Submitted on September 3, 2013 to the Notre Dame Journal of Formal Logic
Volume ??, Number ??,



# *Guessing, Mind-changing, and the Second Ambiguous Class*

Samuel Alexander

**Abstract** In his dissertation, Wadge defined a notion of guessability on subsets of the Baire space and gave two characterizations of guessable sets. A set is guessable iff it is in the second ambiguous class ($\mathbf{\Delta}^0_2$), iff it is eventually annihilated by a certain remainder. We simplify this remainder and give a new proof of the latter equivalence. We then introduce a notion of guessing with an ordinal limit on how often one can change one's mind. We show that for every ordinal $\alpha$, a guessable set is annihilated by $\alpha$ applications of the simplified remainder if and only if it is guessable with fewer than $\alpha$ mind changes. We use guessability with fewer than $\alpha$ mind changes to give a semi-characterization of the Hausdorff difference hierarchy, and indicate how Wadge's notion of guessability can be generalized to higher-order guessability, providing characterizations of $\mathbf{\Delta}^0_\alpha$ for all successor ordinals $\alpha > 1$.

## 1 Introduction

Let $\mathbb{N}^\mathbb{N}$ be the set of sequences $s : \mathbb{N} \to \mathbb{N}$ and let $\mathbb{N}^{<\mathbb{N}}$ be the set $\cup_n \mathbb{N}^n$ of finite sequences. If $s \in \mathbb{N}^{<\mathbb{N}}$, we will write $[s]$ for $\{f \in \mathbb{N}^\mathbb{N} : f \text{ extends } s\}$. We equip $\mathbb{N}^\mathbb{N}$ with a second-countable topology by declaring $[s]$ to be a basic open set whenever $s \in \mathbb{N}^{<\mathbb{N}}$.

Throughout the paper, $S$ will denote a subset of $\mathbb{N}^\mathbb{N}$. We say that $S \in \mathbf{\Delta}^0_2$ if $S$ is simultaneously a countable intersection of open sets and a countable union of closed sets in the above topology. In classic terminology, $S \in \mathbf{\Delta}^0_2$ just in case $S$ is both $G_\delta$ and $F_\sigma$.

The following notion was discovered by Wadge [9] (pp. 141–142) and independently by this author [1]. [1]

2010 Mathematics Subject Classification: Primary 03E15
Keywords: guessability, difference hierarchy

**Definition 1.1** We say $S$ is *guessable* if there is a function $G:\mathbb{N}^{<\mathbb{N}}\to\{0,1\}$ such that for every $f\in\mathbb{N}^{\mathbb{N}}$,

$$\lim_{n\to\infty} G(f\upharpoonright n)=\chi_S(f)=\begin{cases}1, \text{ if } f\in S,\\ 0, \text{ if } f\not\in S.\end{cases}$$

If so, we say $G$ *guesses* $S$, or that $G$ is an $S$-*guesser*.

The intution behind the above notion is captured eloquently by Wadge (p. 142, notation changed):

> Guessing sets allow us to form an opinion as to whether an element $f$ of $\mathbb{N}^{\mathbb{N}}$ is in $S$ or $S^c$, given only a finite initial segment $f\upharpoonright n$ of $f$.

Game theoretically, one envisions an asymmetric game where $II$ (the guesser) has perfect information, $I$ (the sequence chooser) has zero information, and $II$'s winning set consists of all sequences $(a_0,b_0,a_1,b_1,\ldots)$ such that $b_i\to 1$ if $(a_0,a_1,\ldots)\in S$ and $b_i\to 0$ otherwise.

The following result was proved in [9] (pp.144–145) by infinite game-theoretical methods. The present author found a second proof [1] using mathematical logical methods.

**Theorem 1.2** *(Wadge) S is guessable if and only if* $S\in\mathbf{\Delta}^0_2$.

Wadge defined (pp. 113–114) the following remainder operation.

**Definition 1.3** For $A,B\subseteq\mathbb{N}^{\mathbb{N}}$, define $\mathrm{Rm}_0(A,B)=\mathbb{N}^{\mathbb{N}}$. For $\mu>0$ an ordinal, define

$$\mathrm{Rm}_\mu(A,B)=\bigcap_{\nu<\mu}\left(\overline{\mathrm{Rm}_\nu(A,B)\cap A}\cap\overline{\mathrm{Rm}_\nu(A,B)\cap B}\right).$$

(Here $\overline{\bullet}$ denotes topological closure.) Write $\mathrm{Rm}_\mu(S)$ for $\mathrm{Rm}_\mu(S,S^c)$.

By countability considerations, there is some (in fact countable) ordinal $\mu$, depending on $S$, such that $\mathrm{Rm}_\mu(S)=\mathrm{Rm}_{\mu'}(S)$ for all $\mu'\geq\mu$; Wadge writes $\mathrm{Rm}_\Omega(S)$ for $\mathrm{Rm}_\mu(S)$ for such a $\mu$. He then proves the following theorem:

**Theorem 1.4** *(Wadge, attributed to Hausdorff)* $S\in\mathbf{\Delta}^0_2$ *if and only if* $\mathrm{Rm}_\Omega(S)=\emptyset$.

In Section 2, we introduce a simpler remainder $(S,\alpha)\mapsto S_\alpha$ and use it to give a new proof of Theorem 1.4.

In Section 3, we introduce the notion of $S$ being guessable while changing one's mind fewer than $\alpha$ many times ($\alpha\in\mathrm{Ord}$) and show that this is equivalent to $S_\alpha=\emptyset$.

In Section 4, we show that for $\alpha>0$, $S$ is guessable while changing one's mind fewer than $\alpha+1$ many times if and only if at least one of $S$ or $S^c$ is in the $\alpha$th level of the difference hierarchy.

In Section 5, we generalize guessability, introducing the notion of $\mu$th-order guessability ($1\leq\mu<\omega_1$). We show that $S$ is $\mu$th-order guessable if and only if $S\in\mathbf{\Delta}^0_{\mu+1}$.

## 2 Guessable Sets and Remainders

In this section we give a new proof of Theorem 1.4. We find it easier to work with the following remainder[2] which is closely related to the remainder defined by Wadge. For $X\subseteq\mathbb{N}^{<\mathbb{N}}$, we will write $[X]$ to denote the set of infinite sequences all of whose finite initial segments lie in $X$.

**Definition 2.1** Let $S \subseteq \mathbb{N}^{\mathbb{N}}$. We define $S_\alpha \subseteq \mathbb{N}^{<\mathbb{N}}$ ($\alpha \in \mathrm{Ord}$) by transfinite recursion as follows. We define $S_0 = \mathbb{N}^{<\mathbb{N}}$, and $S_\lambda = \cap_{\beta<\lambda} S_\beta$ for every limit ordinal $\lambda$. Finally, for every ordinal $\beta$, we define

$$S_{\beta+1} = \{x \in S_\beta : \exists x', x'' \in [S_\beta] \text{ such that } x \subseteq x',\, x \subseteq x'',\, x' \in S,\, x'' \notin S\}.$$

We write $\alpha(S)$ for the minimal ordinal $\alpha$ such that $S_\alpha = S_{\alpha+1}$, and we write $S_\infty$ for $S_{\alpha(S)}$.

Clearly $S_\alpha \subseteq S_\beta$ whenever $\beta < \alpha$. This remainder notion is related to Wadge's as follows.

**Lemma 2.2** *For each ordinal $\alpha$, $\mathrm{Rm}_\alpha(S) = [S_\alpha]$.*

**Proof** Since $S_\alpha \subseteq S_\beta$ whenever $\beta < \alpha$, for all $\alpha$, we have $S_\alpha = \cap_{\beta<\alpha} S_{\beta+1}$ (with the convention that $\cap\emptyset = \mathbb{N}^{<\mathbb{N}}$). We will show by induction on $\alpha$ that $\mathrm{Rm}_\alpha(S) = [S_\alpha] = [\cap_{\beta<\alpha} S_{\beta+1}]$.

Suppose $f \in [\cap_{\beta<\alpha} S_{\beta+1}]$. Let $\beta < \alpha$. Let $\mathscr{U}$ be an open set around $f$, we can assume $\mathscr{U}$ is basic open, so $\mathscr{U} = [f_0]$, $f_0$ a finite initial segment of $f$. Since $f \in [\cap_{\beta<\alpha} S_{\beta+1}]$, $f_0 \in S_{\beta+1}$. Thus there are $x', x'' \in [S_\beta]$ extending $f_0$ (hence in $\mathscr{U}$), $x' \in S$, $x'' \notin S$. In other words, $x' \in [\cap_{\gamma<\beta} S_{\gamma+1}] \cap S$ and $x'' \in [\cap_{\gamma<\beta} S_{\gamma+1}] \cap S^c$. By induction, $x' \in \mathrm{Rm}_\beta(S) \cap S$ and $x'' \in \mathrm{Rm}_\beta(S) \cap S^c$. By arbitrariness of $\mathscr{U}$, $f \in \overline{\mathrm{Rm}_\beta(S) \cap S} \cap \overline{\mathrm{Rm}_\beta(S) \cap S^c}$. By arbitrariness of $\beta$, $f \in \mathrm{Rm}_\alpha(S)$.

The reverse inclusion is similar. □

Note that Lemma 2.2 does not say that $\mathrm{Rm}_\alpha(S) = \emptyset$ if and only if $S_\alpha = \emptyset$. It is (at least a priori) possible that $S_\alpha \neq \emptyset$ while $[S_\alpha] = \emptyset$. Lemma 2.2 does however imply that $\mathrm{Rm}_\Omega(S) = \emptyset$ if and only if $S_\infty = \emptyset$, since it is easy to see that if $[S_\alpha] = \emptyset$ then $S_{\alpha+1} = \emptyset$. Thus in order to prove Theorem 1.4 it suffices to show that $S$ is guessable if and only if $S_\infty = \emptyset$. The $\Rightarrow$ direction requires no additional machinery.

**Proposition 2.3** *If $S$ is guessable then $S_\infty = \emptyset$.*

**Proof** Let $G : \mathbb{N}^{<\mathbb{N}} \to \{0,1\}$ be an $S$-guesser. Assume (for contradiction) $S_\infty \neq \emptyset$ and let $\sigma_0 \in S_\infty$. We will build a sequence on whose initial segments $G$ diverges, contrary to Definition 1.1. Inductively suppose we have finite sequences $\sigma_0 \subset_{\neq} \cdots \subset_{\neq} \sigma_k$ in $S_\infty$ such that $\forall 0 < i \leq k$, $G(\sigma_i) \equiv i \bmod 2$. Since $\sigma_k \in S_\infty = S_{\alpha(S)} = S_{\alpha(S)+1}$, there are $\sigma', \sigma'' \in [S_\infty]$, extending $\sigma_k$, with $\sigma' \in S$, $\sigma'' \notin S$. Choose $\sigma \in \{\sigma', \sigma''\}$ with $\sigma \in S$ iff $k$ is even. Then $\lim_{n\to\infty} G(\sigma \upharpoonright n) \equiv k+1 \bmod 2$. Let $\sigma_{k+1} \subset \sigma$ properly extend $\sigma_k$ such that $G(\sigma_{k+1}) \equiv k+1 \bmod 2$. Note $\sigma_{k+1} \in S_\infty$ since $\sigma \in [S_\infty]$.

By induction, there are $\sigma_0 \subset_{\neq} \sigma_1 \subset_{\neq} \cdots$ such that for $i > 0$, $G(\sigma_i) \equiv i \bmod 2$. This contradicts Definition 1.1 since $\lim_{n\to\infty} G((\cup_i \sigma_i) \upharpoonright n)$ ought to converge. □

The $\Leftarrow$ direction requires a little machinery.

**Definition 2.4** If $\sigma \in \mathbb{N}^{<\mathbb{N}}$, $\sigma \notin S_\infty$, let $\beta(\sigma)$ be the least ordinal such that $\sigma \notin S_{\beta(\sigma)}$.

Note that whenever $\sigma \notin S_\infty$, $\beta(\sigma)$ is a successor ordinal.

**Lemma 2.5** *Suppose $\sigma \subseteq \tau$ are finite sequences. If $\tau \in S_\infty$ then $\sigma \in S_\infty$. And if $\sigma \notin S_\infty$, then $\beta(\tau) \leq \beta(\sigma)$.*

**Proof** It is enough to show that $\forall\beta\in\text{Ord}$, if $\tau\in S_\beta$ then $\sigma\in S_\beta$. This is by induction on $\beta$, the limit and zero cases being trivial. Assume $\beta$ is successor. If $\tau\in S_\beta$, this means $\tau\in S_{\beta-1}$ and there are $\tau',\tau''\in[S_{\beta-1}]$ extending $\tau$ with $\tau'\in S$, $\tau''\notin S$. Since $\tau'$ and $\tau''$ extend $\tau$, and $\tau$ extends $\sigma$, $\tau'$ and $\tau''$ extend $\sigma$; and since $\sigma\in S_{\beta-1}$ (by induction), this shows $\sigma\in S_\beta$. □

**Lemma 2.6** *Suppose $f:\mathbb{N}\to\mathbb{N}$, $f\notin[S_\infty]$. There is some $i$ such that for all $j\geq i$, $f\restriction j\notin S_\infty$ and $\beta(f\restriction j)=\beta(f\restriction i)$. Furthermore, $f\in[S_{\beta(f\restriction i)-1}]$.*

**Proof** The first part follows from Lemma 2.5 and the well-foundedness of Ord. For the second part we must show $f\restriction k\in S_{\beta(f\restriction i)-1}$ for every $k$. If $k\leq i$, then $f\restriction k\in S_{\beta(f\restriction i)-1}$ by Lemma 2.5. If $k\geq i$, then $\beta(f\restriction k)=\beta(f\restriction i)$ and so $f\restriction k\in S_{\beta(f\restriction i)-1}$ since it is in $S_{\beta(f\restriction k)-1}$ by definition of $\beta$. □

**Definition 2.7** If $S_\infty=\emptyset$ then we define $G_S:\mathbb{N}^{<\mathbb{N}}\to\{0,1\}$ as follows. Let $\sigma\in\mathbb{N}^{<\mathbb{N}}$. Since $S_\infty=\emptyset$, $\sigma\notin S_\infty$, so $\sigma\in S_{\beta(\sigma)-1}\backslash S_{\beta(\sigma)}$. Since $\sigma\notin S_{\beta(\sigma)}$, this means for every two extensions $x',x''$ of $\sigma$ in $[S_{\beta(\sigma)-1}]$, either $x',x''\in S$ or $x',x''\in S^c$. So either all extensions of $\sigma$ in $[S_{\beta(\sigma)-1}]$ are in $S$, or all such extensions are in $S^c$.

(i) If there are no extensions of $\sigma$ in $[S_{\beta(\sigma)-1}]$, and length$(\sigma)>0$, then let $G_S(\sigma)=G_S(\sigma^-)$ where $\sigma^-$ is obtained from $\sigma$ by removing the last term.
(ii) If there are no extensions of $\sigma$ in $[S_{\beta(\sigma)-1}]$, and length$(\sigma)=0$, let $G_S(\sigma)=0$.
(iii) If there are extensions of $\sigma$ in $[S_{\beta(\sigma)-1}]$ and they are all in $S$, define $G_S(\sigma)=1$.
(iv) If there are extensions of $\sigma$ in $[S_{\beta(\sigma)-1}]$ and they are all in $S^c$, define $G_S(\sigma)=0$.

**Proposition 2.8** *If $S_\infty=\emptyset$ then $G_S$ guesses $S$.*

**Proof** Assume $S_\infty=\emptyset$. Let $f\in S$. I will show $G_S(f\restriction n)\to 1$ as $n\to\infty$. Since $f\notin[S_\infty]$, let $i$ be as in Lemma 2.6. I claim $G_S(f\restriction j)=1$ whenever $j\geq i$. Fix $j\geq i$. We have $\beta(f\restriction j)=\beta(f\restriction i)$ by choice of $i$, and $f\in[S_{\beta(f\restriction i)-1}]=[S_{\beta(f\restriction j)-1}]$. Since $f\restriction j$ has one extension (namely $f$ itself) in both $[S_{\beta(f\restriction j)-1}]$ and $S$, $G_S(f\restriction j)=1$.

Identical reasoning shows that if $f\notin S$ then $\lim_{n\to\infty}G_S(f\restriction n)=0$. □

**Theorem 2.9** *$S\in\mathbf{\Delta}^0_2$ if and only if $S_\infty=\emptyset$. That is, Theorem 1.4 is true.*

**Proof** By combining Propositions 2.3 and 2.8 and Theorem 1.2. □

## 3 Guessing without changing one's Mind too often

In this section our goal is to tease out additional information about $\mathbf{\Delta}^0_2$ from the operation defined in Definition 2.1.

**Definition 3.1** For each function $G$ with domain $\mathbb{N}^{<\mathbb{N}}$, if $G(f\restriction(n+1))\neq G(f\restriction n)$ ($f\in\mathbb{N}^\mathbb{N}$, $n\in\mathbb{N}$), we say $G$ *changes its mind on* $f\restriction(n+1)$. Now let $\alpha\in\text{Ord}$. We say $S$ is *guessable with $<\alpha$ mind changes* if there is an $S$-guesser $G$ along with a function $H:\mathbb{N}^{<\mathbb{N}}\to\alpha$ such that the following hold, where $f\in\mathbb{N}^\mathbb{N}$ and $n\in\mathbb{N}$.

(i) $H(f\restriction(n+1))\leq H(f\restriction n)$.
(ii) If $G$ changes its mind on $f\restriction(n+1)$, then $H(f\restriction(n+1))<H(f\restriction n)$.

This notion bears some resemblance to the notion of a set $Z\subseteq\mathbb{N}$ being $f$-c.e. in [4], or $g$-c.a. in [7].

**Theorem 3.2** *For $\alpha \in \mathrm{Ord}$, $S$ is guessable with $< \alpha$ mind changes if and only if $S_\alpha = \emptyset$.*

**Proof**

($\Rightarrow$) Assume $S$ is guessable with $< \alpha$ mind changes. Let $G, H$ be as in Definition 3.1. We claim that for all $\beta \in \mathrm{Ord}$, if $\sigma \in S_\beta$ then $H(\sigma) \geq \beta$. This will prove ($\Rightarrow$) because it implies that if $S_\alpha \neq \emptyset$ then there is some $\sigma$ with $H(\sigma) \geq \alpha$, absurd since codomain$(H) = \alpha$.

We attack the claim by induction on $\beta$. The zero and limit cases are trivial. Assume $\beta = \gamma + 1$. Suppose $\sigma \in S_{\gamma+1}$. There are $x', x'' \in [S_\gamma]$ extending $\sigma$, $x' \in S$, $x'' \notin S$. Pick $x \in \{x', x''\}$ so that $\chi_S(x) \neq G(\sigma)$ and pick $\sigma^+ \in \mathbb{N}^{<\mathbb{N}}$ with $\sigma \subseteq \sigma^+ \subseteq x$ such that $G(\sigma^+) = \chi_S(x)$ (some such $\sigma^+$ exists since $G$ guesses $S$). Since $x \in [S_\gamma]$, $\sigma^+ \in S_\gamma$. By induction, $H(\sigma^+) \geq \gamma$. The fact $G(\sigma^+) \neq G(\sigma)$ implies $H(\sigma^+) < H(\sigma)$, forcing $H(\sigma) \geq \gamma + 1$.

($\Leftarrow$) Assume $S_\alpha = \emptyset$. For all $\sigma \in \mathbb{N}^{<\mathbb{N}}$, define $H(\sigma) = \beta(\sigma) - 1$ (by definition of $\beta(\sigma)$, since $S_\alpha = \emptyset$, clearly $H(\sigma) \in \alpha$). I claim $G_S, H$ witness that $S$ is guessable with $< \alpha$ mind changes.

By Proposition 2.8, $G_S$ guesses $S$. Let $f \in \mathbb{N}^\mathbb{N}$, $n \in \mathbb{N}$. By Lemma 2.5, $H(f \upharpoonright (n+1)) \leq H(f \upharpoonright n)$. Now suppose $G_S$ changes its mind on $f \upharpoonright (n+1)$, we must show $H(f \upharpoonright (n+1)) < H(f \upharpoonright n)$. Assume, for sake of contradiction, that $H(f \upharpoonright (n+1)) = H(f \upharpoonright n)$. Assume $G_S(f \upharpoonright n) = 0$, the other case is similar. By definition of $G_S$, ($*$) for every infinite extension $f'$ of $f \upharpoonright n$, if $f' \in [S_{\beta(f \upharpoonright n)-1}]$ then $f' \in S^c$. Since $G_S$ changes its mind on $f \upharpoonright (n+1)$, $G_S(f \upharpoonright (n+1)) = 1$. Thus ($**$) for every infinite extension $f''$ of $f \upharpoonright (n+1)$, if $f'' \in [S_{\beta(f \upharpoonright (n+1))-1}]$ then $f'' \in S$. And $f \upharpoonright (n+1)$ does actually have some such infinite extension $f''$, because if it had none, that would make $G_S(f \upharpoonright (n+1)) = G_S(f \upharpoonright n)$ by case 1 of the definition of $G_S$ (Definition 2.7). Being an extension of $f \upharpoonright (n+1)$, $f''$ also extends $f \upharpoonright n$; and by the assumption that $H(f \upharpoonright (n+1)) = H(f \upharpoonright n)$, $f'' \in [S_{\beta(f \upharpoonright n)-1}]$. By ($*$), $f'' \in S^c$, and by ($**$), $f'' \in S$. Absurd. $\square$

It is not hard to show $S$ is a Boolean combination of open sets if and only if $S$ is guessable with $< \omega$ mind changes, so Theorem 3.2 and Lemma 2.2 give a new proof of a special case of the main theorem (p. 1348) of [3] (see also [2]).

## 4 Mind Changing and the Difference Hierarchy

We recall the following definition from [5] (p. 175, stated in greater generality—we specialize it to the Baire space). In this definition, $\mathbf{\Sigma}^0_1(\mathbb{N}^\mathbb{N})$ is the set of open subsets of $\mathbb{N}^\mathbb{N}$, and the *parity* of an ordinal $\eta$ is the equivalence class modulo 2 of $n$, where $\eta = \lambda + n$, $\lambda$ a limit ordinal (or $\lambda = 0$), $n \in \mathbb{N}$.

**Definition 4.1** Let $(A_\eta)_{\eta<\theta}$ be an increasing sequence of subsets of $\mathbb{N}^\mathbb{N}$ with $\theta \geq 1$. Define the set $D_\theta((A_\eta)_{\eta<\theta}) \subseteq \mathbb{N}^\mathbb{N}$ by

$$x \in D_\theta((A_\eta)_{\eta<\theta}) \Leftrightarrow x \in \bigcup_{\eta<\theta} A_\eta \text{ \& the least } \eta < \theta \text{ with } x \in A_\eta \text{ has parity opposite to that of } \theta.$$

Let

$$D_\theta(\mathbf{\Sigma}^0_1)(\mathbb{N}^\mathbb{N}) = \{D_\theta((A_\eta)_{\eta<\theta}) : A_\eta \in \mathbf{\Sigma}^0_1(\mathbb{N}^\mathbb{N}), \eta < \theta\}.$$

This hierarchy offers a constructive characterization of $\mathbf{\Delta}^0_2$: it turns out that

$$\mathbf{\Delta}^0_2 = \cup_{1\leq\theta<\omega_1} D_\theta(\mathbf{\Sigma}^0_1)(\mathbb{N}^\mathbb{N})$$

(see Theorem 22.27 of [5], p. 176, attributed to Hausdorff and Kuratowski).

For brevity, we will write $D_\alpha$ for $D_\alpha(\mathbf{\Sigma}^0_1)(\mathbb{N}^\mathbb{N})$.

**Theorem 4.2** *(Semi-characterization of the difference hierarchy) Let $\alpha > 0$. The following are equivalent.*

*(i) $S$ is guessable with $< \alpha+1$ mind changes.*
*(ii) $S \in D_\alpha$ or $S^c \in D_\alpha$.*

We will prove Theorem 4.2 by a sequence of smaller results.

**Definition 4.3** For $\alpha,\beta \in \mathrm{Ord}$, write $\alpha \equiv \beta$ to indicate that $\alpha$ and $\beta$ have the same parity (that is, $2|n-m$, where $\alpha = \lambda + n$ and $\beta = \kappa + m$, $n,m \in \mathbb{N}$, $\lambda$ a limit ordinal or 0, $\kappa$ a limit ordinal or 0).

**Proposition 4.4** *Let $\alpha > 0$. If $S \in D_\alpha$, say $S = D_\alpha((A_\eta)_{\eta<\alpha})$ ($A_\eta \subseteq \mathbb{N}^\mathbb{N}$ open), then $S$ is guessable with $< \alpha+1$ mind changes.*

**Proof** Define $G : \mathbb{N}^{<\mathbb{N}} \to \{0,1\}$ and $H : \mathbb{N}^{<\mathbb{N}} \to \alpha+1$ as follows. Suppose $\sigma \in \mathbb{N}^{<\mathbb{N}}$. If there is no $\eta < \alpha$ such that $[\sigma] \subseteq A_\eta$, let $G(\sigma) = 0$ and let $H(\sigma) = \alpha$. If there is an $\eta < \alpha$ (we may take $\eta$ minimal) such that $[\sigma] \subseteq A_\eta$, then let

$$G(\sigma) = \begin{cases} 0, & \text{if } \eta \equiv \alpha; \\ 1, & \text{if } \eta \not\equiv \alpha, \end{cases} \qquad H(\sigma) = \eta.$$

Let $f : \mathbb{N} \to \mathbb{N}$.

**Claim 1** $\lim_{n\to\infty} G(f \upharpoonright n) = \chi_S(f)$.

If $f \notin \cup_{\eta<\alpha} A_\eta$, then $f \notin D_\alpha((A_\eta)_{\eta<\alpha}) = S$, and $G(f \upharpoonright n)$ will always be 0, so $\lim_{n\to\infty} G(f \upharpoonright n) = 0 = \chi_S(f)$. Assume $f \in \cup_{\eta<\alpha} A_\eta$, and let $\eta < \alpha$ be minimum such that $f \in A_\eta$. Since $A_\eta$ is open, there is some $n_0$ so large that $\forall n \geq n_0$, $[f \upharpoonright n] \subseteq A_\eta$. For all $n \geq n_0$, by minimality of $\eta$, $[f \upharpoonright n] \not\subseteq A_{\eta'}$ for any $\eta' < \eta$, so $G(f \upharpoonright n) = 0$ if and only if $\eta \equiv \alpha$. The following are equivalent.

$$\begin{aligned} f \in S &\text{ iff } f \in D_\alpha((A_\eta)_{\eta<\alpha}) \\ &\text{ iff } \eta \not\equiv \alpha \\ &\text{ iff } G(f \upharpoonright n) \neq 0 \\ &\text{ iff } G(f \upharpoonright n) = 1. \end{aligned}$$

This shows $\lim_{n\to\infty} G(f \upharpoonright n) = \chi_S(f)$.

**Claim 2** $\forall n \in \mathbb{N}$, $H(f \upharpoonright (n+1)) \leq H(f \upharpoonright n)$.

If $H(f \upharpoonright n) = \alpha$, there is nothing to prove. If $H(f \upharpoonright n) < \alpha$, then $H(f \upharpoonright n) = \eta$ where $\eta$ is minimal such that $[f \upharpoonright n] \subseteq A_\eta$. Since $[f \upharpoonright (n+1)] \subseteq [f \upharpoonright n]$, we have $[f \upharpoonright (n+1)] \subseteq A_\eta$, implying $H(f \upharpoonright (n+1)) \leq \eta$.

**Claim 3** $\forall n \in \mathbb{N}$, if $G(f \upharpoonright (n+1)) \neq G(f \upharpoonright n)$, then $H(f \upharpoonright (n+1)) < H(f \upharpoonright n)$.

Assume (for sake of contradiction) $H(f \upharpoonright (n+1)) \geq H(f \upharpoonright n)$. By Claim 2, $H(f \upharpoonright (n+1)) = H(f \upharpoonright n)$. By definition of $H$ this implies that $\forall \eta < \alpha$, $[f \upharpoonright (n+1)] \subseteq A_\eta$ if and only if $[f \upharpoonright n] \subseteq A_\eta$. This implies $G(f \upharpoonright (n+1)) = G(f \upharpoonright n)$, contradiction.

By Claims 1–3, $G$ and $H$ witness that $S$ is guessable with $< \alpha+1$ mind changes. □

**Corollary 4.5** *Let $\alpha > 0$. If $S \in D_\alpha$ or $S^c \in D_\alpha$ then $S$ is guessable with $< \alpha + 1$ mind changes.*

**Proof** If $S \in D_\alpha$ this is immediate by Proposition 4.4. If $S^c \in D_\alpha$ then Proposition 4.4 says $S^c$ is guessable with $< \alpha + 1$ mind changes, and this clearly implies that $S$ is too. □

**Lemma 4.6** *Suppose $S$ is guessable with $< \alpha$ mind changes. Let $G : \mathbb{N}^{<\mathbb{N}} \to \{0,1\}$, $H : \mathbb{N}^{<\mathbb{N}} \to \alpha$ be a pair of functions witnessing as much (Definition 3.1). There is an $H' : \mathbb{N}^{<\mathbb{N}} \to \alpha$ such that $G, H'$ also witness that $S$ is guessable with $< \alpha$ mind changes, with $H'(\emptyset) = H(\emptyset)$, and with the additional property that for every $f : \mathbb{N} \to \mathbb{N}$ and every $n \in \mathbb{N}$,*

$$H(f \upharpoonright (n+1)) \equiv H(f \upharpoonright n) \text{ if and only if } G(f \upharpoonright (n+1)) = G(f \upharpoonright n).$$

**Proof** Define $H'(\sigma)$ by induction on the length of $\sigma$ as follows. Let $H'(\emptyset) = H(\emptyset)$. If $\sigma \neq \emptyset$, write $\sigma = \sigma_0 \frown n$ for some $n \in \mathbb{N}$ ($\frown$ denotes concatenation). If $G(\sigma) = G(\sigma_0)$, let $H'(\sigma) = H'(\sigma_0)$. Otherwise, let $H'(\sigma)$ be either $H(\sigma)$ or $H(\sigma) + 1$, whichever has parity opposite to $H'(\sigma_0)$.

By construction $H'$ has the desired parity properties. A simple inductive argument shows that $(*)$ $\forall \sigma \in \mathbb{N}^{<\mathbb{N}}$, $H(\sigma) \leq H'(\sigma) < \alpha$. I claim that for all $f : \mathbb{N} \to \mathbb{N}$ and $n \in \mathbb{N}$, $H'(f \upharpoonright (n+1)) \leq H'(f \upharpoonright n)$, and if $G(f \upharpoonright (n+1)) \neq G(f \upharpoonright n)$ then $H'(f \upharpoonright (n+1)) < H'(f \upharpoonright n)$.

If $G(f \upharpoonright (n+1)) = G(f \upharpoonright n)$, then by definition $H'(f \upharpoonright (n+1)) = H'(f \upharpoonright n)$ and the claim is trivial. Now assume $G(f \upharpoonright (n+1)) \neq G(f \upharpoonright n)$. If $H'(f \upharpoonright (n+1)) = H(f \upharpoonright (n+1))$ then $H'(f \upharpoonright (n+1)) < H(f \upharpoonright n) \leq H'(f \upharpoonright n)$ and we are done. Assume

$$H'(f \upharpoonright (n+1)) \neq H(f \upharpoonright (n+1)),$$

which forces that $(**)$ $H'(f \upharpoonright (n+1)) = H(f \upharpoonright (n+1)) + 1$. To see that

$$H'(f \upharpoonright (n+1)) < H'(f \upharpoonright n),$$

assume not $(***)$. By Definition 3.1, $H(f \upharpoonright (n+1)) < H(f \upharpoonright n)$, so

$$\begin{aligned}
H(f \upharpoonright n) &\geq H(f \upharpoonright (n+1)) + 1 && \text{(Basic arithmetic)}\\
&= H'(f \upharpoonright (n+1)) && \text{(By } (**))\\
&\geq H'(f \upharpoonright n) && \text{(By } (***))\\
&\geq H(f \upharpoonright n). && \text{(By } (*))
\end{aligned}$$

Equality holds throughout, and $H'(f \upharpoonright (n+1)) = H'(f \upharpoonright n)$. Contradiction: we chose $H'(f \upharpoonright (n+1))$ with parity opposite to $H'(f \upharpoonright n)$. □

**Definition 4.7** For all $G, H$ as in Definition 3.1, $f \in \mathbb{N}^\mathbb{N}$, write $G(f)$ for $\lim_{n\to\infty} G(f \upharpoonright n)$ (so $G(f) = \chi_S(f)$) and write $H(f)$ for $\lim_{n\to\infty} H(f \upharpoonright n)$. Write $G \equiv H$ to indicate that $\forall f \in \mathbb{N}^\mathbb{N}$, $G(f) \equiv H(f)$; write $G \not\equiv H$ to indicate that $\forall f \in \mathbb{N}^\mathbb{N}$, $G(f) \not\equiv H(f)$ (we pronounce $G \not\equiv H$ as "$G$ is anticongruent to $H$").

**Lemma 4.8** *Suppose $G : \mathbb{N}^{<\mathbb{N}} \to \{0,1\}$ and $H : \mathbb{N}^{<\mathbb{N}} \to \alpha$ witness that $S$ is guessable with $< \alpha$ mind changes. There is an $H' : \mathbb{N}^{<\mathbb{N}} \to \alpha$ such that $G, H'$ witness that $S$ is guessable with $< \alpha$ mind changes, and such that the following hold.*

*If $G(\emptyset) \equiv \alpha$ then $H' \not\equiv G$.* *If $G(\emptyset) \not\equiv \alpha$ then $H' \equiv G$.*

**Proof** I claim that without loss of generality, we may assume the following $(*)$:

$$\text{If } G(\emptyset) \equiv \alpha \text{ then } H(\emptyset) \not\equiv G(\emptyset). \qquad \text{If } G(\emptyset) \not\equiv \alpha \text{ then } H(\emptyset) \equiv G(\emptyset).$$

To see this, suppose not: either $G(\emptyset) \equiv \alpha$ and $H(\emptyset) \equiv G(\emptyset)$, or else $G(\emptyset) \not\equiv \alpha$ and $H(\emptyset) \not\equiv G(\emptyset)$. In either case, $H(\emptyset) \equiv \alpha$. If $H(\emptyset) \equiv \alpha$ then $H(\emptyset)+1 \neq \alpha$, and so, since $H(\emptyset) < \alpha$, $H(\emptyset)+1 < \alpha$, meaning we may add 1 to $H(\emptyset)$ to enforce the assumption.

Having assumed $(*)$, we may use Lemma 4.6 to construct $H' : \mathbb{N}^{<\mathbb{N}} \to \alpha$ such that $G, H'$ witness that $S$ is guessable with $< \alpha$ mind changes, $H'(\emptyset) = H(\emptyset)$, and $H'$ changes parity precisely when $G$ changes parity. The latter facts, combined with $(*)$, prove the lemma. ☐

**Proposition 4.9** *Suppose $G : \mathbb{N}^{<\mathbb{N}} \to \{0,1\}$ and $H : \mathbb{N}^{<\mathbb{N}} \to \alpha+1$ witness that $S$ is guessable with $< \alpha+1$ mind changes. If $G(\emptyset) = 0$ then $S \in D_\alpha$.*

**Proof** By Lemma 4.8 we may safely assume the following:

$$\text{If } G(\emptyset) \equiv \alpha+1 \text{ then } H \not\equiv G. \qquad \text{If } G(\emptyset) \not\equiv \alpha+1 \text{ then } H \equiv G.$$

In other words,

$$(*) \text{ If } G(\emptyset) \equiv \alpha \text{ then } H \equiv G. \qquad (**) \text{ If } G(\emptyset) \not\equiv \alpha \text{ then } H \not\equiv G.$$

For each $\eta < \alpha$, let

$$A_\eta = \{f \in \mathbb{N}^{\mathbb{N}} : H(f) \leq \eta\}. \qquad (H(f) \text{ as in Definition 4.7})$$

I claim $S = D_\alpha((A_\eta)_{\eta<\alpha})$, which will prove the proposition since each $A_\eta$ is clearly open.

Suppose $f \in S$, I will show $f \in D_\alpha((A_\eta)_{\eta<\alpha})$. Since $f \in S$, $H(f) \neq \alpha$, because if $H(f)$ were $= \alpha$, this would imply that $G$ never changes its mind on $f$, forcing $\lim_{n\to\infty} G(f \upharpoonright n) = \lim_{n\to\infty} G(\emptyset) = 0$, contradicting the fact that $G$ guesses $S$.

Since $H(f) \neq \alpha$, $H(f) < \alpha$. It follows that for $\eta = H(f)$ we have $f \in A_\eta$ and $\eta$ is minimal with this property.

Case 1: $G(\emptyset) \equiv \alpha$. By $(*)$, $H \equiv G$. Since $f \in S$, $\lim_{n\to\infty} G(f \upharpoonright n) = 1$, so $\eta = \lim_{n\to\infty} H(f \upharpoonright n) \equiv 1$. Since $\alpha \equiv G(\emptyset) = 0$, this shows $\eta \not\equiv \alpha$, putting $f \in D_\alpha((A_\eta)_{\eta<\alpha})$.

Case 2: $G(\emptyset) \not\equiv \alpha$. By $(**)$, $H \not\equiv G$. Since $f \in S$, $\lim_{n\to\infty} G(f \upharpoonright n) = 1$, so $\eta = \lim_{n\to\infty} H(f \upharpoonright n) \equiv 0$. Since $\alpha \not\equiv G(\emptyset) = 0$, this shows $\eta \not\equiv \alpha$, so $f \in D_\alpha((A_\eta)_{\eta<\alpha})$.

Conversely, suppose $f \in D_\alpha((A_\eta)_{\eta<\alpha})$, I will show $f \in S$. Let $\eta$ be minimal such that $f \in A_\eta$ (by definition of $A_\eta$, $\eta = H(f)$). By definition of $D_\alpha((A_\eta)_{\eta<\alpha})$, $\eta \not\equiv \alpha$.

Case 1: $G(\emptyset) \equiv \alpha$. By $(*)$, $H \equiv G$. Since $\lim_{n\to\infty} H(f \upharpoonright n) = H(f) = \eta \not\equiv \alpha \equiv G(\emptyset) = 0$, we see $\lim_{n\to\infty} H(f \upharpoonright n) = 1$. Since $H \equiv G$, $\lim_{n\to\infty} G(f \upharpoonright n) = 1$, forcing $f \in S$ since $G$ guesses $S$.

Case 2: $G(\emptyset) \not\equiv \alpha$. By $(**)$, $H \not\equiv G$. Since

$$\lim_{n\to\infty} H(f \upharpoonright n) = H(f) = \eta \not\equiv \alpha \not\equiv G(\emptyset) = 0,$$

we see $\lim_{n\to\infty} H(f \upharpoonright n) = 0$. Since $H \not\equiv G$, $\lim_{n\to\infty} G(f \upharpoonright n) = 1$, again showing $f \in S$. ☐

**Corollary 4.10** *If $S$ is guessable with $< \alpha+1$ mind changes, then $S \in D_\alpha$ or $S^c \in D_\alpha$.*

**Proof** Let $G,H$ witness that $S$ is guessable with $< \alpha+1$ mind changes. If $G(\emptyset)=0$ then $S \in D_\alpha$ by Proposition 4.9. If not, then $(1-G),H$ witness that $S^c$ is guessable with $< \alpha+1$ mind changes, and $(1-G)(\emptyset)=0$, so $S^c \in D_\alpha$ by Proposition 4.9. ☐

Combining Corollaries 4.5 and 4.10 proves Theorem 4.2.

## 5 Higher-order Guessability

In this section we introduce a notion that generalizes guessability to provide a characterization for $\mathbf{\Delta}^0_{\mu+1}$ $(1 \leq \mu < \omega_1)$. We will show that $S \in \mathbf{\Delta}^0_{\mu+1}$ if and only if $S$ is $\mu$th-order guessable. Throughout this section, $\mu$ denotes an ordinal in $[1,\omega_1)$.

**Definition 5.1** Let $\mathscr{S} = (S_0, S_1, \ldots)$ be a countably infinite tuple of subsets $S_i \subseteq \mathbb{N}^\mathbb{N}$.

(i) For every $f \in \mathbb{N}^\mathbb{N}$, write $\mathscr{S}(f)$ for the sequence $(\chi_{S_0}(f), \chi_{S_1}(f), \ldots) \in \{0,1\}^\mathbb{N}$.
(ii) We say that $S$ is *guessable based on* $\mathscr{S}$ if there is a function

$$G : \{0,1\}^{<\mathbb{N}} \to \{0,1\}$$

(called an *S-guesser based on* $\mathscr{S}$) such that $\forall f \in \mathbb{N}^\mathbb{N}$,

$$\lim_{n\to\infty} G(\mathscr{S}(f) \restriction n) = \chi_S(f).$$

Game theoretically, we envision a game where $I$ (the sequence chooser) has zero information and $II$ (the guesser) has possibly *better-than-perfect* information: $II$ is allowed to ask (once per turn) whether $I$'s sequence lies in various $S_i$. For each $S_i$, player $I$'s act (by answering the question) of committing to play a sequence in $S_i$ or in $S_i^c$ is similar to the act (described in [6], p. 366) of choosing a $I$-imposed subgame.

**Example 5.2** *If $\mathscr{S}$ enumerates the sets of the form $\{f \in \mathbb{N}^\mathbb{N} : f(i) = j\}$, $i,j \in \mathbb{N}$ then it is not hard to show that $S$ is guessable (in the sense of Definition 1.1) if and only if $S$ is guessable based on $\mathscr{S}$.*

**Definition 5.3** We say $S$ is *$\mu$th-order guessable* if there is some $\mathscr{S} = (S_0, S_1, \ldots)$ as in Definition 5.1 such that the following hold.

(i) $S$ is guessable based on $\mathscr{S}$.
(ii) $\forall i$, $S_i \in \mathbf{\Delta}^0_{\mu_i+1}$ for some $\mu_i < \mu$.

**Theorem 5.4** *$S$ is $\mu$th-order guessable if and only if $S \in \mathbf{\Delta}^0_{\mu+1}$.*

In order to prove Theorem 5.4 we will assume the following result, which is a specialization and rephrasing of Exercise 22.17 of [5] (pp. 172–173, attributed to Kuratowski).

**Lemma 5.5** *The following are equivalent.*

*(i)* $S \in \mathbf{\Delta}^0_{\mu+1}$.
*(ii) There is a sequence $(A_i)_{i\in\mathbb{N}}$, each $A_i \in \mathbf{\Delta}^0_{\mu_i+1}$ for some $\mu_i < \mu$, such that*

$$S = \bigcup_n \bigcap_{m\geq n} A_m = \bigcap_n \bigcup_{m\geq n} A_m.$$

**Proof of Theorem 5.4**

($\Rightarrow$) Let $\mathscr{S} = (S_0, S_1, \dots)$ and $G$ witness that $S$ is $\mu$th-order guessable (so each $S_i \in \mathbf{\Delta}^0_{\mu_i+1}$ for some $\mu_i < \mu$). For all $a \in \{0,1\}$ and $X \subseteq \mathbb{N}^\mathbb{N}$, define

$$X^a = \begin{cases} X, & \text{if } a = 1; \\ \mathbb{N}^\mathbb{N} \backslash X, & \text{if } a = 0. \end{cases}$$

For notational convenience, we will write "$G(\vec{a}) = 1$" as an abbreviation for "$0 \le a_0, \dots, a_{m-1} \le 1$ and $G(a_0, \dots, a_{m-1}) = 1$," provided $m$ is clear from context. Observe that for all $f \in \mathbb{N}^\mathbb{N}$ and $m \in \mathbb{N}$, $G(\mathscr{S}(f) \restriction m) = 1$ if and only if

$$f \in \bigcup_{G(\vec{a})=1} \bigcap_{j=0}^{m-1} S_j^{a_j}.$$

Now, given $f : \mathbb{N} \to \mathbb{N}$, $f \in S$ if and only if $G(\mathscr{S}(f) \restriction n) \to 1$, which is true if and only if $\exists n \forall m \ge n$, $G(\mathscr{S}(f) \restriction m) = 1$. Thus

$$\begin{aligned} f \in S &\text{ iff } \exists n \forall m \ge n,\, G(\mathscr{S}(f) \restriction m) = 1 \\ &\text{ iff } \exists n \forall m \ge n,\, f \in \bigcup_{G(\vec{a})=1} \bigcap_{j=0}^{m-1} S_j^{a_j} \\ &\text{ iff } f \in \bigcup_n \bigcap_{m \ge n} \bigcup_{G(\vec{a})=1} \bigcap_{j=0}^{m-1} S_j^{a_j}. \end{aligned}$$

So

$$S = \bigcup_n \bigcap_{m \ge n} \bigcup_{G(\vec{a})=1} \bigcap_{j=0}^{m-1} S_j^{a_j}.$$

At the same time, since $G(\mathscr{S}(f) \restriction m) \to 0$ whenever $f \notin S$, we see $f \in S$ if and only if $\forall n \exists m \ge n$ such that $G(\mathscr{S}(f) \restriction m) = 1$. Thus by similar reasoning to the above,

$$S = \bigcap_n \bigcup_{m \ge n} \bigcup_{G(\vec{a})=1} \bigcap_{j=0}^{m-1} S_j^{a_j}.$$

For each $m$, $\bigcup_{G(\vec{a})=1} \bigcap_{j=0}^{m-1} S_j^{a_j}$ is a finite union of finite intersections of sets in $\mathbf{\Delta}^0_{\mu'+1}$ for various $\mu' < \mu$, thus $\bigcup_{G(\vec{a})=1} \bigcap_{j=0}^{m-1} S_j^{a_j}$ itself is in $\mathbf{\Delta}^0_{\mu_m+1}$ for some $\mu_m < \mu$. Letting $A_m = \bigcup_{G(\vec{a})=1} \bigcap_{j=0}^{m-1} S_j^{a_j}$, Lemma 5.5 says $S \in \mathbf{\Delta}^0_{\mu+1}$.

($\Leftarrow$) Assume $S \in \mathbf{\Delta}^0_{\mu+1}$. By Lemma 5.5, there are $(A_i)_{i \in \mathbb{N}}$, each $A_i \in \mathbf{\Delta}^0_{\mu_i+1}$ for some $\mu_i < \mu$, such that

$$S = \bigcup_n \bigcap_{m \ge n} A_m = \bigcap_n \bigcup_{m \ge n} A_m. \qquad (*)$$

I claim that $S$ is guessable based on $\mathscr{S} = (A_0, A_1, \dots)$. Define $G : \{0,1\}^{<\mathbb{N}} \to \{0,1\}$ by $G(a_0, \dots, a_m) = a_m$, I will show that $G$ is an $S$-guesser based on $\mathscr{S}$.

Suppose $f \in S$. By $(*)$, $\exists n$ s.t. $\forall m \ge n$, $f \in A_m$ and thus $\chi_{A_m}(f) = 1$. For all $m \ge n$,

$$\begin{aligned} G(\mathscr{S}(f) \restriction (m+1)) &= G(\chi_{A_0}(f), \dots, \chi_{A_m}(f)) \\ &= \chi_{A_m}(f) \\ &= 1, \end{aligned}$$

so $\lim_{n\to\infty} G(\mathscr{S}(f) \restriction n) = 1$. A similar argument shows that if $f \notin S$ then $\lim_{n\to\infty} G(\mathscr{S}(f) \restriction n) = 0$. $\square$

Combining Theorems 1.2 and 5.4, we see that $S$ is guessable if and only if $S$ is 1st-order guessable. It is also not difficult to give a direct proof of this equivalence, and having done so, Theorem 5.4 provides yet another proof of Theorem 1.2.

## Notes

1. A third independent usage of the term *guessable*, with similar but not the same meaning, appears in [8] (p. 1280), where a subset $Y \subseteq \mathbb{N}^{\mathbb{N}}$ is called guessable if there is a function $g \in \mathbb{N}^{\mathbb{N}}$ such that for each $f \in Y$, $g(n) = f(n)$ for infinitely many $n$.

2. In general, there seems to be a correspondence between remainders on $\mathbb{N}^{\mathbb{N}}$ and remainders on $\mathbb{N}^{<\mathbb{N}}$ that take trees to trees; in the future we might publish more general work based on this observation.

## References

[1] Alexander, S., "On Guessing Whether a Sequence has a Certain Property," *Journal of Integer Sequences* vol. 14 (2011), 12 pp.

[2] Allouche, J., "Note on the constructible sets of a topological space," *Annals of the New York Academy of Science* vol. 806 (1996), pp. 1–10.

[3] Dougherty, R., and C. Miller, "Definable Boolean combinations of open sets are Boolean combinations of open definable sets," *Illinois Journal of Mathematics* vol. 45 (2001), pp. 1347–1350.

[4] Figueira, S., Hirschfeldt, D., Miller, J., Ng, K., and A. Nies, "Counting the changes of random $\Delta^0_2$ sets," to appear in *Journal of Logic and Computation*, published online 2013, doi: 10.1093/logcom/exs083.

[5] Kechris, A., "Classical Descriptive Set Theory," Springer-Verlag, 1995.

[6] Martin, D., "Borel determinacy," *Annals of Mathematics* vol. 102 (1975), pp. 363–371.

[7] Nies, A., "Calibrating the complexity of $\Delta^0_2$ sets via their changes," preprint, arXiv: `http://arxiv.org/abs/1302.0454`.

[8] Tsaban, B., and L. Zdomskyy, "Combinatorial Images of Sets of Reals and Semifilter Trichotomy," *Journal of Symbolic Logic* vol. 73 (2008), pp. 1278–1288.

[9] Wadge, W., "Reducibility and Determinateness on the Baire Space," PhD dissertation, UC Berkeley (1983).

## Acknowledgments

We acknowledge Tim Carlson, Chris Miller, Dasmen Teh, and Erik Walsberg for many helpful questions and suggestions. We are gratetful to a referee of an earlier manuscript for making us aware of William Wadge's dissertation.

Department of Mathematics
The Ohio State University
231 West 18th Ave
Columbus OH 43210 USA
alexander@math.ohio-state.edu