

Polyhedral completeness of intermediate logics: the Nerve Criterion

Sam Adam-Day¹, Nick Bezhanishvili², David Gabelaia³, and Vincenzo Marra⁴

¹Mathematical Institute, University of Oxford, United Kingdom

²Institute for Logic, Language and Computation, University of Amsterdam, The Netherlands

³A. Razmadze Mathematical Institute of I. Javakishvili Tbilisi State University, Georgia

⁴Dipartimento di Matematica “Federigo Enriques”, Università degli Studi di Milano, Italy

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Abstract

We investigate a recently-devised polyhedral semantics for intermediate logics, in which formulas are interpreted in n -dimensional polyhedra. An intermediate logic is *polyhedrally complete* if it is complete with respect to some class of polyhedra. The first main result of this paper is a necessary and sufficient condition for the polyhedral-completeness of a logic. This condition, which we call the Nerve Criterion, is expressed in terms of Alexandrov’s notion of the nerve of a poset. It affords a purely combinatorial characterisation of polyhedrally-complete logics.

Using the Nerve Criterion we show, easily, that there are continuum many intermediate logics that are not polyhedrally-complete but which have the finite model property. We also provide, at considerable combinatorial labour, a countably infinite class of logics axiomatised by the Jankov-Fine formulas of ‘starlike trees’ all of which are polyhedrally-complete. The polyhedral completeness theorem for these ‘starlike logics’ is the second main result of this paper.

1 Introduction

The genesis of many connections between logic and geometry is rooted in the discovery of topological semantics for intuitionistic and modal logic, as pioneered by Marshall Stone [Sto38], Tang Tsao-Chen [Tsa38], Alfred Tarski [Tar39] and John C. C. McKinsey [McK41]. This semantics is now well-known. In short, one starts with a topological space X , and interprets intuitionistic formulas inside the Heyting algebra of open sets of X , and modal formulas inside the modal algebra of subsets of X with \Box interpreted as the topological interior operator. A celebrated result due to Tarski [Tar39] states that this provides a complete semantics for intuitionistic propositional logic (**IPC**) on the one hand, and the modal logic **S4** on the other. Moreover, one can even obtain completeness with respect to certain individual spaces. Specifically, McKinsey and Tarski showed [MT44] that for any separable metric space X without isolated points, if **IPC** $\not\models \phi$, then ϕ has a countermodel based on X , and similarly with **S4** in place of **IPC**. Later, Helena Rasiowa and Roman Sikorski showed that one can do without the assumption of separability [RS63].

This result traces out an elegant interplay between topology and logic; however, it simultaneously establishes limits on the expressive power of this kind of interpretation. Indeed, examples of separable metric spaces without isolated points are the n -dimensional Euclidean space \mathbb{R}^n and the Cantor space 2^ω . What McKinsey and Tarski’s result shows, then, is that these spaces have the same logic, namely **IPC** (or **S4**). The upshot is that topological semantics does not allow logic to capture much of the geometric content of a space.

A natural idea is that, if we want to remedy the situation and allow for the capture of more information about a space, then we need an algebra finer than the Heyting algebra of open sets, or the modal algebra of arbitrary subsets with the interior operator. For example, Marco Aiello, Johan van Benthem, Guram Bezhanishvili and Mai Gehrke consider the modal logic of *chequered* subsets of \mathbb{R}^n : finite unions of sets of the form $\prod_{i=1}^n C_i$, where each $C_i \subseteq \mathbb{R}$ is convex ([ABB03] and [BBG03]; see also [BB07]). In [Bez+18], [Gab+17] and [Gab+18], this algebra-refinement idea is taken one step further. To be able to capture some of the geometric content of a space, one may restrict attention to topological spaces and subsets which are *polyhedra* (of arbitrary dimension). Indeed, the set $\text{Sub}_o(P)$ of open subpolyhedra of P is a Heyting algebra under \subseteq (and a similar result holds in the modal case). This allows for an interpretation of intuitionistic and modal formulas in $\text{Sub}_o(P)$. The main result of [Bez+18] is that more is true. A polyhedral analogue of Tarski’s theorem holds: these polyhedral semantics are complete for **IPC** and **S4.Grz**. Furthermore, this approach delivers that logic can capture the dimension of the polyhedron in which it is interpreted, via the bounded depth formulas bd_n [CZ97, Sec. 2.4]. In particular, the polyhedron P is n dimensional if, and only if, P validates bd_{n+1} and does not validate bd_{n+2} for $n \in \omega$ [Bez+18].

In this paper we make further advances in the study of polyhedral semantics. We introduce and study polyhedral completeness for intermediate logics. We say that an intermediate logic L is polyhedrally complete if there is a class \mathcal{C} of polyhedra such that L is the logic of \mathcal{C} . It follows from [Bez+18] that **IPC** and the logic **BD_n** of bounded depth n , for each n , are polyhedrally-complete. We construct infinitely many polyhedrally-complete logics, and show that there are continuum many polyhedrally incomplete ones all of which have the finite model property.

To this end we employ a time-honoured tool from combinatorial and polyhedral geometry, the *nerve* of a poset (=partially ordered set). The nerve will be our key concept relating logic with polyhedral geometry. In detail, the nerve $\mathcal{N}(F)$ of the poset F is the collection of finite non-empty chains in F ordered by inclusion. As was already noted in [Bez+18], given a polyhedron P , a triangulation of P corresponds to a validity-preserving map from P onto the poset F of the faces of the triangulation. Through Esakia duality, in turn, this validity-preserving map corresponds to an embedding of the Heyting algebra of upsets of F into the Heyting algebra of open subpolyhedra of P . Nerves are closely related to *barycentric subdivisions* of triangulations. Indeed, if a finite poset F is the face poset of some triangulation Σ of a polyhedron P , then $\mathcal{N}(F)$ corresponds to a barycentric subdivision of Σ .

Applying methods and results from rational polyhedral geometry we present a proof of our first main result, the *Nerve Criterion for polyhedral completeness* (Theorem 4.1): A logic L is complete with respect to some class of polyhedra if and only if it is the logic of a class of finite posets closed under taking nerves. Thus, we obtain that the logic of any given polyhedron is the logic of the iterated nerves of any one of its triangulations. The criterion yields many negative results, showing in particular that there are continuum-many non-polyhedrally-complete logics with the finite model property (Theorem 5.4).

As to positive results, we consider logics defined using *starlike trees* — trees which only branch at the root — as forbidden configurations. *Starlike logics* are then those defined by the Jankov-Fine formulas of a collection of starlike trees. Exploiting the Nerve Criterion, and a result by Zakharyashev [Zak93] that all these logics have the finite model property, we prove our second main result (Theorem 6.15): Every starlike

logic is polyhedrally-complete. This yields a countably infinite class of polyhedrally-complete logics of each finite height and of infinite height. (For instance, Scott’s well-known logic **SL** is in this class.) As forbidden configurations, starlike trees have a natural geometric meaning, expressing connectedness properties of polyhedral spaces.

The paper is organised as follows. In Section 2, we give the required background on intermediate logics and polyhedral geometry. Section 3 presents the polyhedral semantics first defined in [Bez+18]. In Section 4, we present and prove the Nerve Criterion for polyhedral-completeness (Theorem 4.1). Making use of this criterion, Section 5 establishes that all stable logics (as defined in [BB09]) of height at least 2 are polyhedrally-incomplete. Then in Section 6, we define the class of starlike logics, and prove that each one is polyhedrally-complete. The techniques in these two sections are entirely combinatorial.

Finally, let us briefly comment on further research. One major problem already mentioned in [Bez+18] is to characterise the logic of piecewise-linear manifolds of a fixed dimension. Here we announce significant progress on this question; the results will appear in a forthcoming paper. A second relevant goal would be a complete classification of polyhedrally-complete logics. At the time of writing, we do not know how to attain this goal. One might wonder if our results on starlike logics extend to arbitrary trees, or even to a wider class of posets. As to the latter, some negative results are obtained in [Ada19, Corollary 4.12]. For the former, the situation is rather obscure to us at the time of writing; cf. the discussion on ‘general trees’ in [Ada19, p. 61]. Identifying further classes of polyhedrally complete logics beyond the starlike ones introduced in this paper would be the next immediate task in the direction of obtaining a classification of polyhedrally-complete logics.¹

2 Preliminaries

In this section we remind the reader of the relational and algebraic semantics for intermediate logics, and survey the definitions and results which will play their part in the forthcoming. As a main reference we use [CZ97]. We assume rather less familiarity with polyhedral geometry, and thus present in more detail the material we need.

2.1 Posets as Kripke frames

A (*Kripke*) *frame* for intuitionistic logic is simply a poset. We thus use the term ‘frame’ in this paper as a synonym of ‘poset’. The validity relation \models between frames and formulas is defined in the usual way, see, e.g., [CZ97, Ch. 2]. Given a class of frames \mathbf{C} , its *logic* is:

$$\text{Logic}(\mathbf{C}) := \{ \phi \text{ a formula} \mid \forall F \in \mathbf{C}: F \models \phi \}$$

Conversely, given a logic \mathcal{L} , define:

$$\begin{aligned} \text{Frames}(\mathcal{L}) &:= \{ F \text{ a Kripke frame} \mid F \models \mathcal{L} \} \\ \text{Frames}_{\text{fin}}(\mathcal{L}) &:= \{ F \text{ a finite Kripke frame} \mid F \models \mathcal{L} \} \end{aligned}$$

A logic \mathcal{L} has the *finite model property* (fmp) if it is the logic of a class of finite frames. Equivalently, if $\mathcal{L} = \text{Logic}(\text{Frames}_{\text{fin}}(\mathcal{L}))$.

Fix a poset F . For any $x \in F$, its *upset*, *downset*, *strict upset* and *strict downset* are defined, respectively, as follows.

$$\begin{aligned} \uparrow(x) &:= \{ y \in F \mid y \geq x \} \\ \downarrow(x) &:= \{ y \in F \mid y \leq x \} \\ \uparrow\uparrow(x) &:= \{ y \in F \mid y > x \} \\ \downarrow\downarrow(x) &:= \{ y \in F \mid y < x \} \end{aligned}$$

¹This paper is partly based on the fist-named author’s M.Sc. thesis [Ada19].

For any set $S \subseteq F$, its *upset* and *downset* are defined, respectively, as follows.

$$\begin{aligned}\uparrow U &:= \bigcup_{x \in U} \uparrow(x) \\ \downarrow U &:= \bigcup_{x \in U} \downarrow(x)\end{aligned}$$

A *subframe* is a subposet. A subframe $U \subseteq F$ is *upwards-closed* or a *generated subframe* if $U = \uparrow U$, and it is *downwards-closed* if $\downarrow U = U$. The *Alexandrov topology* on F is the set $\text{Up } F$ of its upwards-closed subsets. This constitutes a topology on F . In the sequel, we will freely switch between thinking of F as a poset and as a topological space. Note that the closed sets in this topology correspond to downwards-closed sets.

A *chain* in F is $X \subseteq F$ which as a subposet is linearly-ordered. The *length* of the chain X is $|X|$. A chain $X \subseteq F$ is *maximal* if there is no chain $Y \subseteq F$ such that $X \subset Y$ (i.e. such that X is a proper subset of Y). The *height* of F is the element of $\mathbb{N} \cup \{\infty\}$ defined by:

$$\text{height}(F) := \sup\{|X| - 1 \mid X \subseteq F \text{ is a chain}\}$$

For notational uniformity, say that this value is also the *depth* of F , $\text{depth}(F)$. For any $x \in F$, define its *height* and *depth* as follows.

$$\begin{aligned}\text{height}(x) &:= \text{height}(\downarrow(x)) \\ \text{depth}(x) &:= \text{depth}(\uparrow(x))\end{aligned}$$

The *height* of a logic \mathcal{L} is the element of $\mathbb{N} \cup \{\infty\}$ given by:

$$\text{height}(\mathcal{L}) := \sup\{\text{height}(F) \mid F \in \text{Frames}(\mathcal{L})\}$$

A *top element* of F is $t \in F$ such that $\text{depth}(t) = 0$. For any $x, y \in F$, say that x is an *immediate predecessor* of y , and that y is an *immediate successor* of x , if $x < y$ and there is no $z \in F$ such that $x < z < y$. Write $\text{Succ}(x)$ for the collection of immediate successors of x .

The poset F is *rooted* if it has a minimum element, which is called the *root*, and is usually denoted by \perp . Define:

$$\begin{aligned}\text{Frames}_{\perp}(\mathcal{L}) &:= \{F \in \text{Frames}(\mathcal{L}) \mid F \text{ is rooted}\} \\ \text{Frames}_{\perp, \text{fin}}(\mathcal{L}) &:= \{F \in \text{Frames}_{\text{fin}}(\mathcal{L}) \mid F \text{ is rooted}\}\end{aligned}$$

An *antichain* in F is a subset $Z \subseteq F$ in which no two elements are comparable. The *width*, notation $\text{width}(F)$, of F is the cardinality of the largest antichain in F .

A function $f : F \rightarrow G$ is a *p-morphism* if for every $x \in F$ we have:

$$f(\uparrow(x)) = \uparrow(f(x))$$

Equivalently, f should satisfy the following conditions.

$$\begin{aligned}\forall x, y \in F : (x \leq y \Rightarrow f(x) \leq f(y)) & \quad (\text{Forth}) \\ \forall x \in F : \forall z \in G : (f(x) \leq z \Rightarrow \exists y \in F : (x \leq y \wedge f(y) = z)) & \quad (\text{Back})\end{aligned}$$

An *up-reduction* from F to G is a surjective p-morphism f from an upwards-closed set $U \subseteq F$ to G . Write $f : F \circrightarrow G$.

Proposition 2.1. *If there is an up-reduction $F \circrightarrow G$ then $\text{Logic}(F) \subseteq \text{Logic}(G)$. In other words, if $G \not\models \phi$ then $F \not\models \phi$.*

Proof. See [CZ97, Corollary 2.8, p. 30 and Corollary 2.17, p. 32]. \square

Corollary 2.2. *If \mathbf{C} is any collection of frames and $\mathcal{L} = \text{Logic}(\mathbf{C})$, then:*

$$\mathcal{L} = \text{Logic}(\text{Frames}_{\perp}(\mathcal{L}))$$

Proof. First, $\mathcal{L} \subseteq \text{Logic}(\text{Frames}_{\perp}(\mathcal{L}))$. Conversely, suppose $\mathcal{L} \not\models \phi$. Then there exists $F \in \mathbf{C}$ such that $F \not\models \phi$, hence there is $x \in F$ such that $x \not\models \phi$ (for some valuation on F), meaning that $\uparrow(x) \not\models \phi$. Now, $\uparrow(x)$ is upwards-closed in F , hence $\text{id}_{\uparrow(x)}$ is an up-reduction $F \circrightarrow \uparrow(x)$. Then by Proposition 2.1, we get that $\uparrow(x) \models \mathcal{L}$, so that $\uparrow(x) \in \text{Frames}_{\perp}(\mathcal{L})$. \square

2.2 Heyting algebras, topological semantics

A *Heyting algebra* is a bounded lattice equipped with a *Heyting implication* \rightarrow that satisfies:

$$c \leq a \rightarrow b \iff c \wedge a \leq b$$

The validity relation \models between Heyting algebras and formulas is defined in the usual way; the notation $\text{Logic}(-)$ is extended appropriately. The logic of a Heyting algebra is exactly the logic of its finitely generated subalgebras. Say that A is *locally-finite* if for every $S \subseteq A$ finite, the algebra $\langle S \rangle$ generated by S is finite. If F is any poset, the bounded distributive lattice $\text{Up}F$ is a Heyting algebra, and:

Proposition 2.3. *If F is a poset, $\text{Logic}(F) = \text{Logic}(\text{Up}F)$*

Proof. See [CZ97, Corollary 8.5, p. 238]. \square

Co-Heyting algebras are the order-duals of Heyting algebras. Specifically, a *co-Heyting algebra* is a bounded lattice equipped with a *co-Heyting implication* \leftarrow that satisfies:

$$a \leftarrow b \leq c \iff a \leq b \vee c$$

For more information on co-Heyting algebras, the reader is referred to [MT46, §1] and [Rau74], where they are called ‘Brouwerian algebras’.

Given a topological space X , we regard the collection of open sets $\mathcal{O}(X)$ of X as a Heyting algebra in the standard manner, cf. [CZ97, Proposition 8.31, p. 247]. (Recall that

$$U \rightarrow V = \text{Int}(U^c \cup V)$$

where Int denotes the topological interior operator, and $-^c$ is set-theoretic complement.) We can thus interpret formulas in topological spaces. Write $X \models \phi$ for $\mathcal{O}(X) \models \phi$, and extend the remaining notations accordingly.

The topological space X also comes with a co-Heyting algebra, namely its collection of closed sets $\mathcal{C}(X)$. The co-Heyting implication on $\mathcal{C}(X)$ satisfies:

$$C \leftarrow D := \text{Cl}(C \setminus D)$$

where Cl denotes the topological closure operator. If a Heyting algebra A is regarded as a poset category (A, \leq) , then its opposite category $A^{\text{op}} = (A, \geq)$ is a co-Heyting algebra. In the case of the Heyting algebra $\mathcal{O}(X)$ of open sets of X , $\mathcal{O}(X)^{\text{op}}$ is isomorphic to the co-Heyting algebra $\mathcal{C}(X)$ of closed subsets of X .

2.3 Jankov-Fine formulas as forbidden configurations

To every finite, rooted frame Q , we associate a formula $\chi(Q)$, the *Jankov-Fine* formula of Q (also called its *Jankov-De Jongh formula*). The precise definition of $\chi(Q)$ is somewhat involved, but the exact details of this syntactical form are not relevant for our considerations. What matters to us is its notable semantic property.

Theorem 2.4. *For any frame F , we have that $F \models \chi(Q)$ if and only if F does not up-reduce to Q .*

Proof. See [CZ97, §9.4, p. 310] for a treatment in which Jankov-Fine formulas are considered as specific instances of more general ‘canonical formulas’. An alternative proof can be found in [Bez06, §3.3, p. 56], which gives a complete definition of $\chi(Q)$. See also [BB09] for an algebraic version of this result. \square

Jankov-Fine formulas formalise the intuition of ‘forbidden configurations’. The formula $\chi(Q)$ ‘forbids’ the configuration Q from its frames.

The following consequence of Theorem 2.4 will come in handy later on.

Corollary 2.5. *Let $\mathcal{L} = \text{Logic}(\mathbf{C})$ where \mathbf{C} is a class of frames. Then:*

$$\text{Frames}_{\perp, \text{fin}}(\mathcal{L}) = \{F \text{ finite, rooted frame} \mid \exists G \in \mathbf{C}: G \multimap F\}$$

Proof. First, if F is a finite, rooted frame such that there is $G \in \mathbf{C}$ and an up-reduction $G \circ \rightarrow F$, then by Proposition 2.1 we have that $F \in \text{Frames}_{\perp, \text{fin}}(\mathcal{L})$. Conversely take F finite and rooted, and assume that there is no $G \in \mathbf{C}$ with $G \circ \rightarrow F$. Then by Theorem 2.4, $G \models \chi(F)$ for every $G \in \mathbf{C}$; whence $\mathcal{L} \vdash \chi(F)$. By Theorem 2.4, $F \not\models \chi(F)$ implying $F \notin \mathcal{L}$. This yields $F \notin \text{Frames}_{\perp, \text{fin}}(\mathcal{L})$. \square

2.4 Intermediate logics

The logic **IPC** is intuitionistic propositional logic. An *intermediate logic* is any consistent logic extending **IPC**. Classical logic, **CPC**, is the largest intermediate logic.

Proposition 2.6. *IPC is the logic of the class of all finite frames, i.e. IPC has the fmp.*

Proof. See [CZ97, Theorem 2.57, p. 49]. \square

For every $n \in \mathbb{N}$, let \mathbf{BD}_n be the logic of all finite frames of height at most n . This has the following axiomatisation in terms of Jankov-Fine formulas.

Proposition 2.7. *\mathbf{BD}_n is the logic axiomatised by IPC plus the Jankov-Fine formula of the chain (linear order) on $n + 1$ elements.*

Proof. See [CZ97, Table 9.7, p. 317, and §9]. \square

Scott's Logic, **SL**, is usually axiomatised by the Scott sentence:

$$\mathbf{SL} = \mathbf{IPC} + \mathbf{IPC} + ((\neg\neg p \rightarrow p) \rightarrow p \vee \neg p) \rightarrow \neg p \vee \neg\neg p$$

This logic can also be axiomatised using a forbidden configuration, as follows.

Proposition 2.8. $\mathbf{SL} = \mathbf{IPC} + \chi(\mathfrak{A}_\rho)$.

Proof. See [CZ97, Table 9.7, p. 317, and §9]. \square

2.5 Polytopes, polyhedra, and simplices

Polyhedra are certain subsets of finite-dimensional real affine spaces \mathbb{R}^n . An *affine combination* of $x_0, \dots, x_d \in \mathbb{R}^n$ is a point $r_0x_0 + \dots + r_dx_d$, where $r_0, \dots, r_d \in \mathbb{R}$ are such that $r_0 + \dots + r_d = 1$. A *convex combination* is an affine combination in which additionally each $r_i \geq 0$. Given a set $S \subseteq \mathbb{R}^n$, its *convex hull*, notation $\text{Conv}S$, is the collection of convex combinations of its elements. (We stress that each convex combination involves, by definition, a finite subset of S only.) A subspace $S \subseteq \mathbb{R}^n$ is *convex* if $\text{Conv}S = S$. A *polytope* is the convex hull of a finite set. A *polyhedron* in \mathbb{R}^n is a set which can be expressed as the finite union of polytopes. A *subpolyhedron* of a polyhedron P in \mathbb{R}^n is a subset of P which is itself a polyhedron. Note that every polyhedron is closed and bounded, hence compact, in the canonical (Euclidean) topology carried by the real affine space \mathbb{R}^n . All topological notions pertaining to polyhedra in the following refer to this topology.

A set of points x_0, \dots, x_d is *affinely independent* if whenever:

$$r_0x_0 + \dots + r_dx_d = \mathbf{0} \quad \text{and} \quad r_0 + \dots + r_d = 0$$

we must have that $r_0, \dots, r_d = 0$. This is equivalent to saying that the vectors:

$$x_1 - x_0, \dots, x_d - x_0$$

are linearly independent. A *d-simplex* is the convex hull σ of $d + 1$ affinely independent points x_0, \dots, x_d , which we call its *vertices*. Write $\sigma = x_0 \cdots x_d$; the *dimension* of σ is d .

Proposition 2.9. *Every simplex determines its vertex set: two simplices coincide if and only if they share the same vertex set.*

Proof. See [Mau80, Proposition 2.3.3, p. 32]. \square

A *face* of σ is the convex hull τ of some non-empty subset of $\{x_0, \dots, x_d\}$ (note that τ is then a simplex too). Write $\tau \preceq \sigma$, and $\tau \prec \sigma$ if $\tau \neq \sigma$.

Since x_0, \dots, x_d are affinely independent, every point $x \in \sigma$ can be expressed uniquely as a convex combination $x = r_0x_0 + \dots + r_dx_d$ with $r_0, \dots, r_d \geq 0$ and $r_0 + \dots + r_d = 1$. Call the tuple (r_0, \dots, r_d) the *barycentric coordinates* of x in σ . The *barycentre* $\hat{\sigma}$ of σ is the special point whose barycentric coordinates are $(\frac{1}{d+1}, \dots, \frac{1}{d+1})$. The *relative interior* of σ is defined:

$$\text{Relint } \sigma := \{r_0x_0 + \dots + r_dx_d \in \sigma \mid r_0, \dots, r_d > 0\}$$

The relative interior of σ is ‘ σ without its boundary’ in the following sense. The *affine subspace spanned by σ* is the set of all affine combinations of x_0, \dots, x_d . Then the relative interior of σ coincides with the topological interior of σ inside this affine subspace, the latter being equipped with the subspace topology it inherits from \mathbb{R}^n . Note that $\text{Cl Relint } \sigma = \sigma$, the closure being taken in the ambient space \mathbb{R}^n .

2.6 Triangulations

A *simplicial complex* in \mathbb{R}^n is a finite set Σ of simplices satisfying the following conditions.

- (a) Σ is \prec -downwards-closed: whenever $\sigma \in \Sigma$ and $\tau \prec \sigma$ we have $\tau \in \Sigma$.
- (b) If $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is either empty or a common face of σ and τ .

The *support* of Σ is the set $|\Sigma| := \bigcup \Sigma$. Note that by definition this set is automatically a polyhedron. We say that Σ is a *triangulation* of the polyhedron $|\Sigma|$. The set Σ is a poset under \prec , called the *face poset* of the triangulation. A *subcomplex* of Σ is subset which is itself a simplicial complex. Note that a subcomplex, as a poset, is precisely a downwards-closed set. Given $\sigma \in \Sigma$, its *open star* is defined:

$$o(\sigma) := \bigcup \{\text{Relint}(\tau) \mid \tau \in \Sigma \text{ and } \sigma \preceq \tau\}$$

Proposition 2.10. *The relative interiors of the simplices in a simplicial complex Σ partition $|\Sigma|$. That is, for every $x \in |\Sigma|$, there is exactly one $\sigma \in \Sigma$ such that $x \in \text{Relint } \sigma$.*

Proof. See [Mau80, Proposition 2.3.6, p. 33]. □

In light of Proposition 2.10, for any $x \in |\Sigma|$ let us write σ^x for the unique $\sigma \in \Sigma$ such that $x \in \text{Relint } \sigma$. The simplex σ^x is known as the *carrier* of x .

Proposition 2.11. *Let Σ be a simplicial complex, take $\tau \in \Sigma$ and $x \in \text{Relint } \tau$. Then no proper face $\sigma \prec \tau$ contains x . This means that σ^x is the inclusion-smallest simplex containing x .*

Proof. See [Bez+18, Lemma 3.1]. □

The next result is a basic fact of polyhedral geometry, and is of fundamental importance in its connection with logic. For Σ a triangulation and S a subset of the ambient space \mathbb{R}^n , define:

$$\Sigma_S := \{\sigma \in \Sigma \mid \sigma \subseteq S\}$$

This, being a downwards-closed subset of Σ , is a subcomplex of Σ .

Lemma 2.12 (Triangulation Lemma). *Any polyhedron admits a triangulation which simultaneously triangulates each of any fixed finite set of subpolyhedra. That is, for a collection of polyhedra P, Q_1, \dots, Q_m such that each $Q_i \subseteq P$, there is a triangulation Σ of P such that Σ_{Q_i} triangulates Q_i for each i .*

Proof. See [RS72, Theorem 2.11 and Addendum 2.12, p. 16]. □

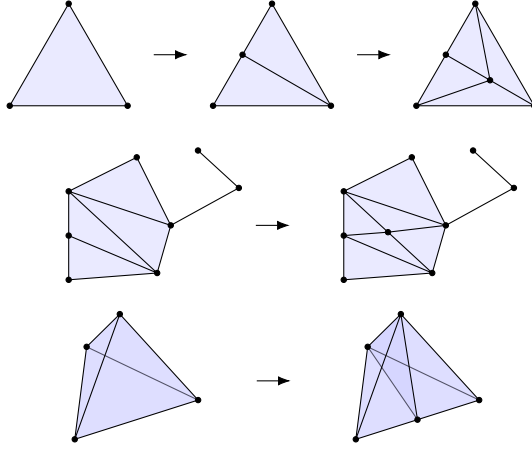


Figure 1: Examples of elementary stellar subdivisions

2.7 Stellar and barycentric subdivisions

For Σ and Δ simplicial complexes, Δ is a *subdivision* or *refinement* of Σ , notation $\Delta \triangleleft \Sigma$, if $|\Sigma| = |\Delta|$ and every simplex of Δ is contained in a simplex of Σ .

Lemma 2.13. *If $\Delta \triangleleft \Sigma$ then for every $\sigma \in \Sigma$ we have:*

$$\sigma = \bigcup \{ \tau \in \Delta \mid \tau \subseteq \sigma \}$$

Proof. Let $S := \{ \tau \in \Delta \mid \tau \subseteq \sigma \}$. Clearly $\bigcup S \subseteq \sigma$. Conversely, for $x \in \sigma$, let $\tau^x \in \Delta$ be such that $x \in \text{Relint } \tau^x$. Since Δ refines Σ , there is some $\rho \in \Sigma$ such that $\tau^x \subseteq \rho$; assume that ρ is inclusion-minimal with this property. It follows from [Spa66, §3, Lemma 3, p. 121] that $\text{Relint } \tau^x \subseteq \text{Relint } \rho$, meaning that $x \in \sigma \cap \text{Relint } \rho$. By condition (b) in the definition of a simplicial complex, we have that $\sigma \cap \rho$ is face of ρ . But then by Proposition 2.11, $\rho \preccurlyeq \sigma$, since otherwise $\sigma \cap \rho$ would be a proper face of ρ containing $x \in \text{Relint } \rho$. Therefore $\tau^x \subseteq \rho \subseteq \sigma$ so that $x \in \bigcup S$. \square

We now introduce a special class of subdivisions, for which the original source [Ale30] remains a fundamental reference. Let Σ be a simplicial complex, and let $c \in |\Sigma|$. The *elementary stellar subdivision of Σ at c* is the set of simplices Δ obtained from Σ via the following transformation: Replace each simplex $\sigma \in \Sigma$ that contains c by the set of all simplices $\text{Conv} \{ \tau \cup \{c\} \}$, where τ ranges over all faces of σ that do not contain c . It can then be proved that Δ is again a triangulation, and a subdivision of Σ . The equality $\Sigma = \Delta$ holds precisely when the chosen c is a vertex of Σ . If Δ is a subdivision of Σ that is obtained via a finite number of successive elementary subdivisions of Σ , then Δ is a *stellar subdivision* of Σ . See Figure 1.

If Δ is obtained from Σ via an elementary stellar subdivision at $c \in |\Sigma|$, and c is moreover the barycentre of the vertices of its carrier simplex $\sigma^c \in \Sigma$ (see Proposition 2.10 and the comments following it), then Δ is an *elementary barycentric subdivision* of Σ (at the barycentre c).

The *barycentric subdivision* $\text{Sd } \Sigma$ of Σ is then defined as the refinement of Σ obtained by successively applying elementary barycentric subdivisions at each simplex of Σ , proceeding in decreasing order of dimension. It can be proved that $\text{Sd } \Sigma$ does not depend on the chosen ordering of the simplices of Σ . See the examples in Figure 2. In the literature, $\text{Sd } \Sigma$ is also often called the *first derived* subdivision of Σ ; cf. e.g. [RS72]. We inductively define, for each $k \in \mathbb{N}$, the k^{th} *derived subdivision* of Σ : $\Sigma^{(0)} := \Sigma$; and $\Sigma^{(k)} = \text{Sd } \Sigma^{(k-1)}$.

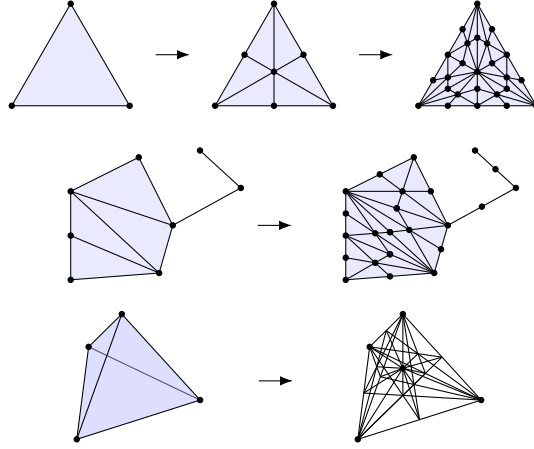


Figure 2: Examples of barycentric subdivision. Each simplex in the simplicial complex is divided at its barycentre, proceeding in decreasing order of dimension. The bottom right tetrahedron is drawn without filled-in faces to aid clarity.

3 The algebra of open subpolyhedra

With the preliminaries in place, we relate intuitionistic logic and polyhedra. For further details please see [Bez+18].

3.1 Polyhedral semantics

Given a polyhedron P , let $\text{Sub}P$ denote the set of its subpolyhedra.

Theorem 3.1. $\text{Sub}P$ is a co-Heyting algebra, and a subalgebra of $\mathcal{C}(P)$.

Proof. See [Bez+18, Corollary 3.4]. The proof makes fundamental use of the Triangulation Lemma. \square

By an *open subpolyhedron* of a polyhedron P in this paper we mean the complement (in P) of a subpolyhedron of P . Denote by Sub_oP the collection of open subpolyhedra in P . Evidently, there is an isomorphism $\text{Sub}_oP \cong (\text{Sub}P)^{\text{op}}$, and Theorem 3.1 yields the following.

Theorem 3.2. For any polyhedron P , Sub_oP is a Heyting algebra, and a subalgebra of $\mathcal{O}(P)$.

For any formula ϕ and polyhedron P , say that $P \models \phi$ if and only if $\text{Sub}_oP \models \phi$ as a Heyting algebra. Call an intermediate logic *polyhedrally-complete* if it is the logic of some class of polyhedra. In [Bez+18], it is shown that **IPC** is polyhedrally-complete, being the logic of all polyhedra, while **BD_n** is the logic of all polyhedra of dimension at most n . It is also shown that all polyhedrally-complete logics must have the finite model property; cf. Theorem 3.7 below.

3.2 Triangulation subalgebras

Let Σ be a triangulation of the polyhedron P . Then $\Sigma \subseteq \text{Sub}P$. Let $P_c(\Sigma)$ be the sublattice of $\text{Sub}P$ generated by Σ .

Lemma 3.3. $P_c(\Sigma)$ is a co-Heyting subalgebra of $\text{Sub}P$.

Proof. See [Bez+18, Lemma 3.6]. \square

Call any algebra of the form $P_c(\Sigma)$ a *triangulation subalgebra*.

Lemma 3.4. *Every finitely-generated subalgebra of $\text{Sub}P$ is contained in some triangulation algebra.*

Proof. See [Bez+18, Lemma 3.2]. Essentially, this is the content of the Triangulation Lemma 2.12. \square

Turning now to the dual, every triangulation Σ of a polyhedron P gives rise to a Heyting subalgebra $\mathbb{P}_0(\Sigma)$ of Sub_0P , which we also call a *triangulation subalgebra*, generated by the complements of the simplices in Σ .

Corollary 3.5. *For any polyhedron P , Sub_0P is a locally-finite Heyting algebra.*

Proof. This follows from the dual of Lemma 3.4 and the fact that triangulation subalgebras are finite. \square

The algebra $\mathbb{P}_0(\Sigma)$, though not necessarily easy to visualise geometrically, is in fact precisely the algebra of upsets of the poset Σ .

Lemma 3.6. *The map:*

$$\begin{aligned} \gamma^\dagger: \text{Up } \Sigma &\rightarrow \mathbb{P}_0(\Sigma) \\ U &\mapsto \bigcup_{\sigma \in U} \text{Relint}(\sigma) \end{aligned}$$

is an isomorphism of Heyting algebras.

Proof. See [Bez+18, Lemma 4.3]. \square

As a consequence of the preceding results, we have:

Theorem 3.7. *The logic of a polyhedron is the logic of its triangulations.*

The following additional facts about triangulation algebras will be useful later on.

- Lemma 3.8.** (1) *Triangulation algebras determine their corresponding triangulations. That is, for any two triangulations Σ and Δ , if $\mathbb{P}_0(\Sigma) = \mathbb{P}_0(\Delta)$ then $\Sigma = \Delta$.*
(2) *If Σ and Δ are triangulations which are isomorphic as posets then $\mathbb{P}_0(\Sigma) \cong \mathbb{P}_0(\Delta)$.*
(3) *If Δ refines Σ , then $\mathbb{P}_0(\Sigma)$ is a subalgebra of $\mathbb{P}_0(\Delta)$.*

Proof. (1) It follows from conditions (a) and (b) on simplicial complexes that $\mathbb{P}_c(\Sigma)$ consists exactly of the unions of elements of Σ , and similarly for Δ . Assume $\mathbb{P}_0(\Sigma) = \mathbb{P}_0(\Delta)$, so that $\mathbb{P}_c(\Sigma) = \mathbb{P}_c(\Delta)$, and take $\sigma \in \Sigma$. Then $\sigma \in \mathbb{P}_c(\Delta)$, so $\sigma = \bigcup S$ for some $S \subseteq \Delta$, and similarly each $\tau \in S$ is $\tau = \bigcup T_\tau$ for some $T_\tau \subseteq \Sigma$. Hence:

$$\sigma = \bigcup_{\tau \in S} T_\tau$$

But then by condition (b) on Σ , every $\rho \in \bigcup_{\tau \in S} T_\tau$ must either be equal to σ or be a proper face of σ . Since $\text{Relint } \sigma$ contains no proper face of σ , we must have $\sigma \in T_\tau$ for some $\tau \in S$. But then $\sigma \subseteq \tau \subseteq \sigma$, and so $\sigma \in \Delta$. Applying this argument also in the other direction, we get that $\Sigma = \Delta$.

(2) This follows from Lemma 3.6.

(3) By Lemma 2.13, every $\sigma \in \Sigma$ is the union of simplices in Δ . Whence $\Sigma \subseteq \mathbb{P}_c(\Delta)$. Therefore, by definition $\mathbb{P}_c(\Sigma) \subseteq \mathbb{P}_c(\Delta)$. From this it follows that $\mathbb{P}_0(\Sigma) \subseteq \mathbb{P}_0(\Delta)$. \square

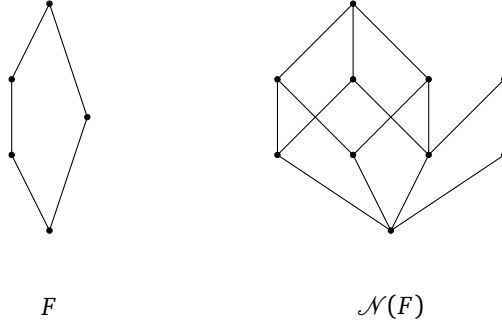


Figure 3: An example showing that the $\text{Frames}_{\text{fin}}(\mathbf{SL})$ is not closed under \mathcal{N} , even though \mathbf{SL} is polyhedrally-complete.

3.3 PL homeomorphisms

Let $P \subseteq \mathbb{R}^m$ and $Q \subseteq \mathbb{R}^n$ be polyhedra. A continuous function $f : P \rightarrow Q$ is *piecewise-linear*, or is a *PL map*, if the graph of f is a polyhedron in the product space $\mathbb{R}^m \times \mathbb{R}^n$. A *PL homeomorphism* is a PL map that is a homeomorphism.

Proposition 3.9. *The inverse of a PL homeomorphism is a PL homeomorphism.*

Proof. See [RS72, p. 6]. □

Proposition 3.10. *If P and Q are PL homeomorphic then $\text{Sub}_o Q$ and $\text{Sub}_o P$ are isomorphic Heyting algebras, and $\text{Logic}(P) = \text{Logic}(Q)$.*

Proof. It is obvious that any homeomorphism between P and Q induces an isomorphism of their open-set lattices by taking inverse images. Since the inverse image of a subpolyhedron under a PL homeomorphism is again a subpolyhedron [RS72, Corollary 2.5, p. 13], and in light of Proposition 3.9, we see that when the homeomorphism is PL this isomorphism of open-set lattices descends to an isomorphism of distributive lattices between $\text{Sub}_o P$ and $\text{Sub}_o Q$. This implies that $\text{Sub}_o P$ and $\text{Sub}_o Q$ are isomorphic as Heyting algebras, too, because the Heyting implication is uniquely determined by the underlying lattice structure, and the proof is complete. □

4 The Nerve Criterion

Given a poset F , its *nerve*, $\mathcal{N}(F)$, is the collection of finite non-empty chains in F ordered by inclusion. The following theorem is the first main contribution of the paper:

Theorem 4.1 (The Nerve Criterion). *A logic is polyhedrally-complete if, and only if, it is the logic of a class of finite posets closed under the nerve operator \mathcal{N} .*

The utility of the Nerve Criterion is that it transforms logico-geometric questions into questions about finite posets, to which finite combinatorial methods are applicable.

Remark 4.2. We cannot strengthen the left-to-right direction to the following. “If a logic \mathcal{L} is polyhedrally-complete then $\text{Frames}_{\text{fin}}(\mathcal{L})$ is closed under the nerve operator \mathcal{N} ”. By Corollary 6.16 below Scott’s Logic \mathbf{SL} is polyhedrally-complete. However $\text{Frames}_{\text{fin}}(\mathbf{SL})$ contains the frame F given in Figure 3. As can be seen there, the nerve $\mathcal{N}(F)$ does not validate \mathbf{SL} , since there is an up-reduction $\mathcal{N}(F) \twoheadrightarrow \mathfrak{Q}_\rho$. Using the terminology introduced in Section 6, the problem is that while F is $(2 \cdot 1)$ -connected, it is not $(2 \cdot 1)$ -diamond-connected.

Achieving a proof of the Nerve Criterion will require considerable work with rational triangulations and their subdivisions. We next state the key intermediate result to be obtained. Let A be a triangulation subalgebra of $\text{Sub}_o P$, for some polyhedron P . By Lemma 3.8 (1), there is a unique triangulation Σ of P such that $A = P_o(\Sigma)$. For

any $k \in \mathbb{N}$, let $A^{(k)} := P_o(\Sigma^{(k)})$, where $\Sigma^{(k)}$ is the k -th derived subdivision of Σ (see Section 2.7).

Theorem 4.3. *Let P be a polyhedron, and let A be any triangulation subalgebra of $\text{Sub}_o P$. For any finitely-generated subalgebra B of $\text{Sub}_o P$, there is $k \in \mathbb{N}$ such that B is isomorphic to a subalgebra of $A^{(k)}$.*

Sections 4.1–4.4 will be devoted to proving Theorem 4.3. The proof of the Nerve Criterion is completed in Section 4.6.

4.1 Rational polyhedra and unimodular triangulations

The geometric intuition behind Theorem 4.3 is that any triangulation can be approximated from any other by taking iterated barycentric subdivisions. One difficulty with spelling out such an intuition is that if we start with a triangulation Σ on vertices with irrational coordinates, and try to approximate it using the iterated barycentric subdivisions of a triangulation on vertices with rational coordinates, the approximations can never quite capture (a refinement of) Σ . The approach taken here is effectively to show that it suffices to restrict attention to the rational case. In order to make this idea precise, we need tools on rational triangulations that go beyond the standard polyhedral topology handbooks, which typically deal with the real case only. For these tools we mainly use [Mun11] as a background reference.

A polytope in \mathbb{R}^n is *rational* if it may be written as the convex hull of finitely many points in $\mathbb{Q}^n \subseteq \mathbb{R}^n$. A polyhedron in \mathbb{R}^n is *rational* if it may be written as a union of a finite collection of rational polytopes. A simplicial complex Σ is *rational* if it consists of rational simplices. Note that when this is the case, $|\Sigma|$ is a rational polyhedron.

For any $x \in \mathbb{Q}^n \subseteq \mathbb{R}^n$, there is a unique way to write out x in coordinates as $x = (\frac{p_1}{q_1}, \dots, \frac{p_n}{q_n})$ such that for each i , we have $p_i, q_i \in \mathbb{Z}$ coprime. The *denominator* of x is defined:

$$\text{Den}(x) := \text{lcm}\{q_1, \dots, q_n\}$$

Thus, $\text{Den}(x) = 1$ if and only if x has integer coordinates. Letting $q = \text{Den}(x)$, the *homogeneous correspondent* of x is defined to be the integer vector:

$$\tilde{x} := \left(\frac{qp_1}{q_1}, \dots, \frac{qp_n}{q_n}, q \right)$$

A rational d -simplex $\sigma = x_0 \cdots x_d$ is *unimodular* if there is an $(n+1) \times (n+1)$ matrix with integer entries whose first $d+1$ columns are $\tilde{x}_0, \dots, \tilde{x}_d$, and whose determinant is ± 1 . This is equivalent to requiring that the set $\{\tilde{x}_0, \dots, \tilde{x}_d\}$ can be completed to a \mathbb{Z} -module basis of \mathbb{Z}^{d+1} . A simplicial complex is *unimodular* if each one of its simplices is unimodular.

4.2 Farey subdivisions

Proposition 4.4. *Given a rational simplex σ with vertices $x_0, \dots, x_d \in \mathbb{Q}^n \subseteq \mathbb{R}^n$, there is a unique $m \in \mathbb{Q}^n$ such that $\tilde{m} = \sum_{i=0}^d \tilde{x}_i$. Moreover, $m \in \text{Relint } \sigma$.*

Proof. Let $H_{n+1} \subseteq \mathbb{R}^{n+1}$ be the hyperplane specified by:

$$H_{n+1} := \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} = 1\}$$

Identify \mathbb{Q}^n with the set of rational points of H_{n+1} via the map $(q_1, \dots, q_n) \mapsto (q_1, \dots, q_n, 1)$. Under this identification, \tilde{m} lies in the rational cone:

$$\left\{ \sum_{i=0}^d c_i \tilde{x}_i \mid c_i \in \mathbb{R}, c_i \geq 0 \right\}$$

A routine computation then proves the geometrically evident fact that m is the point of intersection of the line spanned in \mathbb{R}^{n+1} by the vector \tilde{m} , with the hyperplane H_{n+1} ; from which the result follows. \square

The element $m \in \mathbb{Q}^n$ in Proposition 4.4 is called the *Farey mediant* of (the vertices of) the simplex σ . Note that when $d = 0$, i.e. when σ is a vertex of Σ , then m coincides with the vertex σ . Also observe that the Farey mediant and the barycentre of σ are in general distinct, though both lie in $\text{Relint } \sigma$.

We can now define a specific type of stellar subdivision based on Farey mediants, cf. [Mun11, §5.1, p. 55]. Let Σ be a simplicial complex, let $\sigma \in \Sigma$, and let m be the Farey mediant of σ . The *elementary Farey subdivision* of Σ at m is the elementary stellar subdivision of Σ at m . In general, the triangulation Δ is a *Farey subdivision* of Σ if it is obtained from the latter via finitely many successive elementary Farey subdivisions.

At the combinatorial level, Farey and barycentric subdivisions are indiscernible:

Lemma 4.5. *Let Σ, Δ be simplicial complexes with Σ rational, assume that $\gamma: \Sigma \rightarrow \Delta$ is an isomorphism of Σ and Δ as posets, let $\sigma \in \Sigma$, and let m be the Farey mediant of σ . Then the elementary Farey subdivision of Σ at m and the elementary barycentric subdivision of Δ at the barycentre of $\gamma(\sigma)$ are isomorphic as posets.*

Proof. Indeed, at the level of posets, elementary Farey subdivision and elementary barycentric subdivision are the same operation, as direct inspection of the definitions confirms. For further details see also [Ale30, §III]. \square

However, going beyond the combinatorial level, the construction of universal approximations of arbitrary rational polyhedra does require Farey subdivisions and cannot be done with barycentric ones. This is made precise in the following fundamental fact of rational polyhedral geometry.

Lemma 4.6 (The De Concini-Procesi Lemma). *Let P be a rational polyhedron, and let Σ be a unimodular triangulation of P . There exists a sequence $(\Sigma_i)_{i \in \mathbb{N}}$ of unimodular triangulations of P with $\Sigma_0 = \Sigma$ such that:*

- (a) *For each $i \in \mathbb{N}$, Σ_{i+1} is an elementary Farey subdivision of Σ_i , and*
- (b) *For any rational polyhedron $Q \subseteq P$, there is $i \in \mathbb{N}$ such that Σ_i triangulates Q .*

Proof. See [Mun11, Theorem 5.3, p. 57]. \square

4.3 From \mathbb{R} to \mathbb{Q}

To deploy the power of Lemma 4.6, we need to relate general polyhedra to rational polyhedra, and general triangulations to unimodular ones.

Lemma 4.7. *Let P be a polyhedron, and let Σ be a triangulation of P . There exist an integer $n \in \mathbb{N}$, a rational polyhedron $Q \subseteq \mathbb{R}^n$, and a unimodular triangulation Δ of Q such that P and Q are PL-homeomorphic via a map that induces an isomorphism of Σ and Δ as posets.*

Proof. This is a standard argument. Fix a bijection β from the vertices of Σ to the standard basis of \mathbb{R}^n , where n is the number of vertices in Σ . Take a simplex $\sigma = x_0 \cdots x_d$ in Σ . Note that the points $\beta(x_0), \dots, \beta(x_d)$ are affinely independent; let $\alpha(\sigma)$ be the d -simplex spanned by their convex hull: $\alpha(\sigma) := \text{Conv}\{\beta(x_0), \dots, \beta(x_d)\}$. Since the vertices of $\alpha(\sigma)$ are standard basis elements, $\alpha(\sigma)$ is a unimodular simplex by definition. Let $f_\sigma: \sigma \rightarrow \alpha(\sigma)$ be the linear map determined by $f_\sigma(x_i) = \beta(x_i)$ for each i , and let $g_\sigma: \alpha(\sigma) \rightarrow \sigma$ be its inverse, determined by $g_\sigma(\beta(x_i)) = x_i$.

Now, let $Q := \bigcup_{\sigma \in \Sigma} \alpha(\sigma)$. For any simplices $\sigma \preccurlyeq \tau$, the map f_σ agrees with f_τ on σ . Hence we may glue these maps together to form a map $f: P \rightarrow Q$, i.e. $f(x) = f_\sigma(x)$, where σ is any simplex of Σ containing x . Similarly, we may glue together the maps g_σ for $\sigma \in \Sigma$ to form an inverse to f . By definition f is a PL homeomorphism. Finally, note that $\Delta := \{\alpha(\sigma) \mid \sigma \in \Sigma\}$ is a triangulation of Q , and that f induces the poset isomorphism $\sigma \mapsto \alpha(\sigma)$ between Σ and Δ . \square

Lemma 4.8. *Let Σ be a unimodular triangulation of the rational polyhedron P , and suppose Σ' is a Farey subdivision of Σ . There is a triangulation Δ of P which is isomorphic as a poset to Σ' , and $k \in \mathbb{N}$ such that $\Sigma^{(k)}$ refines Δ .*

Proof. The proof works by replacing each elementary Farey subdivision by an elementary barycentric subdivision. We induct on the number $m \in \mathbb{N}^{>0}$ of elementary Farey subdivisions needed to obtain Σ' from Σ . If $m = 1$, let σ be the simplex of Σ being subdivided at its Farey median. Then the first barycentric subdivision $\Sigma^{(1)}$ of Σ refines the elementary barycentric subdivision Σ^* of Σ at the barycentre of σ . By Lemma 4.5, Σ^* and Σ' are isomorphic.

For the induction step, suppose $m > 1$, and write $(\Sigma_i)_{i=0}^m$ for the finite sequence of triangulations connecting $\Sigma = \Sigma_0$ to $\Sigma' = \Sigma_m$ through elementary Farey subdivisions. By the induction hypothesis, there is $k \in \mathbb{N}$ such that $\Sigma^{(k)}$ refines a triangulation Δ isomorphic to Σ_{m-1} ; let us fix one such isomorphism γ . Let σ be the d -simplex of Σ_{m-1} that must be subdivided through its Farey median in order to obtain Σ_m . Let further δ be the simplex of Δ that corresponds to σ through the isomorphism γ . Since the d -simplices are exactly the height- d elements of Δ , we get that δ is a d -simplex. Then $\Sigma^{(k+1)}$ refines Δ^* , the latter denoting the elementary barycentric subdivision of Δ at the barycentre of δ . But Δ is isomorphic to Σ_{m-1} , and therefore by Lemma 4.5, Δ^* is isomorphic to Σ_m . \square

Finally, we shall need the non-trivial fact that arbitrary triangulations of a rational polyhedron realise no more combinatorial types than its rational triangulations; this is due to Meurig Beynon:

Lemma 4.9 (Beynon's Lemma). *Let P be a rational polyhedron, and let Σ be a triangulation of P . There exists a rational triangulation of P which is isomorphic as a poset to Σ .*

Proof. This is the main result of [Bey77]. \square

4.4 End of proof of Theorem 4.3

Proof of Theorem 4.3. Let Σ be the triangulation of P such that $A = P_0(\Sigma)$. Using Lemma 4.7, Lemma 3.8 (2) and Proposition 3.10 we may assume without loss of generality that P is rational and Σ is unimodular. By Lemma 3.4, there is a triangulation Δ of P such that B is isomorphic to a subalgebra of $P_0(\Delta)$. By Beynon's Lemma 4.9 and Lemma 3.8 (2), we may assume that Δ is rational (and hence each member of B is, too). By the De Concini-Procesi Lemma 4.6, there is a Farey subdivision Σ' of Σ that refines Δ . Therefore by Lemma 3.8 (3), B is isomorphic to a subalgebra of $P_0(\Sigma')$. By Lemma 4.8, there is $k \in \mathbb{N}$ such that $\Sigma^{(k)}$ refines Σ' up to isomorphism. Hence by Lemma 3.8 (3) again, $A^{(k)}$ contains a subalgebra isomorphic to $P_0(\Sigma')$, and therefore also a subalgebra isomorphic to B . This completes the proof. \square

4.5 Nerves, subdivisions, and geometric realisations

The reason that Theorem 4.3 is relevant to the Nerve Criterion is the following classical connection between nerves of posets and derived subdivisions, which is deeply rooted in the work of Pavel S. Alexandrov.

Proposition 4.10. *Let Σ be a simplicial complex, regarded as a poset under inclusion of faces. Then the barycentric subdivision of Σ is isomorphic as a poset to the nerve of Σ :*

$$\text{Sd } \Sigma \cong \mathcal{N}(\Sigma)$$

Proof. The proof flows readily from the definitions, and is in any case available from multiple sources; see e.g. [Ale98, Ch. IV, §2.2] (Alexandrov's own textbook treatment), or [Mau80, Proposition 2.5.10, p. 51], or [RW12, §3]. Details are left to the reader. \square

Corollary 4.11. *For P a polyhedron and Σ a triangulation of P we have:*

$$\text{Logic}(P) = \text{Logic}(\mathcal{N}^k(\Sigma) \mid k \in \mathbb{N})$$

Proof. Indeed:

$$\begin{aligned}
\text{Logic}(P) &= \text{Logic}(\text{Sub}_o P) \\
&= \text{Logic}(A \mid A \text{ finitely-generated subalgebra of } \text{Sub}_o P) \quad ([\text{CZ97, Ch. 7}]) \\
&= \text{Logic}(P_o(\Sigma^{(k)} \mid k \in \mathbb{N})) \quad (\text{Theorem 4.3}) \\
&= \text{Logic}(\Sigma^{(k)} \mid k \in \mathbb{N}) \quad (\text{as above}) \\
&= \text{Logic}(\mathcal{N}^k(\Sigma) \mid k \in \mathbb{N}) \quad (\text{Proposition 4.10})
\end{aligned}$$

□

In order to complete a proof of the Nerve Criterion, we will also need to use geometric realisations of finite posets via nerves, another classical tool. Let $F = \{x_1, \dots, x_m\}$ be a finite poset, and let e_1, \dots, e_m be the standard basis vectors of \mathbb{R}^m . The set:

$$\nabla F := \{\text{Conv}\{e_{i_1}, \dots, e_{i_k}\} \mid \{x_{i_1}, \dots, x_{i_k}\} \in \mathcal{N}(F)\}$$

can be proved to be a triangulation by elementary arguments; its underlying polyhedron $|\nabla F|$ is the *geometric realisation* of F . For us, the key fact about geometric realisations is:

Lemma 4.12. *Let F be a finite poset. The map $\max: \mathcal{N}(F) \rightarrow F$, which sends a chain to its maximum element, is a p -morphism, and $\text{Logic}(|\nabla F|) \subseteq \text{Logic}(F)$.*

Proof. The first statement is easy to verify by direct inspection, and a detailed proof was already given in [Bez+18, p. 389]. For the second statement, observe first that ∇F and $\mathcal{N}(F)$ are isomorphic as posets (under inclusion), by their definitions. Thus, by Proposition 2.1, the surjective p -morphism $\nabla F \rightarrow F$ yields $\text{Logic}(\nabla F) \subseteq \text{Logic}(F)$. But $\text{Up}(\nabla F)$ is a subalgebra of $\text{Sub}_o(|\nabla(F)|)$ by Lemma 3.6 together with Lemma 3.3, so that $\text{Logic}(|\nabla F|) \subseteq \text{Logic}(\nabla F) \subseteq \text{Logic}(F)$, as was to be shown. □

4.6 End of proof of the Nerve Criterion

Proof of Theorem 4.1, the Nerve Criterion. Assume that \mathcal{L} is the logic of a class \mathbf{C} of polyhedra. For each $P \in \mathbf{C}$ fix a triangulation Σ_P , and let:

$$\mathbf{C}^* := \{\mathcal{N}^k(\Sigma_P) \mid P \in \mathbf{C} \text{ and } k \in \mathbb{N}\}$$

Then:

$$\begin{aligned}
\text{Logic}(\mathbf{C}^*) &= \bigcap_{P \in \mathbf{C}} \text{Logic}(\mathcal{N}^k(\Sigma_P) \mid k \in \mathbb{N}) \\
&= \bigcap_{P \in \mathbf{C}} \text{Logic}(P) \quad (\text{Corollary 4.11}) \\
&= \text{Logic}(\mathbf{C}) = \mathcal{L}
\end{aligned}$$

Conversely, assume that $\mathcal{L} = \text{Logic}(\mathbf{D})$, where \mathbf{D} is a class of finite frames closed under \mathcal{N} . Let:

$$\mathbf{D}_* := \{|\nabla(F)| : F \in \mathbf{D}\}$$

where $|\nabla(F)|$ is the geometric realisation of F as in Section 4.5. We will show that $\mathcal{L} = \text{Logic}(\mathbf{D}_*)$. First suppose that $\mathcal{L} \not\vdash \phi$, so that $F \not\vdash \phi$ for some $F \in \mathbf{D}$. Then we have that $|\nabla(F)| \not\vdash \phi$, so that $\text{Logic}(\mathbf{D}_*) \not\vdash \phi$. Conversely, suppose that $\text{Logic}(\mathbf{D}_*) \not\vdash \phi$, so that $|\nabla(F)| \not\vdash \phi$ for some $F \in \mathbf{D}$. By definition $\nabla(F)$ is a triangulation of $|\nabla(F)|$, hence by Corollary 4.11 there is $k \in \mathbb{N}$ such that $\nabla(F)^{(k)} \not\vdash \phi$. But $\nabla(F) \cong \mathcal{N}(F)$ by definition, and so by Proposition 4.10 we get $\mathcal{N}^{k+1}(F) \cong \nabla(F)^{(k)}$. Thus, as \mathbf{D} is closed under \mathcal{N} , we get that $\mathcal{L} \not\vdash \phi$. □

5 Polyhedrally-incomplete logics

In this section, we apply the Nerve Criterion to show that every stable logic other than **IPC** is polyhedrally-incomplete. A logic \mathcal{L} is *stable* if $\text{Frames}_\perp(\mathcal{L})$ is closed under monotone images. (We point out that the original definition of [BB17, Def. 6.6] used Esakia spaces. However, it can be shown that these definitions are equivalent [Ili18, Theorem 3.3.17].)

Proposition 5.1. *The following well-known logics² are all stable.*

- (i) *The logic of weak excluded middle, $\mathbf{KC} = \mathbf{IPC} + (\neg p \vee \neg \neg p)$.*
- (ii) *Gödel-Dummett logic, $\mathbf{LC} = \mathbf{IPC} + (p \rightarrow q) \vee (q \rightarrow p)$.*
- (iii) $\mathbf{LC}_n = \mathbf{LC} + \mathbf{BD}_n$.
- (iv) *The logic of bounded width n , $\mathbf{BW}_n = \mathbf{IPC} + \bigvee_{i=0}^n (p_i \rightarrow \bigvee_{j \neq i} p_j)$.*
- (v) *The logic of bounded top width n , defined:*

$$\mathbf{BTW}_n := \bigwedge_{0 \leq i < j \leq n} \neg(\neg p_i \wedge \neg p_j) \rightarrow \bigvee_{i=0}^n (\neg p_i \rightarrow \bigvee_{j \neq i} \neg p_j)$$

- (vi) *The logic of bounded cardinality n , defined:*

$$\mathbf{BC}_n := p_0 \vee (p_0 \rightarrow p_1) \vee ((p_0 \wedge p_1) \rightarrow p_2) \vee \cdots \vee ((p_0 \wedge \cdots \wedge p_{n-1}) \rightarrow p_n)$$

Proof. See [BB17, Theorem 7.3]. □

In fact:

Theorem 5.2. *There are continuum-many stable logics.*

Proof. See [BB17, Theorem 6.13]. □

Theorem 5.3. *Every stable logic has the finite model property.*

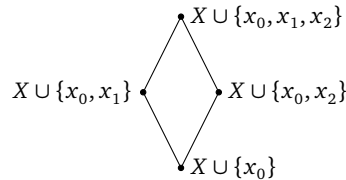
Proof. See [BB17, Theorem 6.8]. □

However, Theorem 5.3 notwithstanding:

Theorem 5.4. *If \mathcal{L} is a stable logic other than **IPC**, and $\text{Frames}(\mathcal{L})$ contains a frame of height at least 2, then \mathcal{L} is not polyhedrally-complete.*

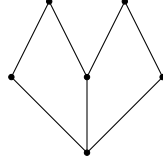
Proof. Let \mathcal{L} be a polyhedrally-complete stable logic of height at least 2. We show that $\mathcal{L} = \mathbf{IPC}$.

By the Nerve Criterion 4.1, there is a class \mathbf{C} of finite frames closed under \mathcal{N} such that $\mathcal{L} = \text{Logic}(\mathbf{C})$. Since $\text{Frames}(\mathcal{L})$ contains a frame of height at least 2, we must have $\mathcal{L} \not\equiv \mathbf{BD}_1$. Since $\mathcal{L} = \text{Logic}(\mathbf{C})$, there is therefore $F \in \mathbf{C}$ such that $\text{height}(F) \geq 2$. This means there are $x_0, x_1, x_2 \in F$ with $x_0 < x_1 < x_2$. Without loss of generality, we may assume that x_2 is a top element and that x_1 is an immediate predecessor of x_2 and x_0 an immediate predecessor of x_1 . Now, by assumption $\mathcal{N}^k(F) \in \mathbf{C}$ for every $k \in \mathbb{N}$. Let us examine the structure of these frames a little. Note that $\{x_0, x_1, x_2\}$ is a chain. Let X be a maximal chain in $\downarrow(x_0)$. We have the following relations occurring in $\mathcal{N}(F)$.

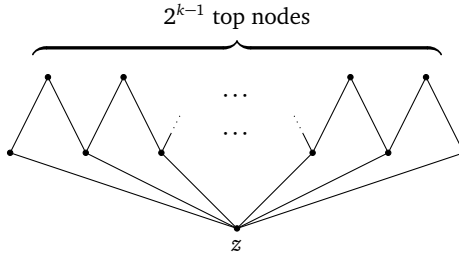


²For more information on these logics see [CZ97, Table 4.1, p. 112].

Moreover, by assumptions on x_0, x_1, x_2 and X , we have that $X \cup \{x_0, x_1, x_2\}$ is a top element of $\mathcal{N}(F)$, with $X \cup \{x_0, x_1\}$ and $X \cup \{x_0, x_2\}$ immediate predecessors, and $X \cup \{x_0\}$ an immediate predecessor of those. So, we may apply this argument once more, to obtain the following structure sitting at the top of $\mathcal{N}^2(F)$.



Iterating, we see that at the top of $\mathcal{N}^k(F)$ we have the following structure.



Let z be the base element of this structure, as indicated. Now, take $k \in \mathbb{N}$ and let $\{t_1, \dots, t_m\}$ be the top nodes of $\mathcal{N}^k(F)$ produced by this construction, where $m = 2^{k-1}$. By Proposition 2.1, $\uparrow(z) \in \text{Frames}_\perp(\mathcal{L})$.

Let now G be an arbitrary poset with up to m elements $\{y_1, \dots, y_m\}$ (possibly with duplicates) plus a root \perp . Define $f : \uparrow(z) \rightarrow G$ as follows.

$$x \mapsto \begin{cases} y_i & \text{if } x = t_i, \\ \perp & \text{otherwise.} \end{cases}$$

Then f is monotonic. Since \mathcal{L} is stable, this means that $G \in \text{Frames}_\perp(\mathcal{L})$. Thus (since, by Proposition 2.6 and Corollary 2.2, IPC is the logic of finite rooted frames) we get that $\mathcal{L} = \text{IPC}$. \square

6 Polyhedrally-complete logics: starlike completeness

In this section, we use the Nerve Criterion to establish a class of logics which are polyhedrally-complete. This constitutes the second main result of the paper.

6.1 Starlike trees

A finite poset T is a *tree* if it has a root \perp , and every other $x \in T \setminus \{\perp\}$ has exactly one immediate predecessor. A *branch* in T is a maximal chain. Say that T is a *starlike tree* if every $x \in T \setminus \{\perp\}$ has at most one immediate successor. (The terminology ‘starlike’ comes from graph theory, see [WS79].) A starlike tree is determined by the multiset of its branch heights, which motivates the following notation.

Let $n_1, \dots, n_k, m_1, \dots, m_k \in \mathbb{N}^{>0}$, with n_1, \dots, n_k distinct. Then let us define $T = \langle n_1^{m_1} \dots n_k^{m_k} \rangle$ as the starlike tree, uniquely determined to within an isomorphism, with the property that if we remove the root \perp we are left with exactly, for each i , m_i chains of length n_i . Let $(\epsilon) = \bullet$, the singleton poset. Call $\alpha = n_1^{m_1} \dots n_k^{m_k}$ (or ϵ) the *signature* of T . We will always assume that $n_1 > n_2 > \dots > n_k$. See Figure 4 for some examples of starlike trees together with their signatures. We will sometimes write 1^0 for ϵ .

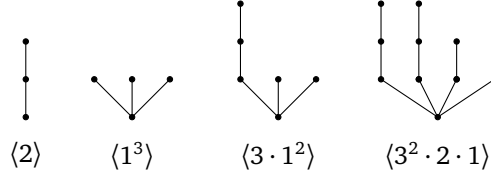


Figure 4: Some examples of starlike trees

The *length* of a signature $\alpha = n_1^{m_1} \cdots n_k^{m_k}$ is defined as $|\alpha| := m_1 + \cdots + m_k$. Let $|\epsilon| := 0$. For $j \leq |\alpha|$, the *j*th height, $\alpha(j)$, is n_i , where:

$$m_1 + \cdots + m_{i-1} \leq j < m_1 + \cdots + m_i$$

Let α and β be signatures. Say that $\alpha \leq \beta$ if $|\alpha| \leq |\beta|$ and for every $j \leq |\alpha|$ we have $\alpha(j) \leq \beta(j)$. Considering the examples in Figure 4, we have the following relations:

$$1^3 < 3 \cdot 1^2 < 3^2 \cdot 2 \cdot 1, \quad 2 < 3 \cdot 1^2$$

Note that if $\alpha = n_1^{m_1} \cdots n_k^{m_k}$, we have $\alpha \leq \beta$ if and only if $|\alpha| \leq |\beta|$ and for every $i \leq k$, we have:

$$\beta(m_1 + \cdots + m_i) \geq n_i$$

Proposition 6.1. *If $\alpha \leq \beta$ then there is a p-morphism $\langle \beta \rangle \rightarrow \langle \alpha \rangle$.*

Proof. We can realise $\langle \alpha \rangle$ as a downwards-closed subset of $\langle \beta \rangle$. The p-morphism $f : \langle \alpha \rangle \rightarrow \langle \beta \rangle$ is then defined as follows. First, f is the identity on $\langle \alpha \rangle$. Second, for any branch of $\langle \beta \rangle$ which contains a branch of $\langle \alpha \rangle$, we let f send any remaining elements to the maximum of the branch of $\langle \alpha \rangle$. Finally, any remaining elements of $\langle \beta \rangle$ are mapped to the maximum element of some fixed branch in $\langle \alpha \rangle$. A routine calculation shows that f is a p-morphism. \square

Note that the starlike tree $\langle k \rangle$ is the chain on $k + 1$ elements; we will use this notation for chains from now on. We will write the signature as k^1 , to disambiguate it from k as a number. For $k \in \mathbb{N}^{>0}$, the *k-fork* is the starlike tree $\langle 1^k \rangle$.

6.2 Starlike logics

We are now in a position to define the principal class of logics that will be investigated in this section. Let $\mathcal{S} := \{\alpha \text{ signature} \mid \alpha \neq 1^2\}$. Take $\Lambda \subseteq \mathcal{S}$ (possibly infinite). The *starlike logic* $\mathbf{SFL}(\Lambda)$ based on Λ is the logic axiomatised by **IPC** plus $\chi(\langle \alpha \rangle)$ for each $\alpha \in \Lambda$. Write $\mathbf{SFL}(\alpha_1, \dots, \alpha_k)$ for $\mathbf{SFL}(\{\alpha_1, \dots, \alpha_k\})$.

Remark 6.2. For an explanation as to why the difork $\langle 1^2 \rangle$ is omitted, see Proposition 6.22 below and the preceding discussion.

Proposition 6.3. $\mathbf{SL} = \mathbf{SFL}(2 \cdot 1)$. *So Scott's Logic is a starlike logic.*

Proof. See [CZ97, §9 and Table 9.7, p. 317]. \square

Let us examine what $\mathbf{SFL}(\Lambda)$ ‘means’ in terms of its class of frames. The formula $\chi(\langle \alpha \rangle)$ turns out to express a kind of connectedness property. We make this precise using the following definitions.

Let F be a finite poset. A *path* in F is a sequence $p = x_0 \cdots x_k$ of elements of F such that for each i we have $x_i < x_{i+1}$ or $x_i > x_{i+1}$. Write $p : x_0 \rightsquigarrow x_k$. The path p is *closed* if $x_0 = x_k$. The poset F is *path-connected* if between any two points there is a path.

Lemma 6.4. *A poset is path-connected if, and only if, it is connected as a topological space.*

Proof. See [BG11, Lemma 3.4]. \square

A *connected component* of F is a subposet $U \subseteq F$ which is connected as a topological subspace and is such that there is no connected V with $U \subset V$.

Lemma 6.5. *Let F be a poset.*

- (1) *The connected components of F partition F .*
- (2) *The connected components of F are downwards-closed and upwards-closed.*

Proof. These results follow straightforwardly from the fact that by Lemma 6.4 the connected components are exactly the equivalence classes under the relation ‘there is a path from x to y ’. \square

Define $\text{ConComps}(F)$ to be the set of connected components of F . The *connectedness type* $\text{ConType}(F)$ of F is the signature $n_1^{m_1} \cdots n_k^{m_k}$ such that $\text{ConComps}(F)$ contains for each i exactly m_i sets of height $n_i - 1$, and nothing else. Let $\text{ConType}(\emptyset) := \epsilon$.

Remark 6.6. Note that when F is connected, $\text{ConType}(F) = n + 1$, where $n = \text{height}(F)$.

Let $\alpha > \epsilon$ be a signature. An α -*partition* of F is an open partition in which the number and heights of the connected components are specified by α . In other words, it is a partition:

$$F = C_1 \sqcup \cdots \sqcup C_{|\alpha|}$$

into open sets such that C_j has height at least $\alpha(j) - 1$. For notational uniformity, say that F has an ϵ -partition if $F = \emptyset$. The following lemma is a straightforward consequence of the definitions.

Lemma 6.7. *A finite poset F has an α -partition if and only if $\alpha \leq \text{ConType}(F)$.*

Corollary 6.8. *When F is connected, F has an α -partition if and only if $\alpha = k^1$, where $k \leq \text{height}(F) + 1$.³*

Let F be a poset and α be a signature. F is α -*connected* if there is no $x \in F$ such that there is an α -partition of $\uparrow(x)$. By Lemma 6.7, this is equivalent to requiring that $\alpha \not\leq \text{ConType}(\uparrow(x))$ for each $x \in F$.

We can now express the meaning of $\chi(\langle \alpha \rangle)$ on frames.

Theorem 6.9. *For F a finite poset and α any signature, $F \models \chi(\langle \alpha \rangle)$ if and only if F is α -connected.*

To prove this result, we make use of the following slight strengthening of Theorem 2.4. Let F and Q be finite posets, and assume that Q has root \perp . An up-reduction $f : F \twoheadrightarrow Q$ is *pointed* with apex $x \in F$ if we have $\text{dom}(f) = \uparrow(x)$ and $f^{-1}\{\perp\} = \{x\}$.

Lemma 6.10. *If there is an up-reduction $F \twoheadrightarrow Q$ then there is a pointed up-reduction $F \twoheadrightarrow Q$.*

Proof. Take $f : F \twoheadrightarrow Q$, and choose $x \in f^{-1}\{\perp\}$ maximal. Then $f|_{\uparrow(x)}$ is still a p-morphism, and is moreover a pointed up-reduction $F \twoheadrightarrow Q$. \square

Corollary 6.11. *Let F, Q be finite posets, with Q rooted. Then $F \models \chi(Q)$ if and only if there is no pointed up-reduction $F \twoheadrightarrow Q$.*

Proof of Theorem 6.9. Assume that $F \not\models \chi(\langle \alpha \rangle)$. Then by Corollary 6.11 there is a pointed up-reduction $f : F \twoheadrightarrow \langle \alpha \rangle$ with apex x . This means that $f^{-1}[\langle \alpha \rangle \setminus \{\perp\}] = \uparrow(x)$. Let C_j be the preimage of the j th branch of $\langle \alpha \rangle \setminus \{\perp\}$ under f , for each $j \leq |\alpha|$. Since f is a p-morphism, C_j is upwards-closed. Note that the C_j 's are disjoint and hence they form an open partition of $\uparrow(x)$. Now, since C_j is the preimage of a chain of length $\alpha(j)$, we can find a chain of the same length inside C_j . From this it follows that C_j has height at least $\alpha(j) - 1$. But then $(C_j \mid j \leq |\alpha|)$ is an α -partition of $\uparrow(x)$, meaning that F is not α -connected.

Conversely, assume that F is not α -connected, so that there is $x \in F$ and an α -partition $(C_j \mid j \leq k)$ of $\uparrow(x)$. For each C_j , we have, by definition, that $\text{height}(C_j) \geq$

³Recall that k^1 is the signature of length 1 which contains the single value k . The starlike tree $\langle k^1 \rangle$ is the chain on $k + 1$ elements.

$\alpha(j) - 1$. Hence by Proposition 2.7 there is a p-morphism $f_j: C_j \rightarrow \langle \alpha(j) - 1 \rangle$. Define $f: \uparrow(x) \rightarrow \langle \alpha \rangle$ as follows.

$$y \mapsto \begin{cases} \perp & \text{if } y = x, \\ f_j(y) & \text{if } y \in C_j \end{cases}$$

Then f is a p-morphism, so an up-reduction $F \circ \rightarrow \langle \alpha \rangle$. □

Remark 6.12. In particular it follows that $\mathbf{BD}_n = \mathbf{IPC} + \chi(\langle n + 1 \rangle)$. This is just Proposition 2.7 of course.

The last matter to resolve before moving on to consider the completeness of starlike logics is their number. For this we make use of Higman's Lemma. A *quasi-well-order* is a preorder which is well-founded and has no infinite antichain. Given a preorder I , let $I^{<\omega}$ be the set of finite sequences of elements of I ordered by $(x_1, \dots, x_n) \leq (y_1, \dots, y_m)$ if and only if there is $f: \{1, \dots, n\} \rightarrow \{1, \dots, m\}$ injective such that for each $k \leq n$ we have $x_k \leq y_{f(k)}$.

Lemma 6.13 (Higman's Lemma, [Hig52]). *If I is a quasi-well-order then so is $I^{<\omega}$.*

Proposition 6.14.

- (1) *Every starlike logic is finitely axiomatizable.*
- (2) *There are exactly countably-many starlike logics.*

Proof. (1) As every starlike logic is axiomatizable by Jankov formulas of starlike trees, it suffices to show that there is no infinite antichain of starlike trees with respect to p-morphic reduction. In light of Proposition 6.1, it therefore suffices to show that there is no infinite antichain of signatures with respect to the ordering defined on them. Now, we can recast signatures as (monotonic decreasing) finite sequences of integers. Indeed, the signature α is determined by the sequence $(\alpha(1), \dots, \alpha(|\alpha|))$. In this way, the set of signatures is seen to be a suborder of $\omega^{<\omega}$. Now, (ω, \leq) is clearly a quasi-well-order, and hence by Higman's Lemma 6.13, so is $\omega^{<\omega}$. Thus there is no infinite antichain of signatures, as required.

- (2) The result follows from (1) as there are only countably many finitely axiomatizable logics. □

6.3 Starlike completeness

The main theorem to be proved in this section is the following.

Theorem 6.15. *Every starlike logic is polyhedrally-complete.*

As an immediate consequence, we obtain:

Corollary 6.16. *Scott's Logic is polyhedrally-complete.*

Remark 6.17. The starlike logic $\mathbf{SFL}(2 \cdot 1, 1^3)$ is particularly important geometrically. In [Ada+20], it is shown that this is the logic of all convex polyhedra.

In order to prove Theorem 6.15, we introduce the following new validity concept on frames. Let F be a poset and ϕ be a formula. F *nerve-validates* ϕ , notation $F \models_{\mathcal{N}} \phi$, if for every $k \in \mathbb{N}$ we have $\mathcal{N}^k(F) \models \phi$.

Remark 6.18. Since for every G we have the p-morphism $\max: \mathcal{N}(G) \rightarrow G$ (see Lemma 4.12), by Proposition 2.1 this is equivalent to requiring that $\mathcal{N}^k(F) \models \phi$ for infinitely-many $k \in \mathbb{N}$.

Lemma 6.19. *A logic \mathcal{L} is polyhedrally-complete if and only if it has the finite model property and every rooted finite frame of \mathcal{L} is the up-reduction of a poset which nerve-validates \mathcal{L} .*

Proof. Assume that \mathcal{L} is polyhedrally-complete. Then by the Nerve Criterion 4.1 it is the logic of a class \mathbf{C} of finite frames which is closed under \mathcal{N} , and so has the fmp. Then by Corollary 2.5, every finite rooted frame F of \mathcal{L} is the up-reduction of some $F' \in \mathbf{C}$. Since $\mathbf{C} \subseteq \text{Frames}(\mathcal{L})$ and is closed under \mathcal{N} , such an F' nerve-validates \mathcal{L} .

Conversely, let \mathbf{C} be the class of all finite rooted frames which nerve-validate \mathcal{L} . Note that \mathbf{C} is closed under \mathcal{N} . Further, clearly $\mathcal{L} \subseteq \text{Logic}(\mathbf{C})$. To see the reverse inclusion, suppose that $\mathcal{L} \not\vdash \phi$. Since \mathcal{L} has the fmp, there is $F \in \text{Frames}_{\perp, \text{fin}}(\mathcal{L})$ such that $F \not\vdash \phi$. By assumption, F is the up-reduction of $F' \in \mathbf{C}$. Then by Proposition 2.1, $F' \not\vdash \phi$, meaning that $\text{Logic}(\mathbf{C}) \not\vdash \phi$. \square

Lemma 6.20. *Every starlike logic has the finite model property.*

Proof. In [Zak93, Corollary 0.11], Zakharyashev shows that every logic axiomatised by the Jankov-Fine formulas of trees has the finite model property. \square

Now, as every finitely axiomatizable logic with the finite model property is decidable we obtain from Proposition 6.14(1) and Lemma 6.20 the following.

Corollary 6.21. *Every starlike logic is decidable.*

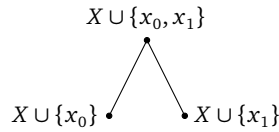
With Lemma 6.20, we can now use Lemma 6.19 to produce a proof of Theorem 6.15. Given a rooted finite frame F of $\text{SFL}(\Lambda)$, we proceed as follows.

- (1) We examine what it means for a frame to nerve-validate $\chi(\langle \alpha \rangle)$.
- (2) We see that it can be assumed that F is *graded* (a structural property of posets defined below).
- (3) Using this additional structure, we construct a frame F' and the p-morphism $F' \rightarrow F$, with the property that $F' \vDash_{\mathcal{N}} \text{SFL}(\Lambda)$

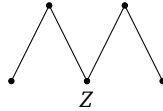
The reader will have noticed that the difork $\langle 1^2 \rangle$ is omitted from the definition of a starlike logic, and consequently from Theorem 6.15. In fact, polyhedral semantics is quite fond of this tree: when we take it as a forbidden configuration, the resulting landscape of polyhedrally-complete logics is as sparse as possible, as is shown below.

Proposition 6.22. *Let \mathcal{L} be a polyhedrally-complete logic containing $\text{SFL}(1^2)$. Then $\mathcal{L} = \text{CPC}$, the maximum logic.*

Proof. Suppose for a contradiction that \mathcal{L} is a polyhedrally-complete logic containing $\text{SFL}(1^2)$ other than CPC . By the Nerve Criterion 4.1, $\mathcal{L} = \text{Logic}(\mathbf{C})$ where \mathbf{C} is a class of finite posets closed under \mathcal{N} . Since $\mathcal{L} \neq \text{CPC}$, there must be $F \in \mathbf{C}$ with $\text{height}(F) \geq 1$. This means that F has a chain $x_0 < x_1$. As in the proof of Theorem 5.4, we may assume that x_1 is a top element of F and that x_0 is an immediate predecessor of x_1 . Take X a maximal chain in $\downarrow(x_0)$. Then, as in that proof, we obtain the following structure lying at the top of $\mathcal{N}(F)$.



Applying the nerve once more, we obtain the following structure at the top of $\mathcal{N}^2(F)$.



Since \mathbf{C} is closed under \mathcal{N} , we get that $\mathcal{N}^2(F) \in \text{Frames}(\mathcal{L})$. But $\uparrow(Z)$ maps p-morphically onto $\langle 1^2 \rangle$, contradicting that $\mathcal{L} \vdash \chi(\langle 1^2 \rangle)$. $\#$ \square

We now proceed with the proof of Theorem 6.15.

6.4 Nerve-validation

While validating $\chi(\langle\alpha\rangle)$ corresponds to α -connectedness (as shown in Theorem 6.9), nerve-validating $\chi(\langle\alpha\rangle)$ corresponds to α -nerve-connectedness. Let F be a poset and $x < y$ in F . The *diamond* and *strict diamond* of x and y are defined, respectively:

$$\begin{aligned}\Downarrow(x, y) &:= \uparrow(x) \cap \downarrow(y) \\ \mathfrak{D}(x, y) &:= \Downarrow(x, y) \setminus \{x, y\}\end{aligned}$$

A poset F is α -*diamond-connected* if there are no $x < y$ in F such that there is an α -partition of $\mathfrak{D}(x, y)$. The poset F is α -*nerve-connected* if it is α -connected and α -diamond-connected.

With a slight conceptual change, α -connectedness and α -diamond-connectedness can be harmonised as follows. For any poset F , we take a new element ∞ , and let $\check{F} := F \cup \{\infty\}$, where ∞ lies above every element of F . Then F is α -nerve-connected if and only if there are no $x < y$ in \check{F} for which there is an α -partition of $\mathfrak{D}(x, y)$.

The following result shows that α -nerve-connectedness is exactly the notion we want.

Theorem 6.23. *Let F be a finite poset and take $\alpha \in \mathcal{S}$. Then $F \vDash_{\mathcal{N}} \chi(\langle\alpha\rangle)$ if and only if F is α -nerve-connected.*

Proof. Assume that F is not α -nerve-connected with the aim of showing $F \not\vDash_{\mathcal{N}} \chi(\langle\alpha\rangle)$. Choose $x < y$ in \check{F} such that $\mathfrak{D}(x, y)$ has an α -partition. That is, there is an open partition $(C_j \mid j \leq |\alpha|)$ of $\mathfrak{D}(x, y)$ such that $\text{height}(C_j) = \alpha(j)$. Choose a chain $X \subseteq F$ such that:

- (i) $x, y \in X \cup \{\infty\}$, and
- (ii) $X \cap \mathfrak{D}(x, y) = \emptyset$,

which is moreover maximal with respect to these requirements. We will show that $\uparrow(X)^{\mathcal{N}(F)}$ has an α -partition. Note that by maximality of X , elements $Y \in \uparrow(X)^{\mathcal{N}(F)}$ are determined by their intersection $Y \cap \mathfrak{D}(x, y)$. For $j \leq |\alpha|$, let:

$$\widehat{C}_j := \{Y \in \uparrow(X)^{\mathcal{N}(F)} \mid Y \cap C_j \neq \emptyset\}$$

Take $j, l \leq |\alpha|$ distinct. Since both C_j and C_l are upwards- and downwards-closed in $\mathfrak{D}(x, y)$, there is no chain $Y \in \uparrow(X)^{\mathcal{N}(F)}$ such that $Y \cap C_j \neq \emptyset$ and $Y \cap C_l \neq \emptyset$. This means that:

- (1) \widehat{C}_j and \widehat{C}_l are disjoint.
- (2) For any $Y \in \uparrow(X)^{\mathcal{N}(F)}$ we have $Y \in \widehat{C}_j$ if and only if $Y \cap \mathfrak{D}(x, y) \subseteq C_j$. Hence each \widehat{C}_j is upwards- and downwards-closed in $\uparrow(X)^{\mathcal{N}(F)}$.

Furthermore, since $(C_j \mid j \leq |\alpha|)$ covers $\mathfrak{D}(x, y)$, we get that $(\widehat{C}_j \mid j \leq |\alpha|)$ covers $\uparrow(X)^{\mathcal{N}(F)}$. Finally, any maximal chain in \widehat{C}_j is a sequence of chains $Y_0 \subset \dots \subset Y_l$ such that $|Y_{i+1} \setminus Y_i| = 1$; this then corresponds to a maximal chain in C_j . Therefore:

$$\text{height}(\widehat{C}_j) = \text{height}(C_j)$$

Ergo $(\widehat{C}_j \mid j \leq |\alpha|)$ is an α -partition of $\uparrow(X)^{\mathcal{N}(F)}$, meaning that $\mathcal{N}(F)$ is not α -connected. Then, by Theorem 6.9, $\mathcal{N}(F) \not\vDash \chi(\langle\alpha\rangle)$, hence by definition $F \not\vDash_{\mathcal{N}} \chi(\langle\alpha\rangle)$.

For the converse direction, we will show that if F is α -nerve-connected, then so is $\mathcal{N}(F)$, which will give the result by induction (note that α -nerve-connectedness is stronger than α -connectedness, and hence by Theorem 6.9 if $\mathcal{N}^k(F)$ is α -nerve-connected then $\mathcal{N}^k(F) \vDash \chi(\langle\alpha\rangle)$). So assume that F is α -nerve-connected. We will first prove α -connectedness. Take $X \in \mathcal{N}(F)$ with the aim of showing that $\uparrow(X)^{\mathcal{N}(F)}$ has no α -partition.

Firstly, assume that X has more than one ‘gap’; that is, there are distinct $w_1, w_2 \in F \setminus X$ such that $X \cup \{w_1\}$ and $X \cup \{w_2\}$ are still chains, but such that there exists $z \in X$

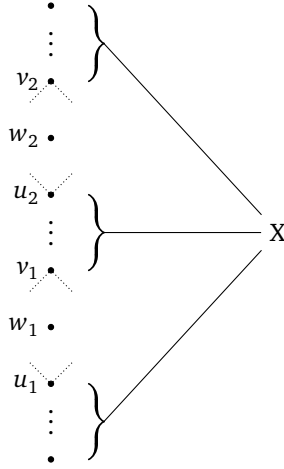


Figure 5: The set-up when X has more than one gap

with $w_1 < z < w_2$. Take $Y, Z \in \uparrow(X)^{\mathcal{N}(F)}$. We will use the two gaps to juggle elements between the two sets so as to provide a path $Y \rightsquigarrow Z$ which never touches X (i.e. lies in $\uparrow(X)^{\mathcal{N}(F)}$). For $i \in \{1, 2\}$, let $u_i \in X \cap \downarrow(w_i)$ be greatest and $v_i \in X \cap \uparrow(w_i)$ be least. See Figure 5 for a representation of the situation. Now, without loss of generality, we may assume that $Y \cap \uparrow(u_1, v_1) \neq \emptyset$ (we may add w_1 to Y , noting that $w_1 \in \uparrow(u_1, v_1)$). Similarly, we may assume that $Y \cap \uparrow(u_2, v_2) \neq \emptyset$, and likewise for Z . We then have the following path in $\uparrow(X)^{\mathcal{N}(F)}$ (note that some of the sets along the path may be equal, but in all cases the path is still there):

$$\begin{array}{ccccccc}
 Y & & (Y \setminus \uparrow(u_1, v_1)) \cup \{w_1\} & & (Z \setminus \uparrow(u_1, v_1)) \cup \{w_1\} & & Z \\
 & \searrow & & \swarrow & & \searrow & \\
 & & Y \setminus \uparrow(u_1, v_1) & & X \cup \{w_1\} & & Z \setminus \uparrow(u_1, v_1)
 \end{array}$$

Here, the gap $\uparrow(u_2, v_2)$ is used to ensure that $Y \setminus \uparrow(u_1, v_1)$ and $Z \setminus \uparrow(u_1, v_1)$ are not equal to X , and the fact that we have $v_1 \leq z \leq u_2$ ensures that all these sets are indeed in $\mathcal{N}(F)$. Hence, $\uparrow(X)^{\mathcal{N}(F)}$ is path-connected so connected. Therefore, by Corollary 6.8, it suffices to show that $\text{height}(\uparrow(X)^{\mathcal{N}(F)}) < \text{height}(F)$. But this is immediate from the definition of \mathcal{N} .

Hence we may assume that X has exactly one gap (when X has no gaps, $\uparrow(X)^{\mathcal{N}(F)} = \emptyset$). This means that there are $x, y \in X$ with $x < y$ such that $X \cap \uparrow(x, y) = \emptyset$ and X is maximal outside of $\uparrow(x, y)$. As before then, elements $Y \in \uparrow(X)^{\mathcal{N}(F)}$ are determined by their intersection $Y \cap \uparrow(x, y)$. Suppose that $\uparrow(X)^{\mathcal{N}(F)}$ has an α -partition $(\widehat{C}_j \mid j \leq |\alpha|)$. For each $j \leq |\alpha|$, let:

$$C_j := \bigcup \widehat{C}_j \cap \uparrow(x, y)$$

Note that $\bigcup_{j \leq |\alpha|} C_j = \uparrow(x, y)$. For each $j \leq |\alpha|$, since \widehat{C}_j is downwards-closed, we have that, for $z \in \uparrow(x, y)$:

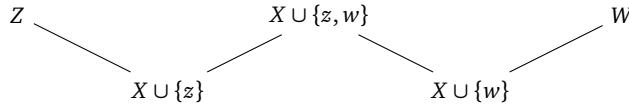
$$z \in C_j \iff \exists Y \in \widehat{C}_j : z \in Y \iff X \cup \{z\} \in \widehat{C}_j$$

This means in particular that the C_j 's are pairwise disjoint. Further, if $z \in C_j$ and $w \in \uparrow(x, y)$ with $w < z$, then $X \cup \{w, z\}$ is a chain, and so as \widehat{C}_j is upwards-closed, we have $X \cup \{w, z\} \in \widehat{C}_j$, meaning that $w \in C_j$; similarly when $w > z$. Whence each C_j is upwards- and downwards-closed. Finally, as above, maximal chains in \widehat{C}_j correspond to maximal chains in C_j of the same length, whence:

$$\text{height}(\widehat{C}_j) = \text{height}(C_j)$$

But then $(C_j \mid j \leq |\alpha|)$ is an α -partition of $\Downarrow(x, y)$, contradicting the fact that F is α -nerve-connected. #

This shows that $\mathcal{N}(F)$ is α -connected. What about α -diamond-connectedness? In fact we can show this without using any assumptions on F . Take $X, Y \in \mathcal{N}(F)$ with $X \subset Y$. We will show that $\Downarrow(X, Y)^{\mathcal{N}(F)}$ has no α -partition. We may assume that $|Y \setminus X| \geq 2$, otherwise $\Downarrow(X, Y)^{\mathcal{N}(F)} = \emptyset$. Note that this means in particular that $\alpha > 1$, since F is α -connected. If $|Y \setminus X| = 2$, then $\Downarrow(X, Y)^{\mathcal{N}(F)}$ is the antichain on two elements, which, since $\alpha \neq 1^2$ by assumption, has no α -partition. So assume that $|Y \setminus X| \geq 3$; we will show that in fact $\Downarrow(X, Y)^{\mathcal{N}(F)}$ is connected. Take distinct $Z, W \in \Downarrow(X, Y)^{\mathcal{N}(F)}$. Choose $z \in Z \setminus X$ and $w \in W \setminus X$. Since $|Y \setminus X| \geq 3$, we have that $X \cup \{z, w\} \in \Downarrow(X, Y)^{\mathcal{N}(F)}$. Hence the following is a path in $\Downarrow(X, Y)^{\mathcal{N}(F)}$:



Therefore, $\Downarrow(X, Y)^{\mathcal{N}(F)}$ is connected. Finally, note that:

$$\text{height}(\Downarrow(X, Y)^{\mathcal{N}(F)}) \leq \text{height}(\mathcal{N}(F)) = \text{height}(F) \quad \square$$

Remark 6.24. Note that the proof shows an interesting property of the formulas $\chi(\langle \alpha \rangle)$: we have $F \models_{\mathcal{N}} \chi(\langle \alpha \rangle)$ if and only if $\mathcal{N}(F) \models \chi(\langle \alpha \rangle)$. This is not true in general. For example, formulas expressing bounded width can take many iterations of the nerve construction to become falsified.

6.5 Graded posets

The next step is to show that we can put $F \in \text{Frames}_{\perp, \text{fin}}(\mathbf{SFL}(\Lambda))$ into a special form. The following definition comes from combinatorics (see e.g. [Sta97, p. 99]).

Definition 6.25 (Graded poset). A *rank function* on a poset F is a map $\rho: F \rightarrow \mathbb{N}$ such that:

- (i) whenever x is minimal in F , we have $\rho(x) = 0$,
- (ii) whenever y is the immediate successor of x , we have $\rho(y) = \rho(x) + 1$.

If F is non-empty and has a rank function, then it is *graded*.

The notion of gradedness has a strong visual connection. When a poset is graded, we can draw it out in well-defined layers such that any element's immediate successors lie entirely in the next layer up.

Lemma 6.26. *Let F be a finite poset.*

- (1) *F is graded if and only if for every $x \in F$, all maximal chains in $\downarrow(x)$ have the same length.*
- (2) *When F is graded, $\rho(x) = \text{height}(x)$ for every $x \in F$, and $\text{height}(F) = \max \rho[F]$.*
- (3) *Rank functions, when they exist, are unique.*

Proof. (1) See [Sta97, p. 99]. Assume that F is graded, and take X a maximal chain in $\downarrow(x)$ for some $x \in F$. Let $k = \rho(x)$. We will show that $|X| = k + 1$. Since X is a chain, the ranks of each of its elements are distinct. Since X is maximal, $x \in X$. Suppose for a contradiction that there is $j < k$ such that there is no $x \in X$ of rank j . We may assume that j is minimal with this property. We can't have $j = 0$, since otherwise X wouldn't contain any minimal element, so wouldn't be a maximal chain. Hence, there is $y \in X$ with $\rho(y) = j - 1$. Let z be next in X after y . Then y has an immediate successor w such that $w \leq z$. By definition, $\rho(w) = j$, so $w \notin X$. But $X \cup \{w\}$ is a chain, contradicting the maximality of X . #Therefore, $|X| = k + 1$.

Conversely, define $\rho : F \rightarrow \mathbb{N}$ by:

$$x \mapsto \text{height}(x)$$

Let us check that ρ is a rank function. (i) Clearly, when x is minimal, $\rho(x) = 0$. (ii) Suppose for a contradiction that there are $x, y \in F$, with y an immediate successor of x , such that $\rho(y) \neq \rho(x) + 1$. First, by definition, $\rho(y) > \rho(x)$, so we must have $\rho(y) > \rho(x) + 1$. Choose maximal chains $X \subseteq \downarrow(x)$, $Y \subseteq \downarrow(y)$. Note that by assumption:

$$|Y| > |X| + 1$$

But now, since y is an immediate successor of x , both $X \cup \{y\}$ and Y are maximal chains in $\downarrow(y)$ of different heights. #

(2) This follows from the proof of (1).

(3) This follows from (2). \square

Corollary 6.27. (1) Every tree is graded.

(2) For any finite poset F , its nerve $\mathcal{N}(F)$ is graded, with rank function given by $\rho(X) = |X| - 1$.

Proof. For (2), note that for any $X \in \mathcal{N}(F)$ we have $\text{height}(X) = |X| - 1$. \square

What we will show in the proceeding two subsections is that any frame $\mathbf{SFL}(\Lambda)$ can be assumed to be graded. In other words, we prove the following ‘gradification’ theorem.

Theorem 6.28. Take $\Lambda \subseteq \mathcal{S}$ and let F be a finite rooted poset such that $F \models \mathbf{SFL}(\Lambda)$. Then there is a finite graded rooted poset F' and a p -morphism $f : F' \rightarrow F$ such that $F' \models \mathbf{SFL}(\Lambda)$.

The proof of the theorem works differently depending on whether we have Scott’s tree $\langle 2 \cdot 1 \rangle$ present. Theorem 6.29 deals with the case $2 \cdot 1 \in \Lambda$, while Theorem 6.33 deals with the case $2 \cdot 1 \notin \Lambda$.

6.6 Gradification in the presence of Scott’s tree

Let us first consider the gradification theorem in the case $2 \cdot 1 \in \Lambda$.

Theorem 6.29. Let $\Lambda \subseteq \mathcal{S}$ be such that $2 \cdot 1 \in \Lambda$. Let F be a finite rooted poset such that $F \models \mathbf{SFL}(\Lambda)$. Then there is a finite graded rooted poset F' and a p -morphism $f : F' \rightarrow F$ such that $F' \models \mathbf{SFL}(\Lambda)$.

To begin with, the following lemmas show us that this case is not too complicated.

Lemma 6.30. Take $\Lambda \subseteq \mathcal{S}$ such that $2 \cdot 1 \in \Lambda$ but $n \notin \Lambda$ for any $n \in \mathbb{N}$.

(1) If there is no $k \in \mathbb{N}^{>0}$ such that $1^k \in \Lambda$, then $\mathbf{SFL}(\Lambda) = \mathbf{SFL}(2 \cdot 1)$.

(2) Otherwise, let $k \in \mathbb{N}^{>0}$ be minimal such that $1^k \in \Lambda$. Then $\mathbf{SFL}(\Lambda) = \mathbf{SFL}(2 \cdot 1, 1^k)$.

Proof. (1) Take $\alpha \in \Lambda$. Then by assumption $\alpha(1) \geq 2$, hence, as $\alpha \neq n$, we have $2 \cdot 1 \leq \alpha$. Then by Proposition 6.1 there is a p -morphism $\langle \alpha \rangle \rightarrow \langle 2 \cdot 1 \rangle$. Hence by the semantic meaning of Jankov-Fine formulas, Theorem 2.4, we have that any frame validating $\chi(\langle 2 \cdot 1 \rangle)$ will also validate $\chi(\langle \alpha \rangle)$. This means that $\mathbf{SFL}(\Lambda) \subseteq \mathbf{SFL}(2 \cdot 1)$. The converse direction is immediate.

(2) Take $\alpha \in \Lambda$. If $\alpha(1) \geq 2$ then by Proposition 6.1 there is a p -morphism $\langle \alpha \rangle \rightarrow \langle 2 \cdot 1 \rangle$. Assume that $\alpha(1) \not\geq 2$. Since $\alpha \neq \epsilon$, we have $\alpha(1) = 1$, meaning that $\alpha = 1^l$ for some $l \in \mathbb{N}^{>0}$. By assumption $k \leq l$. But then $1^k \leq \alpha$, giving that there is a p -morphism $\langle \alpha \rangle \rightarrow \langle 1^k \rangle$. It follows that for any $\alpha \in \Lambda$, $\langle \alpha \rangle$ up-reduces to either $\langle 2 \cdot 1 \rangle$ or $\langle 1^k \rangle$. By Theorem 2.4, any frame validating $\chi(\langle 2 \cdot 1 \rangle)$ and $\chi(\langle 1^k \rangle)$ will also validate $\chi(\langle \alpha \rangle)$. This implies that $\mathbf{SFL}(\Lambda) \subseteq \mathbf{SFL}(2 \cdot 1, 1^k)$. The converse direction is obvious. \square

Corollary 6.31. Take $\Lambda \subseteq \mathcal{S}$ such that $2 \cdot 1 \in \Lambda$ and there is $n \in \mathbb{N}$ with $n \in \Lambda$; assume that n is the minimal such natural number.

- (1) If there is no $k \in \mathbb{N}^{>0}$ such that $1^k \in \Lambda$, then $\mathbf{SFL}(\Lambda) = \mathbf{SFL}(n, 2 \cdot 1)$.
- (2) Otherwise, let $k \in \mathbb{N}^{>0}$ be minimal with $1^k \in \Lambda$. Then $\mathbf{SFL}(\Lambda) = \mathbf{SFL}(n, 2 \cdot 1, 1^k)$.

Proof. This follows from Lemma 6.30 and the fact that when $n_1 < n_2$ every frame validating $\chi(\langle n_1 \rangle)$ also validates $\chi(\langle n_2 \rangle)$. \square

Using this, the ‘meaning’ of $\mathbf{SFL}(\Lambda)$ can be expressed relatively simply. Note that this meaning is expressed in terms of the depth of elements $x \in F$. Up until this point we have mainly been concerned with the height of elements.

Lemma 6.32. Take $\Lambda \subseteq \mathcal{S}$ such that $2 \cdot 1 \in \Lambda$, and let F be a finite poset. Let $n \in \mathbb{N}$ be minimal such that $n \in \Lambda$, or ∞ if no such signature is present. Similarly, let $k \in \mathbb{N}^{>0}$ be minimal with $1^k \in \Lambda$, or ∞ . Then $F \models \mathbf{SFL}(\Lambda)$ if and only if the following three conditions are satisfied for every $x \in F$.

- (i) We have $\text{height}(F) < n$.
- (ii) Whenever $\text{depth}(x) = 1$, we have $|\uparrow(x)| < k$.
- (iii) Whenever $\text{depth}(x) > 1$, the set $\uparrow(x)$ is connected.

Proof. By Corollary 6.31 and the fact that $F \models \chi(\langle n \rangle)$ if and only if $\text{height}(F) \leq n - 1$, it suffices to treat the case $n = \infty$. Now by Lemma 6.30, $\mathbf{SFL}(\Lambda) = \mathbf{SFL}(2 \cdot 1, 1^k)$ when $k < \infty$, and $\mathbf{SFL}(\Lambda) = \mathbf{SFL}(2 \cdot 1)$ otherwise.

Assume that $F \models \mathbf{SFL}(\Lambda)$. (ii) In the case $k < \infty$, take $x \in F$ with $\text{depth}(x) = 1$. Note that $\uparrow(x)$ is an antichain, so $(\{y\} \mid y \in \uparrow(x))$ is an open partition of $\uparrow(x)$. Since $x \models \chi(\langle 1^k \rangle)$, by Lemma 6.7 and Theorem 6.9 we must have $|\uparrow(x)| < k$. (iii) Now take $x \in F$ with $\text{depth}(x) > 1$, and suppose for a contradiction that $\uparrow(x)$ is disconnected. Then we can partition $\uparrow(x)$ into disjoint upwards-closed sets U, V . Since $\text{depth}(x) > 1$, one of U and V (say U) must have height at least 1. But then (U, V) is a $(2 \cdot 1)$ -partition of $\uparrow(x)$, contradicting that $F \models \chi(\langle 2 \cdot 1 \rangle)$ by Theorem 6.9. #

Conversely, assume that $F \not\models \mathbf{SFL}(\Lambda)$. We will show that one of (ii) and (iii) is violated. If $F \not\models \chi(\langle 2 \cdot 1 \rangle)$, then by Theorem 6.9 there is $x \in F$ and a $(2 \cdot 1)$ -partition (U, V) of $\uparrow(x)$. But then $\text{height}(U) \geq 1$, meaning that $\text{depth}(x) > 1$, and furthermore $\uparrow(x)$ is disconnected, violating (iii). So let us assume that $k < \infty$, that $F \models \chi(\langle 2 \cdot 1 \rangle)$ but that $F \not\models \chi(\langle 1^k \rangle)$. Again, we get $x \in F$ and a 1^k -partition (C_1, \dots, C_k) of $\uparrow(x)$. We must have that $\text{height}(C_1) = 0$, otherwise $(C_1, C_2 \cup \dots \cup C_k)$ is a $(2 \cdot 1)$ -partition of $\uparrow(x)$. Similarly $\text{height}(C_i) = 0$ for every $i \leq k$. This means that $\text{depth}(x) = 1$, and that $|\uparrow(x)| \geq k$, violating (ii). \square

Let us turn now to the proof of Theorem 6.29. We first outline the construction before coming to the full proof.

- We first split F up into its tree unravelling $\text{Tree}(F)$ (defined below).
- We then lengthen branches so that every top element has the same height.
- Lastly, we join top nodes of this tree in order to recover any α -connectedness that we lost.

See Figure 6 for an example of this process.

Given any finite, rooted poset F , its *tree unravelling* $\text{Tree}(F)$ is the set of chains X in F such that X is maximal in $\downarrow(\max(X))$, ordered by subset inclusion. Define the function $\text{last}: \text{Tree}(F) \rightarrow F$ by:

$$X \mapsto \max(X)$$

Then $\text{Tree}(F)$ is a tree and last is a p-morphism (see [CZ97, Theorem 2.19, p. 32]).

We make use of the following abbreviations. For any poset F , the set of top elements (i.e. elements of depth 0) in F is denoted by $\text{Top}(F)$; let $\text{Trunk}(F) := F \setminus \text{Top}(F)$.

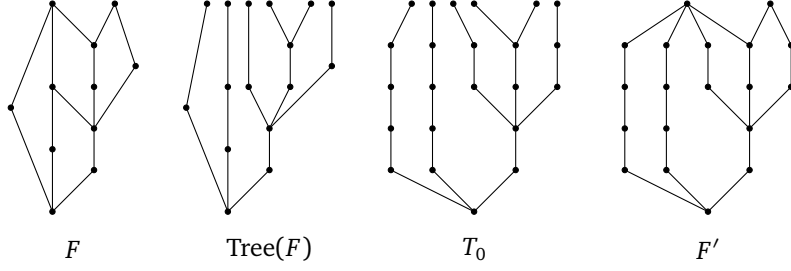


Figure 6: An example of gradification in the presence of Scott's tree

Proof of Theorem 6.29. Let $n := \text{height}(F)$. We may assume $\epsilon \notin \Lambda$. If $2 \in \Lambda$, then by Remark 6.12, $n \leq 1$, so F is already graded. So assume that $2 \notin \Lambda$.

Start with the tree unravelling $T = \text{Tree}(F)$ of F . Form a new tree T_0 by replacing each top node $t \in \text{Top}(T)$ with a chain of new elements $t^*(0), \dots, t^*(m_t)$, where $m_t = n - \text{height}(t)$. The relations between these new elements and the rest of T is as follows:

$$\begin{aligned} t^*(0) &< \dots < t^*(m_t), \\ x < t^*(0) &\iff x < t \quad \forall x \in T \end{aligned}$$

Note that in T_0 all branches have the same length $n + 1$. Define the p-morphism $g: T_0 \rightarrow T$ by:

$$x \mapsto \begin{cases} x & \text{if } x \in \text{Trunk}(T), \\ \text{last}(t) & \text{if } x = t^*(i) \text{ for some } t \in \text{Top}(T) \text{ and } i \leq m_t \end{cases}$$

Form F' from T_0 by identifying, for top nodes $t, s \in \text{Top}(T)$, the elements $t^*(m_t)$ and $s^*(m_s)$ whenever $\text{last}(t) = \text{last}(s)$. That is, let $F' := T_0 / \mathcal{W}$, where:

$$\mathcal{W} := \{\{t^*(m_t) \mid \text{last}(t) = u\} \mid u \in \text{Top}(F)\}$$

Note that we have a p-morphism $f = \text{last} \circ g \circ q_{\mathcal{W}}: F' \rightarrow F$. Furthermore, F is clearly finite and rooted. As to gradedness, take $x \in F'$ with the aim of showing that all maximal chains in $\downarrow(x)$ are of the same length, utilising Lemma 6.26. If $x \in \text{Trunk}(F')$, then $\downarrow(x)^{F'}$ is a linear order. So assume that $x \in \text{Top}(F')$. Then any maximal chain X in $\downarrow(x)$ corresponds to a branch of T_0 , and therefore has length $n + 1$.

Let us now use Lemma 6.32 to verify that our construction preserves α -connectedness for $\alpha \in \Lambda$ and complete the proof. Let $k \in \mathbb{N}^{>0}$ be minimal such that $1^k \in \Lambda$, or ∞ if no such signature is present. For $u \in \text{Top}(F)$ let \hat{u} be the equivalence class of those elements $t^*(m_t)$ such that $\text{last}(t) = u$. Note that by construction, for $x \in \text{Trunk}(T)$ and $u \in \text{Top}(F)$:

$$x < \hat{u} \iff \text{last}(x) < u \quad (\star)$$

We need to check the three conditions of Lemma 6.32.

- (i) Note that $\text{height}(F') = \text{height}(F)$.
- (ii) For any $x \in F'$ with $\text{depth}(x) = 1$, either $x \in \text{Trunk}(T)$ or $x = t^*(n_t - 1)$ for some top node $t \in T$. In the former case, the fact that $|\uparrow(x)| \leq k$ follows from (\star) and the fact that $|\uparrow(\text{last}(x))^F| \leq k$. In the latter case we have $\uparrow(x) = \{\widehat{\text{last}(t)}\}$.
- (iii) Similarly, for any $x \in F'$ with $\text{depth}(x) > 1$, either $x \in \text{Trunk}(T)$ or $x = t^*(r)$ for some top node $t \in T$ and $r < n_t - 1$. In the latter case, $\uparrow(x)$ is a chain, so connected. For the former case, it suffices to show that any two top elements $\hat{u}, \hat{v} \in \uparrow(x)$ are connected by a path in $\uparrow(x)$. Note that $\text{depth}(\text{last}(x))^F > 1$. Now, since $F \models \chi(\langle 2 \cdot 1 \rangle)$, by Lemma 6.32 there is a path $u \rightsquigarrow v$ in $\uparrow(\text{last}(x))^F$. We may assume that this path is of form given in Figure 7 (a), where w_0, \dots, w_k

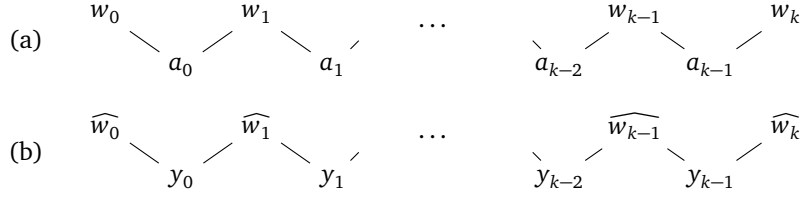


Figure 7: The form of the paths in $\uparrow(\text{last}(x))^F$ and $\uparrow(x)^{F'}$

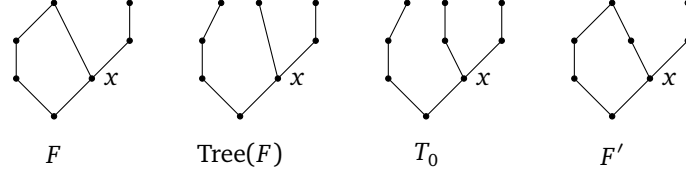


Figure 8: The technique in the proof of Theorem 6.29 does not work in general

are top nodes in F . Using (\star) , this path then translates into a path $\widehat{u} \rightsquigarrow \widehat{v}$ as in Figure 7 (b), where $y_i \in \text{last}^{-1}\{a_i\} \cap \uparrow(x)$ for each i . \square

6.7 Gradification without Scott's tree

Now that the situation $2 \cdot 1 \in \Lambda$ has been dealt with, let us turn to the case $2 \cdot 1 \notin \Lambda$.

Theorem 6.33. *Let $\Lambda \subseteq \mathcal{S}$ be such that $2 \cdot 1 \notin \Lambda$. Let F be a finite, rooted poset such that $F \models \mathbf{SFL}(\Lambda)$. Then there is a finite, graded, rooted poset F' and a p -morphism $f : F' \rightarrow F$ such that $F' \models \mathbf{SFL}(\Lambda)$.*

Unfortunately, the proof of Theorem 6.29 crucially relied on the fact that the original frame F was $(2 \cdot 1)$ -connected. Consider for instance the frame F given in Figure 8, which at x is not $(2 \cdot 1)$ -connected. If we apply the construction to F , we end up with a frame F' in which x sits below two connected components of height 1, that is⁴, $\text{ConType}(\uparrow(x)^{F'}) = 2^2$. Hence F' is not 2^2 -connected, while F is. Taking $2 \cdot 1$ away from Λ is a double-edged sword however, since it allows for more complex constructions in F' .

The following reusable lemma will come in handy a couple of times.

Lemma 6.34. *Let $f : F' \rightarrow F$ be a surjective p -morphism between finite posets, and take $x \in F'$. Assume that for any $y, z \in \text{Succ}(x)$ there is a path $y \rightsquigarrow z$ in $\uparrow(x)$ whenever there is a path $f(y) \rightsquigarrow f(z)$ in $\uparrow(f(x))$. Then:*

$$\text{ConComps}(\uparrow(x)) = \{f^{-1}[C] \mid C \in \text{ConComps}(\uparrow(f(x)))\}$$

In particular, if $\text{height}(f^{-1}[C]) = \text{height}(C)$ for any $C \in \text{ConComps}(\uparrow(f(x)))$ then:

$$\text{ConType}(\uparrow(x)) = \text{ConType}(\uparrow(f(x)))$$

Proof. Note that, since f is a p -morphism and F and F' are finite, $\{f^{-1}[C] \mid C \in \text{ConComps}(\uparrow(f(x)))\}$ is a partition of $\uparrow(x)$ into upwards- and downwards-closed sets. So it suffices to show that $f^{-1}[C]$ is connected for every $C \in \text{ConComps}(\uparrow(f(x)))$. Take $y_0, z_0 \in f^{-1}[C]$. Since $f^{-1}[C]$ is downwards-closed in $\uparrow(x)$, there are $y, z \in \text{Succ}(x) \cap f^{-1}[C]$ such that $y \leq y_0$ and $z \leq z_0$. Then $f(y), f(z) \in C$, so by assumption there is a path $f(y) \rightsquigarrow f(z)$ in $\uparrow(f(x))$. But then by assumption there is a path $y \rightsquigarrow z$ in $\uparrow(x)$, which lies in $f^{-1}[C]$ since the latter is upwards- and downwards-closed. This yields a path $y_0 \rightsquigarrow z_0$. \square

⁴Recall that $\text{ConComps}(F)$ is the set of connected components of F and that $\text{ConType}(F)$ of F is the signature $n_1^{m_1} \cdots n_k^{m_k}$ such that $\text{ConComps}(F)$ contains for each i exactly m_i sets of height $n_i - 1$, and nothing else.

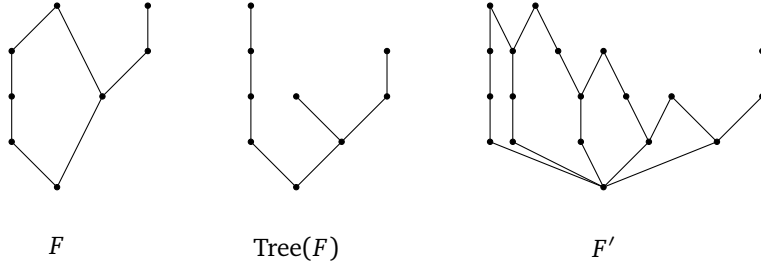


Figure 9: An example of gradification in the absence of Scott's tree.

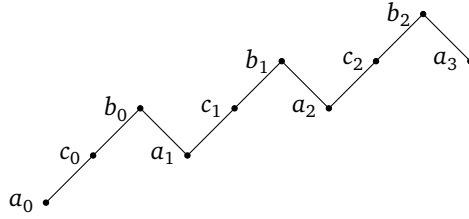


Figure 10: The relations between the zigzag points in case $l = 3$.

Let us turn now to the proof of Theorem 6.33. The construction works in two steps as follows (see Figure 9 for an example).

- Again, we start by splitting F up into its tree unravelling $\text{Tree}(F)$.
- Then, in order to connect the frame back up again while ensuring that it remains graded, we construct 'zigzag roller-coasters' connecting top nodes of different heights.

Proof of Theorem 6.33. As in the proof of Theorem 6.29, we may assume that $\epsilon, 1, 2 \notin \Lambda$.

Start with $T = \text{Tree}(F)$. For every two distinct $p, q \in \text{Top}(T)$ such that $\text{last}(p) = \text{last}(q) = t$, we will build a 'roller-coaster' structure $Z(p, q)$, which will furnish a bridge between p and q . Every such structure $Z(p, q)$ is independent, so that they can all be added to T at the same time. First note that by Corollary 6.27, T is graded; let $\rho : T \rightarrow \mathbb{N}$ be its rank function.

Now, take distinct $p, q \in \text{Top}(T)$ such that $\text{last}(p) = \text{last}(q) = t$. Let $l := \rho(q) - \rho(p)$. By swapping p and q , we may assume that $l \geq 0$. We need to join p to q with a path which ascends in grade. We do this using a zigzagging path, which consists of lower points a_0, \dots, a_l , upper points b_0, \dots, b_{l-1} and intermediate points c_0, \dots, c_{l-1} . The relations between these points are as follows (see Figure 10).

$$a_i < c_i < b_i, \quad a_{i+1} < b_i$$

Consider $p \wedge q$ (i.e. the intersection of p and q , regarded as chains), and let $k := \rho(p) - \rho(p \wedge q) - 1$. Note that $k \geq 0$ since p and q are incomparable. Moreover, $k \geq 1$ as follows. Suppose for a contradiction that $k = 0$, so that p is an immediate successor of $p \wedge q$. Then $\text{last}(p)$ is an immediate successor of $\text{last}(p \wedge q)$. But $\text{last}(q) = \text{last}(p)$, so we have, as chains:

$$p = (p \wedge q) \cup \{\text{last}(p)\} = (p \wedge q) \cup \{\text{last}(q)\} = q$$

contradicting that p and q are distinct. #

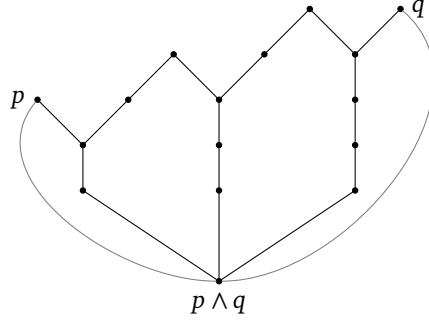


Figure 11: The zigzag path and the ladder structure in place.

To ensure that the new poset F' is still graded, we need to dangle some scaffolding down from the zigzag path to $p \wedge q$. Below each lower point a_i we will dangle a chain of $k + i - 1$ points $d(i, 1), \dots, d(i, k + i - 1)$. The relations are as follows:

$$d(i, 1) < d(i, 2) < \dots < d(i, k + i - 1) < a_i$$

Finally, let $Z(p, q)$ denote the whole structure of the zigzag path plus the dangling scaffolding. Attach $Z(p, q)$ to T by adding the following relations and closing under transitivity (see Figure 11).

$$a_0 < p, \quad a_i < q, \quad \forall i: p \wedge q < d(i, 1)$$

Let F' be the result of adding $Z(p, q)$ to T for every pair p, q , and define the function $f: F' \rightarrow F$ by:

$$f(x) := \begin{cases} \text{last}(x) & \text{if } x \in T \\ \text{last}(p) & \text{if } x \in Z(p, q) \text{ for some } p, q \end{cases}$$

First, let us see that f is a p-morphism. The (Forth) condition follows from the fact that last is monotonic, and that:

- if $x \leq y$ with $x \in T$ and $y \in Z(p, q)$, then by construction $x \leq p \wedge q$, meaning that $f(x) = \text{last}(x) \leq \text{last}(p \wedge q) \leq \text{last}(p) = f(y)$, and
- if $x \leq y$ with $x \in Z(p, q)$ and $y \in T$, then by construction $y \in \{p, q\}$, so that $f(x) = \text{last}(p) = f(y)$.

The (Back) condition follows from the fact that last is open, and that each $Z(p, q)$ maps to a top node.

Second, for any pair p, q , we can extend the rank function ρ to the new structure $Z(p, q)$ as follows (as indicated by the heights of the nodes in Figure 11):

$$\begin{aligned} \rho(a_i) &= \rho(p) + i - 1 \\ \rho(b_i) &= \rho(p) + i + 1 \\ \rho(c_i) &= \rho(p) + i \\ \rho(d(i, j)) &= \rho(p \wedge q) + j \end{aligned}$$

To see that, thus extended, ρ is still a rank function, it suffices to check that the newly-ranked $Z(p, q)$ fits into T as a ranked structure. That is, we need to check the following equations.

$$\begin{aligned} \rho(p) &= \rho(a_0) + 1 \\ \rho(q) &= \rho(a_i) + 1 \\ \rho(d(i, 1)) &= \rho(p \wedge q) + 1 \end{aligned}$$

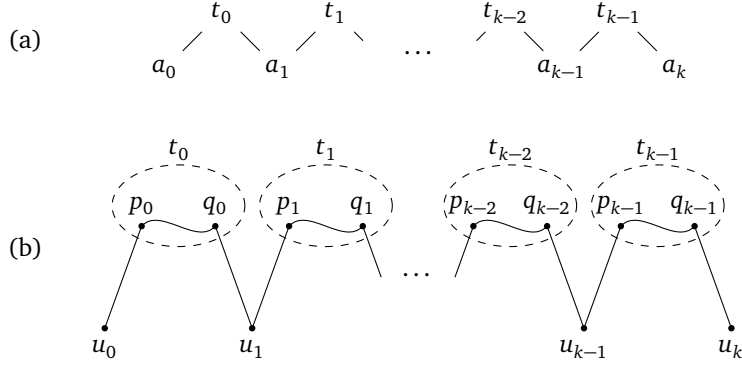


Figure 12: The form of the paths in $\uparrow(\text{last}(x))$ and $\uparrow(x)$

But these follow by definition. In this way we see that F' is graded.

Finally, it remains to be shown that $F \models \mathbf{SFL}(\Lambda)$. So take $x \in F$. First, whenever $x \in Z(p, q)$ for some p, q , by construction $\uparrow(x)$ is α -connected for every signature other than ϵ , 1^2 , $2 \cdot 1$ and k where $k \geq \text{height}(F) + 1$. Hence we may assume that $x \in T$. Let us use Lemma 6.34. Take $y, z \in \text{Succ}(x)$ such that there is a path $f(y) \rightsquigarrow f(z)$ in $\uparrow(\text{last}(x))$, with the aim of finding a path $y \rightsquigarrow z$ in $\uparrow(x)$.

First assume that $y \in Z(p, q)$ for some p, q . Then since $y \in \text{Succ}(x)$ and $x \in T$, by construction $x = p \wedge q$. All of $Z(p, q)$ is connected in $\uparrow(x)$, hence there is a path $y \rightsquigarrow p$. Let $p' \in T$ be the immediate successor of x which lies below p (this exists since T is a tree). Then we have a path $y \rightsquigarrow p'$ in $\uparrow(x)$. With this case thus dealt with, we may now assume that $y \in T$, and similarly that $z \in T$.

So, we have a path $\text{last}(y) \rightsquigarrow \text{last}(z)$. We now proceed in a similar fashion to the proof of Theorem 6.29. We may assume that the path $\text{last}(y) \rightsquigarrow \text{last}(z)$ has the form in Figure 12 (a), where t_0, \dots, t_k are top nodes in F . Let $u_0 := y$ and $u_k := z$. For each $i \in \{1, \dots, k-1\}$, choose $u_i \in \text{last}^{-1}\{a_i\}$. For $i \in \{0, \dots, k-1\}$, take $p_i, q_i \in \text{last}^{-1}\{t_i\}$ such that $u_i \leq p_i$ and $u_{i+1} \leq q_i$. For each such i , since $\text{last}(p_i) = \text{last}(q_i)$, there is a path $p_i \rightsquigarrow q_i$ which lies in $Z(p_i, q_i)$, and hence lies in $\uparrow(x)$. Compose all these paths as in Figure 12 to form a path $y \rightsquigarrow z$ in $\uparrow(x)$ as required.

It now remains to show that if $C \in \text{ConComps}(\uparrow(\text{last}(x)))$, then $\text{height}(f^{-1}[C]) = \text{height}(C)$. First, since f is a p-morphism, $\text{height}(f^{-1}[C]) \geq \text{height}(C)$. Conversely, let $X \subseteq f^{-1}[C]$ be a maximal chain. Assume X intersects with some $Z(p, q)$. Then we can replace the part $X \cap (Z(p, q) \cup \{p, q\})$ with the unique maximal chain in $\uparrow(p \wedge q)^T$ containing q (this exists since T is a tree). Then by construction this does not decrease the length of X nor does it move X outside of $f^{-1}[C]$ (since the latter is upwards- and downwards-closed). Therefore, we may assume that $X \subseteq T$, so X corresponds to a chain $\text{last}[X]$ of the same length in C .

Therefore, by Lemma 6.34 we get that $\text{ConType}(\uparrow(x)) = \text{ConType}(\uparrow(\text{last}(x)))$. Applying Lemma 6.7, we have that $\uparrow(x)$ has an α -partition if and only if $\uparrow(\text{last}(x))$ has an α -partition. \square

6.8 Nervification

We now find ourselves, having suitably prepared F , in a position to make use of its additional graded structure. The general method of the final construction, in which we transform F into a frame which nerve-validates $\mathbf{SFL}(\Lambda)$, is the same as in Theorem 6.29 and Theorem 6.33. We begin with the tree unravelling $\text{Tree}(F)$, perform some alterations, then rejoin top nodes. A key difference here is that we won't rejoin every top node to every other top node whose 'last' value is the same. Instead, we line up all the top nodes mapping to the same element and link each top node to at

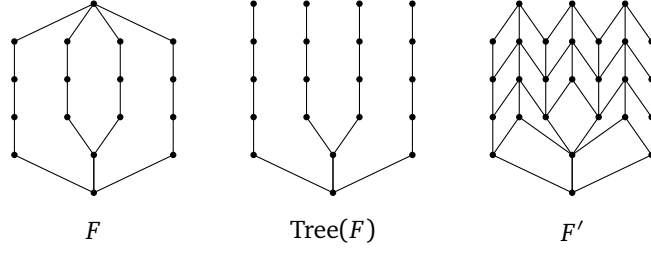


Figure 13: An example of nervification, using the graded structure of F

most two other top nodes, which we think of as its neighbours. See Figure 13 for an example of the construction.

Definition 6.35. Let T be a finite tree. Then for each $x \in T$, we have that $\downarrow(x)$ is a chain. For $k \leq \text{height}(x)$, let $x^{(k)}$ be the element of this chain which has height k . Let $x^{(-k)}$ be the element which has height $\text{height}(x) - k$.

Definition 6.36. For $n \in \mathbb{N}$, let $\mathcal{S}_n := \mathcal{S} \setminus \{1^k \mid k < n\}$.

Theorem 6.37. Take $\Lambda \subseteq \mathcal{S}$ and let F be a finite, graded, rooted poset of height n such that $F \models \mathbf{SFL}(\Lambda)$. Then there is a poset F' and a p -morphism $f : F' \rightarrow F$ such that $F' \models \mathbf{SFL}(\Lambda)$ and such that F' is α -diamond-connected for every $\alpha \in \mathcal{S}_n$.

Proof of Theorem 6.37. We may assume that $\epsilon, 1 \notin \Lambda$. Further, if $2 \in \Lambda$, then $\text{height}(F) = 1$, so F is already α -diamond-connected for every $\alpha \in \mathcal{S}_n$. Hence we may assume that $2 \notin \Lambda$.

Once more, start with $T = \text{Tree}(F)$. Chop off the top nodes: let $T' := \text{Trunk}(T)$. For each $t \in \text{Top}(F)$, we will add a new structure $W(t)$, which lies only above elements of T' . Let $\rho : F \rightarrow \mathbb{N}$ be the rank function on F . Note that $\rho \circ \text{last} : T \rightarrow \mathbb{N}$ is the rank function on T .

Take $t \in \text{Top}(F)$. Enumerate $\text{last}^{-1}\{t\} = \{p_1, \dots, p_m\}$. For each $i \leq m - 1$, define:

$$\begin{aligned} r_i &:= p_i \wedge p_{i+1} \\ l_i &:= \rho(\text{last}(r_i)) \\ k_i &:= \rho(t) - \rho(\text{last}(r_i)) - 1 \end{aligned}$$

Note that $k_i \geq 1$ just as in the proof of Theorem 6.33. Since F is graded and T is a tree, we have that:

$$|\Downarrow(r_i, p_i)^T| = |\Downarrow(r_i, p_{i+1})^T| = k_i$$

In other words, $p_i^{(l_i)} = p_{i+1}^{(l_i)} = r_i$. We will construct a ‘chevron’ structure which joins $p_i^{(-1)}$ to $p_{i+1}^{(-1)}$. For each $i \leq m - 1$, take new elements $a(i, 1), \dots, a(i, k_i)$, and add them to T' using the following relations.

$$a(i, 1) < \dots < a(i, k_i), \quad \forall j \leq k_i : p_i^{(l+j)}, p_{i+1}^{(l+j)} < a(i, j)$$

Let $W(t)$ be this new structure (i.e. the chain $\{a(i, 1) < \dots < a(i, k_i)\}$ in place). See Figure 14 and Figure 15 for examples of this process of adding chevrons.

The process of adding $W(t)$ is independent for each $t \in \text{Top}(F)$. Let F' be the result of adding every $W(t)$ to T' . Define $f : F' \rightarrow F$ by:

$$f(x) := \begin{cases} \text{last}(x) & \text{if } x \in T' \\ t & \text{if } x \in W(t) \text{ for some } t \in \text{Top}(F) \end{cases}$$

Since we have made sure that each $W(t)$ contains, for each $p_i \in \text{last}^{-1}\{t\}$, a node above $p_i^{(-1)}$ which maps to t , and that all of the new structure maps to a top node, f is a p -morphism.

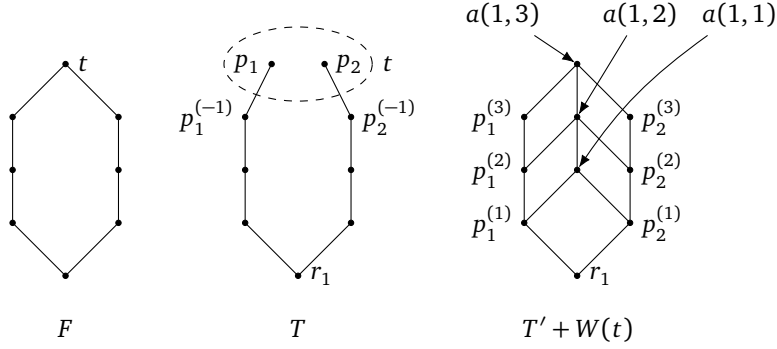


Figure 14: The chevron structure in a case with two branches.

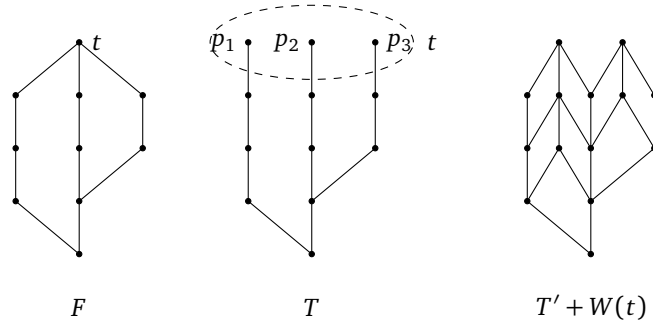


Figure 15: The chevron structure in a more complex case involving three branches.

Let us see that $F' \models \mathbf{SFL}(\Lambda)$. Take $x \in F'$. If $x \in W(t)$ for some t , then $\uparrow(x)$ is either empty or a chain, hence $\uparrow(x) \models \mathbf{SFL}(\Lambda)$. So we assume that $x \in T'$. The verification is now very similar to that in Theorem 6.33, making use of Lemma 6.34. Take $y, z \in \text{Succ}(x)$ such that there is a path $f(y) \rightsquigarrow f(z)$ in $\uparrow(\text{last}(x))$. As in the proof of Theorem 6.33, by construction of $W(t)$ we may assume that $y, z \in T'$. Just as in that proof, we can construct a path $y \rightsquigarrow z$ from the path $f(y) \rightsquigarrow f(z)$, using the fact that whenever $t \in \uparrow(\text{last}(x)) \cap \text{Top}(F)$, any $w, v \in f^{-1}\{t\}$ are connected by a path in $\uparrow(x)^{F'}$ (this is how we constructed F'). It is straightforward then to check that if $C \in \text{ConComps}(\uparrow(\text{last}(x)))$ we have $\text{height}(f^{-1}[C]) = \text{height}(C)$, giving that:

$$\text{ConType}(\uparrow(x)) = \text{ConType}(\uparrow(\text{last}(x)))$$

To complete the proof, let us see that F' is α -diamond-connected for every $\alpha \in \mathcal{S}_n$. Take $x, y \in F'$ with $x < y$ and consider $\uparrow(x, y)$. There are several cases.

- (a) Case $y \in T'$. We have that $\uparrow(x, y)^{F'} = \uparrow(x, y)^{T'}$, which is linearly-ordered since T' is a tree; hence it is connected and of height at most $n - 2$.

Hence $y = a(i, j)$ for $a(i, j) \in W(t)$ a new element. Let p_i, p_{i+1}, r_i, l_i be as above.

- (b) Case $x \in W(t)$. Note that by construction $\uparrow(x, y)$ is linearly-ordered.
- (c) Case $x = p_i^{(l+e)}$ for some e . If we have $\text{height}(\uparrow(x, y)) = 1$, then $e = i - 1$ and $\uparrow(x, y)$ is the antichain on two elements, which is α -connected. Otherwise, by construction, $a(i, j - 1) \in \uparrow(x, y)$ which is connected to everything.
- (d) Case $x = p_{i+1}^{(l+e)}$ for some e . This is symmetric.
- (e) Case $x = r_i$. Again, if $\text{height}(\uparrow(x, y)) = 1$ then $j = 1$ and $\uparrow(x, y)$ is the antichain on two elements, otherwise $a(i, 1) \in \uparrow(x, y)$ which is connected to everything.

- (f) Otherwise, $x < r_i$ (since T' is a tree). Then $r_i \in \mathfrak{F}(x, y)$ which is connected to everything. \square

6.9 End of Proof of Theorem 6.15

We can now prove our second main result:

Proof of Theorem 6.15. By Lemma 6.19 and Lemma 6.20, we need to show that every finite, rooted frame of $\mathbf{SFL}(\Lambda)$ is the up-reduction of one which nerve-validates $\mathbf{SFL}(\Lambda)$; in fact this up-reduction is just a p-morphism. So take such a frame F . We may assume that F is graded: when we have $2 \cdot 1 \in \Lambda$, apply Theorem 6.29, otherwise apply Theorem 6.33. Then by Theorem 6.37, there is a frame F' and a p-morphism $f : F' \rightarrow F$ such that F' is α -nerve-connected for every $\alpha \in \Lambda$ (note that by Remark 6.12 we must have $\Lambda \subseteq \mathcal{S}_n$ where $n = \text{height}(F)$). Then, by Theorem 6.23, F' nerve-validates $\mathbf{SFL}(\Lambda)$, which completes the proof. \square

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