fourth section, Ackermann establishes the equivalence of the two formulations of the **RI** logic, namely Σ' and Π'. The theorem demonstrated is as follows: if a formula is provable in Π', then it is derivable in Σ', and conversely, if a formula is derivable in Σ', it is also provable in Π'. In the concluding remarks of the section, Ackermann observes that Π' contains the entirety of the two-valued propositional calculus (*enthält den vollen zweiwertigen Aussagenkalkül*), meaning that all classically valid formulas constructed using the three connectives &, $\vee$, and $\overline{\cdot}$ are derivable. He further notes that disjunction can be defined in terms of conjunction, or vice versa, in the standard way, and that, consequently, the system Π' could be simplified by omitting certain axioms.

§5. Introducing the modalities: The fifth section of Ackermann's paper addresses two main topics: the introduction of the falsity constant, "$\curlywedge$", referred to as the "absurd" (*das Absurde*) and its subsequent use in presenting the modalities. Firstly, Ackermann introduces the "absurd", drawing inspiration from I. Johansson's treatment of negation in the minimal calculus. He extends Σ' by incorporating an additional inference rule and Π' by introducing two axioms and a supplementary rule. Secondly, Ackermann defines the modal concepts of necessary (*notwendig*), possible (*möglich*) and impossible (*unmöglich*) using rigorous implication and the falsity constant (with negation included where necessary). The section concludes with a discussion on the potential extension of modalities to aid in the formalization of theories requiring notions of necessity that go beyond mere logical necessity.

§6. Final remarks: The final section of Ackermann's paper includes a brief conclusion, presenting a proof of the invalidity of formulas that do not adhere to the criterion of logical connection between antecedent and consequent, using five-valued matrices. The unprovable formula in question is $\mathfrak{A} \to (\mathfrak{B} \to \mathfrak{C})$, where $\mathfrak{A}$ contains neither $\to$ nor $\curlywedge$. The corresponding truth tables are arranged as follows (p. 18 and [1, 127], [6, 136]):

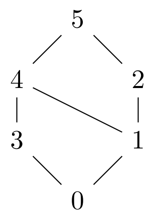
	$\overline{\cdot}$
0	5
1	4
2	3
3	2
4	1
5	0

&	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	0	1	1
2	0	1	2	0	1	2
3	0	0	0	3	3	3
4	0	1	1	3	4	4
5	0	1	2	3	4	5

$\vee$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	2	4	4	5
2	2	2	2	5	5	5
3	3	4	5	3	4	5
4	4	4	5	4	4	5
5	5	5	5	5	5	5

$\to$	0	1	2	3	4	5
0	3	3	3	3	3	3
1	0	3	3	0	3	3
2	0	0	3	0	0	3
3	0	0	0	3	3	3
4	0	0	0	0	3	3
5	0	0	0	0	0	3

Each propositional formula is assigned the value 1 and the constant $\curlywedge$ is assigned the value 2; & and $\vee$ correspond to the greatest lower bound and least upper bound, respectively. The resulting Hasse diagram looks as follows:



In terms of matrices, the falsification of $\mathfrak{A} \to (\mathfrak{B} \to \mathfrak{C})$ (p. 18 and [1, 128]) is given as follows:

$\mathfrak{A}$	$\mathfrak{B}$	$\mathfrak{C}$	$\mathfrak{B} \to \mathfrak{C}$	$\mathfrak{A} \to (\mathfrak{B} \to \mathfrak{C})$	$\mathfrak{A}$	$\mathfrak{B}$	$\mathfrak{C}$	$\mathfrak{B} \to \mathfrak{C}$	$\mathfrak{A} \to (\mathfrak{B} \to \mathfrak{C})$
1	0	0	3	0	4	0	0	3	0
1	0	1	3	0	4	0	1	3	0
1	0	2	3	0	4	0	2	3	0
1	0	3	3	0	4	0	3	3	0
1	0	4	3	0	4	0	4	3	0
1	0	5	3	0	4	0	5	3	0
1	1	0	0	0	4	1	0	0	0
1	1	1	3	0	4	1	1	3	0
1	1	2	0	0	4	1	2	0	0
1	1	3	0	0	4	1	3	0	0
1	1	4	0	0	4	1	4	0	0
1	1	5	0	0	4	1	5	0	0
1	2	0	0	0	4	2	0	0	0
1	2	1	0	0	4	2	1	0	0
1	2	2	3	0	4	2	2	3	0
1	2	3	0	0	4	2	3	0	0
1	2	4	0	0	4	2	4	0	0
1	2	5	0	0	4	2	5	0	0
1	3	0	0	0	4	3	0	0	0
1	3	1	0	0	4	3	1	0	0
1	3	2	0	0	4	3	2	0	0
1	3	3	3	0	4	3	3	3	0
1	3	4	3	0	4	3	4	3	0
1	3	5	3	0	4	3	5	3	0
1	4	0	0	0	4	4	0	0	0
1	4	1	0	0	4	4	1	0	0
1	4	2	0	0	4	4	2	0	0
1	4	3	0	0	4	4	3	0	0
1	4	4	3	0	4	4	4	3	0
1	4	5	3	0	4	4	5	3	0
1	5	0	0	0	4	5	0	0	0
1	5	1	0	0	4	5	1	0	0
1	5	2	0	0	4	5	2	0	0
1	5	3	0	0	4	5	3	0	0
1	5	4	0	0	4	5	4	0	0
1	5	5	3	0	4	5	5	3	0

Note that a logic in which $\mathfrak{A} \to (\mathfrak{B} \to \mathfrak{C})$ is unprovable was later characterized by Anderson and Belnap as exhibiting the "Ackermann Property" (cf. [4, 95–96]).

Finally, Ackermann concludes the paper with a short discussion on the potential extension of the **RI** logic to the first-order level, proposing additional axioms to be incorporated into Π'.

References

[1] Ackermann, W. (1956). Begründung einer strengen Implikation. *The Journal of Symbolic Logic*, *21*(2), pp. 113–128.

[2] Ackermann, W. (1958). Über die Beziehung zwischen strikter und strenger Implikation. *Dialectica*, *12*(3–4), 213–222.

[3] Belnap, N.D. and Anderson, A.R. (1959). Modalities in Ackermann's 'Rigorous Implication'. *The Journal of Symbolic Logic*. *24*(2), 107–111.

[4] Belnap, N.D. and Anderson, A.R. (1975). *Entailment: The Logic of Relevance and Necessity* (Vol. 1). Princeton: Princeton University Press.

[5] Belnap, N.D., Anderson, A.R. and Dunn, J.M. (1992). *Entailment: The Logic of Relevance and Necessity* (Vol. 2). Princeton: Princeton University Press.

[6] Belnap, N.D., Anderson, A.R. and Dunn, J.M. (1992). Ackermann's *strenge Implikation.* In [5, Chapter VIII, 129–141].

English Translation

Ackermann, W. (1956). Begründung einer strengen Implikation. *The Journal of Symbolic Logic*, *21*(2), 113–128.

[113]

Foundation of a Rigorous Implication

Willhelm Ackermann

1 Introduction

The motivations that led C. I. Lewis [5], [6] to introduce strict implication in addition to the usual implication are well known. In a similar vein, this paper presents a rigorous implication that represents a concept narrower than that of strict implication.

My work shares little in common with that of Arnold Schmidt [7], as he focuses on strict implication. For strict implication, a relatively simple axiom system is provided, demonstrating how familiar modal systems can be derived through appropriate definitions of necessity and possibility, and, if necessary, by adding further axioms.

The primary focus of my work is the introduction of the concept of rigorous implication; building on this foundation, the second part of the work will introduce the modalities.

To motivate the manner in which the calculus is introduced, we will briefly outline what we have in mind when we refer to rigorous implication. The rigorous implication, represented as $A \to B$, is intended to convey that there is a logical connection between A and B. It signifies that the content of B is part of the content of A, or however one might express it. This has no bearing on the correctness or incorrectness of A and B. Consequently, one would reject the general validity of the formula $A \to (B \to A)$, as it includes the inference from A to $B \to A$, and the correctness of A has no relation to whether there is a logical connection between B and A. For the same reasons, one would not consider the formulas $A \to (B \to A \,\&\, B)$, $A \to (\overline{A} \to B)$, and $A \to ((A \to B) \to B)$ to be universally valid. The same applies to $B \to (A \to A)$, as $A \to A$ is valid independently of the correctness of B.

In rejecting the last formula, my implication differs from the strict one, just as it does in rejecting $(A \,\&\, \overline{A}) \to B$ as a universally valid formula, since the existence of a proposition that is implied by all or that implies all other propositions does not adequately represent the concept of implication as a logical connection between two propositions.

On the other hand, we would recognize a formula like $(A \to B) \to ((B \to C) \to (A \to C))$ as universally valid, because, given the assumption $A \to B$, the inference from $B \to C$ to $A \to C$ is logically compelling.

2 A system for the [classical] propositional calculus

First, we establish a new and systematic framework for [classical] propositional logic, beginning with the simplest rules, [114] in which the roles of each connective become clearly evident.

Although this calculus bears a certain formal resemblance to the sequent calculus of G. Gentzen [1], they differ significantly. The system we are about to introduce is particularly suitable for our considerations, as it can be easily adapted into a calculus for rigorous implication.

In what follows, A, B, C, and other capital Latin letters will be used as propositional variables, from which further formulas are constructed using "$\supset$" (follows), "&" (and), "$\vee$" (or), and "$\overline{\cdot}$" (not) in the usual way. $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, and other capital German letters denote any formula of this kind. To minimize the use of parentheses, we specify that "$\vee$" and "&" bind more tightly than "$\supset$", [where] "$\supset$" denotes the standard [material] implication, not rigorous implication, which does not appear at all in this section.

Rather than beginning with specific axioms, we start from some fundamental inference rules, for which we employ a specific symbolism.

We express the fact that one can infer $\mathfrak{B}$ from $\mathfrak{A}$ by $\mathfrak{A} \vdash \mathfrak{B}$. $\mathfrak{A}, \mathfrak{B} \vdash \mathfrak{C}$ denotes the inference of $\mathfrak{C}$ from $\mathfrak{A}$ and $\mathfrak{B}$. We also use $\vdash \mathfrak{A}$ to indicate that $\mathfrak{A}$ is, independently of any assumptions and for purely logical reasons, correct.

As basic inference rules, we consider the following:

(1) $\vdash \mathfrak{A} \supset \mathfrak{A}$
(2) $\mathfrak{A} \,\&\, \mathfrak{B} \vdash \mathfrak{A}$
(3) $\mathfrak{A} \,\&\, \mathfrak{B} \vdash \mathfrak{B}$
(4) $\mathfrak{A} \vdash \mathfrak{A} \vee \mathfrak{B}$
(5) $\mathfrak{B} \vdash \mathfrak{A} \vee \mathfrak{B}$

These inference schemas have one or no premise. Additionally, there are inference schemas with two (but no more) premises, which are interpreted in the sense described above. Furthermore, there are schemas where the conclusion is absent. These are to be interpreted as indicating that the two premises are not logically compatible, or, in the case of a single premise, that it leads to a contradiction.

We now derive several additional schemas from the basic schemas (1)–(5). In this manner, these additional schemas become part of a calculus, taking on the role typically played by formulas.

We define:

By a schema, we mean any of the following formal structures:

$$\mathfrak{A} \vdash \mathfrak{B}; \qquad \mathfrak{A}, \mathfrak{B} \vdash \mathfrak{C}; \qquad \mathfrak{A} \vdash; \qquad \mathfrak{A}, \mathfrak{B} \vdash; \qquad \vdash \mathfrak{A}.$$

The rules by which we derive further schemas from the basic schemas are as follows:

I*a*. If we have a provable schema with at least one premise and one conclusion, we can derive another provable schema by adding $\mathfrak{D} \supset$ in front of the conclusion and in front of one or all premises, where $\mathfrak{D}$ represents an arbitrary formula. For example, starting with the schema $\mathfrak{A} \supset \mathfrak{B}, \mathfrak{A} \vdash \mathfrak{B}$, we can derive the following schemas: (i) $\mathfrak{D} \supset (\mathfrak{A} \supset \mathfrak{B}), \mathfrak{A} \vdash \mathfrak{D} \supset \mathfrak{B}$, (ii) $\mathfrak{A} \supset \mathfrak{B}, \mathfrak{D} \supset \mathfrak{A} \vdash \mathfrak{D} \supset \mathfrak{B}$, and (iii) $\mathfrak{D} \supset (\mathfrak{A} \supset \mathfrak{B}), \mathfrak{D} \supset \mathfrak{A} \vdash \mathfrak{D} \supset \mathfrak{B}$. **[115]**

I*b*. If $\mathfrak{A} \vdash \mathfrak{B} \vee \mathfrak{C}$ is provable, then $\mathfrak{B} \supset \mathfrak{D}, \mathfrak{C} \supset \mathfrak{D} \vdash \mathfrak{A} \supset \mathfrak{D}$ is a new provable schema.

II. If $\Psi, \mathfrak{B} \vdash \mathfrak{C}$ and $\vdash \mathfrak{B}$ are provable, then $\Psi \vdash \mathfrak{C}$ is also provable.

III. From $\Psi \vdash \mathfrak{B} \supset \mathfrak{C}$, we obtain $\Psi, \mathfrak{B} \vdash \mathfrak{C}$.

In II, III, and the subsequent rules, Ψ is used to represent either the empty sign, a formula, or, depending on the context, a sequence of two premises. In II and III, it can only represent the empty sign or a formula.

IV. In a schema with two premises, their positions can be interchanged.

V. From $\mathfrak{A} \,\&\, \mathfrak{B} \vdash \mathfrak{C}$, we obtain $\mathfrak{A}, \mathfrak{B} \vdash \mathfrak{C}$ (where $\mathfrak{C}$ can also be absent).

VI*a*. In a schema, one may interchange a premise with the conclusion after marking both with the negation overline. This also applies if the conclusion or the relevant premise is empty. (That is, we allow the switch from $\mathfrak{A} \vdash$ to $\vdash \overline{\mathfrak{A}}$, from $\mathfrak{A}, \mathfrak{B} \vdash$ to $\mathfrak{A} \vdash \overline{\mathfrak{B}}$, from $\vdash \mathfrak{A}$ to $\overline{\mathfrak{A}} \vdash$, and from $\mathfrak{A} \vdash \overline{\mathfrak{B}}$ to $\mathfrak{A}, \mathfrak{B} \vdash$.)

VI*b*. In a schema, one can replace a premise $\mathfrak{A}$ with $\overline{\overline{\mathfrak{A}}}$, and vice versa.

We denote the system (1)–(5), I–VI as Σ.

We now aim to demonstrate that Σ is adequate for the [classical] propositional calculus in the sense that, for every universally valid formula $\mathfrak{A}$, $\vdash \mathfrak{A}$ is provable. For comparison, we utilize the axiom system of propositional calculus as established by Hilbert and Bernays [3, 66], where the role of each connective is clearly delineated. The following proof and the observations made by Hilbert and Bernays at the referenced location illustrate that Σ, without VI, is suitable for so-called positive logic or intuitionistic logic without negation.

By modifying the Hilbert-Bernays system to include axiom schemas, it will have the following basic formulas, which I will number as indicated in the specified reference:

I,1	$\mathfrak{A} \supset (\mathfrak{B} \supset \mathfrak{A})$,
I,2	$(\mathfrak{A} \supset (\mathfrak{A} \supset \mathfrak{B})) \supset (\mathfrak{A} \supset \mathfrak{B})$,
I,3	$(\mathfrak{A} \supset \mathfrak{B}) \supset ((\mathfrak{A} \supset \mathfrak{C}) \supset (\mathfrak{A} \supset \mathfrak{C}))$,
II,1	$\mathfrak{A} \,\&\, \mathfrak{B} \supset \mathfrak{A}$,
II,2	$\mathfrak{A} \,\&\, \mathfrak{B} \supset \mathfrak{B}$,
II,3	$(\mathfrak{A} \supset \mathfrak{B}) \supset ((\mathfrak{A} \supset \mathfrak{C}) \supset (\mathfrak{A} \supset \mathfrak{B} \,\&\, \mathfrak{C}))$,
III,1	$\mathfrak{A} \supset \mathfrak{A} \vee \mathfrak{B}$,
III,2	$\mathfrak{B} \supset \mathfrak{A} \vee \mathfrak{B}$,
III,3	$(\mathfrak{A} \supset \mathfrak{C}) \supset ((\mathfrak{B} \supset \mathfrak{C}) \supset (\mathfrak{A} \vee \mathfrak{B} \supset \mathfrak{C}))$,
V,1	$(\mathfrak{A} \supset \mathfrak{B}) \supset (\overline{\mathfrak{B}} \supset \overline{\mathfrak{A}})$,
V,2	$\mathfrak{A} \supset \overline{\overline{\mathfrak{A}}}$,
V,3	$\overline{\overline{\mathfrak{A}}} \supset \mathfrak{A}$.

The only inference rule is modus ponens. We first prove the following theorem for Σ:

Theorem H1. If $\Psi, \mathfrak{A} \vdash \mathfrak{B}$ is provable, so is $\Psi \vdash \mathfrak{A} \supset \mathfrak{B}$. [116]

Proof.

$$\dfrac{\dfrac{\Psi, \mathfrak{A} \vdash \mathfrak{B}}{\Psi, \mathfrak{A} \supset \mathfrak{A} \vdash \mathfrak{A} \supset \mathfrak{B}}\,(\mathrm{I}a) \qquad \dfrac{}{\vdash \mathfrak{A} \supset \mathfrak{A}}\,(1)}{\Psi \vdash \mathfrak{A} \supset \mathfrak{B}}\,(\mathrm{II})$$

II,1, II,2, III,1 and III,2 result from the schemas (2), (3), (4), (5), respectively. □

Theorem H2. $\mathfrak{A} \vdash \mathfrak{A}$ is provable.

Proof. From $\vdash \mathfrak{A} \supset \mathfrak{A}$ (1), one proves the proposition by using III.

[Derivation of] I,1:

$$\begin{array}{cl} \hline \mathfrak{A} \,\&\, \mathfrak{B} \vdash \mathfrak{A} & (2) \\ \hline \mathfrak{A}, \mathfrak{B} \vdash \mathfrak{A} & (\mathrm{V}) \\ \hline \vdash \mathfrak{A} \supset (\mathfrak{B} \supset \mathfrak{A}) & (\mathbf{H1}) \end{array}$$

[Derivation of] I,2:

$$\begin{array}{cl} \hline \mathfrak{A} \supset \mathfrak{B} \vdash \mathfrak{A} \supset \mathfrak{B} & (\mathbf{H2}) \\ \hline \mathfrak{A} \supset \mathfrak{B}, \mathfrak{A} \vdash \mathfrak{B} & (\mathrm{III}) \\ \hline \mathfrak{A} \supset (\mathfrak{A} \supset \mathfrak{B}), \mathfrak{A} \supset \mathfrak{A} \vdash \mathfrak{A} \supset \mathfrak{B} & (\mathrm{I}a) \\ \hline \mathfrak{A} \supset (\mathfrak{A} \supset \mathfrak{B}) \vdash \mathfrak{A} \supset \mathfrak{B} & (\mathrm{II}) \\ \hline \vdash (\mathfrak{A} \supset (\mathfrak{A} \supset \mathfrak{B})) \supset (\mathfrak{A} \supset \mathfrak{B}) & (\mathbf{H1}) \end{array}$$

[Derivation of] I,3:

$$\begin{array}{cl} \hline \mathfrak{B} \supset \mathfrak{C} \vdash \mathfrak{B} \supset \mathfrak{C} & (\mathbf{H2}) \\ \hline \mathfrak{B} \supset \mathfrak{C}, \mathfrak{B} \vdash \mathfrak{C} & (\mathrm{III}) \\ \hline \mathfrak{B} \supset \mathfrak{C}, \mathfrak{A} \supset \mathfrak{B} \vdash \mathfrak{A} \supset \mathfrak{C} & (\mathrm{I}a) \\ \hline \mathfrak{A} \supset \mathfrak{B}, \mathfrak{B} \supset \mathfrak{C} \vdash \mathfrak{A} \supset \mathfrak{C} & (\mathrm{IV}) \\ \hline \vdash (\mathfrak{A} \supset \mathfrak{B}) \supset ((\mathfrak{B} \supset \mathfrak{C}) \supset (\mathfrak{A} \supset \mathfrak{C})) & (\mathbf{H1}) \end{array}$$

[Derivation of] II,3:

$$\begin{array}{cl} \hline \mathfrak{B} \,\&\, \mathfrak{C} \vdash \mathfrak{B} \,\&\, \mathfrak{C} & (\mathbf{H2}) \\ \hline \mathfrak{B}, \mathfrak{C} \vdash \mathfrak{B} \,\&\, \mathfrak{C} & (\mathrm{V}) \\ \hline \mathfrak{A} \supset \mathfrak{B}, \mathfrak{A} \supset \mathfrak{C} \vdash \mathfrak{A} \supset \mathfrak{B} \,\&\, \mathfrak{C} & (\mathrm{I}a) \\ \hline \vdash (\mathfrak{A} \supset \mathfrak{B}) \supset ((\mathfrak{A} \supset \mathfrak{C}) \supset (\mathfrak{A} \supset \mathfrak{B} \,\&\, \mathfrak{C})) & (\mathbf{H1}) \end{array}$$

[Derivation of] III,3:

$$\begin{array}{cl} \hline \mathfrak{A} \vee \mathfrak{B} \vdash \mathfrak{A} \vee \mathfrak{B} & (\mathbf{H2}) \\ \hline \mathfrak{A} \supset \mathfrak{C}, \mathfrak{B} \supset \mathfrak{C} \vdash \mathfrak{A} \vee \mathfrak{B} \supset \mathfrak{C} & (\mathrm{I}b) \\ \hline \vdash (\mathfrak{A} \supset \mathfrak{C}) \supset ((\mathfrak{B} \supset \mathfrak{C}) \supset (\mathfrak{A} \vee \mathfrak{B} \supset \mathfrak{C})) & (\mathbf{H1}) \end{array}$$

[Derivation of] V,I:

$$\begin{array}{cl} \hline \mathfrak{A} \supset \mathfrak{B} \vdash \mathfrak{A} \supset \mathfrak{B} & (\mathbf{H2}) \\ \hline \mathfrak{A} \supset \mathfrak{B}, \mathfrak{A} \vdash \mathfrak{B} & (\mathrm{III}) \\ \hline \mathfrak{A} \supset \mathfrak{B}, \overline{\mathfrak{B}} \vdash \overline{\mathfrak{A}} & (\mathrm{VI}a) \\ \hline \vdash (\mathfrak{A} \supset \mathfrak{B}) \supset (\overline{\mathfrak{B}} \supset \overline{\mathfrak{A}}) & (\mathbf{H1}) \end{array}$$

[Derivation of] V,2:

$$\begin{array}{cl} \hline \overline{\overline{\mathfrak{A}}} \vdash \overline{\overline{\mathfrak{A}}} & (\mathbf{H2}) \\ \hline \mathfrak{A} \vdash \overline{\overline{\mathfrak{A}}} & (\mathrm{VI}b) \\ \hline \vdash \mathfrak{A} \supset \overline{\overline{\mathfrak{A}}} & (\mathbf{H1}) \end{array}$$

[Derivation of] V,3:

$$\begin{array}{cl} \hline \mathfrak{A} \vdash \mathfrak{A} & (\mathbf{H2}) \\ \hline \overline{\overline{\mathfrak{A}}} \vdash \mathfrak{A} & (\mathrm{VI}b) \\ \hline \vdash \overline{\overline{\mathfrak{A}}} \supset \mathfrak{A} & (\mathbf{H1}) \end{array}$$

□

In this manner, we have established all the basic formulas. If both $\vdash \mathfrak{A}$ and $\vdash \mathfrak{A} \supset \mathfrak{B}$ are provable, we derive $\mathfrak{A} \vdash \mathfrak{B}$ from III and $\vdash \mathfrak{B}$ from II.

In the future, the system comprising the basic formulas I,1–V,3 will be referred to as Π.

3 The RI-calculus

We now seek to construct the calculus for rigorous implication in a manner akin to our development of the [classical] propositional calculus in Section 2. This will involve specific modifications to the rules I–VI presented in that section. I will begin by outlining this new calculus and subsequently explain the rationale behind the adjustments made.

The symbols $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, and so forth, maintain the same meanings as in Section 2, with the sole distinction that the formulas are constructed using "$\rightarrow$", the symbol for rigorous implication, instead [117] of "$\supset$".

The *basic schemas* remain consistent with those in Section 2, with the addition of (6′):

(1′)	$\vdash \mathfrak{A} \rightarrow \mathfrak{A}$,
(2′)	$\mathfrak{A} \,\&\, \mathfrak{B} \vdash \mathfrak{A}$,
(3′)	$\mathfrak{A} \,\&\, \mathfrak{B} \vdash \mathfrak{B}$,
(4′)	$\mathfrak{A} \vdash \mathfrak{A} \vee \mathfrak{B}$,
(5′)	$\mathfrak{B} \vdash \mathfrak{A} \vee \mathfrak{B}$,
(6′)	$\mathfrak{A}, \mathfrak{B} \vee \mathfrak{C} \vdash \mathfrak{B} \vee (\mathfrak{A} \,\&\, \mathfrak{C})$.

(6′) is essential because, without it, the distributive law – which was derived in Section 2 from the other schemas using the inference rules – cannot be obtained.

For schemas with two premises, we differentiate them based on whether the premises are marked, despite their structural similarity. The basic schemas do not feature any marked premises. In the inference rules, we indicate which premises are marked in the derived schemas, specifically for those with two premises. In the presentation of proofs, we denote the marked premises with an asterisk (*). The rules I–VI are thus modified as follows:

- I′*a*. If $\mathfrak{S}$ is a provable schema with at least one premise and one conclusion, another provable schema can be derived by adding $\mathfrak{D}\rightarrow$ in front of the conclusion and in front of one or more premises, ensuring it is always prefixed to each unmarked premise. If $\mathfrak{S}$ has two premises, then after applying I′*a*, both premises of the new schema will be marked unless $\mathfrak{S}$ originally contained two unmarked premises. In this latter case, no premise of the new schema will be marked. For example, starting with the schema $\mathfrak{A} \rightarrow \mathfrak{B}^*, \mathfrak{A} \vdash \mathfrak{B}$, one obtains the schemas $\mathfrak{A} \rightarrow \mathfrak{B}^*, \mathfrak{D} \rightarrow \mathfrak{A}^* \vdash \mathfrak{D} \rightarrow \mathfrak{B}$ and $\mathfrak{D} \rightarrow (\mathfrak{A} \rightarrow \mathfrak{B})^*, \mathfrak{D} \rightarrow \mathfrak{A}^* \vdash \mathfrak{D} \rightarrow \mathfrak{B}$; furthermore, from $\mathfrak{A}, \mathfrak{B} \vdash \mathfrak{A} \,\&\, \mathfrak{B}$, one obtains the schema $\mathfrak{D} \rightarrow \mathfrak{A}, \mathfrak{D} \rightarrow \mathfrak{B} \vdash \mathfrak{D} \rightarrow \mathfrak{A} \,\&\, \mathfrak{B}$.
- I′*b*. If $\mathfrak{A} \vdash \mathfrak{B} \vee \mathfrak{C}$ is provable, then $\mathfrak{B} \rightarrow \mathfrak{D}, \mathfrak{C} \rightarrow \mathfrak{D} \vdash \mathfrak{A} \rightarrow \mathfrak{D}$ is also provable. The premises of the new schema remain unmarked.
- II′*a*. If the schemas $\vdash \mathfrak{A}$, $\vdash \mathfrak{B}$, and $\mathfrak{A}, \mathfrak{B} \vdash \mathfrak{C}$ are provable, then $\vdash \mathfrak{C}$ is also provable. The marking of the premises $\mathfrak{A}$ and $\mathfrak{B}$ in the second-to-last schema is irrelevant.
- II′*b*. If the schemas $\Psi, \mathfrak{A} \vdash \mathfrak{B}$ and $\vdash \mathfrak{A}$ are provable, and if Ψ is either empty or marked, then $\Psi \vdash \mathfrak{B}$ is provable. The marking of $\mathfrak{A}$ in the first schema does not affect this result.
- II′*c*. If the schemas $\mathfrak{A}, \mathfrak{B} \vdash$ and $\vdash \mathfrak{A}$ are provable, then $\mathfrak{B} \vdash$ is also provable. Any marking of the premises in the first schema is irrelevant.
- III′. From $\Psi \vdash \mathfrak{B} \rightarrow \mathfrak{C}$, we derive $\Psi, \mathfrak{B} \vdash \mathfrak{C}$. In $\Psi, \mathfrak{B} \vdash \mathfrak{C}$, Ψ is marked if it is not empty; $\mathfrak{B}$, on the other hand, is not.
- IV′. In a schema with two premises, their positions can be exchanged if both premises are either marked or unmarked. [118] After the exchange, the marking of the premises remains unchanged.

V′. From $\mathfrak{A} \,\&\, \mathfrak{B} \vdash \mathfrak{C}$, we derive $\mathfrak{A}, \mathfrak{B} \vdash \mathfrak{C}$. The premises $\mathfrak{A}$ and $\mathfrak{B}$ in the last schema remain unmarked. This rule also applies if the conclusion $\mathfrak{C}$ is absent.

VI′*a*. In a schema, one can switch a premise with the conclusion after marking both of them with the negation overline. This applies even if the conclusion or the relevant premise is empty. Schemas with two unmarked premises are excluded from this transformation. If a marked premise retains its position, it remains marked if and only if it occupies the first position. New formulas that appear as premises are never marked.

VI′*b*. In schemas, a premise $\mathfrak{A}$ can be replaced by $\overline{\overline{\mathfrak{A}}}$, and vice versa. Any marking of the premises remains unchanged, even if they are affected by the replacement.

In what follows, the system (1′)–(6′), I′–VI′ will be denoted by Σ'. To justify the variations from the system Σ, we need to provide an interpretation of both the schemas and the marking of the premises. A schema $\mathfrak{A} \vdash \mathfrak{B}$ is understood as the inference from $\mathfrak{A}$ to $\mathfrak{B}$; a schema $\mathfrak{A}, \mathfrak{B} \vdash \mathfrak{C}$ as the inference from both $\mathfrak{A}$ and $\mathfrak{B}$ to $\mathfrak{C}$; and a schema $\mathfrak{A}^*, \mathfrak{B} \vdash \mathfrak{C}$ is interpreted in the same way as $\mathfrak{A} \vdash \mathfrak{B} \to \mathfrak{C}$, meaning it is the inference from $\mathfrak{A}$ to $\mathfrak{B} \to \mathfrak{C}$. The last inference concludes $\mathfrak{C}$ from $\mathfrak{A}$ and $\mathfrak{B}$ since it is possible to infer $\mathfrak{C}$ from both $\mathfrak{B}$ and $\mathfrak{B} \to \mathfrak{C}$, but not vice versa. For example, one can infer $\mathfrak{A} \,\&\, \mathfrak{B}$ from $\mathfrak{A}$ and $\mathfrak{B}$, but not from the inference from $\mathfrak{A}$ to $\mathfrak{B} \to \mathfrak{A} \,\&\, \mathfrak{B}$ (cf. introduction). The validity of $\mathfrak{A}^*, \mathfrak{B}^* \vdash \mathfrak{C}$ is understood as the validity of both inferences: from $\mathfrak{A}$ to $\mathfrak{B} \to \mathfrak{C}$ and from $\mathfrak{B}$ to $\mathfrak{A} \to \mathfrak{C}$. (Schemas like $\mathfrak{A}, \mathfrak{B}^* \vdash \mathfrak{C}$ never occur.) A schema $\vdash \mathfrak{A}$ signifies that $\mathfrak{A}$ is valid for purely logical reasons, while a schema $\vdash \overline{\mathfrak{A}}$ indicates that $\overline{\mathfrak{A}}$ is valid for purely logical reasons, i.e., that $\mathfrak{A}$ is logically contradictory. A schema $\mathfrak{A}, \mathfrak{B} \vdash$ implies that $\mathfrak{A}$ and $\mathfrak{B}$ are incompatible, whereas a schema $\mathfrak{A}^*, \mathfrak{B} \vdash$ denotes the inference from $\mathfrak{A}$ to $\overline{\mathfrak{B}}$, and a schema $\mathfrak{A}^*, \mathfrak{B}^* \vdash$ encompasses both inferences: from $\mathfrak{A}$ to $\overline{\mathfrak{B}}$ and from $\mathfrak{B}$ to $\overline{\mathfrak{A}}$.

According to this interpretation, the basic schemas are evidently correct, as are the rules II′*a*, II′*b*, III′, IV′, and V′, along with the restrictions established in IV′ compared to IV in §2. II′*c* states that if $\mathfrak{A}$ is incompatible with $\mathfrak{B}$ and $\mathfrak{A}$ is valid for purely logical reasons, then $\mathfrak{B}$ must be a contradiction.

In contrast to II of §2, we have ruled out the possibility of inferring $\mathfrak{A} \vdash \mathfrak{C}$ from $\mathfrak{A}, \mathfrak{B} \vdash \mathfrak{C}$ and $\vdash \mathfrak{B}$. In fact, such an inference is not permissible. If it were possible to derive $\mathfrak{B} \to \mathfrak{B}$ from $\mathfrak{A}$ and $\mathfrak{B} \to \mathfrak{B}$, then, given the [general] validity of $\mathfrak{B} \to \mathfrak{B}$, one could also infer $\mathfrak{B} \to \mathfrak{B}$ from $\mathfrak{A}$. However, our inferences are always **[119]** conceived in a way that ensures a logical connection exists between premises and conclusion – something that does not hold in the aforementioned example.

In I′*a*, we have not permitted the inference from $\mathfrak{A}, \mathfrak{B} \vdash \mathfrak{C}$ to $\mathfrak{D} \to \mathfrak{A}^*, \mathfrak{D} \to \mathfrak{B}^* \vdash \mathfrak{D} \to \mathfrak{C}$. Otherwise, it would be possible to derive $D \to A^*, D \to B^* \vdash D \to A \,\&\, B$ from the provable schema $A, B \vdash A \,\&\, B$. Moreover, since $A \,\&\, B \vdash A$ holds, we could also obtain $D \to A^*, D \to B^* \vdash D \to A$.

However, there is no justification for accepting the inference from $D \to A$ to $(D \to B) \to (D \to A)$ as valid, since it is not possible to infer $\mathfrak{B} \to \mathfrak{A}$ from $\mathfrak{A}$.

Concerning rule VI′*a*, we have rightly disallowed the inferences from $\mathfrak{A}, \mathfrak{B} \vdash$ to $\mathfrak{A} \vdash \overline{\mathfrak{B}}$ and from $\mathfrak{A}, \mathfrak{B} \vdash \mathfrak{C}$ to $\mathfrak{A}, \overline{\mathfrak{C}} \vdash \overline{\mathfrak{B}}$. Otherwise, it would be possible to pass from $\mathfrak{A} \,\&\, \overline{\mathfrak{A}}, \overline{\mathfrak{B}} \vdash$ – a schema which would be valid according to the interpretation – to both $\mathfrak{A} \,\&\, \overline{\mathfrak{A}} \vdash \overline{\overline{\mathfrak{B}}}$ and $\mathfrak{A} \,\&\, \overline{\mathfrak{A}} \vdash \mathfrak{B}$, and according to the second rule from $\mathfrak{A}, \overline{\mathfrak{B}} \vdash \mathfrak{A}$ to either $\mathfrak{A}, \overline{\mathfrak{A}} \vdash \overline{\overline{\mathfrak{B}}}$ or $\mathfrak{A}, \overline{\mathfrak{A}} \vdash \mathfrak{B}$. However, we rejected the formula $\mathfrak{A} \,\&\, \overline{\mathfrak{A}} \to \mathfrak{B}$ as universally valid in the introduction.

[Therefore,] the admitted inference rules do not contain anything that contradicts the notion of rigorous implication. This is further corroborated by the fact that the rules can be interpreted within the system Π' (§4) and that the fundamental formulas and rules of Π' align with the concept of rigorous implication.

The system Π' corresponds to the system Π described in §2. It contains the following basic formulas:

(1) $\mathfrak{A} \to \mathfrak{A}$,
(2) $(\mathfrak{A} \to \mathfrak{B}) \to ((\mathfrak{B} \to \mathfrak{C}) \to (\mathfrak{A} \to \mathfrak{C}))$,
(3) $(\mathfrak{A} \to \mathfrak{B}) \to ((\mathfrak{C} \to \mathfrak{A}) \to (\mathfrak{C} \to \mathfrak{B}))$,
(4) $(\mathfrak{A} \to (\mathfrak{A} \to \mathfrak{B})) \to (\mathfrak{A} \to \mathfrak{B})$,
(5) $\mathfrak{A} \,\&\, \mathfrak{B} \to \mathfrak{A}$,
(6) $\mathfrak{A} \,\&\, \mathfrak{B} \to \mathfrak{B}$,
(7) $(\mathfrak{A} \to \mathfrak{B}) \,\&\, ((\mathfrak{A} \to \mathfrak{C}) \to (\mathfrak{A} \to \mathfrak{B} \,\&\, \mathfrak{C}))$,
(8) $\mathfrak{A} \to \mathfrak{A} \vee \mathfrak{B}$,
(9) $\mathfrak{B} \to \mathfrak{A} \vee \mathfrak{B}$,
(10) $(\mathfrak{A} \to \mathfrak{C}) \,\&\, (\mathfrak{B} \to \mathfrak{C}) \to (\mathfrak{A} \vee \mathfrak{B} \to \mathfrak{C})$,
(11) $\mathfrak{A} \,\&\, (\mathfrak{B} \vee \mathfrak{C}) \to \mathfrak{B} \vee (\mathfrak{A} \,\&\, \mathfrak{C})$,
(12) $(\mathfrak{A} \to \mathfrak{B}) \to (\overline{\mathfrak{B}} \to \overline{\mathfrak{A}})$,
(13) $\mathfrak{A} \,\&\, \overline{\mathfrak{B}} \to (\overline{\mathfrak{B} \to \mathfrak{A}})$,
(14) $\mathfrak{A} \to \overline{\overline{\mathfrak{A}}}$,
(15) $\overline{\overline{\mathfrak{A}}} \to \mathfrak{A}$.

We include the following inference rules:

(α) infer $\mathfrak{B}$ from $\mathfrak{A}$ and $\mathfrak{A} \to \mathfrak{B}$,
(β) infer $\mathfrak{A} \,\&\, \mathfrak{B}$ from $\mathfrak{A}$ and $\mathfrak{B}$,
(γ) infer $\mathfrak{B}$ from $\mathfrak{A}$ and $\overline{\mathfrak{A}} \vee \mathfrak{B}$,
(δ) infer $\mathfrak{A} \to \mathfrak{C}$ from $\mathfrak{A} \to (\mathfrak{B} \to \mathfrak{C})$ and $\mathfrak{B}$.

For the validity of (δ), in contrast to (α), (β), and (γ), it is essential [120] that $\mathfrak{B}$ is a logical identity. Consequently, rule (δ) cannot remain in this form if additional non-logical axioms are introduced alongside (1) to (15). In a later extension of the axiom system in §5, rule (δ) will become unnecessary.

4 Equivalence of Systems Σ' and Π' of §3.

We prove the equivalence between the systems Σ' and Π' in the following manner: If a formula $\mathfrak{A}$ is provable in Π', then $\vdash \mathfrak{A}$ is also provable in Σ', and conversely.

We begin by establishing the first part of the claim. For the basic formula (1), this is evident due to the schema (1′) of Σ'. For the basic formulas (2), (4), (5), (6), (8), (9), (12), (14), (15), this follows in the same way as for the corresponding formulas of system Π' in §2 with respect to system Σ', since the necessary markings of the premises are always present. Formula (3) is proved in the same way as (2), except that the interchange of premises is not carried out. The auxiliary theorems **H1** and **H2** from §2 also apply in this context (**H1** with the restriction that Ψ is empty or marked).

Theorem H3. If $\mathfrak{A}, \mathfrak{B} \vdash \mathfrak{C}$ is derivable in Σ', then so is $\vdash \mathfrak{A} \,\&\, \mathfrak{B} \to \mathfrak{C}$.

Proof.

$$\dfrac{\dfrac{\mathfrak{A}, \mathfrak{B} \vdash \mathfrak{C}}{\mathfrak{A} \,\&\, \mathfrak{B} \to \mathfrak{A}, \mathfrak{A} \,\&\, \mathfrak{B} \to \mathfrak{B} \vdash \mathfrak{A} \,\&\, \mathfrak{B} \to \mathfrak{C}}\;(\mathrm{I}'a)}{\vdash \mathfrak{A} \,\&\, \mathfrak{B} \to \mathfrak{C}}\;(\mathrm{II}'a)$$

Proof of (7).

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathfrak{B} \,\&\, \mathfrak{C} \vdash \mathfrak{B} \,\&\, \mathfrak{C}}\;(\mathbf{H2})}{\mathfrak{B}, \mathfrak{C} \vdash \mathfrak{B} \,\&\, \mathfrak{C}}\;(\mathrm{V}')}{\mathfrak{A} \to \mathfrak{B}, \mathfrak{A} \to \mathfrak{C} \vdash \mathfrak{A} \to \mathfrak{B} \,\&\, \mathfrak{C}}\;(\mathrm{I}'a)}{\vdash (\mathfrak{A} \to \mathfrak{B}) \,\&\, (\mathfrak{A} \to \mathfrak{C}) \to (\mathfrak{A} \to \mathfrak{B} \,\&\, \mathfrak{C})}\;(\mathbf{H3})$$

Proof of (10).

$$\dfrac{\dfrac{\dfrac{}{\mathfrak{A} \vee \mathfrak{B} \vdash \mathfrak{A} \vee \mathfrak{B}}\;(\mathbf{H2})}{\mathfrak{A} \to \mathfrak{C}, \mathfrak{B} \to \mathfrak{C} \vdash \mathfrak{A} \vee \mathfrak{B} \to \mathfrak{C}}\;(\mathrm{I}'b)}{\vdash (\mathfrak{A} \to \mathfrak{C}) \,\&\, (\mathfrak{B} \to \mathfrak{C}) \to (\mathfrak{A} \vee \mathfrak{B} \to \mathfrak{C})}\;(\mathbf{H3})$$

Proof of (13).

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathfrak{A} \to \mathfrak{B}^*, \mathfrak{A} \vdash \mathfrak{B}}\;(\text{from } (1') \text{ and } (\mathrm{III}'))}{\overline{\mathfrak{B}}, \mathfrak{A} \vdash \overline{\mathfrak{A} \to \mathfrak{B}}}\;(\mathrm{VI}'a)}{\mathfrak{A}, \overline{\mathfrak{B}} \vdash \overline{\mathfrak{A} \to \mathfrak{B}}}\;(\mathrm{IV}')}{\vdash \mathfrak{A} \,\&\, \overline{\mathfrak{B}} \to \overline{\mathfrak{A} \to \mathfrak{B}}}\;(\mathbf{H3})$$

(11) follows from $(6')$ and **H3**. □

In this manner, each basic formula of Π' is proved in Σ'. Furthermore, if both $\vdash \mathfrak{A}$ and $\vdash \mathfrak{A} \to \mathfrak{B}$ are provable, then $\mathfrak{A} \vdash \mathfrak{B}$ can be derived (by III$'$), and consequently, $\vdash \mathfrak{B}$ is provable (by II$'b$). If $\vdash \mathfrak{A}$ and $\vdash \mathfrak{B}$ are provable, then $\vdash \mathfrak{A} \,\&\, \mathfrak{B}$ is also provable. Since from $\mathfrak{A} \,\&\, \mathfrak{B} \vdash \mathfrak{A} \,\&\, \mathfrak{B}$ (**H2**), it follows that by V$'$, we have $\mathfrak{A}, \mathfrak{B} \vdash \mathfrak{A} \,\&\, \mathfrak{B}$ and, ultimately, $\vdash \mathfrak{A} \,\&\, \mathfrak{B}$ by II$'a$. If $\vdash \mathfrak{A}$ and $\vdash \overline{\mathfrak{A}} \vee \mathfrak{B}$ are provable, then $\vdash \mathfrak{B}$ is also provable. From $\mathfrak{A} \,\&\, \overline{\mathfrak{B}} \vdash \mathfrak{A}$ (by $2'$), we can derive both $\overline{\mathfrak{A}} \vdash \overline{\mathfrak{A} \,\&\, \overline{\mathfrak{B}}}$ (VI$'a$) and $\vdash \overline{\mathfrak{A}} \to \overline{\mathfrak{A} \,\&\, \overline{\mathfrak{B}}}$ (**H1**). From $\mathfrak{A} \,\&\, \overline{\mathfrak{B}} \vdash \overline{\mathfrak{B}}$ (by $3'$), we obtain [121] $\overline{\overline{\mathfrak{B}}} \vdash \overline{\mathfrak{A} \,\&\, \overline{\mathfrak{B}}}$ (VI$'a$), $\mathfrak{B} \vdash \overline{\mathfrak{A} \,\&\, \overline{\mathfrak{B}}}$ (VI$'b$), and also $\vdash \mathfrak{B} \to \overline{\mathfrak{A} \,\&\, \overline{\mathfrak{B}}}$ (**H1**). Furthermore (cf. the proof of (10)):

$$\overline{\mathfrak{A}} \to \overline{\mathfrak{A} \,\&\, \overline{\mathfrak{B}}}, \mathfrak{B} \to \overline{\mathfrak{A} \,\&\, \overline{\mathfrak{B}}} \vdash \overline{\mathfrak{A}} \vee \mathfrak{B} \to \overline{\mathfrak{A} \,\&\, \overline{\mathfrak{B}}}$$

is provable. From II$'a$, we obtain $\vdash \overline{\mathfrak{A}} \vee \mathfrak{B} \to \overline{\mathfrak{A} \,\&\, \overline{\mathfrak{B}}}$. Then, by III$'$, we derive $\overline{\mathfrak{A}} \vee \mathfrak{B} \vdash \overline{\mathfrak{A} \,\&\, \overline{\mathfrak{B}}}$. Using II$'b$, we further obtain $\vdash \overline{\mathfrak{A} \,\&\, \overline{\mathfrak{B}}}$. From VI$'a$, it follows that $\overline{\overline{\mathfrak{A} \,\&\, \overline{\mathfrak{B}}}} \vdash$, and by VI$'b$, we derive $\mathfrak{A} \,\&\, \overline{\mathfrak{B}} \vdash$. Finally, applying V$'$, we conclude that $\mathfrak{A}, \overline{\mathfrak{B}} \vdash$. From the last schema, together with $\vdash \mathfrak{A}$, one can prove $\overline{\mathfrak{B}} \vdash$ by II$'c$, and thus, by VI$'a$, we have $\vdash \overline{\overline{\overline{\mathfrak{B}}}}$. Finally, using the provable schema $\overline{\overline{\overline{\mathfrak{B}}}} \vdash \mathfrak{B}$ and applying II$'b$, we obtain $\vdash \mathfrak{B}$.

Moreover, if $\vdash \mathfrak{A} \to (\mathfrak{B} \to \mathfrak{C})$ and $\vdash \mathfrak{B}$ are provable, then so is also $\vdash \mathfrak{A} \to \mathfrak{C}$.

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathfrak{B}\to\mathfrak{C}\vdash\mathfrak{B}\to\mathfrak{C}}\,(\mathbf{H2})}{\mathfrak{B}\to\mathfrak{C}^*,\mathfrak{B}\vdash\mathfrak{C}}\,(\mathrm{III}')}{\mathfrak{B}\to\mathfrak{C}\vdash\mathfrak{C}}\,(\mathrm{II}'b)}{\mathfrak{A}\to(\mathfrak{B}\to\mathfrak{C})\vdash\mathfrak{A}\to\mathfrak{C}}\,(\mathrm{I}'a)}{\vdash\mathfrak{A}\to\mathfrak{C}}\,(\mathrm{II}'b)$$

□

Thus, we have shown that, even in the case of inference rules of Π', the provability of formulas is preserved in the system Σ'. Consequently, for every provable formula $\mathfrak{A}$ in Π', it follows that $\vdash\mathfrak{A}$ is provable in Σ'.

Let us now demonstrate the converse and prove a more general theorem: If $\vdash\mathfrak{A}$, $\mathfrak{A}\vdash\mathfrak{B}$, $\mathfrak{A},\mathfrak{B}\vdash\mathfrak{C}$, $\mathfrak{A}^*,\mathfrak{B}\vdash\mathfrak{C}$, $\mathfrak{A}^*,\mathfrak{B}^*\vdash\mathfrak{C}$, $\mathfrak{A}\vdash$, $\mathfrak{A},\mathfrak{B}\vdash$, and $\mathfrak{A}^*,\mathfrak{B}\vdash$ are provable, then so are the corresponding Π' formulas: $\mathfrak{A}$, $\mathfrak{A}\to\mathfrak{B}$, $\mathfrak{A}\mathbin{\&}\mathfrak{B}\to\mathfrak{C}$, $\mathfrak{A}\to(\mathfrak{B}\to\mathfrak{C})$, both formulas $\mathfrak{A}\to(\mathfrak{B}\to\mathfrak{C})$ and $\mathfrak{B}\to(\mathfrak{A}\to\mathfrak{C})$, $\overline{\mathfrak{A}}$, $\overline{\mathfrak{A}\mathbin{\&}\mathfrak{B}}$, and $\mathfrak{A}\to\overline{\mathfrak{B}}$.

For the basic schemas, this is evident since the corresponding formulas are basic formulas of Π'. To demonstrate that inference rules also preserve these properties of schemas, we establish the following properties of Π':

(a) If $\mathfrak{A}\to\mathfrak{B}$ and $\mathfrak{B}\to\mathfrak{C}$ are provable, then so is $\mathfrak{A}\to\mathfrak{C}$.

(b) If $\mathfrak{A}\to\mathfrak{B}$ is provable, then so is $(\mathfrak{C}\to\mathfrak{A})\to(\mathfrak{C}\to\mathfrak{B})$.

(c) If $\mathfrak{A}\to\mathfrak{B}$ is provable, then so is $(\mathfrak{B}\to\mathfrak{C})\to(\mathfrak{A}\to\mathfrak{C})$.

(a), (b), and (c) follow from basic formula (2) or (3) by applying (α).

(d) If $\mathfrak{A}\to(\mathfrak{B}\to\mathfrak{C})$ and $\mathfrak{C}\to\mathfrak{D}$ are provable, then so is $\mathfrak{A}\to(\mathfrak{B}\to\mathfrak{D})$. In fact, from $\mathfrak{C}\to\mathfrak{D}$, one can derive $(\mathfrak{B}\to\mathfrak{C})\to(\mathfrak{B}\to\mathfrak{D})$ using (b), and consequently, the desired result follows from (a).

(e) If $\mathfrak{A}\mathbin{\&}\mathfrak{B}\to\mathfrak{C}$ is provable, then $(\mathfrak{D}\to\mathfrak{A})\mathbin{\&}(\mathfrak{D}\to\mathfrak{B})\to(\mathfrak{D}\to\mathfrak{C})$ is also provable. Since $(\mathfrak{D}\to\mathfrak{A})\mathbin{\&}(\mathfrak{D}\to\mathfrak{B})\to(\mathfrak{D}\to(\mathfrak{A}\mathbin{\&}\mathfrak{B}))$ is a basic formula, the result follows from (d).

(f) If $\mathfrak{A}\to(\mathfrak{B}\to\mathfrak{C})$ is provable, then so are $\mathfrak{A}\to((\mathfrak{D}\to\mathfrak{B})\to(\mathfrak{D}\to\mathfrak{C}))$ and $(\mathfrak{D}\to\mathfrak{B})\to(\mathfrak{A}\to(\mathfrak{D}\to\mathfrak{C}))$. Since $(\mathfrak{D}\to\mathfrak{C})\to((\mathfrak{D}\to\mathfrak{B})\to(\mathfrak{D}\to\mathfrak{C}))$ is a basic formula, we obtain [122] $\mathfrak{A}\to((\mathfrak{D}\to\mathfrak{B})\to(\mathfrak{D}\to\mathfrak{C}))$ by applying (a). Moreover, from the basic formula $(\mathfrak{D}\to\mathfrak{B})\to((\mathfrak{B}\to\mathfrak{C})\to(\mathfrak{D}\to\mathfrak{C}))$, we derive $(\mathfrak{D}\to\mathfrak{B})\to((\mathfrak{A}\to(\mathfrak{B}\to\mathfrak{C}))\to(\mathfrak{A}\to(\mathfrak{D}\to\mathfrak{C})))$, and from this, we conclude $(\mathfrak{D}\to\mathfrak{B})\to(\mathfrak{A}\to(\mathfrak{D}\to\mathfrak{C}))$ using (δ).

(g) If $\mathfrak{A}\to(\mathfrak{B}\to\mathfrak{C})$ is provable, then so are $(\mathfrak{D}\to\mathfrak{A})\to((\mathfrak{D}\to\mathfrak{B})\to(\mathfrak{D}\to\mathfrak{C}))$ and $(\mathfrak{D}\to\mathfrak{B})\to((\mathfrak{D}\to\mathfrak{A})\to(\mathfrak{D}\to\mathfrak{C}))$. According to (f), $(\mathfrak{D}\to\mathfrak{B})\to(\mathfrak{A}\to(\mathfrak{B}\to\mathfrak{C}))$ is provable, as is $(\mathfrak{D}\to\mathfrak{A})\to((\mathfrak{D}\to\mathfrak{B})\to(\mathfrak{D}\to(\mathfrak{D}\to\mathfrak{C})))$. Since $(\mathfrak{D}\to(\mathfrak{D}\to\mathfrak{C}))\to(\mathfrak{D}\to\mathfrak{C})$ is a basic formula, we can apply (d) to obtain $(\mathfrak{D}\to\mathfrak{A})\to((\mathfrak{D}\to\mathfrak{B})\to(\mathfrak{D}\to\mathfrak{C}))$. Furthermore, from $\mathfrak{A}\to(\mathfrak{B}\to\mathfrak{C})$, by (f), we get both $(\mathfrak{D}\to\mathfrak{B})\to(\mathfrak{A}\to(\mathfrak{B}\to\mathfrak{C}))$ and $(\mathfrak{D}\to\mathfrak{B})\to((\mathfrak{D}\to\mathfrak{A})\to(\mathfrak{D}\to(\mathfrak{D}\to\mathfrak{C})))$, leading to $(\mathfrak{D}\to\mathfrak{B})\to((\mathfrak{D}\to\mathfrak{A})\to(\mathfrak{D}\to\mathfrak{C}))$.

With (b) and (e) to (g), it is shown that the provability of the schemas in the specified sense is preserved under the application of I$'a$.

I$'b$ states: If $\mathfrak{A} \to \mathfrak{B} \lor \mathfrak{C}$ is provable, then so is $(\mathfrak{B} \to \mathfrak{D}) \,\&\, (\mathfrak{C} \to \mathfrak{D}) \to (\mathfrak{A} \to \mathfrak{D})$. From $\mathfrak{A} \to \mathfrak{B} \lor \mathfrak{C}$, one can derive, using (c), $(\mathfrak{B} \lor \mathfrak{C} \to \mathfrak{D}) \to (\mathfrak{A} \to \mathfrak{D})$, and combined with the basic formula $(\mathfrak{B} \to \mathfrak{D}) \,\&\, (\mathfrak{C} \to \mathfrak{D}) \to (\mathfrak{B} \lor \mathfrak{C} \to \mathfrak{D})$, we conclude the desired result by applying (a).

II$'a$ states: If $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{A} \,\&\, \mathfrak{B} \to \mathfrak{C}$ – that is $\mathfrak{A} \to (\mathfrak{B} \to \mathfrak{C})$ – are provable, then so is $\mathfrak{C}$. This follows with the assistance of (α) and (β).

II$'b$ follows from applications of (α) and (β).

II$'c$ states: Given $\mathfrak{A}$ and $\overline{\mathfrak{A} \,\&\, \mathfrak{B}}$ – that is $\mathfrak{A} \to \overline{\mathfrak{B}}$ – one can prove $\overline{\mathfrak{B}}$. The latter holds due to (α). If $\overline{\mathfrak{A} \,\&\, \mathfrak{B}}$ is given, then it is possible to derive from the basic formulas $\overline{\mathfrak{A}} \to \overline{\mathfrak{A}} \lor \overline{\mathfrak{B}}$ and $\overline{\mathfrak{B}} \to \overline{\mathfrak{A}} \lor \overline{\mathfrak{B}}$. By applying (12) and (15), both formulas lead to $\overline{\overline{\mathfrak{A}} \lor \overline{\mathfrak{B}}} \to \mathfrak{A}$ and $\overline{\overline{\mathfrak{A}} \lor \overline{\mathfrak{B}}} \to \mathfrak{B}$. Using (β) and the basic formula (7), we obtain $\overline{\overline{\mathfrak{A}} \lor \overline{\mathfrak{B}}} \to \mathfrak{A} \,\&\, \mathfrak{B}$. Furthermore, by applying (12) and (15), we get $\overline{\mathfrak{A} \,\&\, \mathfrak{B}} \to \overline{\mathfrak{A}} \lor \overline{\mathfrak{B}}$, and finally, by rule (α), we also obtain $\overline{\mathfrak{A}} \lor \overline{\mathfrak{B}}$. From $\mathfrak{A}$ and $\overline{\mathfrak{A}} \lor \overline{\mathfrak{B}}$, we can apply (γ) to conclude $\overline{\mathfrak{B}}$.

III$'$ to V$'$ are self-evident according to the interpretation. VI$'b$ follows easily with the help of (14) and (15). The special cases of VI$'a$ that are not self-evident according to the interpretation are the following:

$$\text{S1}\ \frac{\mathfrak{A}^*, \mathfrak{B} \vdash}{\mathfrak{B} \vdash \overline{\mathfrak{A}}}\,; \qquad \text{S2}\ \frac{\vdash \mathfrak{A}}{\overline{\mathfrak{A}} \vdash}\,; \qquad \text{S3}\ \frac{\mathfrak{A} \vdash \mathfrak{B}}{\overline{\mathfrak{B}} \vdash \overline{\mathfrak{A}}}\,;$$

$$\text{S4}\ \frac{\mathfrak{A}^*, \mathfrak{B} \vdash \mathfrak{C}}{\mathfrak{A}^*, \overline{\mathfrak{C}} \vdash \overline{\mathfrak{B}}}\,; \qquad \text{S5}\ \frac{\mathfrak{A}^*, \mathfrak{B} \vdash \mathfrak{C}}{\overline{\mathfrak{C}}, \mathfrak{B} \vdash \overline{\mathfrak{A}}}\,; \qquad \text{S6}\ \frac{\mathfrak{A} \vdash \mathfrak{B}}{\mathfrak{A}, \overline{\mathfrak{B}} \vdash}\,.$$

In S1, S4, and S5, $\mathfrak{B}$ could also be marked with an asterisk. The meaning of the rules in Π' results from the following provable propositions, each followed by the number of the basic formula from Π' that is essential to its proof.

If $\mathfrak{A} \to \overline{\mathfrak{B}}$ is provable, then so is $\mathfrak{B} \to \overline{\mathfrak{A}}$ (12), (14). If $\mathfrak{A}$ is provable, then so is $\overline{\overline{\mathfrak{A}}}$ (14). If $\mathfrak{A} \to \mathfrak{B}$ is provable, then so is $\overline{\mathfrak{B}} \to \overline{\mathfrak{A}}$ (12). If $\mathfrak{A} \to (\mathfrak{B} \to \mathfrak{C})$ is provable, then so is $\mathfrak{A} \to (\overline{\mathfrak{C}} \to \overline{\mathfrak{B}})$ (12). If $\mathfrak{A} \to (\mathfrak{B} \to \mathfrak{C})$ is provable, then so is $\overline{\mathfrak{C}} \,\&\, \mathfrak{B} \to \overline{\mathfrak{A}}$. From $\mathfrak{A} \to (\mathfrak{B} \to \mathfrak{C})$, [123] one can derive $\overline{\mathfrak{B} \to \overline{\mathfrak{C}}} \to \overline{\mathfrak{A}}$ (12), and consequently, $\mathfrak{B} \,\&\, \overline{\mathfrak{C}} \to \overline{\mathfrak{A}}$ (13). Since $\overline{\mathfrak{C}} \,\&\, \mathfrak{B} \to \mathfrak{B} \,\&\, \overline{\mathfrak{C}}$ is provable, the claim follows. If $\mathfrak{A} \to \mathfrak{B}$ is provable, then so is $\overline{\mathfrak{A} \,\&\, \overline{\mathfrak{B}}}$. Firstly, one obtains $\overline{\overline{\mathfrak{A} \to \mathfrak{B}}} \to \overline{\mathfrak{A} \,\&\, \overline{\mathfrak{B}}}$ (12), (13), and then concludes with the claim according to (14). In this way, the equivalence between Σ' and Π' is established.

We make one more remark about the system Π'.

Π' contains the full two-valued propositional calculus meaning that all general formulas constructed using the three connectives $\&$, $\lor$, and $\overline{\cdot}$ can be derived.

Let's outline the proof. First, we recognize through induction based on the structure of formulas that a substitution theorem holds for the formulas of the calculus. Specifically, if $\mathfrak{A} \to \mathfrak{B}$ and $\mathfrak{A} \to \mathfrak{B}$ are provable, then one can obtain another provable formula by replacing $\mathfrak{A}$ within the formula with $\mathfrak{B}$. According to (14) and (15), one can always replace $\overline{\overline{\mathfrak{A}}}$ with $\mathfrak{A}$ and vice versa.

Furthermore, the commutative and associative laws for conjunction and disjunction, as well as the distributive laws, are derivable in the sense that the formulas $\mathfrak{A} \,\&\, \mathfrak{B} \to \mathfrak{B} \,\&\, \mathfrak{A}$, etc., are provable. Moreover, one can derive the formulas $\overline{\mathfrak{A} \,\&\, \mathfrak{B}} \to \overline{\mathfrak{A}} \lor \overline{\mathfrak{B}}$, $\overline{\mathfrak{A} \lor \mathfrak{B}} \to \overline{\mathfrak{A}} \,\&\, \overline{\mathfrak{B}}$, and $\overline{\mathfrak{A}} \,\&\, \overline{\mathfrak{B}} \to \overline{\mathfrak{A} \lor \mathfrak{B}}$, possibly using contraposition (12) repeatedly.

If $\mathfrak{A}$ is a formula constructed using $\&$, $\vee$, and $\overline{\cdot}$, and $\mathfrak{A}'$ is a certain conjunctive normal form of it, then $\mathfrak{A}' \to \mathfrak{A}$ can be proved (cf. Hilbert-Ackermann, [3, 10]). Furthermore, $\overline{\mathfrak{B}} \vee \mathfrak{B}$ is provable. From $\mathfrak{B} \to \mathfrak{B}$, we obtain $\overline{\mathfrak{B} \,\&\, \overline{\mathfrak{B}}}$ (cf. the proof of S6 above) and, with the help of the above transformations, $\overline{\mathfrak{B}} \vee \overline{\overline{\mathfrak{B}}}$ and $\overline{\mathfrak{B}} \vee \mathfrak{B}$.

It can then be easily recognized that every universally valid formula can be derived in conjunctive normal form, and thus that every universally valid formula is derivable.

In accordance with the above remarks, disjunction can be defined through conjunction, or vice versa, in the usual way. Hence, the system Π' could be presented more simply by omitting three primitive formulas.

5 Introducing the modalities

In what follows, we introduce modal concepts like "necessary", "possible", etc., into the calculus. The option to define modalities through other logical operations, such as interpreting $\overline{\mathfrak{A}} \to \mathfrak{A}$ as "$\mathfrak{A}$ is necessary" and $\mathfrak{A} \to \overline{\mathfrak{A}}$ as "$\mathfrak{A}$ is impossible", is not considered here. Indeed, a proposition must always be regarded as impossible if it implies any kind of contradiction, that is, if it implies a formula like $\mathfrak{B} \,\&\, \overline{\mathfrak{B}}$. However, since $\mathfrak{B} \,\&\, \overline{\mathfrak{B}} \to \overline{\mathfrak{A}}$ is not a universally valid formula, we cannot deduce $\mathfrak{A} \to \overline{\mathfrak{A}}$ from $\overline{\mathfrak{A}} \to \mathfrak{B} \,\&\, \overline{\mathfrak{B}}$. Thus, the aforementioned definition of impossibility does not apply in this context.

Additional axioms **[124]** cannot be introduced into the calculus into the current framework, as doing so would render universally valid formulas that we have deliberately excluded. On the other hand, within the system Σ', we already possess a specific way to express necessity and impossibility. In this system, which does not allow the consideration of additional non-logical axioms, $\vdash \mathfrak{A}$ simultaneously means "$\mathfrak{A}$ is necessary", and $\mathfrak{A} \vdash$ signifies "$\mathfrak{A}$ is impossible". However, logical operations cannot be carried out using necessity in this manner.

Considering this characteristic of the system Σ', we propose the following approach. We define impossibility in a manner analogous to I. Johansson's [4] introduction of negation in the minimal calculus. A new symbol "$\curlywedge$" is introduced into the calculus, functioning as a propositional symbol and signifying "the absurd". The phrase "$\mathfrak{A}$ is impossible" is understood as "$\mathfrak{A} \to \curlywedge$". We add the following inference rule to the system Σ':

VII′. If $\Psi \vdash \curlywedge$ is derivable, then so is $\Psi \vdash$, and vice versa. Here, Ψ represents one or two premises. Any marked premise is retained during the first transformation but not during the second.

We note in this regard that it was necessary to exclude the derivation of $\mathfrak{A}^*, \mathfrak{B} \vdash \curlywedge$ from $\mathfrak{A}^*, \mathfrak{B} \vdash$, and instead allow only the derivation of $\mathfrak{A}, \mathfrak{B} \vdash \curlywedge$ from $\mathfrak{A}^*, \mathfrak{B} \vdash$. This is because from $\mathfrak{A} \to \overline{\mathfrak{B}}$, it does not follow that $\mathfrak{A}$ implies the impossibility of $\mathfrak{B}$; however, it does follow that $\mathfrak{A} \,\&\, \mathfrak{B}$ is impossible. Examples of derivations are as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathfrak{A} \to \curlywedge \vdash \mathfrak{A} \to \curlywedge}\ (\mathbf{H2})}{\mathfrak{A} \to \curlywedge^*, \mathfrak{A} \vdash \curlywedge}\ (\mathrm{III}')}{\mathfrak{A} \to \curlywedge^*, \mathfrak{A} \vdash}\ (\mathrm{VII}')}{\mathfrak{A} \to \curlywedge \vdash \overline{\mathfrak{A}}}\ (\mathrm{VI}'a)}{\vdash (\mathfrak{A} \to \curlywedge) \to \overline{\mathfrak{A}}}\ (\mathbf{H1})
\qquad\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{\mathfrak{A} \vdash \mathfrak{A}}\ (\mathbf{H2})}{\mathfrak{A}, \overline{\mathfrak{A}} \vdash}\ (\mathrm{VI}'a)}{\mathfrak{A}, \overline{\mathfrak{A}} \vdash \curlywedge}\ (\mathrm{VII}')}{\vdash \mathfrak{A} \,\&\, \overline{\mathfrak{A}} \to \curlywedge}\ (\mathbf{H3})$$

If $\vdash \mathfrak{A} \to \mathfrak{B}$ and $\vdash (\mathfrak{A} \to \mathfrak{B}) \,\&\, \mathfrak{C} \to \curlywedge$ are provable, then so is $\vdash \mathfrak{C} \to \curlywedge$. From $\vdash (\mathfrak{A} \to \mathfrak{B}) \,\&\, \mathfrak{C} \to \curlywedge$ one obtains $(\mathfrak{A} \to \mathfrak{B}) \,\&\, \mathfrak{C} \vdash \curlywedge$ by III′, $\mathfrak{A} \to \mathfrak{B}, \mathfrak{C} \vdash \curlywedge$ by V′, $\mathfrak{A} \to \mathfrak{B}, \mathfrak{C} \vdash$ by VII′, $\mathfrak{C} \vdash$ by II′c, $\mathfrak{C} \vdash \curlywedge$ by VII′, and, finally, $\vdash \mathfrak{C} \to \curlywedge$ by **H1**.

We now aim to introduce corresponding additions to the system Π′. We incorporate into Π′ the basic formulas:

(16) $(\mathfrak{A} \to \curlywedge) \to \overline{\mathfrak{A}}$ and (17) $\mathfrak{A} \,\&\, \overline{\mathfrak{A}} \to \curlywedge$

which are both provable in Σ′, along with following inference rule:

(ε) If $\vdash \mathfrak{A} \to \mathfrak{B}$ and $\vdash (\mathfrak{A} \to \mathfrak{B}) \,\&\, \mathfrak{C} \to \curlywedge$ are provable, then so is $\vdash \mathfrak{C} \to \curlywedge$.

It is natural to ask why I did not introduce (ε) in the simpler form corresponding to the inference from $\mathfrak{A}$ and $\mathfrak{A} \,\&\, \mathfrak{B} \to \curlywedge$ to $\mathfrak{B} \to \curlywedge$. The reason is that we aim to formulate all rules in such a way that they yield correct results even when additional non-logical axioms are introduced. [125] (The rule that constitutes an exception to this, (δ), will be removed shortly.) However, this is not the case with the aforementioned rule.

Let $\mathfrak{A}$ be a formula provable through additional axioms, but not a logical identity. From $\mathfrak{A}$ and the provable formula $\mathfrak{A} \,\&\, \overline{\mathfrak{A}} \to \curlywedge$, we could deduce $\overline{\mathfrak{A}} \to \curlywedge$ using the simpler form of the rule, implying that $\mathfrak{A}$ is necessary, which of course should not be the case.

In the extension of Π′, the following theorem holds: *If $\mathfrak{A}$ is provable, then $\overline{\mathfrak{A}} \to \curlywedge$ is also provable.*

This is true if $\mathfrak{A}$ has the form $\mathfrak{G} \to \mathfrak{H}$. Indeed, according to (17), $(\mathfrak{G} \to \mathfrak{H}) \,\&\, \overline{\mathfrak{G} \to \mathfrak{H}} \to \curlywedge$ is provable, and by using (ε) one obtains $\overline{\mathfrak{G} \to \mathfrak{H}} \to \curlywedge$. The claim is therefore true for all basic formulas as well as for all formulas derived using (δ) or (ε). Furthermore, let a formula $\mathfrak{B}$ be derived by using (α), that is concluded from $\mathfrak{A}$ and $\mathfrak{A} \to \mathfrak{B}$. By assumption $\overline{\mathfrak{A}} \to \curlywedge$ and $\overline{\mathfrak{A} \to \mathfrak{B}} \to \curlywedge$ can be shown. From these two formulas, using (10), one obtains $\overline{\mathfrak{A}} \vee \overline{\mathfrak{A} \to \mathfrak{B}} \to \curlywedge$. Since (cf. §4) $\overline{\mathfrak{A} \,\&\, (\mathfrak{A} \to \mathfrak{B})} \to \overline{\mathfrak{A}} \vee \overline{(\mathfrak{A} \to \mathfrak{B})}$ is provable, we get $\overline{\mathfrak{A} \,\&\, (\mathfrak{A} \to \mathfrak{B})} \to \curlywedge$. From $(\mathfrak{A} \to \mathfrak{B}) \to (\mathfrak{A} \to \mathfrak{B})$ we obtain, using (g) of §4:

$$(\mathfrak{A} \,\&\, (\mathfrak{A} \to \mathfrak{B}) \to (\mathfrak{A} \to \mathfrak{B})) \to ((\mathfrak{A} \,\&\, (\mathfrak{A} \to \mathfrak{B}) \to \mathfrak{A}) \to (\mathfrak{A} \,\&\, (\mathfrak{A} \to \mathfrak{B}) \to \mathfrak{B}))$$

and then $\mathfrak{A} \,\&\, (\mathfrak{A} \to \mathfrak{B}) \to \mathfrak{B}$ by applying (α) twice and, thus, $\overline{\mathfrak{B}} \to \overline{\mathfrak{A} \,\&\, (\mathfrak{A} \to \mathfrak{B})}$ by (12). From $\overline{\mathfrak{A} \,\&\, (\mathfrak{A} \to \mathfrak{B})} \to \curlywedge$ we obtain $\overline{\mathfrak{B}} \to \curlywedge$. Within rule ($\alpha$), the property of the formulas expressed in the theorem is thus retained. The same applies to rule (β). Since from $\overline{\mathfrak{A}} \to \curlywedge$ and $\overline{\mathfrak{B}} \to \curlywedge$ one gets $\overline{\mathfrak{A}} \vee \overline{\mathfrak{B}} \to \curlywedge$, and, given $\overline{\mathfrak{A} \,\&\, \mathfrak{B}} \to \overline{\mathfrak{A}} \vee \overline{\mathfrak{B}}$, also $\overline{\mathfrak{A} \,\&\, \mathfrak{B}} \to \curlywedge$ follows. Now, let $\mathfrak{B}$ be derived from $\mathfrak{A}$ and $\overline{\mathfrak{A}} \vee \mathfrak{B}$ using (γ). By assumption $\overline{\mathfrak{A}} \to \curlywedge$ and $\overline{\overline{\mathfrak{A}} \vee \mathfrak{B}} \to \curlywedge$ are provable and, because of $\mathfrak{A} \,\&\, \overline{\mathfrak{B}} \to \overline{\overline{\mathfrak{A}} \vee \mathfrak{B}}$, so is $\mathfrak{A} \,\&\, \overline{\mathfrak{B}} \to \curlywedge$. From the basic formula $(\overline{\mathfrak{A}} \to \curlywedge) \to \overline{\overline{\mathfrak{A}}}$ one obtains also $(\overline{\mathfrak{A}} \to \curlywedge) \to \mathfrak{A}$ by $\overline{\overline{\mathfrak{A}}} \to \mathfrak{A}$. From this one gets $(\overline{\mathfrak{A}} \to \curlywedge) \,\&\, \overline{\mathfrak{B}} \to \mathfrak{A} \,\&\, \overline{\mathfrak{B}}$, and, hence, $(\overline{\mathfrak{A}} \to \curlywedge) \,\&\, \overline{\mathfrak{B}} \to \curlywedge$. Applying rule ($\varepsilon$) one gets $\overline{\mathfrak{B}} \to \curlywedge$. This universally proves the above theorem.

We can now dispense with rule (δ). If $\mathfrak{B}$ and $\mathfrak{A} \to (\mathfrak{B} \to \mathfrak{C})$ are provable, then according to the theorem just demonstrated, $\overline{\mathfrak{B}} \to \curlywedge$ is also provable. Using (12), we derive $\mathfrak{A} \to (\overline{\mathfrak{C}} \to \overline{\mathfrak{B}})$, and consequently, $\mathfrak{A} \to (\overline{\mathfrak{C}} \to \curlywedge)$ (cf. (d) of §4). By applying (16), we further deduce $\mathfrak{A} \to \mathfrak{C}$.

The remaining rules, (α), (β), and (γ), are formulated such that they remain valid even when additional non-logical axioms are introduced. Of course, the implicit dependence of rule (δ) on the other rules is lost when such axioms are added. However, this is acceptable. We already observed during the formulation of the axiom system for Π' that rule (δ) ceases to be correct when such axioms are introduced. This can be explained as follows: Let $\mathfrak{A}$ be a formula derivable from the additional axioms, [126] but not a logical identity. From $\mathfrak{A}$ and $(\mathfrak{A} \to \mathfrak{A}) \to (\mathfrak{A} \to \mathfrak{A})$, rule ($\delta$) would allow us to derive $(\mathfrak{A} \to \mathfrak{A}) \to \mathfrak{A}$, and by (12), $\overline{\mathfrak{A}} \to \overline{\mathfrak{A} \to \mathfrak{A}}$. Since $\overline{\mathfrak{A} \to \mathfrak{A}} \to \curlywedge$ is provable, we obtain $\overline{\mathfrak{A}} \to \curlywedge$, meaning $\mathfrak{A}$ is necessary, which, of course, cannot be the case.

If no additional non-logical axioms are introduced, then (γ) is also dispensable. This is because, with $\mathfrak{A}$, also $\overline{\mathfrak{A}} \to \curlywedge$ is provable, and with $\overline{\mathfrak{A}} \vee \mathfrak{B}$, also $\overline{\overline{\mathfrak{A}} \vee \mathfrak{B}} \to \curlywedge$ is provable. As shown earlier, this leads to $\overline{\mathfrak{B}} \to \curlywedge$ directly without the need to use (γ). Then, by applying (16) and (15), $\mathfrak{B}$ can be derived.

When additional non-logical axioms are involved, the implicit dependence of (γ) on other rules is lost. However, (γ) remains valid and must be included; otherwise, the inferences of the [classical] propositional calculus involving these additional axioms cannot be carried out.

Let us further demonstrate that, even after adding VII$'$, the schemas derivable in Σ' remain provable in Π' in the previously established sense. VII$'$ introduces the transitions from $\mathfrak{A} \vdash$ to $\mathfrak{A} \vdash \curlywedge$ and from $\mathfrak{A}, \mathfrak{B} \vdash$ to $\mathfrak{A}, \mathfrak{B} \vdash \curlywedge$. These transitions correspond in Π' to the following propositions: if $\overline{\mathfrak{A}}$ is provable, then so is $\mathfrak{A} \to \curlywedge$, and if $\overline{\mathfrak{A} \,\&\, \mathfrak{B}}$ is provable, then $\mathfrak{A} \,\&\, \mathfrak{B} \to \curlywedge$ is provable as well. These results follow straightforwardly from the theorem mentioned earlier.

The transition from $\mathfrak{A}^*, \mathfrak{B} \vdash$ or $\mathfrak{A}^*, \mathfrak{B}^* \vdash$ to $\mathfrak{A}, \mathfrak{B} \vdash \curlywedge$ is justified since $\mathfrak{A} \,\&\, \mathfrak{B} \to \mathfrak{C}$ follows from $\mathfrak{A} \to (\mathfrak{B} \to \mathfrak{C})$. The transition from $\Psi \vdash \curlywedge$ to $\Psi \vdash$ is proved within Π' using the basic formula (16).

We now define the following modal operators $U\mathfrak{A}$ ("$\mathfrak{A}$ is impossible") as $\mathfrak{A} \to \curlywedge$, $N\mathfrak{A}$ ("$\mathfrak{A}$ is necessary") as $\overline{\mathfrak{A}} \to \curlywedge$, and $M\mathfrak{A}$ ("$\mathfrak{A}$ is possible") as $\overline{\mathfrak{A} \to \curlywedge}$. All common formulas of modal logic can be proven within this system, including: $N\mathfrak{A} \to \mathfrak{A}$, $(\mathfrak{A} \to \mathfrak{B}) \to (N\mathfrak{A} \to N\mathfrak{B})$, $(\mathfrak{A} \to \mathfrak{B}) \to (M\mathfrak{A} \to M\mathfrak{B})$, $N(\mathfrak{A} \,\&\, \mathfrak{B}) \to N\mathfrak{A} \,\&\, N\mathfrak{B}$, $\mathfrak{A} \to M\mathfrak{A}$, $M(\mathfrak{A} \vee \mathfrak{B}) \to M\mathfrak{A} \vee M\mathfrak{B}$.

Moreover, the so-called "paradoxes" of strict implication, such as $N\mathfrak{A} \to (\mathfrak{B} \to \mathfrak{A})$ and $U\mathfrak{A} \to (\overline{\mathfrak{A}} \to \mathfrak{B})$, are not provable, and this is in line with our position.

It is possible to define alternative implicative relations, such as $(\mathfrak{A} \to \mathfrak{B}) \vee U\mathfrak{A} \vee N\mathfrak{B}$ or $\mathfrak{A} \,\&\, \overline{\mathfrak{B}} \to \curlywedge$, which possess slightly different properties. While some modal systems include axioms like $N\mathfrak{A} \to NN\mathfrak{A}$ and $M\mathfrak{A} \to NM\mathfrak{A}$, these are not provable here. Instead, we have the theorem: if $\mathfrak{A}$ is provable, so is $N\mathfrak{A}$. This result applies only to systems without non-logical additional axioms. With such axioms, we instead have: if $N\mathfrak{A}$ is provable, so is $NN\mathfrak{A}$.

Finally, a remark: necessity, impossibility, and related notions are understood here as purely logical. In specific theories, a broader concept of necessity might be employed. [127] Incorporating such a concept poses no particular difficulty. For instance, if $\mathfrak{G}_1, \ldots, \mathfrak{G}_n$ are certain foundational formulas of a theory regarded as necessary along with their consequences, it suffices to include $\overline{\mathfrak{G}}_1 \to \curlywedge, \ldots, \overline{\mathfrak{G}}_n \to \curlywedge$ in Π', rather than merely adding the formulas $\mathfrak{G}_1, \ldots, \mathfrak{G}_n$, to provide the necessary basis for derivations.

6 Final remarks

We will now summarize the complete axiom system once again. The basic formulas are those labeled (1)–(15) of §3, along with the basic formulas (16) and (17) of §5. Moreover, the inference rules are as follows:

From $\mathfrak{A}$ and $\mathfrak{A} \to \mathfrak{B}$, one obtains $\mathfrak{B}$. From $\mathfrak{A}$ and $\mathfrak{B}$, one obtains $\mathfrak{A} \mathbin{\&} \mathfrak{B}$. From $\mathfrak{A} \to \mathfrak{B}$ and $(\mathfrak{A} \to \mathfrak{B}) \mathbin{\&} \mathfrak{C} \to \curlywedge$, one obtains $\mathfrak{C} \to \curlywedge$.

A remark should be made about the axiom system, confirming that despite the numerous derivations possible from it – including, for instance, the derivation of all universally valid formulas of the [classical] propositional calculus – it fully aligns with the substantive standpoint we have adopted [in this paper]. In the introduction, specific formulas like $A \to (B \to A)$, $A \to (\overline{A} \to B)$, $A \to ((A \to B) \to B)$, $B \to (A \to A)$, $A \to (A \to A)$, $A \mathbin{\&} \overline{A} \to B$ were dismissed as universally valid. We aim to demonstrate that, no matter how intricate the derivations, these formulas cannot be proved.

We establish a more general theorem: *A formula of the form* $\mathfrak{A} \to (\mathfrak{B} \to \mathfrak{C})$ *is unprovable if* $\mathfrak{A}$ *contains neither* $\to$ *nor* $\curlywedge$. The proof proceeds as follows: we introduce a method that uniquely transforms each formula into one of the characters 0, 1, 2, 3, 4, 5. We replace each propositional variable in the formula with 1 and the symbol $\curlywedge$ with 2, and then reduce the formula with the help of the following equations until it becomes one of the characters 0, 1, 2, 3, 4, 5 (as long as it does not consist of a single character): $\overline{0} = 5$, $\overline{1} = 4$, $\overline{2} = 3$, $\overline{3} = 2$, $\overline{4} = 1$, $\overline{5} = 0$.

Furthermore, let a and b be any of the digits $0, \ldots, 5$. If $a = b$, $a = 0$, or $b = 5$, then $a \to b = 3$. Additionally, $1 \to 2 = 1 \to 4 = 3 \to 4 = 3$. In all other cases, $a \to b = 0$. Moreover, $a \mathbin{\&} b$ is the largest number c such that $c \to a = 3$ and $c \to b = 3$, while $a \vee b$ is the smallest number c such that $a \to c = 3$ and $b \to c = 3$. Such numbers exist because $0 \to a = 0 \to b = a \to 5 = b \to 5 = 3$.

All provable formulas are transformed into one of the digits 3, 4, or 5 through this process. Firstly, one can ascertain that all basic formulas transition to 3. Furthermore, if $a = 3$, 4, or 5 and $a \to b = 3$, then also $b = 3$, 4, or 5, ensuring that this property is preserved under rule (α). Since $a \mathbin{\&} b$ also takes on one of the values 3, 4, or 5 when a and b are restricted to these values, rule (β) also produces only formulas of this kind. Finally, if $a \to b = 3$ and $(a \to b) \mathbin{\&} c \to 2 = 3$, as well as **[128]** $3 \mathbin{\&} c \to 2 = 3$, then $c \to 2 = 3$. Therefore, rule (ε) also preserves this property. Now, let us consider the expression $\mathfrak{A} \to (\mathfrak{B} \to \mathfrak{C})$, where $\mathfrak{A}$ contains neither $\to$ nor $\curlywedge$. During the evaluation, $\mathfrak{A}$ receives a value of 1 or 4, while $\mathfrak{B} \to \mathfrak{C}$ has a value of 3 or 0. However, $1 \to 3$, $1 \to 0$, $4 \to 3$, and $4 \to 0$ all have a value of 0, not 3. Therefore, $\mathfrak{A} \to (\mathfrak{B} \to \mathfrak{C})$ does not belong to the set of provable formulas.

To show that $A \mathbin{\&} \overline{A} \to B$ cannot be derived, we proceed as follows. Suppose we have a proof for $A \mathbin{\&} \overline{A} \to B$. We then replace the propositional variable B everywhere in this proof with $A \to A$, which evidently leaves the proof structure unchanged. If $A \mathbin{\&} \overline{A} \to B$ were provable, then so would $A \mathbin{\&} \overline{A} \to (A \to A)$, which is ruled out according to the theorem we have proved. Furthermore, $\curlywedge \to B$ is not provable, as this formula is assigned the value 0.

It should only be briefly mentioned that the axiom system can also be extended to first-order predicate calculus by expanding the formula domain to include individual variables, predicate variables, and quantifiers in the usual manner. Let $\mathfrak{A}(x)$ be any formula containing the free individual variable x but not y, and let $\mathfrak{B}$ be any formula that does not contain x. Accordingly, one can add to (1)–(17) the following basic formulas:

(18) $\forall(x)\mathfrak{A}(x) \rightarrow \mathfrak{A}(y)$,
(19) $\mathfrak{A}(y) \rightarrow \exists(x)\mathfrak{A}(x)$,
(20) $\forall(x)(\mathfrak{B} \rightarrow \mathfrak{A}(x)) \rightarrow (\mathfrak{B} \rightarrow \forall(x)\mathfrak{A}(x))$,
(21) $\forall(x)(\mathfrak{A}(x) \rightarrow \mathfrak{B}) \rightarrow (\exists(x)\mathfrak{A}(x) \rightarrow \mathfrak{B})$,
(22) $\forall(x)(\mathfrak{A}(x) \vee \mathfrak{B}) \rightarrow \forall(x)\mathfrak{A}(x) \vee \mathfrak{B}$,
(23) $\mathfrak{B} \,\&\, \exists(x)\mathfrak{A}(x) \rightarrow \exists(x)(\mathfrak{B} \,\&\, \mathfrak{A}(x))$.

As an additional inference rule, the transition from $\mathfrak{A}(x)$ to $\forall(x)\mathfrak{A}(x)$ is added, along with a renaming rule for the free and bound individual variables. The axiom system can also be simplified by defining $\exists(x)\mathfrak{A}(x)$ as $\overline{\forall(x)\overline{\mathfrak{A}(x)}}$.

References

[1] G. Gentzen. (1935). Untersuchungen über das logische Schließen. *Mathematische Zeitschrift*, *39*, pp. 176–210, 405–431.

[2] D. Hilbert and P. Bernays. (1934). *Grundlagen der Mathematik*. Vol. 1. Berlin: Springer.

[3] D. Hilbert and W. Ackermann. (1949). *Grundzüge der theoretischen Logik*. 3rd Ed. Berlin-Göttingen-Heidelberg: Springer.

[4] I. Johansson. (1936). Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus. *Compositio mathematica*, *4*, 119-136.

[5] C. I. Lewis. (1918). *A Survey of Symbolic Logic*. Berkeley: University of California Press.

[6] C. I. Lewis and C. H. Langford. (1932). *Symbolic Logic*. New York & London: The Century Company.

[7] A. Schmidt. (1954). Ein aussagenlogischer Zugang zu den Modalitäten der strikten Logik. *Proceedings of the International Congress of Mathematicians*. Vol. 2, pp. 407-408.

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