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Combining Algebraizable Logics

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Abstract The general methodology of "algebraizing" logics is used here for combining different logics. The combination of logics is represented as taking the colimit of the constituent logics in the category of algebraizable logics. The cocompleteness of this category as well as its isomorphism to the corresponding category of certain first-order theories are proved.

1 Introduction In this paper we translate the "combining logics" problem to the problem of "combining" certain theories of usual first-order logic. We prove that the category of a special class of logics, called *algebraizable logical systems* (see Definition 2.1 below), is isomorphic to the category of the corresponding first-order theories. We also show that these categories are cocomplete. Some directions in which the approach chosen can perhaps be generalized are pointed out in the last section.

2 Preliminaries As a set theoretic framework we presume any set theory which is suitable for the foundation of category theory. For basic category theoretical notions such as category, object, morphism, small diagram, cocone, coproduct, colimit, coequalizer, etc., we follow the usage of MacLane [8].

Our terminology follows the usual standards concerning classical first-order logic and basics of universal algebra. For notions not defined but used here, see Monk [9], and Burris and Sankappanavar [6].

ω denotes the set of natural numbers. An *algebraic similarity type* is a function t mapping some nonempty set into ω. An element f of the domain $dom(t)$ of t with $t(f) = k$ is called a *k-ary function symbol of type t*. *t-type algebras* are structures (in the usual sense) of the algebraic similarity type t. Throughout the paper we fix an infinite set $X = \{x_0, x_1, x_2, \ldots\}$ of variables. x, y will always denote one of these variables. The sets Trm_t of *t-type terms*, and $Fmla_t$ of *t-type* (first-order) *formulas*, having variables from X, are defined as usual. A *k-ary term* is a term containing at most k-many distinct variables. $\tau(x_{i_1}, \ldots, x_{i_k})$ denotes that the variables occurring in τ are among $x_{i_1}, \ldots, x_{i_k}$. Substitutions are functions $\sigma : X \to Trm_t$ as usual, which extend to maps from terms to terms the natural way. For any substitution σ and term $\tau(x_{i_1}, \ldots, x_{i_k})$, $\sigma(\tau)$ will also be denoted by $\tau(x_{i_1}/\sigma(x_{i_1}), \ldots, x_{i_k}/\sigma(x_{i_k}))$. A binary term $\Delta(x, y)$ will also be written as $x\Delta y$. $\underline{Trm}_t$ denotes the *t-type world-algebra* (absolutely free algebra) generated by set X.

We will use symbol "$\models$" for both validity (in models) and (semantical) consequence relation of standard first-order logic. For any set $\Gamma \subseteq Fmla_t$,

$$Mod_t(\Gamma) =_{def} \{\underline{A} : \underline{A} \text{ is a } t\text{-type algebra and } (\forall\varphi \in \Gamma)\ \underline{A} \models \varphi\}.$$

A *t-type quasi-equation* is a t-type formula of form $(\tau_1 = \tau_1' \wedge \cdots \wedge \tau_k = \tau_k' \to \tau_0 = \tau_0')$, where $\tau_0, \tau_0', \ldots, \tau_k, \tau_k' \in Trm_t$. A *t-type quasi-variety* is a class K of t-type algebras such that $\mathsf{K} = Mod_t(\Gamma)$ for some set Γ of t-type quasi-equations. For any class K of t-type algebras, $Qvar(\mathsf{K})$ denotes the *generated quasi-variety* i.e., the smallest quasi-variety including K.

Algebraizable logical systems defined below are the same as "algebraizable deductive systems" of Blok and Pigozzi [4], or "algebraizable 1-deductive systems" of Blok and Pigozzi [5], or the semantical consequence relation of "consequence compact strongly nice general logics" of Andréka et al. [2].

Definition 2.1 A pair $\mathcal{L} = \langle Cn(\mathcal{L}), \approx_{\mathcal{L}}\rangle$ is called an *algebraizable logical system* iff $Cn(\mathcal{L})$ is an algebraic similarity type and $\approx_{\mathcal{L}}$ is a binary relation between sets of $Cn(\mathcal{L})$-type terms and $Cn(\mathcal{L})$-type terms, satisfying conditions (1–6) below. Elements of the domain of $Cn(\mathcal{L})$ are called the *logical connectives of* $\mathcal{L}$. The elements of set X (of variables) are called in this context *atomic formulas* (or *propositional variables*) *of* $\mathcal{L}$. Similarly, if φ is a (k-ary) term of type $Cn(\mathcal{L})$ then φ is also called a (k-ary) *formula of* $\mathcal{L}$, and the set $Trm_{Cn(\mathcal{L})}$ is also called as $Fm(\mathcal{L})$ when it is regarded as the *set of all formulas of* $\mathcal{L}$. $\approx_{\mathcal{L}}$ is called the *consequence relation of* $\mathcal{L}$.

1. $(\forall\varphi \in Fm(\mathcal{L}))(\forall\Gamma \subseteq Fm(\mathcal{L}))\varphi \in \Gamma \Rightarrow \Gamma \approx_{\mathcal{L}} \varphi$.
2. $(\forall\varphi \in Fm(\mathcal{L}))(\forall\Gamma, \Delta \subseteq Fm(\mathcal{L}))\Gamma \subseteq \Delta$ and $\Gamma \approx_{\mathcal{L}} \varphi \Rightarrow \Delta \approx_{\mathcal{L}} \varphi$.
3. $(\forall\varphi \in Fm(\mathcal{L}))(\forall\Gamma, \Delta \subseteq Fm(\mathcal{L}))\Gamma \approx_{\mathcal{L}} \varphi$ and $(\forall\psi \in \Gamma)\Delta \approx_{\mathcal{L}} \psi \Rightarrow \Delta \approx_{\mathcal{L}} \varphi$.
4. $(\forall\varphi \in Fm(\mathcal{L}))(\forall\Gamma \subseteq Fm(\mathcal{L}))\Gamma \approx_{\mathcal{L}} \varphi \Rightarrow (\exists$ finite $\Gamma' \subseteq \Gamma)\Gamma' \approx_{\mathcal{L}} \varphi$.
5. $(\forall\varphi \in Fm(\mathcal{L}))(\forall\Gamma \subseteq Fm(\mathcal{L}))(\forall$ substitution $\sigma)\Gamma \approx_{\mathcal{L}} \varphi \Rightarrow$ $\{\sigma(\psi) : \psi \in \Gamma\} \approx_{\mathcal{L}} \sigma(\varphi)$.
6. There are some $m, n \in \omega$, unary formulas $\varepsilon_0, \ldots, \varepsilon_{m-1}$ and $\delta_0, \ldots, \delta_{m-1}$, and binary formulas $\Delta_0 \ldots, \Delta_{n-1}$ of $\mathcal{L}$ such that properties (a–e) below hold for any $\varphi, \varphi_1, \ldots, \varphi_k, \psi, \psi_1, \ldots, \psi_k, \chi \in Fm(\mathcal{L})$, and for any $i < n$:

   (a) $\approx_{\mathcal{L}} \varphi\Delta_i\varphi$,

   (b) $\{\varphi\Delta_j\psi : j < n\} \approx_{\mathcal{L}} \psi\Delta_i\varphi$,

   (c) $\{\varphi\Delta_j\psi, \psi\Delta_j\chi : j < n\} \approx_{\mathcal{L}} \varphi\Delta_i\chi$,

   (d) $(\forall$ k-ary $c \in dom(Cn(\mathcal{L})))$,
   $\{\varphi_1\Delta_j\psi_1, \ldots, \varphi_k\Delta_j\psi_k : j < n\} \approx_{\mathcal{L}} c(\varphi_1, \ldots, \varphi_k)\Delta_i c(\psi_1, \ldots, \psi_k)$,

   (e) $(\forall s < m)\{\varphi\} \approx_{\mathcal{L}} \varepsilon_s(\varphi)\Delta_i\delta_s(\varphi)$ and
   $\{\varepsilon_s(\varphi)\Delta_j\delta_s(\varphi) : s < m, j < n\} \approx_{\mathcal{L}} \varphi$.

   A sequence $\langle\varepsilon_0, \ldots, \varepsilon_{m-1}, \delta_0, \ldots, \delta_{m-1}, \Delta_0, \ldots, \Delta_{n-1}\rangle$ satisfying (6)(a–e) is called an *algebraizator for* $\mathcal{L}$.

Some simple examples of algebraizable logical systems are inconsistent logics (where $\Gamma \approx_{\mathcal{L}} \varphi$ holds for any Γ, φ), and usual propositional logic (with algebraizator $\varepsilon_0(\varphi) = (\varphi \to \varphi)$, $\delta_0(\varphi) = \varphi$ and $\varphi \Delta_0 \psi = (\varphi \leftrightarrow \psi)$). Other examples (also for nonalgebraizable logical systems) can be found, e.g., in [4], [2], Andréka et al. [3], and Németi and Andréka [12].

Notation 2.2 For any $\Gamma, \Delta \subseteq Fm(\mathcal{L})$, if $\Delta \neq \varnothing$ then

$$\Gamma \approx_{\mathcal{L}} \Delta \iff_{def} (\forall \psi \in \Delta)\ \Gamma \approx_{\mathcal{L}} \psi.$$

We shall use $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ as an abbreviation for $\langle \varepsilon_0, \ldots, \varepsilon_{m-1}, \delta_0, \ldots, \delta_{m-1}, \Delta_0, \ldots, \Delta_{n-1} \rangle$. Similarly, e.g. $\bar{\varepsilon}(\varphi)\bar{\Delta}\bar{\delta}(\psi)$ abbreviates the set $\{\varepsilon_i(\varphi)\Delta_j\delta_i(\psi) : i < m,\ j < n\}$ of formulas. Or, on the first-order logic side, we write e.g. $\bar{\varepsilon}(x) = \bar{\delta}(x) \to \bar{\varepsilon}(y) = \bar{\delta}(y)$ instead of the set

$$\{\bigwedge_{i<m} \varepsilon_i(x) = \delta_i(x) \to \varepsilon_j(y) = \delta_j(y)\ :\ j < m\}$$

of quasi-equations. Related abbreviations will also be used without further explanation.

Definition 2.3 Let $\mathcal{L}$ be an algebraizable logical system and let $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ be an algebraizator for $\mathcal{L}$. For any $\Gamma \cup \{\varphi, \psi\} \subseteq Fm(\mathcal{L})$, let

$$\varphi \equiv_\Gamma \psi \quad \iff_{def} \quad \Gamma \approx_{\mathcal{L}} \varphi \bar{\Delta} \psi.$$

Then, by (6)(a–d) of Definition 2.1, $\equiv_\Gamma$ is a congruence relation on $\underline{Trm}_{Cn(\mathcal{L})}$. Let

$$\mathsf{Alg}(\mathcal{L}) =_{def} Qvar(\{\underline{Trm}_{Cn(\mathcal{L})} / \equiv_\Gamma :\ \Gamma \subseteq Fm(\mathcal{L})\}).$$

That is, $\mathsf{Alg}(\mathcal{L})$ is a class of algebras (set of sentences) of type $Cn(\mathcal{L})$. The definition of $\mathsf{Alg}(\mathcal{L})$ does not depend on the choice of the algebraizator $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ as the following proposition shows.

Proposition 2.4 (cf. [4], Theorem 2.15) *Let $\mathcal{L}$ be an algebraizable logical system and let both $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ and $\langle \bar{\varepsilon}', \bar{\delta}', \bar{\Delta}' \rangle$ be algebraizators for $\mathcal{L}$. Then for any formulas φ, ψ of $\mathcal{L}$,*

$$\varphi \bar{\Delta} \psi \approx_{\mathcal{L}} \varphi \bar{\Delta}' \psi \quad \textit{and} \quad \varphi \bar{\Delta}' \psi \approx_{\mathcal{L}} \varphi \bar{\Delta} \psi.$$

Thus, for any algebraizable logical system $\mathcal{L}$ there is a uniquely determined quasi-variety $\mathsf{Alg}(\mathcal{L})$. In the other direction, there are different algebraizable logical systems with the same "corresponding" quasi-variety, see e.g. [4], Chapter 5.2.4 for an example.

The following "back and forth" theorem establishes the basic connection between a logic $\mathcal{L}$ and its algebraic (i.e., usual first-order) "translation" $\mathsf{Alg}(\mathcal{L})$.

Theorem 2.5 (cf. [4] Thms.2.4, 4.7, 4.10 and [2] Thm.3.2.1) *Let $\mathcal{L}$ be an algebraizable logical system, and let $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ be an algebraizator for $\mathcal{L}$. Then*

1. *for any formulas $\varphi_0, \varphi_1, \ldots, \varphi_k$ of $\mathcal{L}$,*

$$\{\varphi_1, \ldots, \varphi_k\} \approx_{\mathcal{L}} \varphi_0 \iff \mathsf{Alg}(\mathcal{L}) \models \bigwedge_{1 \leq s \leq k} \bar{\varepsilon}(\varphi_s) = \bar{\delta}(\varphi_s) \ \to\ \bar{\varepsilon}(\varphi_0) = \bar{\delta}(\varphi_0);$$

2. *for any formulas* $\tau_0, \tau_1, \dots, \tau_k, \tau'_0, \tau'_1, \dots, \tau'_k$ *of* $\mathcal{L}$,

$$\mathsf{Alg}(\mathcal{L}) \models \tau_1 = \tau'_1 \wedge \cdots \wedge \tau_k = \tau'_k \ \rightarrow\ \tau_0 = \tau'_0 \quad \Longleftrightarrow \quad \{\tau_1 \bar{\Delta} \tau'_1, \dots, \tau_k \bar{\Delta} \tau'_k\} \approx_{\mathcal{L}} \tau_0 \bar{\Delta} \tau'_0.$$

3 *The category of algebraizable logical systems*

Definition 3.1

1. Let $\mathcal{L}_1, \mathcal{L}_2$ be algebraizable logical systems. A function $I : dom(Cn(\mathcal{L}_1)) \to Fm(\mathcal{L}_2)$ is called a *logic-translation of* $\mathcal{L}_1$ *into* $\mathcal{L}_2$ iff for any k-ary connective $c \in dom(Cn(\mathcal{L}_1))$, $I(c)$ is a k-ary formula of $\mathcal{L}_2$. A logic-translation always induces a function $\hat{I} : Fm(\mathcal{L}_1) \to Fm(\mathcal{L}_2)$ in the following natural way:
   (a) for any propositional variable x, $\hat{I}(x) =_{def} x$;
   (b) if c is a k-ary connective and $\varphi_0, \dots, \varphi_{k-1}$ are formulas of $\mathcal{L}_1$ then
   $$\hat{I}(c(\varphi_0, \dots, \varphi_{k-1})) =_{def} I(c)(x_0/\hat{I}(\varphi_0), \dots, x_{k-1}/\hat{I}(\varphi_{k-1})).$$
   $\hat{I}$ can be extended to any set Γ of formulas of $\mathcal{L}_1$ by taking $\hat{I}(\Gamma) =_{def} \{\hat{I}(\varphi) : \varphi \in \Gamma\}$.
2. A logic-translation I is called an $(\mathcal{L}_1, \mathcal{L}_2)$-*interpretation* iff
   (a) for any $\Gamma \cup \{\varphi\} \subseteq Fm(\mathcal{L}_1)$,
   $$\Gamma \approx_{\mathcal{L}_1} \varphi \Longrightarrow \hat{I}(\Gamma) \approx_{\mathcal{L}_2} \hat{I}(\varphi);$$
   (b) if $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ is an algebraizator for $\mathcal{L}_1$ then $\langle \hat{I}(\bar{\varepsilon}), \hat{I}(\bar{\delta}), \hat{I}(\bar{\Delta}) \rangle$ is an algebraizator for $\mathcal{L}_2$.
3. We define an equivalence relation on $(\mathcal{L}_1, \mathcal{L}_2)$-interpretations as follows.
   $$I \sim J \quad \Longleftrightarrow_{def} \quad (\forall \varphi \in Fm(\mathcal{L}_1)) \ \approx_{\mathcal{L}_2} \hat{I}(\varphi) \bar{\Delta}_2 \hat{J}(\varphi).$$
   (Here $\langle \bar{\varepsilon}_2, \bar{\delta}_2, \bar{\Delta}_2 \rangle$ is an arbitrary algebraizator for $\mathcal{L}_2$. By Proposition 2.4 and Definition 2.1.3, the definition of $\sim$ does not depend on the choice of the algebraizator.) Let $[I]$ denote the $\sim$-equivalence class of I.
4. For any algebraizable logical system $\mathcal{L}$, let $id_{\mathcal{L}}$ be the logic-translation of $\mathcal{L}$ into $\mathcal{L}$ defined by $id_{\mathcal{L}}(c) =_{def} c(x_0, \dots, x_{k-1})$, for each k-ary connective $c \in dom(Cn(\mathcal{L}))$.

Lemma 3.2

1. *Let* $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ *be algebraizable logical systems, let* I, I' *be* $(\mathcal{L}_1, \mathcal{L}_2)$*-interpretations and let* J, J' *be* $(\mathcal{L}_2, \mathcal{L}_3)$*-interpretations such that* $I \sim I'$ *and* $J \sim J'$ *hold. Then* $\hat{J} \circ I$ *and* $\hat{J}' \circ I'$ *are* $(\mathcal{L}_1, \mathcal{L}_3)$*-interpretations, and* $\hat{J} \circ I \sim \hat{J}' \circ I'$ *(where* $\circ$ *is the usual composition of functions).*
2. *For any algebraizable logical system* $\mathcal{L}$, $id_{\mathcal{L}}$ *is an* $(\mathcal{L}, \mathcal{L})$*-interpretation and for any* $(\mathcal{L}, \mathcal{L}')$*-interpretation* I, $\hat{I} \circ id_{\mathcal{L}} \sim I \sim \hat{id}_{\mathcal{L}'} \circ I$.

Proof: Since $(\hat{J} \circ I)\hat{} = \hat{J} \circ \hat{I}$, it is easy to check that $\hat{J} \circ I$ is an $(\mathcal{L}_1, \mathcal{L}_3)$-interpretation. To prove (1), let $\langle \bar{\varepsilon}_i, \bar{\delta}_i, \bar{\Delta}_i \rangle$ be an algebraizator for $\mathcal{L}_i$ (i=1,2,3), and let φ be an arbitrary formula of $\mathcal{L}_1$. Then, by $I \sim I'$,

$$\begin{aligned} \approx_{\mathcal{L}_2} \hat{I}(\varphi)\bar{\Delta}_2\hat{I}'(\varphi) \quad &\Longrightarrow \quad (J \text{ is an interpretation}) \\ \approx_{\mathcal{L}_3} \hat{J}(\hat{I}(\varphi)\bar{\Delta}_2\hat{I}'(\varphi)) \quad &\Longleftrightarrow \\ \approx_{\mathcal{L}_3} (\hat{J} \circ \hat{I})(\varphi)\hat{J}(\bar{\Delta}_2)(\hat{J} \circ \hat{I}')(\varphi) \quad &\Longrightarrow \quad (\text{Proposition 2.4 and Definition 2.1.3}) \\ \approx_{\mathcal{L}_3} (\hat{J} \circ \hat{I})(\varphi)\bar{\Delta}_3(\hat{J} \circ \hat{I}')(\varphi). \quad & \end{aligned}$$

On the other hand, by $J \sim J'$,

$$\approx_{\mathcal{L}_3} \hat{J}(\hat{I}'(\varphi))\bar{\Delta}_3\hat{J}'(\hat{I}'(\varphi)) \Longleftrightarrow \approx_{\mathcal{L}_3} (\hat{J} \circ \hat{I}')(\varphi)\bar{\Delta}_3(\hat{J}' \circ \hat{I}')(\varphi).$$

Thus, by Definition 2.1.3 and 2.1.6c, $\approx_{\mathcal{L}_3} (\hat{J} \circ \hat{I})(\varphi)\bar{\Delta}_3(\hat{J}' \circ \hat{I}')(\varphi)$ follows.

The proof of (2) is obvious. □

Definition 3.3 The *category* ALOG *of algebraizable logical systems* is defined as follows.

$$\begin{aligned} Obj_{\mathbf{ALOG}} \quad &=_{def} \quad \{\mathcal{L} : \mathcal{L} \text{ is an algebraizable logical system}\} \\ Mor_{\mathbf{ALOG}}(\mathcal{L}_1, \mathcal{L}_2) \quad &=_{def} \quad \{[I] : I \text{ is an } (\mathcal{L}_1, \mathcal{L}_2)\text{-interpretation}\}, \\ & \qquad\quad \text{for any } \mathcal{L}_1, \mathcal{L}_2 \in Obj_{\mathbf{ALOG}} \\ ID_{\mathcal{L}} \quad &=_{def} \quad [id_{\mathcal{L}}], \text{ for any } \mathcal{L} \in Obj_{\mathbf{ALOG}} \\ [J][I] \quad &=_{def} \quad [\hat{J} \circ I], \text{ for any } \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \in Obj_{\mathbf{ALOG}}, \\ & \qquad\quad [I] \in Mor_{\mathbf{ALOG}}(\mathcal{L}_1, \mathcal{L}_2),\ [J] \in Mor_{\mathbf{ALOG}}(\mathcal{L}_2, \mathcal{L}_3). \end{aligned}$$

Then, by Lemma 3.2, ALOG is indeed a category.

Now we proceed with making preparations to formulate the "algebraic" counterpart of category ALOG.

Definition 3.4

1. Let t be an algebraic similarity type and let K be a t-type quasi-variety. Let $\varepsilon_0, \ldots, \varepsilon_{m-1}, \delta_0, \ldots, \delta_{m-1}$ be unary and $\Delta_0 \ldots, \Delta_{n-1}$ be binary t-type terms for some $m, n \in \omega$. Then $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ is called a *deductivizator of* K iff

$$\mathsf{K} \models \bar{\varepsilon}(x\bar{\Delta}y) = \bar{\delta}(x\bar{\Delta}y) \leftrightarrow x = y$$

   holds.
2. We define an equivalence relation on deductivizators of K as follows:

$$\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle \simeq_{\mathbf{K}} \langle \bar{\varepsilon}', \bar{\delta}', \bar{\Delta}' \rangle \quad \Longleftrightarrow_{def} \quad \mathsf{K} \models \bar{\varepsilon}(x) = \bar{\delta}(x) \leftrightarrow \bar{\varepsilon}'(x) = \bar{\delta}'(x).$$

   Let $[\bar{\varepsilon}, \bar{\delta}, \bar{\Delta}]_{\mathbf{K}}$ denote the $\simeq_{\mathbf{K}}$-equivalence class of $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$.

Proposition 3.5 *Let $\mathcal{L}$ be an algebraizable logical system and let $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$, $\langle \bar{\varepsilon}', \bar{\delta}', \bar{\Delta}' \rangle$ be two algebraizators for $\mathcal{L}$. Then*

1. *$\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ and $\langle \bar{\varepsilon}', \bar{\delta}', \bar{\Delta}' \rangle$ are both deductivizators of* $\mathsf{Alg}(\mathcal{L})$*;*

2. *(cf. [4], Theorem 2.15)*
$\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle \simeq_{\mathbf{Alg}(\mathcal{L})} \langle \bar{\varepsilon}', \bar{\delta}', \bar{\Delta}' \rangle$.

Proof:

1. By Definition 2.1.6e,

$$x\bar{\Delta}y \approx\!\!\!\!\approx_{\mathcal{L}} \bar{\varepsilon}(x\bar{\Delta}y)\bar{\Delta}\bar{\delta}(x\bar{\Delta}y) \quad \text{and} \quad \bar{\varepsilon}(x\bar{\Delta}y)\bar{\Delta}\bar{\delta}(x\bar{\Delta}y) \approx\!\!\!\!\approx_{\mathcal{L}} x\bar{\Delta}y.$$

Thus, by Theorem 2.5.2, $\mathsf{Alg}(\mathcal{L}) \models \bar{\varepsilon}(x\bar{\Delta}y) = \bar{\delta}(x\bar{\Delta}y) \leftrightarrow x = y$.

2. By Definition 2.1.2 and 2.1.6e,

$$\bar{\varepsilon}(x)\bar{\Delta}\bar{\delta}(x) \approx\!\!\!\!\approx_{\mathcal{L}} \bar{\varepsilon}'(x)\bar{\Delta}'\bar{\delta}'(x) \quad \text{and} \quad \bar{\varepsilon}'(x)\bar{\Delta}'\bar{\delta}'(x) \approx\!\!\!\!\approx_{\mathcal{L}} \bar{\varepsilon}(x)\bar{\Delta}\bar{\delta}(x).$$

Thus, by Definition 2.1.3 and Proposition 2.4,

$$\bar{\varepsilon}(x)\bar{\Delta}\bar{\delta}(x) \approx\!\!\!\!\approx_{\mathcal{L}} \bar{\varepsilon}'(x)\bar{\Delta}\bar{\delta}'(x) \quad \text{and} \quad \bar{\varepsilon}'(x)\bar{\Delta}\bar{\delta}'(x) \approx\!\!\!\!\approx_{\mathcal{L}} \bar{\varepsilon}(x)\bar{\Delta}\bar{\delta}(x).$$

Therefore, by Theorem 2.5.2, $\mathsf{Alg}(\mathcal{L}) \models \bar{\varepsilon}(x) = \bar{\delta}(x) \leftrightarrow \bar{\varepsilon}'(x) = \bar{\delta}'(x)$. □

□

Definition 3.6 Let $t_1,\ t_2$ be algebraic similarity types. A function $\imath : dom(t_1) \to Trm_{t_2}$ is called a *term-translation of* t_1 *into* t_2 iff for any k-ary t_1-type function symbol f, $\imath(f)$ is a k-ary term of type t_2. A term-translation always induces a function $\hat{\imath} : Trm_{t_1} \to Trm_{t_2}$ and a function $\tilde{\imath} : Fmla_{t_1} \to Fmla_{t_2}$ as follows:

- for any variable $x \in X$, $\hat{\imath}(x) =_{def} x$;
- if f is a k-ary function symbol of type t_1 and $\tau_0, \ldots, \tau_{k-1} \in Trm_{t_1}$ then

$$\hat{\imath}(f(\tau_0, \ldots, \tau_{k-1})) =_{def} \imath(f)(x_0/\hat{\imath}(\tau_0), \ldots, x_{k-1}/\hat{\imath}(\tau_{k-1}));$$

- for any $\tau_0, \tau_1 \in Trm_{t_1}$, $\tilde{\imath}(\tau_0 = \tau_1) =_{def} (\hat{\imath}(\tau_0) = \hat{\imath}(\tau_1))$;
- for any $\varphi, \psi \in Fmla_{t_1}$,

$$\tilde{\imath}(\neg\varphi) =_{def} \neg\tilde{\imath}(\varphi), \quad \tilde{\imath}(\varphi \vee \psi) =_{def} \tilde{\imath}(\varphi) \vee \tilde{\imath}(\psi), \quad \tilde{\imath}(\exists x\varphi) =_{def} \exists x\, \tilde{\imath}(\varphi).$$

Similarly, the functions $\hat{\imath}$ and $\tilde{\imath}$ can be extended to sets of terms and formulas, respectively, by stipulating that for $\bar{\tau} \subseteq Trm_{t_1}$, $\hat{\imath}(\bar{\tau}) =_{def} \{\hat{\imath}(\tau) : \tau \in \bar{\tau}\}$, and for $\Gamma \subseteq Fmla_{t_1}$, $\tilde{\imath}(\Gamma) =_{def} \{\tilde{\imath}(\varphi) : \varphi \in \Gamma\}$.

Remark 3.7 A logic-translation I of some logic $\langle Cn(\mathcal{L}_1), \approx\!\!\!\!\approx_{\mathcal{L}_1} \rangle$ into some logic $\langle Cn(\mathcal{L}_2), \approx\!\!\!\!\approx_{\mathcal{L}_2} \rangle$ is in fact a term-translation of similarity type $Cn(\mathcal{L}_1)$ into $Cn(\mathcal{L}_2)$. Moreover, since formulas of $\mathcal{L}_i$ $(i = 1, 2)$ can be considered as $Cn(\mathcal{L}_i)$-type terms, the function $\hat{I}$ induced by I as a logic-translation is the same as $\hat{I}$ induced by I as a term-translation.

Lemma 3.8 *If* $\imath$ *is a term-translation of* t_1 *into* t_2 *then for any* $\Gamma \cup \{\varphi\} \subseteq Fmla_{t_1}$,

$$\Gamma \models \varphi \Longrightarrow \tilde{\imath}(\Gamma) \models \tilde{\imath}(\varphi).$$

Proof: It is easy to check that $\tilde{\imath}$ "preserves" the axioms and rules of any calculus for first-order logic. □

Definition 3.9

1. For $n = 1, 2$, let t_n be an algebraic similarity type, let K_n be a t_n-type quasi-variety and let $\langle \bar{\varepsilon}_n, \bar{\delta}_n, \bar{\Delta}_n \rangle$ be a deductivizator of K_n. Let
$$\mathcal{A}_n =_{def} \langle t_n, \mathsf{K}_n, [\bar{\varepsilon}_n, \bar{\delta}_n, \bar{\Delta}_n]_{\mathbf{K}_n} \rangle (n = 1, 2).$$
A term-translation $\imath$ from t_1 into t_2 is called an $(\mathcal{A}_1, \mathcal{A}_2)$-*interpretation* iff
   (a) for any $\varphi \in Fmla_{t_1}$, $\mathsf{K}_1 \models \varphi \quad \Longrightarrow \quad \mathsf{K}_2 \models \tilde{\imath}(\varphi)$;
   (b) $\langle \hat{\imath}(\bar{\varepsilon}_1), \hat{\imath}(\bar{\delta}_1), \hat{\imath}(\bar{\Delta}_1) \rangle \simeq_{\mathbf{K}_2} \langle \bar{\varepsilon}_2, \bar{\delta}_2, \bar{\Delta}_2 \rangle$.

   We note that this definition is sensible because, by (1a), $\langle \hat{\imath}(\bar{\varepsilon}_1), \hat{\imath}(\bar{\delta}_1), \hat{\imath}(\bar{\Delta}_1) \rangle$ is a deductivizator of K_2.
2. We define an equivalence relation on $(\mathcal{A}_1, \mathcal{A}_2)$-interpretations as follows:
$$\imath \approx \jmath \Longleftrightarrow_{def} \text{ for any } \tau \in Trm_{t_1}, \quad \mathsf{K}_2 \models \hat{\imath}(\tau) = \hat{\jmath}(\tau).$$
Let $[[\imath]]$ denote the $\approx$-equivalence class of $\imath$.
3. Let $\mathcal{A} =_{def} \langle t, \mathsf{K}, [\bar{\varepsilon}, \bar{\delta}, \bar{\Delta}]_{\mathbf{K}} \rangle$ as above. Let $id_{\mathcal{A}}$ be the term-translation of t into t defined by $id_{\mathcal{A}}(f) =_{def} f(x_0, \ldots, x_{k-1})$, for each k-ary function symbol $f \in dom(t)$.

We note that the function $\tilde{\imath}$ induced by an $(\mathcal{A}_1, \mathcal{A}_2)$-interpretation is a special case of the well-investigated notion of "interpretation between first-order theories," cf. [9], Andréka et al. [1], van Benthem and Pearce [13], Gergely [7], and Németi [10] and [11].

The following lemma is an easy consequence of basic properties of equational logic.

Lemma 3.10

1. *Let $\imath, \imath'$ be $(\mathcal{A}_1, \mathcal{A}_2)$-interpretations and let $\jmath, \jmath'$ be $(\mathcal{A}_2, \mathcal{A}_3)$-interpretations such that $\imath \approx \imath'$ and $\jmath \approx \jmath'$ hold. Then $\hat{\jmath} \circ \imath$ and $\hat{\jmath}' \circ \imath'$ are $(\mathcal{A}_1, \mathcal{A}_3)$-interpretations, and $\hat{\jmath} \circ \imath \approx \hat{\jmath}' \circ \imath'$.*
2. *$id_{\mathcal{A}}$ is an $(\mathcal{A}, \mathcal{A})$-interpretation, and for any $(\mathcal{A}, \mathcal{A}')$-interpretation $\imath$, $\hat{\imath} \circ id_{\mathcal{A}} \approx \imath \approx \widehat{id}_{\mathcal{A}'} \circ \imath$.*

Definition 3.11 The *category* QVAR *of logic-generated quasi-varieties* is defined as follows.

$$\begin{aligned}
Obj_{\mathbf{QVAR}} &=_{def} \{\mathcal{A} : \mathcal{A} = \langle t, \mathsf{K}, [\bar{\varepsilon}, \bar{\delta}, \bar{\Delta}]_{\mathbf{K}} \rangle, t \text{ is an algebraic} \\
&\qquad \text{similarity type, } \mathsf{K} \text{ is a } t\text{-type quasi-variety, and} \\
&\qquad \langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle \text{ is a deductivizator of } \mathsf{K}\} \\
Mor_{\mathbf{QVAR}}(\mathcal{A}_1, \mathcal{A}_2) &=_{def} \{[[\imath]] : \imath \text{ is an } (\mathcal{A}_1, \mathcal{A}_2)\text{-interpretation}\}, \\
&\qquad \text{for any } \mathcal{A}_1, \mathcal{A}_2 \in Obj_{\mathbf{QVAR}} \\
ID_{\mathcal{A}} &=_{def} [[id_{\mathcal{A}}]], \text{ for any } \mathcal{A} \in Obj_{\mathbf{QVAR}} \\
[[\jmath]][[\imath]] &=_{def} [[\hat{\jmath} \circ \imath]], \text{ for any } \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \in Obj_{\mathbf{QVAR}}, \\
&\qquad [[\imath]] \in Mor_{\mathbf{QVAR}}(\mathcal{A}_1, \mathcal{A}_2),\ [[\jmath]] \in Mor_{\mathbf{QVAR}}(\mathcal{A}_2, \mathcal{A}_3).
\end{aligned}$$

Then, by Lemma 3.10, QVAR is indeed a category.

4 Isomorphism

Theorem 4.1 ALOG *and* QVAR *are isomorphic categories.*

Proof: To prove the theorem, we define functors $F_1 : \mathsf{ALOG} \to \mathsf{QVAR}$ and $F_2 : \mathsf{QVAR} \to \mathsf{ALOG}$, and prove that (1–4) below hold.

1. for any $\mathcal{L} \in Obj_{\mathbf{ALOG}}$, $F_2(F_1(\mathcal{L})) = \mathcal{L}$;
2. for any $\mathcal{A} \in Obj_{\mathbf{QVAR}}$, $F_1(F_2(\mathcal{A})) = \mathcal{A}$;
3. for any $\mathcal{L}_1, \mathcal{L}_2 \in Obj_{\mathbf{ALOG}}$, $[I] \in Mor_{\mathbf{ALOG}}(\mathcal{L}_1, \mathcal{L}_2)$, $F_2(F_1([I])) = [I]$;
4. for any $\mathcal{A}_1, \mathcal{A}_2 \in Obj_{\mathbf{QVAR}}$, $[[\imath]] \in Mor_{\mathbf{QVAR}}(\mathcal{A}_1, \mathcal{A}_2)$, $F_1(F_2([[\imath]])) = [[\imath]]$.

Step 1. The definition of functors F_1, F_2 on objects.

First, let $\mathcal{L}$ be an algebraizable logical system and let $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ be an algebraizator for $\mathcal{L}$. Then let

$$F_1(\mathcal{L}) =_{def} \langle Cn(\mathcal{L}), \mathsf{Alg}(\mathcal{L}), [\bar{\varepsilon}, \bar{\delta}, \bar{\Delta}]_{\mathbf{Alg}(\mathcal{L})} \rangle.$$

Note that this definition is sensible by Proposition 3.5.

Second, to define functor F_2, let $\mathcal{A} \in Obj_{\mathbf{QVAR}}$, $\mathcal{A} = \langle t, \mathsf{K}, [\bar{\varepsilon}, \bar{\delta}, \bar{\Delta}]_{\mathbf{K}} \rangle$. Then

$$F_2(\mathcal{A}) =_{def} \langle t, \approx_{F_2(\mathcal{A})} \rangle,$$

where for any $\Gamma \cup \{\varphi\} \subseteq Trm_t$,

$$\Gamma \approx_{F_2(\mathcal{A})} \varphi \iff_{def} \text{there is some finite } \Gamma' \subseteq \Gamma \text{ such that } \mathsf{K} \models \bigwedge_{\psi \in \Gamma'} \bar{\varepsilon}(\psi) = \bar{\delta}(\psi) \ \to \ \bar{\varepsilon}(\varphi) = \bar{\delta}(\varphi).$$

By Definition 3.4.2, this definition is independent from the choice of representative $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ from the class $[\bar{\varepsilon}, \bar{\delta}, \bar{\Delta}]_{\mathbf{K}}$.

We show that $F_2(\mathcal{A})$ is an algebraizable logical system, and

$$\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle \text{ is an algebraizator for } F_2(\mathcal{A}). \tag{1}$$

Indeed, conditions (1–5) of Definition 2.1 hold for $F_2(\mathcal{A})$ by some basic properties of first-order logic. Since $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ is a deductivizator of K, condition (6) of Definition 2.1 holds for $F_2(\mathcal{A})$ and $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ because of basic properties of equational logic.

Step 2. The proofs of statements (1–2).

For (1): We show that for any algebraizable logical system $\mathcal{L} = \langle Cn(\mathcal{L}), \approx_{\mathcal{L}} \rangle$, $F_2(F_1(\mathcal{L})) = \mathcal{L}$ holds. Let $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ be an algebraizator for $\mathcal{L}$, and let

$$F_2(F_1(\mathcal{L})) =_{def} \langle Cn(\mathcal{L}), \approx' \rangle.$$

Then for any $\Gamma \cup \{\varphi\} \subseteq Fm(\mathcal{L})$,

$\Gamma \approx' \varphi \iff$ (by definition of F_1, F_2)
$(\exists \Gamma' \subseteq \Gamma,\ \Gamma' \text{ is finite})\ \mathsf{Alg}(\mathcal{L}) \models \bigwedge_{\psi \in \Gamma'} \bar{\varepsilon}(\psi) = \bar{\delta}(\psi) \to \ \bar{\varepsilon}(\varphi) = \bar{\delta}(\varphi) \iff$
(by Theorem 2.5.1)
$(\exists \Gamma' \subseteq \Gamma,\ \Gamma' \text{ is finite})\ \Gamma' \approx_{\mathcal{L}} \varphi \iff$ (by Definition 2.1.2, 2.1.4)
$\Gamma \approx_{\mathcal{L}} \varphi$.

For (2): Let $\mathcal{A} = \langle t, \mathsf{K}, [\bar{\varepsilon}, \bar{\delta}, \bar{\Delta}]_{\mathbf{K}}\rangle$. We show that $F_1(F_2(\mathcal{A})) = \mathcal{A}$. By (1) above, it is enough to show that $\mathsf{K} = \mathsf{Alg}(F_2(\mathcal{A}))$ holds. To this end, let q be an arbitrary t-type quasi-equation of form $\tau_1 = \tau_1' \wedge \cdots \wedge \tau_k = \tau_k' \;\rightarrow\; \tau_0 = \tau_0'$. Then, by Theorem 2.5.2,

$$\begin{aligned}&\mathsf{Alg}(F_2(\mathcal{A})) \models q \iff \{\tau_1\bar{\Delta}\tau_1', \ldots, \tau_k\bar{\Delta}\tau_k'\} \approx_{F_2(\mathcal{A})} \tau_0\bar{\Delta}\tau_0' \\ &\overset{\text{def. of } F_2}{\iff} \mathsf{K} \models \bigwedge_{1\le i\le k} \bar{\varepsilon}(\tau_i\bar{\Delta}\tau_i') = \bar{\delta}(\tau_i\bar{\Delta}\tau_i') \rightarrow \bar{\varepsilon}(\tau_0\bar{\Delta}\tau_0') = \bar{\delta}(\tau_0\bar{\Delta}\tau_0') \\ &\iff \mathsf{K} \models q,\end{aligned}$$

since $\langle\bar{\varepsilon}, \bar{\delta}, \bar{\Delta}\rangle$ is a deductivizator of K.

Step 3. The definition of functors F_1, F_2 on morphisms.

First, for any $(\mathcal{L}_1, \mathcal{L}_2)$-interpretation I, let $F_1([I]) =_{def} [[I]]$. We have to show that this definition is sensible, that is,

(a) if I is an $(\mathcal{L}_1, \mathcal{L}_2)$-interpretation then I is also an $(F_1(\mathcal{L}_1), F_1(\mathcal{L}_2))$-interpretation;

(b) for any $(\mathcal{L}_1, \mathcal{L}_2)$-interpretations I, J, if $I \sim J$ then also $I \approx J$.

Let $\langle\bar{\varepsilon}_j, \bar{\delta}_j, \bar{\Delta}_j\rangle$ be an algebraizator for $\mathcal{L}_j$ $(j = 1, 2)$.

For (a): First, we have to show that for any $\varphi \in Fmla_{Cn(\mathcal{L}_1)}$, "$\mathsf{Alg}(\mathcal{L}_1) \models \varphi \Rightarrow \mathsf{Alg}(\mathcal{L}_2) \models \tilde{I}(\varphi)$" holds. By Lemma 3.8, it is enough to prove this statement for quasi-equations, since $\mathsf{Alg}(\mathcal{L}_1) \models \varphi$ implies that there is some set Γ of quasi-equations such that $\mathsf{Alg}(\mathcal{L}_1) \models \Gamma$ and $\Gamma \models \varphi$ hold. Thus, assume that $\mathsf{Alg}(\mathcal{L}_1) \models (\tau_1 = \tau_1' \wedge \cdots \wedge \tau_k = \tau_k' \rightarrow \tau_0 = \tau_0')$. Then, by Theorem 2.5.2,

$$\begin{aligned}\{\tau_1\bar{\Delta}_1\tau_1', \ldots, \tau_k\bar{\Delta}_1\tau_k'\} \approx_{\mathcal{L}_1} \tau_0\bar{\Delta}_1\tau_0' &\quad\Longrightarrow \\ \{\hat{I}(\tau_1\bar{\Delta}_1\tau_1'), \ldots, \hat{I}(\tau_k\bar{\Delta}_1\tau_k')\} \approx_{\mathcal{L}_2} \hat{I}(\tau_0\bar{\Delta}_1\tau_0') &\quad\Longleftrightarrow \\ \{\hat{I}(\tau_1)\hat{I}(\bar{\Delta}_1)\hat{I}(\tau_1'), \ldots, \hat{I}(\tau_k)\hat{I}(\bar{\Delta}_1)\hat{I}(\tau_k')\} \approx_{\mathcal{L}_2} \hat{I}(\tau_0)\hat{I}(\bar{\Delta}_1)\hat{I}(\tau_0') &\quad\Longleftrightarrow \\ \text{(by Proposition 2.4)} & \\ \{\hat{I}(\tau_1)\bar{\Delta}_2\hat{I}(\tau_1'), \ldots, \hat{I}(\tau_k)\bar{\Delta}_2\hat{I}(\tau_k')\} \approx_{\mathcal{L}_2} \hat{I}(\tau_0)\bar{\Delta}_2\hat{I}(\tau_0') &\quad\Longleftrightarrow \\ \text{(by Theorem 2.5.2)} & \\ \mathsf{Alg}(\mathcal{L}_2) \models \tilde{I}(\tau_1 = \tau_1' \wedge \cdots \wedge \tau_k = \tau_k' \rightarrow \tau_0 = \tau_0'). &\end{aligned}$$

Second, by Definition 3.1.2b, $\langle\hat{I}(\bar{\varepsilon}_1), \hat{I}(\bar{\delta}_1), \hat{I}(\bar{\Delta}_1)\rangle$ is an algebraizator for $\mathcal{L}_2$. Therefore, by Proposition 3.5,

$$\langle\hat{I}(\bar{\varepsilon}_1), \hat{I}(\bar{\delta}_1), \hat{I}(\bar{\Delta}_1)\rangle \simeq_{\mathbf{Alg}(\mathcal{L}_2)} \langle\bar{\varepsilon}_2, \bar{\delta}_2, \bar{\Delta}_2\rangle$$

holds, as needed.

For (b): Assume $I \sim J$; then $\approx_{\mathcal{L}_2} \hat{I}(\tau)\bar{\Delta}_2\hat{J}(\tau)$ for any $\tau \in Fm(\mathcal{L}_1) = Trm_{Cn(\mathcal{L}_1)}$. Then, by Theorem 2.5.2, $\mathsf{Alg}(\mathcal{L}_2) \models \hat{I}(\tau) = \hat{J}(\tau)$ holds, proving $I \approx J$.

Next, let $\mathcal{A}_1, \mathcal{A}_2 \in Obj_{\mathbf{QVAR}}$, $\mathcal{A}_k = \langle t_k, \mathsf{K}_k, [\bar{\varepsilon}_k, \bar{\delta}_k, \bar{\Delta}_k]_{\mathbf{K}_k}\rangle$ $(k = 1, 2)$. For any $(\mathcal{A}_1, \mathcal{A}_2)$-interpretation $\imath$, let $F_2([[\imath]]) =_{def} [\imath]$.

We have to show that this definition is sensible, that is,

(c) if $\imath$ is an $(\mathcal{A}_1, \mathcal{A}_2)$-interpretation then $\imath$ is also an $(F_2(\mathcal{A}_1), F_2(\mathcal{A}_2))$-interpretation;

(d) for any $(\mathcal{A}_1, \mathcal{A}_2)$-interpretations $\imath$, $\jmath$, if $\imath \approx \jmath$ then also $\imath \sim \jmath$.

For (c): First, by Remark 3.7, we must show that for any $\Gamma \cup \{\varphi\} \subseteq Fm(F_2(\mathcal{A}_1)) = Trm_{t_1}$, $\Gamma \approx_{F_2(\mathcal{A}_1)} \varphi \Longrightarrow \hat{\imath}(\Gamma) \approx_{F_2(\mathcal{A}_2)} \hat{\imath}(\varphi)$ holds. Now assume that $\Gamma \approx_{F_2(\mathcal{A}_1)} \varphi$. Then, by definition, there is some finite $\Gamma' \subseteq \Gamma$ such that

$$
\begin{aligned}
& \mathsf{K}_1 \models \bigwedge_{\psi \in \Gamma'} \bar{\varepsilon}_1(\psi) = \bar{\delta}_1(\psi) \ \rightarrow \ \bar{\varepsilon}_1(\varphi) = \bar{\delta}_1(\varphi) \\
\Longrightarrow \quad & \mathsf{K}_2 \models \tilde{\imath}\Big(\bigwedge_{\psi \in \Gamma'} \bar{\varepsilon}_1(\psi) = \bar{\delta}_1(\psi) \ \rightarrow \ \bar{\varepsilon}_1(\varphi) = \bar{\delta}_1(\varphi)\Big) \\
\Longleftrightarrow \quad & \mathsf{K}_2 \models \bigwedge_{\psi \in \Gamma'} \hat{\imath}(\bar{\varepsilon}_1(\psi)) = \hat{\imath}(\bar{\delta}_1(\psi)) \ \rightarrow \ \hat{\imath}(\bar{\varepsilon}_1(\varphi)) = \hat{\imath}(\bar{\delta}_1(\varphi)) \\
\Longleftrightarrow \quad & \mathsf{K}_2 \models \bigwedge_{\psi \in \Gamma'} \hat{\imath}(\bar{\varepsilon}_1)(\hat{\imath}(\psi)) = \hat{\imath}(\bar{\delta}_1)(\hat{\imath}(\psi)) \ \rightarrow \ \hat{\imath}(\bar{\varepsilon}_1)(\hat{\imath}(\varphi)) = \hat{\imath}(\bar{\delta}_1)(\hat{\imath}(\varphi)) \\
\Longleftrightarrow \quad & \mathsf{K}_2 \models \bigwedge_{\psi \in \Gamma'} \bar{\varepsilon}_2(\hat{\imath}(\psi)) = \bar{\delta}_2(\hat{\imath}(\psi)) \ \rightarrow \ \bar{\varepsilon}_2(\hat{\imath}(\varphi)) = \bar{\delta}_2(\hat{\imath}(\varphi)) \\
\Longleftrightarrow \quad & \hat{\imath}(\Gamma) \approx_{F_2(\mathcal{A}_2)} \hat{\imath}(\varphi).
\end{aligned}
$$

Second, let $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ be an arbitrary algebraizator for $F_2(\mathcal{A}_1)$. We have to show that $\langle \hat{\imath}(\bar{\varepsilon}), \hat{\imath}(\bar{\delta}), \hat{\imath}(\bar{\Delta}) \rangle$ is an algebraizator for $F_2(\mathcal{A}_2)$. By (1) above, $\langle \bar{\varepsilon}_1, \bar{\delta}_1, \bar{\Delta}_1 \rangle$ is also an algebraizator for $F_2(\mathcal{A}_1)$, thus, by Proposition 3.5, $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle$ and $\langle \bar{\varepsilon}_1, \bar{\delta}_1, \bar{\Delta}_1 \rangle$ are both deductivizators of $\mathsf{Alg}(F_2(\mathcal{A}_1))$ with

$$\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle \simeq_{\mathbf{Alg}(F_2(\mathcal{A}_1))} \langle \bar{\varepsilon}_1, \bar{\delta}_1, \bar{\Delta}_1 \rangle.$$

By statement (2) above, $\mathsf{Alg}(F_2(\mathcal{A}_1)) = \mathsf{K}_1$, thus $\langle \bar{\varepsilon}, \bar{\delta}, \bar{\Delta} \rangle \simeq_{\mathbf{K}_1} \langle \bar{\varepsilon}_1, \bar{\delta}_1, \bar{\Delta}_1 \rangle$ holds. Since $\imath$ is an $(\mathcal{A}_1, \mathcal{A}_2)$-interpretation, this implies that $\langle \hat{\imath}(\bar{\varepsilon}), \hat{\imath}(\bar{\delta}), \hat{\imath}(\bar{\Delta}) \rangle$ and $\langle \hat{\imath}(\bar{\varepsilon}_1), \hat{\imath}(\bar{\delta}_1), \hat{\imath}(\bar{\Delta}_1) \rangle$ are both deductivizators of K_2 and $\langle \hat{\imath}(\bar{\varepsilon}), \hat{\imath}(\bar{\delta}), \hat{\imath}(\bar{\Delta}) \rangle \simeq_{\mathbf{K}_2} \langle \hat{\imath}(\bar{\varepsilon}_1), \hat{\imath}(\bar{\delta}_1), \hat{\imath}(\bar{\Delta}_1) \rangle$. Now, by (1) again, it follows that $\langle \hat{\imath}(\bar{\varepsilon}), \hat{\imath}(\bar{\delta}), \hat{\imath}(\bar{\Delta}) \rangle$ is an algebraizator for $F_2(\mathcal{A}_2)$.

For (d): Assume $\imath \approx \jmath$, and let $\varphi \in Trm_{t_1} = Fm(F_2(\mathcal{A}_1))$. Then $\mathsf{K}_2 \models \hat{\imath}(\varphi) = \hat{\jmath}(\varphi)$ holds. Thus, by $\langle \bar{\varepsilon}_2, \bar{\delta}_2, \bar{\Delta}_2 \rangle$ being a deductivizator, $\mathsf{K}_2 \models \bar{\varepsilon}_2(\hat{\imath}(\varphi)\bar{\Delta}_2\hat{\jmath}(\varphi)) = \bar{\delta}_2(\hat{\imath}(\varphi)\bar{\Delta}_2\hat{\jmath}(\varphi))$ follows. Then, by the definition of F_2, $\approx_{F_2(\mathcal{A}_2)} \hat{\imath}(\varphi)\bar{\Delta}_2\hat{\jmath}(\varphi)$, proving $\imath \sim \jmath$.

The proofs of statements (3) and (4) above are immediate from the definitions of F_1 and F_2.

We have proved that ALOG and QVAR are isomorphic categories. □

5 Cocompleteness

Theorem 5.1 QVAR *is a small-cocomplete category (i.e., all small colimits exist in it).*

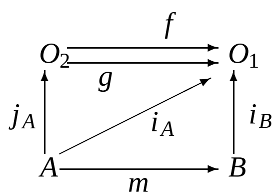
The proof uses the following lemma.

Lemma 5.2 (cf. [8], p. 109) *If a category has all coequalizers and all small coproducts then it is small-cocomplete.*

Proof of Lemma 5.2: Here we give the sketch of the proof in order to illustrate that colimits in general are indeed "computable" if coequalizers and coproducts are given.

Let a small diagram $\mathcal{D}$ be given. Let $\langle O_1, i_A\rangle_{A\in Obj_{\mathcal{D}}}$ be the coproduct cocone of all the objects of $\mathcal{D}$. Let $\mathcal{M}$ denote the set of those objects of $\mathcal{D}$ which are domains of some morphisms of $\mathcal{D}$, and let $\langle O_2, j_A\rangle_{A\in\mathcal{M}}$ be the coproduct of $\mathcal{M}$. Then the two cocones $\langle O_1, i_A\rangle_{A\in\mathcal{M}}$ and $\langle O_1, i_B m\rangle_{A\in\mathcal{M}, B\in Obj_{\mathcal{D}}, m\in Mor_{\mathcal{D}}(A,B)}$ induce two morphisms f and g from O_2 to O_1.

$$(\exists! f)(\forall A\in\mathcal{M})\ f j_A = i_A$$
$$(\exists! g)(\forall A\in\mathcal{M})(\forall B\in Obj_{\mathcal{D}})(\forall m\in Mor_{\mathcal{D}}(A,B))\ g j_A = i_B m$$



It is proved in MacLane [8] that the coequalizer of diagram $\langle O_1, O_2, f, g\rangle$ equals to the colimit of diagram $\mathcal{D}$. □

Proof of Theorem 5.1: We give the small coproducts and the coequalizers in category QVAR.

Let $\mathcal{D}$ be a small diagram in QVAR with

$$Obj_{\mathcal{D}} = \{\mathcal{A}_s : s\in S\} = \{\langle t_s, \mathsf{K}_s, [\bar\varepsilon_s, \bar\delta_s, \bar\Delta_s]_{\mathbf{K}_s}\rangle : s\in S\},$$

for some set S, and having no morphisms. For each $s\in S$, let $Ax_s \subseteq Fmla_{t_s}$ be a set of t_s-type quasi-equations such that $Mod_{t_s}(Ax_s) = \mathsf{K}_s$. Let

$$t =_{def} \biguplus_{s\in S} t_s \qquad (\biguplus \text{ denotes disjoint union})$$
$$Ax =_{def} \biguplus_{s\in S} Ax_s \cup \{(\bar\varepsilon_{s_1}(x) = \bar\delta_{s_1}(x)) \leftrightarrow (\bar\varepsilon_{s_2}(x) = \bar\delta_{s_2}(x))\ :\ s_1, s_2\in S\})$$
$$\mathsf{K} =_{def} Mod_t(Ax).$$

Then for any $s_1, s_2\in S$, $\langle\bar\varepsilon_{s_1}, \bar\delta_{s_1}, \bar\Delta_{s_1}\rangle \simeq_{\mathbf{K}} \langle\bar\varepsilon_{s_2}, \bar\delta_{s_2}, \bar\Delta_{s_2}\rangle$. Now let $s\in S$ be arbitrary and let

$$[\bar\varepsilon, \bar\delta, \bar\Delta]_{\mathbf{K}} =_{def} [\bar\varepsilon_s, \bar\delta_s, \bar\Delta_s]_{\mathbf{K}}.$$

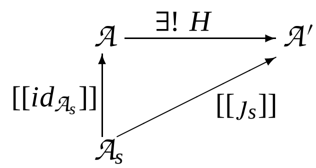
Claim 5.3 *$\langle\langle t, \mathsf{K}, [\bar\varepsilon, \bar\delta, \bar\Delta]_{\mathbf{K}}\rangle, [[id_{\mathcal{A}_s}]]\rangle_{s\in S}$ is the coproduct of $\mathcal{D}$.*

Proof of Claim 5.3: Let $\mathcal{A} =_{def} \langle t, \mathsf{K}, [\bar\varepsilon, \bar\delta, \bar\Delta]_{\mathbf{K}}\rangle$ and $\mathcal{A}' =_{def} \langle t', \mathsf{K}', [\bar\varepsilon', \bar\delta', \bar\Delta']_{\mathbf{K}'}\rangle$. Assume that $\langle\mathcal{A}', [[\jmath_s]]\rangle_{s\in S}$ is a cocone of $\mathcal{D}$. We have to prove that there is a unique $H\in Mor_{\mathbf{QVAR}}(\mathcal{A}, \mathcal{A}')$ such that $(\forall s\in S)\ H[[id_{\mathcal{A}_s}]] = [[\jmath_s]]$.

To this end, let $h : dom(t) \to Trm_{t'}$ be the following function. For any $s\in S$, $f\in dom(t_s)$,

$$h(f) =_{def} \jmath_s(f).$$

Then h is a term-translation of t into t' with $\hat h \circ id_{\mathcal{A}_s} = \jmath_s$, for any $s\in S$. We prove that



(a) h is an $(\mathcal{A}, \mathcal{A}')$-interpretation;
(b) for any $(\mathcal{A}, \mathcal{A}')$-interpretation h' with $\hat{h}' \circ id_{\mathcal{A}_s} \approx \jmath_s$ ($s \in S$), $h \approx h'$ holds.

For (a): Since $\jmath_s$ is an $(\mathcal{A}_s, \mathcal{A}')$-interpretation,

$$\langle \bar{\varepsilon}', \bar{\delta}', \bar{\Delta}' \rangle \simeq_{\mathbf{K}'} \langle \hat{\jmath}_s(\bar{\varepsilon}_s), \hat{\jmath}_s(\bar{\delta}_s), \hat{\jmath}_s(\bar{\Delta}_s) \rangle$$

holds, for any $s \in S$. Therefore, for any $s_1, s_2 \in S$,

$$\begin{array}{l} \langle \hat{\jmath}_{s_1}(\bar{\varepsilon}_{s_1}), \hat{\jmath}_{s_1}(\bar{\delta}_{s_1}), \hat{\jmath}_{s_1}(\bar{\Delta}_{s_1}) \rangle \simeq_{\mathbf{K}'} \langle \hat{\jmath}_{s_2}(\bar{\varepsilon}_{s_2}), \hat{\jmath}_{s_2}(\bar{\delta}_{s_2}), \hat{\jmath}_{s_2}(\bar{\Delta}_{s_2}) \rangle, \quad \text{i.e.,} \\ \mathsf{K}' \models (\hat{\jmath}_{s_1}(\bar{\varepsilon}_{s_1})(x) = \hat{\jmath}_{s_1}(\bar{\delta}_{s_1})(x)) \leftrightarrow (\hat{\jmath}_{s_2}(\bar{\varepsilon}_{s_2})(x) = \hat{\jmath}_{s_2}(\bar{\delta}_{s_2})(x)). \end{array} \tag{2}$$

Now let $\varphi \in Fmla_t$ and assume $\mathsf{K} \models \varphi$. Then $Ax \models \varphi$ thus, by Lemma 3.8,

$$\tilde{h}(Ax) \models \tilde{h}(\varphi). \tag{3}$$

By definition,

$$\begin{array}{rcl} \tilde{h}(Ax) & = & \biguplus_{s \in S} \tilde{h}(\tilde{id}_{\mathcal{A}_s}(Ax_s)) \cup \\ & & \{(\hat{\jmath}_{s_1}(\bar{\varepsilon}_{s_1})(x) = \hat{\jmath}_{s_1}(\bar{\delta}_{s_1})(x)) \leftrightarrow (\hat{\jmath}_{s_2}(\bar{\varepsilon}_{s_2})(x) = \hat{\jmath}_{s_2}(\bar{\delta}_{s_2})(x)) : s_1, s_2 \in S\}. \end{array}$$

Now, since $(\forall s \in S)$ $\tilde{h} \circ \tilde{id}_{\mathcal{A}_s} = \tilde{\jmath}_s$ and $\jmath_s$ is an $(\mathcal{A}_s, \mathcal{A}')$-interpretation, (2) implies that $\mathsf{K}' \models \tilde{h}(Ax)$. Thus, by (3), $\mathsf{K}' \models \tilde{h}(\varphi)$ follows, as needed.

For (b): Let h' be an $(\mathcal{A}, \mathcal{A}')$-interpretation with $\hat{h}' \circ id_{\mathcal{A}_s} \approx \jmath_s$ ($s \in S$). Then for any $s \in S$, $\tau_s \in Trm_{t_s}$,

$$\mathsf{K}' \models (\hat{h}' \circ id_{\mathcal{A}_s})\hat{\ }(\tau_s) = \hat{\jmath}_s(\tau_s).$$

In particular, for any k-ary $f \in dom(t_s)$,

$$\mathsf{K}' \models \hat{h}'(f(x_0, \ldots, x_{k-1})) = \hat{\jmath}_s(f(x_0, \ldots, x_{k-1})).$$

By the definition of h, for any $s \in S$, for any k-ary $f \in dom(t_s)$,

$$\mathsf{K}' \models \hat{h}(f(x_0, \ldots, x_{k-1})) = \hat{\jmath}_s(f(x_0, \ldots, x_{k-1}))$$

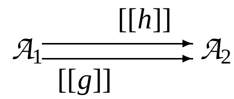
also holds. Now, by induction on the structure of t-type terms, it follows that for any $\tau \in Trm_t$,

$$\mathsf{K}' \models \hat{h}'(\tau) = \hat{h}(\tau),$$

proving $h' \approx h$.

Thus, by (a) and (b), $H =_{def} [[h]]$ is the unique morphism with $H[[id_{\mathcal{A}_s}]] = [[\jmath_s]]$ ($s \in S$), proving Claim 5.3. □

Now let $\mathcal{A}_i = \langle t_i, \mathsf{K}_i, [\bar{\varepsilon}_i, \bar{\delta}_i, \bar{\Delta}_i]_{\mathbf{K}_i}\rangle$ $(i = 1, 2)$ be two objects of QVAR, and let $[[h]], [[g]] \in Mor_{\mathbf{QVAR}}(\mathcal{A}_1, \mathcal{A}_2)$. Consider the following diagram $\mathcal{E}$.



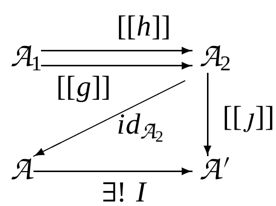
Let $Ax_2 \subseteq Fmla_{t_2}$ be a set of t_2-type quasi-equations such that $Mod_{t_2}(Ax_2) = \mathsf{K}_2$, and let

$$\begin{aligned} Ax \;&=_{def}\; Ax_2 \cup \\ &\quad \{\hat{h}(f(x_0, \ldots, x_{k-1})) = \hat{g}(f(x_0, \ldots, x_{k-1})) : f \in dom(t_1)\ k\text{-ary}\} \\ \mathsf{K} \;&=_{def}\; Mod_{t_2}(Ax). \end{aligned}$$

Claim 5.4 *$\langle\langle t_2, \mathsf{K}, [\bar{\varepsilon}_2, \bar{\delta}_2, \bar{\Delta}_2]_{\mathbf{K}}\rangle, [[id_{\mathcal{A}_2}]]\rangle$ is the colimit of $\mathcal{E}$.*

Proof of Claim 5.4: First, it can be proved by induction on the structure of t_1-type terms that for any $\tau \in Trm_{t_1}$, $\mathsf{K} \models \hat{h}(\tau) = \hat{g}(\tau)$. Therefore, since $(id_{\mathcal{A}_2} \circ h)\hat{} = \hat{h}$ and $(id_{\mathcal{A}_2} \circ g)\hat{} = \hat{g}$, $[[id_{\mathcal{A}_2}]][[h]] = [[id_{\mathcal{A}_2}]][[g]]$ follows.

Second, let $\mathcal{A} =_{def} \langle t_2, \mathsf{K}, [\bar{\varepsilon}_2, \bar{\delta}_2, \bar{\Delta}_2]_{\mathbf{K}}\rangle$, and take an object $\mathcal{A}' =_{def} \langle t', \mathsf{K}', [\bar{\varepsilon}', \bar{\delta}', \bar{\Delta}']_{\mathbf{K}'}\rangle$ of QVAR and some $[[\jmath]] \in Mor_{\mathbf{QVAR}}(\mathcal{A}_2, \mathcal{A}')$ with $[[\jmath]][[h]] = [[\jmath]][[g]]$. We have to show that there is a unique $I \in Mor_{\mathbf{QVAR}}(\mathcal{A}, \mathcal{A}')$ such that $I[[id_{\mathcal{A}_2}]] = [[\jmath]]$.



We show that $I =_{def} [[\jmath]]$ is an appropriate choice that is,

(c) $\jmath$ is an $(\mathcal{A}, \mathcal{A}')$-interpretation;

(d) for any $(\mathcal{A}, \mathcal{A}')$-interpretation $\jmath'$ with $\hat{\jmath}' \circ id_{\mathcal{A}_2} \approx \jmath$, $\jmath' \approx \jmath$ holds.

For (c): First, since $\jmath$ is an $(\mathcal{A}_2, \mathcal{A}')$-interpretation,

$$\langle \hat{\jmath}(\bar{\varepsilon}_2), \hat{\jmath}(\bar{\delta}_2), \hat{\jmath}(\bar{\Delta}_2)\rangle \simeq_{\mathbf{K}'} \langle \bar{\varepsilon}', \bar{\delta}', \bar{\Delta}'\rangle \quad \text{and} \quad \mathsf{K}' \models \tilde{\jmath}(Ax_2). \tag{4}$$

Second, since $[[\jmath]][[h]] = [[\jmath]][[g]]$, thus for any k-ary function symbol of type t_1,

$$\begin{aligned} &\mathsf{K}' \models \hat{\jmath}(\hat{h}(f(x_0, \ldots, x_{k-1}))) = \hat{\jmath}(\hat{g}(f(x_0, \ldots, x_{k-1}))) \iff \\ &\mathsf{K}' \models \tilde{\jmath}(\hat{h}(f(x_0, \ldots, x_{k-1})) = \hat{g}(f(x_0, \ldots, x_{k-1}))). \end{aligned} \tag{5}$$

Now let $\varphi \in Fmla_{t_2}$ and assume $\mathsf{K} \models \varphi$. By Lemma 3.8, $\tilde{\jmath}(Ax) \models \tilde{\jmath}(\varphi)$ holds. Therefore, by (4) and (5), $\mathsf{K}' \models \tilde{\jmath}(\varphi)$ follows.

Item (d) can be proved analogously to item (b) in the proof of Claim 5.3 above.

□

We have proved that small coproducts and coequalizers exist in category QVAR. Now, by Lemma 5.2, all small colimits exist in QVAR. □

Corollary 5.5 ALOG *is a small-cocomplete category.*

We note that though colimits always exist in ALOG, they are not always "interesting." E.g. if $\mathcal{L}_1$ and $\mathcal{L}_2$ are two different algebraizable logical systems with $\mathsf{Alg}(\mathcal{L}_1) = \mathsf{Alg}(\mathcal{L}_2)$ then their coproduct in ALOG is an inconsistent logic.

The proof of Theorem 5.1 also yields the following result.

Corollary 5.6 *Let $\mathcal{D}$ be a small diagram of* QVAR*, having objects $\langle t_s, \mathsf{K}_s, [\bar{\varepsilon}_s, \bar{\delta}_s, \bar{\Delta}_s]_{\mathbf{K}_s}\rangle_{s\in S}$ for some set S, and having arbitrary morphisms. Let $\langle t, \mathsf{K}, [\bar{\varepsilon}, \bar{\delta}, \bar{\Delta}]_{\mathbf{K}}\rangle$ be the colimit of $\mathcal{D}$. If for each $s \in S$, K_s is a finitely axiomatizable quasi-variety then* K *is also finitely axiomatizable.*

From the point of view of logics, this corollary means that any combination of *finitely axiomatizable logics* ("logics admitting finite Hilbert-style inference systems" in [2], or "finite deductive systems" in [4]) is also finitely axiomatizable.

6 Discussion In this paper only the first steps have been taken toward a systematic study of combining arbitrary logics by translating them into usual first-order logic. Investigation can be extended to the study of categories of logics, where e.g. the consequence relation is *not compact* ((4) of Definition 2.1 is missing); or where condition (6e) of Definition 2.1 is missing (called *congruential logics* in [4]); or where condition (6) of Definition 2.1 is missing altogether (called *structural logics* in [4]).

An even more ambitious task is to develop the category theoretic "reconstruction" of combining logics which are given not merely with their consequence relations but also together with their semantics. (Algebraization of these kinds of logics is given e.g. [2], [3], [12].) This kind of "modeling" should be capable to reconstruct how the semantics of a combined logic is built up from the semantics of its "components." A means of treating the "combination of semantics" problem without translating the consituent logics into first-order logic is Gabbay's fibred semantics.

There is also an "inward" direction, i.e., towards the subcategories of QVAR. In this terrain, mostly the category of varieties and its subcategories have been studied in the literature. However, the investigation of the cocompleteness conditions in the subcategories of QVAR is still largely open, notwithstanding that the cocompleteness of a subcategory can be considered a kind of methodological test of the "autonomy" of the corresponding class of logics.

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