Self-referential theories

Samuel A. Alexander*

Department of Mathematics, the Ohio State University

August 11, 2020

Abstract

We study the structure of families of theories in the language of arithmetic extended to allow these families to refer to one another and to themselves. If a theory contains schemata expressing its own truth and expressing a specific Turing index for itself, and contains some other mild axioms, then that theory is untrue. We exhibit some families of true self-referential theories that barely avoid this forbidden pattern.

1 Introduction

This is a paper about families of r.e. theories, each capable of referring to itself and the others. Many of this paper’s results first appeared in the author’s dissertation [1]. There, they were stated in terms of families of interacting mechanical knowing agents. Here, we will speak instead of families of self-referential r.e. theories. We hope this will more directly expose the underlying mathematics.

In epistemology, it is well-known that a (suitably idealized) truthful knowing machine capable of arithmetic, logic, and self-reflection, cannot know its own truth and its own code. This is due, in various guises, to authors such as Lucas [8], Benacerraf [3], Reinhardt [11], Penrose [9], and Putnam [10]. In terms of self-referential theories, a true theory satisfying certain assumptions cannot contain schemata stating its own truth and its own G"odel number (if such a theory did exist, we could program a machine knower that knows precisely its consequences). Reinhardt conjectured, and Carlson proved [5], a truthful machine knower can know (in a local sense, i.e., expressed by infinite schemata rather than a single axiom) that it is truthful and has some code, without knowing which. A true self-referential theory can (in a local sense) state its own truth and recursive enumerability. We showed [2] that, alternatively, a truthful machine can (in a local sense) exactly know its own code, if not required to know its own truth. A true theory can state (in a local sense) its own G"odel number.

Our goal is to generalize the above consistency results to multiple theories. The paper contains four main findings. In the following list of promises, except where otherwise stated, \(\prec\) is an r.e. well-founded partial-order on \(\omega\), and expresses is meant in the local (infinite schema) sense.

1. There are true theories \((T_i)_{i \in \omega}\) such that \(T_i\) expresses a G"odel number of \(T_j\) (all \(i, j\)) and \(T_i\) expresses the truth of \(T_j\) (all \(j \prec i\)).

2. There are true theories \((T_i)_{i \in \omega}\) such that \(T_i\) expresses a G"odel number of \(T_j\) (\(j \prec i\)), the truth of \(T_j\) (\(j \preceq i\)), and the fact that \(T_j\) has some G"odel number (all \(i, j\)).

3. If \(\prec\) is ill-founded, and if we extend the base language to include a predicate for computable ordinals and require the theories to include rudimentary facts about them, then 1 and 2 fail.

4. Finally, if we do not extend the base language as in 3, then there do exist ill-founded r.e. partial orders \(\prec\) such that 1 and 2 hold.

Our proofs of 1 and 2 are constructive, but the proof of 4 is nonconstructive. In short, if 4 were false, either of 1 or 2 could be used to define the set \(WF\) of r.e. well-founded partial orders of \(\omega\) using nothing but arithmetic and a truth predicate \(Tr\) for arithmetic. This is impossible since \(WF\) is \(\Pi^1_1\)-complete and \(Tr\) is \(\Delta^1_1\).

*Email: alexander@math.ohio-state.edu
2 Preliminaries

To us, theory and schema mean set of sentences (a sentence is a formula with no free variables).

Definition 1. (Standard Definitions)

1. When a first-order structure is clear from context, an assignment is a function s mapping first-order variables into the universe of that structure. If x is a variable and u is an element of the universe, s(x|u) is the assignment that agrees with s except that it maps x to u.

2. We write \( M \models \phi[s] \) to indicate that the first-order structure \( M \) satisfies the formula \( \phi \) relative to the assignment \( s \). We write \( M \models \phi \) just in case \( M \models \phi[s] \) for every assignment \( s \). If \( T \) is a theory, \( M \models T \) means that \( M \models \phi \) for every \( \phi \in T \).

3. We write FV(\( \phi \)) for the set of free variables of \( \phi \).

4. We write \( \phi(x|t) \) for the result of substituting term \( t \) for variable \( x \) in \( \phi \).

5. \( \mathcal{L}_{PA} \) is the language of Peano arithmetic, with constant symbol 0 and function symbols \( S, +, \cdot \) with the usual arities. If \( \mathcal{L} \) extends \( \mathcal{L}_{PA} \), an \( \mathcal{L} \)-structure has standard first-order part if it has universe \( \mathbb{N} \) and interprets \( 0, S, +, \cdot \) as intended.

6. We define \( \mathcal{L}_{PA} \)-terms \( \pi \) \((n \in \mathbb{N})\), called numerals, so that \( \bar{0} \equiv 0 \) and \( \bar{n + 1} \equiv S(n) \).

7. We fix a computable bijection \((\bullet, \bullet, \bullet) : \mathbb{N}^3 \to \mathbb{N} \). Being computable, this is \( \mathcal{L}_{PA} \)-definable, so we may freely act as if \( \mathcal{L}_{PA} \) contained a function symbol for this bijection. Similarly we may act as if \( \mathcal{L}_{PA} \) contained a binary predicate symbol \( \bullet \in W_n \) for membership in the \( n \)th r.e. set \( W_n \).

8. Whenever a computable language is clear from context, \( \phi \mapsto \neg \phi \) denotes Gödel numbering.

9. A valid formula is one that is true in every structure.

10. A universal closure of \( \phi \) is a sentence \( \forall x_1 \cdots \forall x_n \phi \) where \( \text{FV}(\phi) \subseteq \{x_1, \ldots, x_n\} \). We write ucl(\( \phi \)) to denote a generic universal closure of \( \phi \).

Note that if \( M \) is a structure and \( \psi \) is a universal closure of \( \phi \), in order to prove \( M \models \psi \) it suffices to let \( s \) be an arbitrary assignment and show \( M \models \phi[s] \).

To formalize self-referential theories, we employ an extension of first-order logic where languages may contain new unary connective symbols. This logic is borrowed from [2].

Definition 2. (The Base Logic) A language \( \mathcal{L} \) of the base logic is a first-order language \( \mathcal{L}_0 \) together with a class of symbols called operators. Formulas of \( \mathcal{L} \) are defined as usual, with the clause that \( T_i \models \phi \) is a formula whenever \( \phi \) is a formula and \( T_i \models \phi \) is an operator. Syntactic parts of Definition 1 extend to the base logic in obvious ways (we define \( \text{FV}(T_i \models \phi) = \text{FV}(\phi) \)). An \( \mathcal{L} \)-structure \( M \) is a first-order \( \mathcal{L}_0 \)-structure \( M_0 \) together with a function that takes one operator \( T_i \models \), one \( \mathcal{L} \)-formula \( \phi \), and one assignment \( s \), and outputs either True or False—in which case we write \( M \models T_i \models \phi[s] \) or \( M \not\models T_i \models \phi[s] \), respectively—satisfying the following three requirements.

1. Whether or not \( M \models T_i \models \phi[s] \) does not depend on \( s(x) \) if \( x \notin \text{FV}(\phi) \).

2. If \( \phi \) and \( \psi \) are alphabetic variants (meaning that one is obtained from the other by renaming bound variables so as to respect the binding of the quantifiers), then \( M \models T_i \models \phi[s] \) if and only if \( M \models T_i \models \psi[s] \).

3. For variables \( x \) and \( y \) such that \( y \) is substitutable for \( x \) in \( T_i \models \phi \), \( M \models T_i \models \phi(x|y)[s] \) if and only if \( M \models T_i \models \phi(s(x)s(y))[s] \).

The definition of \( M \models \phi[s] \) for arbitrary \( \mathcal{L} \)-formulas is obtained from this by induction. Semantic parts of Definition 1 extend to the base logic in obvious ways.

Traditionally the operator \( T_i \models \) would be written \( K_i \), and the formula \( K_i \phi \) would be read like “agent \( i \) knows \( \phi \)”. For the present paper, the added intuition would not be worth the philosophical distraction.
Theorem 3. (Completeness and compactness) Suppose $\mathcal{L}$ is an r.e. language in the base logic.

1. The set of valid $\mathcal{L}$-formulas is r.e.

2. For any r.e. $\mathcal{L}$-theory $\Sigma$, $\{ \phi : \Sigma \models \phi \}$ is r.e.

3. There is an effective procedure, given (a Gödel number of) an r.e. $\mathcal{L}$-theory $\Sigma$, to find (a Gödel number of) $\{ \phi : \Sigma \models \phi \}$.

4. If $\Sigma$ is an $\mathcal{L}$-theory and $\Sigma \models \phi$, there are $\sigma_1, \ldots, \sigma_n \in \Sigma$ such that $\sigma_1 \to \cdots \to \sigma_n \to \phi$ is valid.

Proof. By interpreting the base logic within first-order logic (for details see [1]).

Definition 4. If $\mathcal{L}$ is a first-order language and $I$ is an index set, let $\mathcal{L}(I)$ be the language (in the base logic) consisting of $\mathcal{L}$ along with operators $T_i$ for all $i \in I$.

In case $I$ is a singleton, $\mathcal{L}_{PA}(I)$ is a form of Shapiro’s [12] language of Epistemic Arithmetic.

Definition 5.

- For any $\mathcal{L}_{PA}(I)$-formula $\phi$ with $\text{FV}(\phi) = \{x_1, \ldots, x_n\}$, and for assignment $s$ (into $\mathbb{N}$), let $\phi^s$ be the sentence
  \[ \phi^s \equiv \phi(x_1|s(x_1)) \cdots (x_n|s(x_n)) \]
  obtained by replacing all free variables in $\phi$ by numerals for their $s$-values.

- For any language $\mathcal{L}$ extending $\mathcal{L}_{PA}$, if $\mathcal{M}$ is an $\mathcal{L}$-structure, then $\mathcal{M}$ is said to interpret formulas by substitution if $\mathcal{M}$ has standard first-order part and the following property holds: for every $\mathcal{L}$-formula $\phi$ and assignment $s$, $\mathcal{M} \models \phi[s]$ if and only if $\mathcal{M} \models \phi^s$.

For example, if $s(x) = 0$ and $s(y) = 2$ then $(\forall z (x = y + z))^s \equiv \forall z (0 = S(S(0)) + z)$.

Definition 6. If $T = (T_i)_{i \in I}$ is an $I$-indexed family of $\mathcal{L}_{PA}(I)$-theories and $\mathcal{N}$ is an $\mathcal{L}_{PA}(I)$-structure, we say $\mathcal{N} \models T$ if $\mathcal{N} \models T_i$ for all $i \in I$.

Definition 7. Suppose $T = (T_i)_{i \in I}$ is an $I$-indexed family of $\mathcal{L}_{PA}(I)$-theories. The intended structure for $T$ is the $\mathcal{L}_{PA}(I)$-structure $\mathcal{M}_T$ with standard first-order part, interpreting the operators $T_i$ for all $i \in I$ as follows:

\[ \mathcal{M}_T \models T_i \models \phi[s] \text{ if and only if } T_i \models \phi^s. \]

If $\mathcal{M}_T \models T$, we say $T$ is true.

Lemma 8. For any family $T = (T_i)_{i \in I}$ of $\mathcal{L}_{PA}(I)$-theories, $\mathcal{M}_T$ interprets formulas by substitution.

Proof. In other words, we must show that for every $\mathcal{L}_{PA}(I)$-formula $\phi$ and assignment $s$, $\mathcal{M}_T \models \phi[s]$ if and only if $\mathcal{M}_T \models \phi^s$. The proof is a straightforward induction.

Definition 9. By the axioms of Peano arithmetic for $\mathcal{L}_{PA}(I)$ we mean the axioms of Peano arithmetic, with induction extended to $\mathcal{L}_{PA}(I)$.

Lemma 10. For any $\mathcal{L}_{PA}(I)$-structure $\mathcal{M}$, if $\mathcal{M}$ interprets formulas by substitution, then $\mathcal{M}$ satisfies the axioms of Peano arithmetic for $\mathcal{L}_{PA}(I)$.

Proof. Let $\mathcal{M}$ be any $\mathcal{L}_{PA}(I)$-structure which interprets formulas by substitution. This means $\mathcal{M}$ has standard-first order part and for every formula $\phi$ and assignment $s$, $\mathcal{M} \models \phi[s]$ if and only if $\mathcal{M} \models \phi^s$.

Let $\sigma$ be an axiom of Peano arithmetic for $\mathcal{L}_{PA}(I)$. If $\sigma$ is not an instance of induction, then $\mathcal{M} \models \sigma$ since $\mathcal{M}$ has standard first-order part. But suppose $\sigma$ is true in $\mathcal{M}$, let $s$ be an arbitrary assignment and assume $\mathcal{M} \models \phi(x|0)[s]$ and $\mathcal{M} \models \forall x (\phi(x|S(x)))[s]$. By assumption, $\mathcal{M} \models \phi^s[x|m]$ and for all $m \in \mathbb{N}$, $\mathcal{M} \models \phi^{s(x|m)}$. Evidently $\phi(x|S(x))^{s(x|m)} \equiv \phi^{s(x|m+1)}$. By mathematical induction, $\forall m \in \mathbb{N}, \mathcal{M} \models \phi^{s(x|m)}$. By assumption, $\mathcal{M} \models \forall x \phi[s]$.

\[ ^1 \text{We write } A \to B \to C \text{ for } A \to (B \to C), \text{ and likewise for longer chains. } \]
Definition 11. Suppose $T = (T_i)_{i \in I}$ is a family $\mathcal{L}_{PA}(I)$-theories. If $T^+ = (T_i^+)_{i \in I}$ is another such family, we say $T \subseteq T^+$ if $T_i \subseteq T_i^+$ for every $i \in I$. If $T$ is a single $\mathcal{L}_{PA}(I)$-theory, we say $T \subseteq T_i$ if $T \subseteq T_i$ for all $i \in I$. If $T^1 = (T^1_i)_{i \in I}$ and $T^2 = (T^2_i)_{i \in I}$ are families of $\mathcal{L}_{PA}(I)$-theories, $T^1 \cup T^2$ is the family $T' = (T'_i)_{i \in I}$ where each $T'_i = T^1_i \cup T^2_i$. Arbitrary unions $\bigcup_{n \in X} T^n$ are defined similarly.

Definition 12. Suppose $T = (T_i)_{i \in I}$ is a family of $\mathcal{L}_{PA}(I)$-theories. For each $i \in I$, we say $T_i$ is $T_i$-closed if $T_i \models \phi \in T_i$ whenever $\phi \in T_i$. We say $T$ is closed if each $T_i$ is $T_i$-closed.

Definition 13. If $I$ is an r.e. index set, a family $T = (T_i)_{i \in I}$ is r.e. just in case $\{(\phi, i) : \phi \in T_i\}$ is r.e.

3 Generic Axioms

If $T$ is a family of theories whose truth was in doubt, and if we state a theorem removing that doubt, we often state more: that $T \cup S$ is true, where $S$ is some background theory of provability, including non-controversial things like Peano arithmetic or the schema $\text{ucl}(T_i \models (\phi \to \psi)) \to T_i \models \phi \to T_i \models \psi$). The choice of $S$ is somewhat arbitrary, or at best based on tradition. We will avoid this arbitrary choice by stating results in the form: 

“$T$ is true together with any background theory of provability such that…”

Definition 14. A family $T$ of $\mathcal{L}_{PA}(\omega)$-theories is closed-r.e.-generic if $T$ is r.e. and $\mathcal{M}_U \models T$ for every closed r.e. family $U \supseteq T$ of $\mathcal{L}_{PA}(\omega)$-theories.

Lemma 15. If $T$ is a union of closed-r.e.-generic families and $T$ is r.e., then $T$ is closed-r.e.-generic.

Proof. Straightforward.

Definition 16. For $i \in I$ and for $T$ an $\mathcal{L}_{PA}(I)$-theory, we write $[T]_i$ for the family $T = (T_k)_{k \in I}$ where $T_i = T$ and $T_k = \emptyset$ for all $k \neq i$.

3.1 Closed-r.e.-generic Building Blocks

In this subsection, we will exhibit some examples of closed-r.e.-generic families. They can be combined in diverse ways, via Lemma 15, to form background theories of provability. This will allow us to state Theorem 24 below in a generalized way, essentially saying that a certain doubted theory is consistent with any background theory of provability made up of closed-r.e.-generic building blocks. The alternative would be for us to arbitrarily choose one such background theory and build it directly into Theorem 24, which would cause the core details in the proof of Theorem 24 to get jumbled up with unimportant distractions.

It is common for a theory to state its own closure under modus ponens. When there are multiple theories, it is less clear whether each individual theory should only state its own closure thereunder, or the closure of all the other theories, or of some subset thereof. With the following lemma, we avoid arbitrarily imposing a decision along these lines.

Lemma 17. For any $i, j \in \omega$, the following family is closed-r.e.-generic:

- $[S]_i$ where $S$ is: ($j$-Deduction) the schema $\text{ucl}(T_j \models (\phi \to \psi) \to T_j \models \phi \to T_j \models \psi)$.

Proof. Let $U = (U_k)_{k \in \omega}$ be any closed r.e. family of $\mathcal{L}_{PA}(\omega)$-theories such that $U \supseteq [S]_i$ where $S$ is $j$-Deduction. We must show $\mathcal{M}_U \models [S]_i$. In other words we must show $\mathcal{M}_U \models \text{ucl}(T_j \models (\phi \to \psi) \to T_j \models \phi \to T_j \models \psi)$ for any $\phi, \psi$. Let $s$ be an assignment and assume $\mathcal{M}_U \models T_j \models (\phi \to \psi)[s]$ and $\mathcal{M}_U \models T_j \models \phi[s]$, we must show $\mathcal{M}_U \models T_j \models \psi[s]$. By Definition of $\mathcal{M}_U$, $U_j \models (\phi \to \psi)[s]$ and $U_j \models \phi[s]$. Clearly $(\phi \to \psi)[s] \equiv \phi[s] \to \psi[s]$ so by modus ponens $U_j \models \psi[s]$, that is, $\mathcal{M}_U \models T_j \models \psi[s]$. 

It might not be controversial to require that a theory express its own ability to prove valid sentences, but in a multi-theory context, should we require each theory to express that much about all its fellow theories? The following lemma allows us to avoid arbitrarily declaring the right answer to that question. Part 2 of this lemma illustrates an interesting combinatorial property of closed-r.e.-generic building blocks. Some schemas would not be suitable building blocks by themselves, but when paired with other schemas, the combination can become a suitable building block.
Lemma 18. For any \( i, j \in \omega \), the following families are closed-r.e.-generic:

1. \([S]_i\) where \( S \) is: (Assigned Validity) the schema \( \phi^* \) (\( \phi \) valid, \( s \) an assignment).

2. \([\text{Assigned Validity}]_i \cup [S]_j\) where \( S \) is: (\( i \)-Validity) \( \text{ucl}(T_i \vDash \phi) \) for \( \phi \) valid.

Proof. Both (1) and (2) are r.e. by Theorem 3.

(1) Let \( U = (U_k)_{k \in \omega} \) be a closed r.e. superset of \([S]_i\), where \( S \) is Assigned Validity. We must show \( \mathcal{M}_U \models \phi^* \) for some valid \( \phi \) and some assignment \( s \). Since \( \phi^* \) is valid, \( \mathcal{M}_U \models \phi^*[s] \). By Lemma 8, \( \mathcal{M}_U \models \phi^0 \).

(2) Let \( U = (U_k)_{k \in \omega} \) be any closed r.e. family of \( \mathcal{L}_{\text{PA}}(\omega)\) theories such that \( U_i \) contains Assigned Validity and \( U_j \) contains \( i \)-Validity. By (1), \( \mathcal{M}_U \) satisfies Assigned Validity. It remains to show \( \mathcal{M}_U \) satisfies \( i \)-Validity. Let \( \phi \) be valid and \( s \) an assignment. Since \( U_i \) contains Assigned Validity, \( U_i \models \phi^* \), so by definition of \( \mathcal{M}_U \), \( \mathcal{M}_U \models \langle T_i \vDash \phi \rangle[s] \).

In modal logic, some papers treat the so-called positive introspection axiom (also known as the KK axiom) as one of the fundamental axioms of knowledge, and some do not. Rather than join either side, we prefer instead to study the combinatorial structure of the axiom, asking: are there other schemas we can add to it to make the combination closed-r.e.-generic?

Lemma 19. For any \( i, j \in \omega \), the following family is closed-r.e.-generic:

- \([\text{Assigned Validity}]_i \cup [\text{i-Validity}]_i \cup [\text{i-Deduction}]_i \cup [S]_j\) where \( S \) is:
  
  (\( i \)-Introspection) the schema \( \text{ucl}(T_i \vDash \phi \rightarrow T_i \vDash T_j \vDash \phi) \).

Proof. Recursive enumerability is by Theorem 3. Let \( U = (U_k)_{k \in \omega} \) be any closed r.e. family of \( \mathcal{L}_{\text{PA}}(\omega) \) theories such that \( U_i \) contains Assigned Validity, \( i \)-Validity and \( i \)-Deduction, and \( U_j \) contains \( i \)-Introspection. Then \( \mathcal{M}_U \) satisfies Assigned Validity and \( i \)-Validity by Lemma 18. By Lemma 17, \( \mathcal{M}_U \) satisfies \( i \)-Deduction. For \( i \)-Introspection, let \( s \) be an assignment and assume \( \mathcal{M}_U \models T_i \vDash \phi[s] \), we will show \( \mathcal{M}_U \models T_i \vDash T_j \vDash \phi[s] \). Since \( \mathcal{M}_U \models T_i \vDash \phi[s] \), \( U_i \models \phi^* \). By Theorem 3 there are \( \sigma_1, \ldots, \sigma_n \in U_i \) such that \( \sigma_1 \rightarrow \cdots \rightarrow \sigma_n \rightarrow \phi^* \) is valid. Since \( U_i \) contains \( i \)-Validity, \( U_i \models T_i \vDash \sigma_1 \rightarrow \cdots \rightarrow \sigma_n \rightarrow \phi^* \). By repeated applications of \( i \)-Deduction contained in \( U_i \), \( U_i \models T_i \vDash \sigma_1 \rightarrow \cdots \rightarrow T_i \vDash \sigma_n \rightarrow T_i \vDash \phi^* \). Since \( U \) is closed, \( U_i \) is \( T_i \vDash \)-closed and so contains \( T_j \vDash \sigma_1, \ldots, T_i \vDash \sigma_n \). So \( U \models (T_i \vDash \phi)^* \) and \( \mathcal{M}_U \models T_i \vDash T_j \vDash \phi[s] \).

The following lemma shows that arithmetic is generic, which will enable us to state a later result (Theorem 24) in such a way that it is clear that the result is neither contingent on the presence, nor the absence, of arithmetic in the theories in question.

Lemma 20. For any \( i \in \omega \), \([S]_i\) is closed-r.e.-generic, where \( S \) is the set of axioms of Peano arithmetic for \( \mathcal{L}_{\text{PA}}(\omega) \).

Proof. By Lemmas 8 and 10.

Carlson proved [5] that it is consistent for an idealized knowing machine to know “I am a machine” (without knowing which specific machine it is). The following lemma sheds additional light: not only is it consistent for a knowing machine to know “I am a machine”, in fact that knowledge is generic: it does not depend heavily on specific arbitrary decisions about the background theory of provability.

Lemma 21. For any \( i, j \in \omega \), the following family is closed-r.e.-generic.

- \([S]_i\) where \( S \) is: (\( j \)-SMT) (See 3 and 11) \( \text{ucl}(\exists e \forall x (T_j \vDash \phi \leftrightarrow x \in W_e)) \), \( e \notin \text{FV}(\phi) \).

Proof. Suppose \( U = (U_k)_{k \in \omega} \) is a closed r.e. family of \( \mathcal{L}_{\text{PA}}(\omega) \) theories and \( U \supseteq [S]_i \), where \( S \) is \( j \)-SMT. We must show \( \mathcal{M}_U \models [S]_i \). That is, given \( \phi \) with \( e \notin \text{FV}(\phi) \), we must show \( \mathcal{M}_U \models \text{ucl}(\exists e \forall x (T_j \vDash \phi \leftrightarrow x \in W_e)) \).

Let \( s \) be an assignment and let \( x_1, \ldots, x_k = \text{FV}(\phi) \setminus \{x\} \). Since \( U_j \) is r.e., by the \( S \)-m-n theorem there is some \( n \) such that \( W_n = \{m : U_j \models \phi(x[m]) (x_1[s(x_1)]) \cdots (x_k[s(x_k)])\} \). Since \( e \notin \text{FV}(\phi) \), and \( \mathcal{M}_U \) has standard first-order part, it follows that \( \mathcal{M}_U \models \forall x (T_j \vDash \phi \leftrightarrow x \in W_e)[s(e[n])] \).
Finally, the following lemma offers a way to obtain new building blocks from old. This can be combined with Lemma 21 to advance from “I am a machine” to “I know I am a machine”.

Lemma 22. For any \( i, j \in \omega \) and any closed-r.e.-generic family \( U = (U_i)_{i \in \omega} \supseteq T \cup [S]_i \) is closed-r.e.-generic, where \( S \) is the schema: \( T_j \models \phi (\phi \in T_j) \).

Proof. Suppose \( U = (U_i)_{i \in \omega} \supseteq T \cup [S]_i \) is closed and r.e. Right away \( \mathcal{M}_U \models T \) because \( T \) is closed-r.e.-generic. It remains to show that \( \mathcal{M}_U \models [S]_i \), i.e., that \( \mathcal{M}_U \models S \). Fix \( \phi \in T_j \) and let \( s \) be any assignment. Since \( \phi \) is a sentence, \( \phi \equiv \phi^s \) and thus \( T_j \models \phi^s \). Since \( U_j \supseteq T_j, U_j \models \phi^s \). By definition of \( \mathcal{M}_U \), \( \mathcal{M}_U \models T_j \models \phi[s] \).

We gather Lemmas 17–22 together into the following summary.

Corollary 23. For any \( i, j \in \omega \), each of the following families is closed-r.e.-generic.

1. \([j]^{-}\text{Deduction})_i\).
2. \([\text{Assigned Validity}]_i\).
3. \([\text{Assigned Validity}])_i \cup [i]^{-}\text{Validity})_j\).
4. \([\text{Assigned Validity}]_i \cup [i]^{-}\text{Validity})_j \cup [i]^{-}\text{Deduction})_i \cup [i]^{-}\text{Introspection})_j\).
5. \([S]_i \) where \( S \) is the set of axioms of Peano arithmetic for \( \mathcal{L}_{PA} (\omega) \).
6. \([j]^{-}\text{SMT})_i\).
7. \( T \cup [S]_i \), for any closed-r.e.-generic \( T \), where \( S \) is the schema: \( T_j \models \phi (\phi \in T_j) \).

The above building blocks are not exhaustive. In choosing building blocks, a primary concern was to facilitate creation of background provability theories strong enough to make our consistency result (Theorem 24) generalize Carlson’s consistency result [5]. If that were our lone motivation, we could restrict Corollary 23 to only those families where \( i = j \), but a secondary motivation was to provide inter-theory versions of those restricted building blocks. It would be interesting to investigate questions about whether the above building-blocks are minimal. For example, in Lemma 19 is it really necessary to bundle \( j \)-Introspection with all three other schemas? For now, we will leave those questions open.

4 First Consistency Result: Prioritizing Exact Codes

The following theorem fulfills the first promise from the introduction.

Theorem 24. Suppose \( \prec \) is an r.e. well-founded partial order on \( \omega \) and \( T^0 = (T^0_i)_{i \in \omega} \) is closed-r.e.-generic. For each \( n \in \mathbb{N} \), let \( T(n) = (T_i(n))_{i \in \omega} \) where each \( T_i(n) \) is the smallest \( T_i \)-closed theory containing the following:

1. The axioms in \( T^0_i \).
2. \( \forall x(T_j \models \phi \leftrightarrow (\overline{x} \phi^j, j, x) \in \mathcal{W}_n) \) whenever \( j \in \omega, \mathcal{F}(\phi) \subseteq \{x\} \).
3. \( \mathcal{U}(T_j \models \phi \rightarrow \phi) \) whenever \( j \prec i \).

There is some \( n \in \mathbb{N} \) such that \( T(n) \) is true.

Proof. By the \( S \)-m-n Theorem, there is a total computable \( f : \mathbb{N} \rightarrow \mathbb{N} \) such that \( \forall n \in \mathbb{N} \),

\[ W_{f(n)} = \{\langle \overline{x} \phi^i, i, m \rangle : \mathcal{F}(\phi) \subseteq \{x\} \text{ and } T_i(n) \models \phi(x|m) \} \].

Using the Recursion Theorem, fix \( n \in \mathbb{N} \) such that \( W_{f(n)} = \mathcal{W}_n \). For brevity write \( T \) for \( T(n) \) and \( T_i \) for \( T_i(n) \). We will show \( \mathcal{M}_T \models T \). This is a self-referential statement: to show \( T_i \) true includes showing \( \mathcal{M}_T \models \mathcal{U}(T_j \models \phi \rightarrow \phi) \), which is essentially the statement that \( T_j \) is true. Hence the restriction \( j \prec i \), which allows induction since \( \prec \) is well founded. We will show, by \( \prec \)-induction on \( i \), that \( \mathcal{M}_T \models T_i \) for every \( i \in \omega \). Fix \( i \in \omega \) and assume \( \mathcal{M}_T \models T_j \) for all \( j < i \). Suppose \( \sigma \in T_i \), we will show \( \mathcal{M}_T \models \sigma \).
Case 1: $\sigma \in T_0$. Then $\mathcal{M}_T \models \sigma$ because $T^0$ is closed r.e.-generic and $T \supseteq T^0$ is closed r.e.

Case 2: $\sigma$ is $\forall \nu(T_j \vdash \phi \leftrightarrow \langle \overline{i} \phi \rangle \in W_\nu)$ for some $j \in \omega$, $\text{FV}(\phi) \subseteq \{x\}$. Let $s$ be an assignment, $m \in \mathbb{N}$. The following are equivalent.

\[
\mathcal{M}_T \models T_j \vdash (\overline{i} \phi)(s[m])
\]

\[
T_j \models (\overline{i} \phi)(s[m])
\quad \text{(Definition of $\mathcal{M}_T$)}
\]

\[
T_j \models (\overline{i} \phi)(s[m])
\quad \text{(Since $\text{FV}(\phi) \subseteq \{x\}$)}
\]

\[
\mathcal{M}_T \models (\overline{i} \phi)(s[m]) \in W_\nu
\quad \text{(By definition of $n$)}
\]

\[
\mathcal{M}_T \models (\overline{i} \phi)(s[m]) \in W_\nu
\quad \text{($\mathcal{M}_T$ has standard first-order part)}
\]

\[
\mathcal{M}_T \models (\overline{i} \phi)(s[m]) \in W_\nu[\{s[x|m]\}]
\quad \text{(Lemma 8)}
\]

Case 3: $\sigma$ is $\text{uc}(T_j \vdash \phi \rightarrow \phi)$ for some $j \prec i$. Let $s$ be an assignment and assume $\mathcal{M}_T \models T_j \vdash \phi[s]$. This means $T_j \models \phi^s$. By our $\prec$-induction hypothesis, $\mathcal{M}_T \models T_j$, so $\mathcal{M}_T \models \phi^s$. By Lemma 8, $\mathcal{M}_T \models \phi[s]$.

Case 4: $\sigma$ is present in $T_i$ because of the clause that $T_i$ is $T^i$-closed. Then $\sigma$ is $T_i \vdash \sigma_0$ for some $\sigma_0 \in T_i$. Being in $T_i$, $\sigma_0$ is a sentence, so for any assignment $s$, $\sigma_0 \equiv \sigma_0^s$, $T_i \models \sigma_0^s$, and finally $\mathcal{M}_T \models T_i \vdash \sigma_0^s$.

By $\prec$-induction, $\mathcal{M}_T \models T_i$ for all $i \in \omega$. This shows $\mathcal{M}_T \models T$, that is, $T$ is true.

The first promise from the introduction is met: for any r.e. well-founded partial order $\prec$ on $\omega$, there are theories $(T_n)_{n \in \omega}$ such that $\forall i, j, k \in \omega$ with $j \prec i$, $T_j$ expresses the truth of $T_j$, and $T_i$ expresses a Gödel number of $T_k$.

In order to fulfill the second promise we will extend Carlson’s notion of stratification to the case of multiple operators, and introduce stratifiers, a tool used to deal with subtleties that arise when multiple self-referential theories refer to one another.

In [2] the technique behind Theorem 24 was used to exhibit a machine that knows its own code.

## 5 Stratification

For the second promise from the introduction, we need to prove a result like Theorem 24 where $T_i$ includes $\text{uc}(T_j \vdash \phi \rightarrow \phi)$ for all $j \leq i$, not just $j \prec i$. This rules out the direct $\prec$-induction of the type used above. Induction on formula complexity will not work either: we would need to show all of $T_i$ consistent just to show $\mathcal{M}_T \models T_i \vdash (1 = 0) \rightarrow (1 = 0)$. Instead, we will use ordinal induction. But there are no ordinals anywhere in sight. To obtain ordinals to induct on, we will modify the theories we care about, in a process called stratification. We will start with some informal motivational remarks. Readers who would like to advance directly to the formal definitions can safely skip Subsection 5.1.

### 5.1 Motivation for Stratification

As explained above, we would like to invoke ordinal induction, but there are no ordinals in sight. In order to make ordinal induction relevant, we will do the following. We will extend the background language to contain not only the operators $T_i \vdash (i \in \omega)$, but also operators $T^\alpha_i \vdash (i \in \omega, \alpha \in e_0 \cdot \omega)$. And instead of focusing directly on $T_i$, we will focus on a theory $U_i$ such that the result $U_i^\beta$ of erasing superscripts from $U_i$ is $U_i^\beta = T_i$. The intended interpretation of $T^\alpha_i \vdash \phi[s]$ will be $U_i \cap \alpha \vdash \phi^s$, where $U_i \cap \alpha$ is the set of axioms of $U_i$ whose superscripts are $\prec \alpha$. Thus, we may think of $U_i$ as a version of $T_i$ with extra information about the structure of $T_i$. We will show (Theorem 20), for certain formulas $\phi$ whose superscripts are positive multiples of $e_0$, that $\phi$ holds (in the intended interpretation) if and only if $\phi^\beta$ holds. We will use this, after proving that $U_i$ holds, to conclude that $T_i$ also holds.

Suppose we would like $T_i$ to contain the axiom $T_i \vdash (1 + 1 = 2)$. Then, as we carry out the procedure in the above paragraph, we would ensure that $U_i$ contain all sentences of the form $T^\alpha_i \vdash (1 + 1 = 2)$. This would have the side effect that for any $\beta > \alpha$, $U_i \cap \beta \vdash T^\beta_i \vdash (1 + 1 = 2)$, so that $T^\beta_i \vdash T^\alpha_i \vdash (1 + 1 = 2)$ would hold in structures with the intended interpretation.

Next, suppose that for every arithmetical sentence $\phi$, we would like $T_i$ to include

\[
T_i \vdash \phi \rightarrow T_i \vdash T_i \vdash \phi.
\]

7
Then we would arrange that $U_i$ contain
\[ T_i^\beta \models \phi \rightarrow T_i^\beta \models T_i^\alpha \models \phi \]
whenever $\beta > \alpha$. The reason for the $\beta$ is as follows. The intended interpretation of $T_i^\alpha \models \phi$ shall be $U_i \cap \alpha \models \phi$. Thus, it would make no sense to put the axiom $T_i^\alpha \models \phi \rightarrow T_i^\beta \models T_i^\alpha \models \phi$ into $U_i$: the fact that $U_i \cap \alpha \models \phi$ does not generally imply that $U_i \cap \alpha \models T_i^\alpha \models \phi$, since $U_i \cap \alpha$ is limited to formulas in which all superscripts are $< \alpha$. At least $T_i^\alpha \models \phi \rightarrow T_i^\beta \models T_i^\alpha \models \phi$ is plausible.

Again, suppose that for some $j \prec i$, we would like for $T_i$ to include
\[ T_i \models (T_j \models (1 = 0) \rightarrow (1 = 0)). \]
We would arrange that $U_i$ contain (for all $\alpha$):
\[ T_i \models (T_j \models (1 = 0) \rightarrow (1 = 0)). \]
Note the lack of superscript on $T_j \models$. The intuition is that $U_i$ is a version of $T_i$ with extra information about the structure of $T_i$ (namely, that said structure arises from an increasing family of theories), but without any additional information about the structure of $T_j$.

Similarly, suppose we would like $T_i$ to include
\[ T_i \models (T_j \models (1 = 0) \rightarrow T_i \models (1 = 0)). \]
We would arrange that $U_i$ contain (for each $\alpha$):
\[ T_i \models (T_j \models (1 = 0) \rightarrow T_i \models (1 = 0)). \]
Note the lack of superscript on the $T_j \models$ within the scope of $T_i \models$. As above, the intuition is that $U_i$ is a version of $T_i$ with extra information about the structure of $T_i$. It does not have any extra information about the structure of $T_j$—not even about what $T_j$ says about $T_i$. This is important because, when $j \prec i$, we would like $T_i$ to contain axioms declaring, essentially, the Gödel number of $T_j$. This Gödel number would be hardcoded into such axioms, and thus there would be no hope of such axioms remaining true if $T_j$ were changed.

5.2 Stratification Formal Details
To get a foothold for induction, instead of considering a particular theory $T_i$, we will be considering copies of $T_i$ with ordinal-number superscripts added. To recover information about the original $T_i$ from these modified theories, we will need to use sophisticated results from [4] about the structure of the ordinals.

**Definition 25.** We define a binary relation $\leq_1$ on $\text{Ord}$ by transfinite recursion so that for all $\alpha, \beta \in \text{Ord}$, $\alpha \leq_1 \beta$ if and only if $\alpha \leq \beta$ and $(\alpha, \leq_1, \leq_1)$ is a $\Sigma_1$-elementary substructure of $(\beta, \leq, \leq_1)$.

The following theorem is based on calculations from [4]. It was used by Carlson to prove Reinhardt’s conjecture [5]. We state it here without proof.

**Theorem 26.**

1. The binary relation $\leq_1$ is a recursive partial ordering on $\epsilon_0 \cdot \omega$.

2. For all positive integers $m \leq n$, $\epsilon_0 \cdot m \leq_1 \epsilon_0 \cdot n$.

3. For any $\alpha \leq \beta \in \text{Ord}$, $\alpha \leq_1 \beta$ if and only if the following statement is true. For every finite set $X \subseteq \alpha$ and every finite set $Y \subseteq [\alpha, \beta)$, there is a set $X < Y \prec \alpha$ such that $X \cup Y \equiv_{(\leq_1, \leq_1)} X \cup Y$.

The usefulness of Theorem 26 will appear in Theorem 38, but first we need some machinery.

**Definition 27.** Let $I = ((\epsilon_0 \cdot \omega) \times \omega) \sqcup \omega$. Thus $\mathcal{L}_{PA}(I)$ contains operators $T_{(\alpha, i)} \models$ for all $\alpha \in \epsilon_0 \cdot \omega$, $i \in \omega$, along with operators $T_i \models$ for all $i \in \omega$. As abbreviation, we write $T_i^\alpha \models$ for $T_{(\alpha, i)} \models$, and refer to $\alpha$ as its superscript.
Definition 31. Suppose $X \subseteq \epsilon_0 \cdot \omega$ and $h : X \rightarrow \epsilon_0 \cdot \omega$ is order preserving. For each $\mathcal{L}_{PA}(\mathcal{I})$-formula $\phi$, define an $\mathcal{L}_{PA}(\mathcal{I})$-formula $h(\phi)$ inductively as follows:

1. If $\phi$ is $\neg \phi_0$, $\phi_1 \rightarrow \phi_2$, or $\forall x \phi_0$, then $h(\phi)$ is $\neg h(\phi_0)$, $h(\phi_1) \rightarrow h(\phi_2)$, or $\forall x h(\phi_0)$, respectively.

2. If $\phi$ is atomic or $\mathcal{T}_j \models \phi_0$, then $h(\phi) \equiv \phi$.

3. If $\phi$ is $\mathcal{T}_i^\alpha \models \phi_0$ where $\alpha \in X$, then $h(\phi) \equiv \mathcal{T}_i^{h(\alpha)} \models h(\phi_0)$.

4. If $\phi$ is $\mathcal{T}_i^\alpha \models \phi_0$ where $\alpha \notin X$, then $h(\phi) \equiv \mathcal{T}_i^\alpha \models h(\phi_0)$.

In practice, we will mainly be interested in $\phi$ when $\phi$ is $i$-stratified for some $i$, in which case $\mathcal{T}_i^\alpha \models$ cannot occur within the scope of $\mathcal{T}_i^\alpha \models$ in $\phi$ for any $k, j$. For such $\phi$, $h(\phi)$ is simply the result of applying $h$ to every superscript in $\phi$ that is in $X$.

For example if $X = \{1, \omega\}$, $h(1) = 0$, and $h(\omega) = \omega \cdot 2 + 1$, then

$$h \left( \mathcal{T}_1^1 \models (1 = 0) \rightarrow \mathcal{T}_1^1 \models (1 = 0) \rightarrow \mathcal{T}_i^\omega \models (1 = 0) \right) \equiv \mathcal{T}_1^0 \models (1 = 0) \rightarrow \mathcal{T}_1^0 \models (1 = 0) \rightarrow \mathcal{T}_i^{\omega^2 + 1} \models (1 = 0).$$

In practice, we will primarily be interested in applying Definition 31 in the case where $\text{On}(\phi) \subseteq X$. 

---

**Definition 28.** For any $\mathcal{L}_{PA}(\mathcal{I})$-formula $\phi$, $\text{On}(\phi) \subseteq \epsilon_0 \cdot \omega$ denotes the set of superscripts appearing in $\phi$.

**Definition 29.** Suppose $i \in \omega$. The $i$-stratified formulas of $\mathcal{L}_{PA}(\mathcal{I})$ are defined as follows (where $\phi$ ranges over $\mathcal{L}_{PA}(\mathcal{I})$-formulas):

1. If $\phi$ is $\mathcal{T}_j \models \phi_0$ for some $j \neq i$, then $\phi$ is $i$-stratified if and only if $\phi$ is a $\mathcal{L}_{PA}(\omega)$-formula.

2. If $\phi$ is $\mathcal{T}_j^0 \models \phi_0$ for some $j \neq i$, then $\phi$ is not $i$-stratified.

3. If $\phi$ is $\mathcal{T}_i \models \phi_0$, then $\phi$ is not $i$-stratified.

4. If $\phi$ is $\mathcal{T}_i^\alpha \models \phi_0$, then $\phi$ is $i$-stratified if and only if $\phi_0$ is $i$-stratified and $\alpha > \text{On}(\phi_0)$.

5. If $\phi$ is $\neg \phi_0$, $\phi_1 \rightarrow \phi_2$, or $\forall x \phi_0$, then $\phi$ is $i$-stratified if and only if its immediate subformula(s) are.

6. If $\phi$ is atomic, then $\phi$ is $i$-stratified.

An $\mathcal{L}_{PA}(\mathcal{I})$-theory $T$ is $i$-stratified if $\phi$ is $i$-stratified whenever $\phi \in T$. An $\mathcal{L}_{PA}(\mathcal{I})$-formula $\phi$ is very $i$-stratified if $\phi$ is $i$-stratified and $\text{On}(\phi) \subseteq \{\epsilon_0 \cdot 1, \epsilon_0 \cdot 2, \ldots\}$.

**Lemma 30.** Suppose $\phi$ is an $\mathcal{L}_{PA}(\mathcal{I})$-formula, $i \in \omega$. Then $\phi$ is $i$-stratified if and only if all of the following conditions hold.

1. For all $j \in \omega$ and $\alpha \in \epsilon_0 \cdot \omega$, if $\mathcal{T}_i^\alpha \models$ occurs in $\phi$, then $j = i$.

2. Every occurrence of $\mathcal{T}_i^\alpha \models$ in $\phi$ is inside the scope of $\mathcal{T}_j^\beta \models$ for some $j \neq i$.

3. $\mathcal{T}_i^\alpha \models$ never occurs in $\phi$ inside the scope of $\mathcal{T}_j^\beta \models$, for any $\alpha \in \epsilon_0 \cdot \omega$ or any $j \in \omega$.

4. For all $\alpha, \beta \in \epsilon_0 \cdot \omega$, if $\mathcal{T}_i^\alpha \models$ occurs in $\phi$ inside the scope of $\mathcal{T}_j^\beta \models$, then $\beta > \alpha$. 

**Proof.** Straightforward. 

---

**Definition 31.** Suppose $X \subseteq \epsilon_0 \cdot \omega$ and $h : X \rightarrow \epsilon_0 \cdot \omega$ is order preserving. For each $\mathcal{L}_{PA}(\mathcal{I})$-formula $\phi$, define an $\mathcal{L}_{PA}(\mathcal{I})$-formula $h(\phi)$ inductively as follows:

1. If $\phi$ is $\neg \phi_0$, $\phi_1 \rightarrow \phi_2$, or $\forall x \phi_0$, then $h(\phi)$ is $\neg h(\phi_0)$, $h(\phi_1) \rightarrow h(\phi_2)$, or $\forall x h(\phi_0)$, respectively.

2. If $\phi$ is atomic or $\mathcal{T}_i \models \phi_0$, then $h(\phi) \equiv \phi$.

3. If $\phi$ is $\mathcal{T}_i^\alpha \models \phi_0$ where $\alpha \in X$, then $h(\phi) \equiv \mathcal{T}_i^{h(\alpha)} \models h(\phi_0)$.

4. If $\phi$ is $\mathcal{T}_i^\alpha \models \phi_0$ where $\alpha \notin X$, then $h(\phi) \equiv \mathcal{T}_i^\alpha \models h(\phi_0)$.
Definition 32. Suppose $X \subseteq \varepsilon_0 \cdot \omega$ and $h : X \rightarrow \varepsilon_0 \cdot \omega$ is order preserving. For any $\mathcal{L}_{PA}(\mathcal{I})$-structure $\mathcal{N}$, we define an $\mathcal{L}_{PA}(\mathcal{I})$-structure $h(\mathcal{N})$ that has the same universe as $\mathcal{N}$, agrees with $\mathcal{N}$ on $\mathcal{L}_{PA}(\omega)$, and interprets $\mathcal{L}_{PA}(\mathcal{I}) \setminus \mathcal{L}_{PA}(\omega)$ so that

$$h(\mathcal{N}) \models T^\mathcal{N}_i \models \phi[s] \text{ if and only if } \mathcal{N} \models h(T^\mathcal{N}_i \models \phi)[s].$$

Lemma 33. Suppose $X \subseteq \varepsilon_0 \cdot \omega$, $h : X \rightarrow \varepsilon_0 \cdot \omega$ is order preserving, and $\mathcal{N}$ is an $\mathcal{L}_{PA}(\mathcal{I})$-structure. For any $\mathcal{L}_{PA}(\mathcal{I})$-formula $\phi$ and assignment $s$, $h(\mathcal{N}) \models \phi[s]$ if and only if $\mathcal{N} \models h(\phi)[s]$.

Proof. By induction.

Corollary 34. Suppose $X \subseteq \varepsilon_0 \cdot \omega$ and $h : X \rightarrow \varepsilon_0 \cdot \omega$ is order preserving. For any valid $\mathcal{L}_{PA}(\mathcal{I})$-formula $\phi$, $h(\phi)$ is valid.

Proof. For any $\mathcal{L}_{PA}(\mathcal{I})$-structure $\mathcal{N}$ and assignment $s$, $h(\mathcal{N}) \models \phi[s]$ by validity, so $\mathcal{N} \models h(\phi)[s]$ by Lemma 33.

Definition 35. If $X \subseteq \text{Ord}$ and $h : X \rightarrow \text{Ord}$, we call $h$ a covering if $h$ is order preserving and whenever $x, y \in X$ and $x \leq y$, $h(x) \leq h(y)$.

Definition 36. Suppose $i \in \omega$. An $\mathcal{L}_{PA}(\mathcal{I})$-theory $T$ is $i$-unistratified if the following conditions hold:

1. $T$ is $i$-stratified.
2. (Uniformity) Whenever $\phi \in T$, $X \subseteq \varepsilon_0 \cdot \omega$, $\text{On}(\phi) \subseteq X$, and $h : X \rightarrow \varepsilon_0 \cdot \omega$ is a covering, then $h(\phi) \in T$.

Definition 37. If $T$ is an $\mathcal{L}_{PA}(\mathcal{I})$-theory and $\alpha \in \varepsilon_0 \cdot \omega$, let $T \cap \alpha$ be the set $\{ \phi \in T : \text{On}(\phi) \subseteq \alpha \}$ of sentences in $T$ that do not contain any superscripts $\geq \alpha$.

Theorem 38. (The Collapse Theorem) Suppose $T$ is an $i$-unistratified $\mathcal{L}_{PA}(\mathcal{I})$-theory.

1. If $n$ is a positive integer and \text{On}(\phi) \subseteq \varepsilon_0 \cdot n$, then $T \models \phi$ if and only if $T \cap (\varepsilon_0 \cdot n) \models \phi$.
2. If $\alpha \leq_1 \beta$ and \text{On}(\phi) \subseteq \alpha$, then $T \cap \alpha \models \phi$ if and only if $T \cap \beta \models \phi$.

Proof. Note that since $T$ is $i$-unistratified, in particular $T$ is $i$-stratified. We will prove (1), the proof of (2) is similar.

($\Leftarrow$) Immediate since $T \cap (\varepsilon_0 \cdot n) \subseteq T$.

($\Rightarrow$) Assume $T \models \phi$. By Theorem 2, there are $\sigma_1, \ldots, \sigma_k \in T$ such that

$$\Phi \equiv \sigma_1 \rightarrow \cdots \rightarrow \sigma_k \rightarrow \phi$$

is valid. Let $X = \text{On}(\Phi) \cap (\varepsilon_0 \cdot n)$, $Y = \text{On}(\Phi) \cap [\varepsilon_0 \cdot n, \infty)$, note $\left| X \right|, \left| Y \right| < \infty$.

Since $Y$ is finite, there is some integer $n' > n$ such that $Y \subseteq \varepsilon_0 \cdot n'$. By Theorem 26, $\varepsilon_0 \cdot n \leq \varepsilon_0 \cdot n'$.

By Theorem 26, there is some $X < Y < \varepsilon_0 \cdot n$ such that $X \cup \tilde{Y} \cong (\leq, \leq_1) X \cup \tilde{Y}$.

Let $h : X \cup \tilde{Y} \rightarrow X \cup \tilde{Y}$ be a $(\leq, \leq_1)$-isomorphism. Since $\text{On}(\phi) \subseteq \varepsilon_0 \cdot n$, $h(\phi) = \phi$. By Corollary 34,

$$h(\Phi) \equiv h(\sigma_1) \rightarrow \cdots \rightarrow h(\sigma_k) \rightarrow h(\phi)$$

is valid. Since $T$ is $i$-unistratified, $h(\sigma_1), \ldots, h(\sigma_k) \in T$. Finally since range($h) < \varepsilon_0 \cdot n$, $h(\sigma_1), \ldots, h(\sigma_k) \in T \cap (\varepsilon_0 \cdot n)$, showing $T \cap (\varepsilon_0 \cdot n) \models \phi$.

Loosely speaking, what we have done in Theorem 38 is we have taken a proof of $\phi$ and we have collapsed the proof, shrinking its ordinals by using Theorem 26 part 3.

Definition 39. For every $i \in \omega$ we define the following $\mathcal{L}_{PA}(\mathcal{I})$-schema:

- ($i$-Collapse) ucl ($T^\mathcal{N}_i \models \phi \leftrightarrow T^\mathcal{N}_i \models \phi$) whenever $T^\mathcal{N}_i \models \phi$ is $i$-stratified and $\alpha \leq_1 \beta$.

Definition 40. For any $\mathcal{L}_{PA}(\mathcal{I})$-formula $\phi$, $\phi^-$ is the result of erasing all superscripts from $\phi$. If $T$ is an $\mathcal{L}_{PA}(\mathcal{I})$-theory, $T^- = \{ \sigma^- : \sigma \in T \}$. 

10
For example, if \( \phi \) is \( T_S^{x=0} \rightarrow T_S^{x+1=0} \rightarrow T_S^{y=0} \), then \( \phi^- \) is \( T_S^{x=0} \rightarrow T_S^{y=0} \).

**Lemma 41.** If \( T \) is \( i \)-unistratified then for every \( \phi \in T \) there is some \( \psi \in T \) such that \( \psi \) is very \( i \)-stratified and \( \psi^- \equiv \phi^- \).

**Proof.** Let \( X = \Omega(\phi) = \{ \alpha_1 < \cdots < \alpha_n \} \), \( Y = \{ \epsilon_0 \cdot 1, \ldots, \epsilon_0 \cdot n \} \), and define \( h : X \rightarrow Y \) by \( h(\alpha_j) = \epsilon_0 \cdot j \).

Clearly \( h \) is order preserving; by Theorem 26 part 2, \( h \) is a covering. Since \( T \) is \( i \)-unistratified, \( T \) contains \( \psi \equiv h(\phi) \). Clearly \( \psi \) is very \( i \)-stratified and \( \psi^- \equiv \phi^- \). \( \square \)

**Definition 42.** For any \( L_{PA}(\omega) \)-structure \( \mathcal{N} \), we define an \( L_{PA}(\mathcal{I}) \)-structure \( \mathcal{N}^- \) that has the same universe as \( \mathcal{N} \), agrees with \( \mathcal{N} \) on \( L_{PA}(\omega) \), and interprets \( L_{PA}(\mathcal{I}) \setminus L_{PA}(\omega) \) as follows. For any \( L_{PA}(\mathcal{I}) \)-formula \( \phi \), \( \alpha \in \epsilon_0 \cdot \omega \), \( i \in \mathbb{N} \), and assignment \( s \),

\[ \mathcal{N}^- \models T_i^{\phi}[s] \text{ if and only if } \mathcal{N} \models (T_i^{\phi})^-[s] \]

**Lemma 43.** Suppose \( \mathcal{N} \) is an \( L_{PA}(\omega) \)-structure. For every \( L_{PA}(\mathcal{I}) \)-formula \( \phi \) and assignment \( s \), \( \mathcal{N}^- \models \phi[s] \) if and only if \( \mathcal{N} \models \phi^-[s] \).

**Proof.** By induction. \( \square \)

**Corollary 44.** If \( \phi \) is a valid \( L_{PA}(\mathcal{I}) \)-formula, then \( \phi^- \) is a valid \( L_{PA}(\omega) \)-formula.

**Proof.** Similar to the proof of Corollary 34.

A converse-like statement holds for Corollary 43 as well.

**Lemma 45.** For any valid \( L_{PA}(\omega) \)-sentence \( \phi \) and \( i \in \omega \), there is a valid very \( i \)-stratified \( L_{PA}(\mathcal{I}) \)-sentence \( \psi \) such that \( \psi^- \equiv \phi \).

**Proof.** Let \( \psi \mapsto \psi^+ \) be the function taking \( L_{PA}(\omega) \)-formulas to \( L_{PA}(\mathcal{I}) \)-formulas defined as follows.

1. If \( \psi \) is atomic, or of the form \( T_j^{\psi_0} \) with \( j \neq i \), then \( \psi^+ \equiv \psi \).
2. If \( \psi = T_i^{\psi_0} \), then \( \psi^+ \equiv T_i^{\psi^+_n} \equiv \psi^+ \), where \( n = \text{min} \{ m \in \mathbb{N} : \epsilon_0 \cdot m > \Omega(\psi^+_0) \} \).
3. If \( \psi = \neg \psi_0, \psi_1 \), or \( \forall x \psi_0 \), then \( \psi^+ = \neg \psi^+_0, \psi^+_1 \rightarrow \psi^+_2 \), or \( \forall x \psi^+_2 \), respectively.

It is straightforward to show \( \phi^+ \) is very \( i \)-stratified. We claim \( \phi^+ \) is valid. Let \( \mathcal{M} \) be any \( L_{PA}(\mathcal{I}) \)-structure, we will show \( \mathcal{M} \models \phi^+ \). Let \( \mathcal{M}^+ \) be the \( L_{PA}(\omega) \)-structure with the same universe as \( \mathcal{M} \), which agrees with \( \mathcal{M} \) on the interpretation of arithmetic and of \( T_j^{\phi} \) for \( j \neq i \), and which interprets \( T_i^{\phi} \) as follows:

\[ \mathcal{M}^+ \models T_i^{\psi[s]} \text{ if and only if } \mathcal{M} \models (T_i^{\psi})^+[s] \]

Since \( \phi \) is valid, \( \mathcal{M}^+ \models \phi \). It follows that \( \mathcal{M} \models \phi^+ \). \( \square \)

**Definition 46.** Let \( i \in \omega \). We define the following \( L_{PA}(\mathcal{I}) \)-schemas.

- \( (i \text{-Strativality}) \) \( ucl(T_i^{\phi}) \) whenever \( \phi \) is a valid \( L_{PA}(\mathcal{I}) \)-formula and \( T_i^{\phi} \) is \( i \)-stratified.
- \( (i \text{-Stratiduction}) \) \( ucl(T_i^{\phi \rightarrow \psi}) \rightarrow T_i^{\phi \rightarrow \psi} \rightarrow T_i^{\phi \rightarrow \psi} \) whenever this formula is \( i \)-stratified.

**Definition 47.** An \( L_{PA}(\mathcal{I}) \)-theory \( T \) is \( i \)-straticolosed if the following conditions hold:

1. \( T \) is \( i \)-unistratified.
2. \( T \) includes \( i \)-Strativality, \( i \)-Stratiduction and \( i \)-Collapse.
3. For every \( \phi \in T \), if \( T_i^{\phi} \) is \( i \)-stratified then \( T_i^{\phi} \in T \).

A family \( T = (T_i)_{i \in \omega} \) is straticolosed if each \( T_i \) is \( i \)-straticolosed.

The following theorem serves as an omnibus of results from Section 5 of [5].
Theorem 48. (Proof Stratification) Suppose $T$ is an $i$-stratclosed $\mathcal{L}_{PA}(I)$-theory. Then:

1. Whenever $T \cap \alpha \models \phi$, $T_i^\alpha \models \phi$ is an $i$-stratified sentence, and $\beta > \alpha$, then $T \cap \beta \models T_i^\beta \models \phi$.

2. For any very $i$-stratified $\mathcal{L}_{PA}(I)$-sentences $\rho$ and $\sigma$, if $\rho^- \equiv \sigma^-$ then $T \models \rho \leftrightarrow \sigma$.

3. For any very $i$-stratified $\mathcal{L}_{PA}(I)$-sentence $\phi$, $T \models \phi$ if and only if $T^- \models \phi^-$.

Proof. Note that since $T$ is $i$-stratclosed, in particular $T$ is $i$-unistratified and hence, $i$-stratified.

Claim 0: Any time $T \models T_i^\alpha \models (\rho \leftrightarrow \sigma)$ and this is $i$-stratified, $T \models T_i^\alpha \models \rho \leftrightarrow T_i^\alpha \models \sigma$.

Assume the hypotheses. By $i$-Strativalidity, $T \models T_i^\alpha \models ((\rho \leftrightarrow \sigma) \rightarrow (\rho \rightarrow \sigma))$. By $i$-Straticeduction,

$$T \models T_i^\alpha \models ((\rho \leftrightarrow \sigma) \rightarrow (\rho \rightarrow \sigma)) \rightarrow T_i^\alpha \models (\rho \leftrightarrow \sigma) \rightarrow T_i^\alpha \models (\rho \rightarrow \sigma)$$

and

$$T \models T_i^\alpha \models (\rho \rightarrow \sigma) \rightarrow T_i^\alpha \models \rho \rightarrow T_i^\alpha \models \sigma.$$

It follows that $T \models T_i^\alpha \models \rho \rightarrow T_i^\alpha \models \sigma$. The reverse implication is similar.

Claim 1: If $T \cap \alpha \models \phi$, $T_i^\alpha \models \phi$ is an $i$-stratified sentence, and $\beta > \alpha$, then $T \cap \beta \models T_i^\beta \models \phi$.

Given $T \cap \alpha \models \phi$, there are $\sigma_1, \ldots, \sigma_n \in T \cap \alpha$ such that $\sigma_1 \rightarrow \cdots \rightarrow \sigma_n \rightarrow \phi$ is valid. By instances of $i$-Strativalidity and $i$-Straticeduction contained in $T \cap \beta$, $T \cap \beta \models T_i^\beta \models \phi$.

Claim 2: If $\rho$ and $\sigma$ are very $i$-stratified $\mathcal{L}_{PA}(I)$-sentences and $\rho^- \equiv \sigma^-$, then $T \models \rho \leftrightarrow \sigma$.

By induction on $\rho$. Note that $\rho$ is not of the form $T_i^\alpha \models \rho_0$ (with $j \neq i$), as that is not $i$-stratified. If $\rho$ is $T_j^\alpha \models \rho_0$ then $\rho \equiv \rho^- \equiv \sigma^- \equiv \sigma$ and the claim is immediate.

The only nontrivial remaining case is when $\rho$ is $T_i^\alpha \models \rho_0$. Since $\rho$ is very $i$-stratified, this implies $\alpha = \epsilon_0 \cdot n$ (some positive integer $n$) and $\rho_0$ is very $i$-stratified. Since $\sigma^- \equiv \rho^- \equiv \sigma$ is very stratified, this implies $\sigma \equiv T_i^\alpha \models \sigma_0$ for some positive integer $m$ and very $i$-stratified $\sigma_0$ with $\sigma_0^- \equiv \rho_0^-$. Assume $m \leq n$, the other case is similar.

By induction, $T \models \rho_0 \leftrightarrow \sigma_0$. By compactness, there is a natural $\ell \geq n$ such that $T \cap (\epsilon_0 \cdot \ell) \models \rho_0 \leftrightarrow \sigma_0$.

By Claim 1, $T \models T_i^\alpha \models (\rho_0 \leftrightarrow \sigma_0)$; Claim 0 then gives $T \models T_i^\alpha \models \rho_0 \leftrightarrow T_i^\alpha \models \sigma_0$. The claim now follows since $T$ contains $i$-Collapse and $\epsilon_0 \cdot m \leq 1 \leq \epsilon_0 \cdot n \leq 1 \epsilon_0 \cdot \ell$ (Theorem 26 part 2).

Claim 3: If $\phi$ is an $i$-stratified $\mathcal{L}_{PA}(I)$-sentence and $T \models \phi$, then $T^- \models \phi^-$. By compactness, find $\sigma_1, \ldots, \sigma_n \in T$ such that $\sigma_1 \rightarrow \cdots \rightarrow \sigma_n \rightarrow \phi$ is valid. By Corollary 44 so is $\sigma_1^- \rightarrow \cdots \rightarrow \sigma_n^- \rightarrow \phi^-$, witnessing $T^- \models \phi^-$. By compactness, there is a valid sentence $\Phi \equiv \sigma_1^- \rightarrow \cdots \rightarrow \sigma_n^- \rightarrow \phi^-$ where each $\sigma_j \in T$. By Lemma 45 there is a valid very $i$-stratified $\mathcal{L}_{PA}(I)$-sentence $\Psi$ such that $\Psi^- \equiv \Phi$.

And because $\Psi^- \equiv \Phi$, this implies $\Psi \equiv \sigma_1^* \rightarrow \cdots \rightarrow \sigma_n^* \rightarrow \phi^*$ where each $(\sigma_j^*)^- \equiv \sigma_j^-$, $(\phi^*)^- \equiv \phi^-$, and $\sigma_1^*, \ldots, \sigma_n^*, \phi^*$ are very $i$-stratified.

By Lemma 41, there are very $i$-stratified $\sigma_1^{**}, \ldots, \sigma_n^{**} \in T$ with each $(\sigma_j^{**})^- \equiv \sigma_j^- \equiv (\sigma_j^*)^-$. By Claim 2, $T \models \phi^* \leftrightarrow \phi$, and for $j = 1, \ldots, n$, $T \models \sigma_j^{**} \leftrightarrow \sigma_j^*$. Thus

$$T \models (\sigma_1^{**} \rightarrow \cdots \rightarrow \sigma_n^{**} \rightarrow \phi) \leftrightarrow \Psi,$$

and since $\Psi$ is valid and the $\sigma_j^{**} \in T$, this shows $T \models \phi$.

Definition 49. If $T = \langle T_i \rangle_{i \in \omega}$ is a stratclosed family of $\mathcal{L}_{PA}(I)$-theories, its stratification, written $\text{Str}(T)$, is the family $\text{Str}(T) = \langle S_i \rangle_{i \in \omega}$, where for every $i \in \omega$, $S_i = T_i^-$ and $\forall \alpha \in \epsilon_0 \cdot \omega, S_{i(\alpha)} = T_i \cap \alpha$.

Theorem 50. (The Stratification Theorem) Suppose $T = \langle T_i \rangle_{i \in \omega}$ is a stratclosed family of $\mathcal{L}_{PA}(I)$-theories. For any $i \in \omega$, any very $i$-stratified $\mathcal{L}_{PA}(I)$-formula $\phi$, and any assignment $s$, $\mathcal{M}_{\text{Str}(T)} \models \phi[s]$ if and only if $\mathcal{M}_{\text{Str}(T)} \models \phi^- [s]$. 

\[ \square \]
Proof. By induction on $\phi$. The only nontrivial case is when $\phi$ is $T^i_k \models \psi$. Since $\phi$ is very $i$-stratified, $\psi$ is very $i$-stratified and we may write $\alpha = \epsilon_0 \cdot n$ for some positive integer $n$, $\text{On}(\psi) \subseteq \epsilon_0 \cdot n$. The following are equivalent.

$$\mathcal{M}_{\text{Str}(T)} \models T^\alpha_0 \models \psi[s]$$

(Definition of $\mathcal{M}_{\text{Str}(T)}$)

$T_i \cap (\epsilon_0 \cdot n) \models \psi^*$

(Theorem 35)

$T_i \models \psi^*$

(Theorem 48)

$T_i^- \models (\psi^*)^-$

(Clearly $(\psi^*)^- \equiv (\psi^-)^*$)

$T_i^- \models (\psi^-)^s$

(Definition of $\mathcal{M}_{\text{Str}(T)}$)

\[
\square
\]

6 Stratifiers

In order to apply theorems from the previous section, it is necessary to work with families $T = (T_i)_{i \in \omega}$ where each $T_i$ is $i$-stratified. If we want $T_i^-$ to (locally) express the truthfulness of $T_i^-$, we cannot simply add a schema like $ucl(T_i \models \phi \rightarrow \phi)$ to $T_i$, because this is not necessarily $i$-stratified: for example, the particular instance $T_j \models T_i \models (1 = 0) \rightarrow T_i \models (1 = 0)$ is not $i$-stratified. But neither is, say, $T_j \models T_i \models (1 = 0) \rightarrow T_i \models (1 = 0)$, where $T_i \models$ occurs within the scope of $T_j$. We will use a schema $ucl(T_i \models \phi \rightarrow \phi^+)$, where $\phi^+$ varies over what we call $i$-stratifiers.

**Definition 51.** Suppose $X \subseteq \epsilon_0 \cdot \omega$, $|X| = \infty$, and $i \in \omega$. The $i$-stratifier given by $X$ is the function $\phi \mapsto \phi^+$ taking $\mathcal{L}_{PA}(\omega)$-formulas to $\mathcal{L}_{PA}(\mathcal{I})$-formulas as follows.

1. If $\phi$ is atomic or of the form $T_j \models \phi_0$ with $j \neq i$, then $\phi^+ \equiv \phi$.
2. If $\phi$ is $T_i \models \phi_0$ then $\phi^+ \equiv T_i^\alpha \models \phi_0^+$ where $\alpha = \min \{x \in X : x > \text{On}(\phi_0^+)\}$.
3. If $\phi = \neg \psi$, $\psi \rightarrow \rho$, or $\forall x \psi$, then $\phi^+ = \neg \psi^+$, $\psi^+ \rightarrow \rho^+$ or $\forall x \psi^+$, respectively.

By an $i$-stratifier we mean an $i$-stratifier given by some $X$. By the $i$-veristratifier we mean the $i$-stratifier given by $X = \{\epsilon_0 \cdot 1, \epsilon_0 \cdot 2, \ldots\}$.

For example, if $\bullet^*$ is the $i$-veristratifier and $j \neq i$ then

$$(T_j \models T_i \models (1 = 0) \rightarrow T_i \models T_i \models (1 = 0))^+ \equiv T_j \models T_i \models (1 = 0) \rightarrow T_i^\alpha_0 \models T_i^\alpha_0 (1 = 0) = 1.$$

**Lemma 52.** Suppose $Z \subseteq \epsilon_0 \cdot \omega$, $h : Z \rightarrow \epsilon_0 \cdot \omega$ is order preserving, $i \in \omega$, and $\bullet^*$ is an $i$-stratifier. For any $\mathcal{L}_{PA}(\omega)$-formula $\theta$ with $\text{On}(\theta^+) \subseteq Z$, there is a computable $i$-stratifier $\bullet^*$ with $\theta^+ \equiv h(\theta^+)$.  

**Proof.** Let $X_0 = \{h(\alpha) : \alpha \in \text{On}(\theta^+)\}$, let $X = X_0 \cup \{\alpha \in \epsilon_0 \cdot \omega : \alpha > X_0\}$, and let $\bullet^*$ be the $i$-stratifier given by $X$. By induction, for every subformula $\theta_0$ of $\theta$, $\theta_0^+ \equiv h(\theta_0^+)$. \(\square\)

**Definition 53.** By a stratifier-set, we mean a finite set

$$I = \{\bullet^+, \ldots, \bullet^+\}$$

where each $\bullet^+_p$ is an $i_p$-stratifier for some $i_p \in \omega$, and $i_1, \ldots, i_k$ are distinct. With $I$ as above, we write $\text{Indices}(I)$ for $\{i_1, \ldots, i_k\}$. We say $I$ is computable if each $\bullet^+_p$ is computable.

For example, if $\bullet^{+1}$ is a 1-stratifier, $\bullet^{+2}$ is a 5-stratifier, and $\bullet^{+3}$ is a 2-stratifier, then $I = \{\bullet^{+1}, \bullet^{+2}, \bullet^{+3}\}$ is a stratifier-set and $\text{Indices}(I) = \{1, 5, 2\}$. For a non-example, if $\bullet^{+1}$ and $\bullet^{+2}$ are distinct 1-stratifiers, then $\{\bullet^{+1}, \bullet^{+2}\}$ is not a stratifier-set, because it fails the distinctness condition.
**Definition 54.** 1. Suppose $\mathcal{N}$ is an $\mathcal{L}_{PA}(\mathcal{I})$-structure and $I$ is a stratifier-set. We define an $\mathcal{L}_{PA}(\mathcal{I})$-structure $\mathcal{N}'$ as follows. The universe and interpretation of arithmetic of $\mathcal{N}'$ agree with those of $\mathcal{N}$, as do the interpretations of $T_i^\neq (i \notin \text{Indices}(I))$ and $T_i^\alpha$ (any $\alpha$, $i$). For each $i \in \text{Indices}(I)$, let $i^+ \in I$ be the corresponding $i$-stratifier, and let $\mathcal{N}'$ interpret $T_i^\neq$ as follows. For any $\mathcal{L}_{PA}(\mathcal{I})$-formula $\phi$ and assignment $s$, we consider two cases.

(a) If $\phi$ is an $\mathcal{L}_{PA}(\omega)$-formula, then $\mathcal{N}' \models T_i^\neq \phi[s]$ if and only if $\mathcal{N} \models T_i^\neq \phi^+[s]$.

(b) If $\phi$ is not an $\mathcal{L}_{PA}(\omega)$-formula, then $\mathcal{N}' \models T_i^\neq \phi[s]$ if and only if $\mathcal{N} \models T_i^\neq \phi[s]$.

2. For any $i \in \omega$, any $i$-stratifier $i^+$, and any $\mathcal{L}_{PA}(\mathcal{I})$-structure $\mathcal{N}$, let $\mathcal{N}^+ = \mathcal{N}' I$ where $I = \{i^+\}$ is the stratifier-set containing only $i^+$.

Case 1b in Definition 54 is somewhat arbitrary. We will only ever really care about whether $\mathcal{N}' \models T_i^\neq \phi[s]$ when $T_i^\neq \phi$ is $j$-stratified for some $j$. If $\phi$ is not an $\mathcal{L}_{PA}(\omega)$-formula then $T_i^\neq \phi$ is not $j$-stratified for any $j$.

**Lemma 55.** (Compare Lemma 43) Suppose $\mathcal{N}$ is an $\mathcal{L}_{PA}(\mathcal{I})$-structure, $i \in \omega$, and $i^+$ is an $i$-stratifier. For every $\mathcal{L}_{PA}(\omega)$-formula $\phi$ and assignment $s$, $\mathcal{N}^+ \models \phi[s]$ if and only if $\mathcal{N} \models \phi^+[s]$.

**Proof.** By induction.

**Lemma 56.** For any $\mathcal{L}_{PA}(\omega)$-formula $\phi$, any $i \in \omega$, and any $i$-stratifier $i^+$, $\phi$ is valid if and only if $\phi^+$ is valid.

**Proof.**

($\Rightarrow$) Assume $\phi$ is valid. For any $\mathcal{L}_{PA}(\mathcal{I})$-structure $\mathcal{N}$ and assignment $s$, $\mathcal{N}^+ \models \phi[s]$ by Lemma 55.

($\Leftarrow$) By Corollary 44.

**Lemma 57.** Suppose $\mathcal{M}$ is an $\mathcal{L}_{PA}(\mathcal{I})$-structure, $I_0$ is a stratifier-set, $i \in \omega$, $i \notin \text{Indices}(I_0)$, and $i^+$ is an $i$-stratifier. Let $I = I_0 \cup \{i^+\}$. Then $\mathcal{M}' = (\mathcal{M}^{I_0})^+$. Furthermore, $\mathcal{M}^+$ and $\mathcal{M}'$ agree on the interpretation of $T_i^\neq$.

**Proof.** Straightforward.

**Lemma 58.** Suppose $i \in \omega$ and suppose $\mathcal{M}$ is an $\mathcal{L}_{PA}(\mathcal{I})$-structure with the property that for every very $i$-stratified $\mathcal{L}_{PA}(\mathcal{I})$-formula $\phi$ and assignment $s$, $\mathcal{M} \models \phi[s]$ if and only if $\mathcal{M} \models \phi^-[s]$. Suppose $I$ is a stratifier-set such that $i \notin \text{Indices}(I)$. Then for every very $i$-stratified $\mathcal{L}_{PA}(\mathcal{I})$-formula $\phi$ and assignment $s$, $\mathcal{M}^+ \models \phi[s]$ if and only if $\mathcal{M}' \models \phi^-[s]$.

**Proof.** By induction on $\phi$. Let $s$ be an assignment. The only interesting cases are the following.

**Case 1:** $\phi$ is $T_i^\psi$ for some $j$. Then $\phi^+ \equiv \phi$ and the claim is trivial.

**Case 2:** $\phi$ has the form $T_j^\alpha$ for some $j \neq i$. Impossible, this is not $i$-stratified.

**Case 3:** $\phi$ has the form $T_i^\alpha$. The following are equivalent:

$\mathcal{M}' \models T_i^\alpha \psi[s]$  
$\mathcal{M} \models T_i^\alpha \psi[s]$  
$\mathcal{M} \models (T_i^\alpha \psi)^- [s]$  
$\mathcal{M}' \models (T_i^\alpha \psi)^- [s]$.

(Since $i \notin \text{Indices}(I)$, $\mathcal{M}$ and $\mathcal{M}'$ agree on $T_i^\neq$)

**Lemma 59.** Suppose $\mathcal{L}_{PA}(\mathcal{I})$-structure $\mathcal{M}$ is an instance of Definition 7 and suppose $I$ is a stratifier-set. Then $\mathcal{M}'$ interprets formulas by substitution.
Proof. By induction on \(|I|\). If \(|I| = 0\), we are done by Lemma 8. Otherwise, we may decompose \(I\) as \(I = I_0 \cup \{\ast\}\) where \(\ast\) is an \(i\)-stratifier. By induction, \(\mathcal{M}^I\) interprets formulas by substitution \((\ast)\). By Lemma 57, \(\mathcal{M}^I = (\mathcal{M}^I)^+\).

By definition of interpreting formulas by substitution, for every \(\mathcal{L}_{PA}(I)\)-formula \(\phi\) and assignment \(s\), \(\mathcal{M}^I \models \phi[s]\) if and only if \(\mathcal{M}^I \models \phi^s\). We must show that for every such \(\phi\) and \(s\), \((\mathcal{M}^I)^+ \models \phi[s]\) if and only if \((\mathcal{M}^I)^+ \models \phi^s\).

We induct on \(\phi\). By Definition 54, \((\mathcal{M}^I)^+\) and \((\mathcal{M}^I)^+\) agree on all symbols except \(T, \vdash\), and they agree on \(T, \models \phi_0\) if \(\phi_0\) is not an \(\mathcal{L}_{PA}(\omega)\)-formula. Thus the only nontrivial case is when \(\phi\) is of the form \(T, \models \phi_0\) for some \(\mathcal{L}_{PA}(\omega)\)-formula \(\phi_0\). Any such \(\phi\) is itself an \(\mathcal{L}_{PA}(\omega)\)-formula and thus susceptible to Lemma 55. The following are equivalent.

\[
(\mathcal{M}^I)^+ \models \phi[s] \\
\mathcal{M}^I \models \phi^+ [s] \\
\mathcal{M}^I \models (\phi^+)^s \\
(\mathcal{M}^I)^+ \models (\phi^+)^* \\
(\mathcal{M}^I)^+ \models \phi^s.
\]

\(\square\)

7 Generic Stratified Axioms

We now have enough technical machinery to fulfill the second promise from the Introduction. We will fulfill it in a general way, essentially saying: “The theories in question, whose truth were in doubt, are true together with any background theory of provability such that...” Just like in Section 3, we do this by introducing a notion of genericness. Throughout this section, \(\prec\) is an r.e. well-founded partial-order of \(\omega\).

Definition 60. If \(i \in \omega\), we say that a stratifier-set \(I\) is above \(i\) if \(\forall j \in \text{Indices}(I), i \prec j\). We adopt the following convention: if \(I\) is above \(i\) then we will write \(I\) as \(I(i)\) in order to remind ourselves that \(I\) is above \(i\).

Definition 61. (Compare Definition 14) Suppose \(T = (T_i)_{i \in \omega}\) is an r.e. family of \(\mathcal{L}_{PA}(I)\)-theories and each \(T_i\) is \(i\)-unistratified. We say \(T\) is \(\prec\)-stratified-r.e.-generic (or stratified-r.e.-generic, if \(\prec\) is clear from context) if for every stratified r.e. family \(U \supseteq T\), every \(i \in \omega\), and every computable stratifier-set \(I(i)\) above \(i\), \(\mathcal{M}^I(i) \models T_i\).

Lemma 62. If the family \(T = (T_i)_{i \in \omega}\) of \(\mathcal{L}_{PA}(I)\)-sets is r.e. and is a union of stratified-r.e.-generic families, then \(T\) is stratified-r.e.-generic.

Proof. Straightforward.

7.1 Stratified-r.e.-generic Building Blocks

As in Section 3.1, we exhibit some examples of stratified-r.e.-generic families, which can be combined (via Lemma 52) to form background theories of provability. This will allow us to state Theorem 72 below in a generalized way, essentially saying that certain doubted theories are consistent with any background theory of provability built up from such blocks. This saves us from having to arbitrarily impose any particular background theory of provability.

In the following lemma, for part 3, the intuition is that for the purpose of stratified-r.e.-genericness, what things \(T_i\) says about \(T_j\) need not merely be true, but must even remain true when a \(j\)-stratifier is applied to them. \(T_j \models (\phi \rightarrow \psi) \rightarrow T_j \models (\phi \rightarrow T_i \models \psi)\) lacks this property, because it could be that \((T_j \models \phi)^+ \equiv T^j_{\beta} \models \phi^+,\) \((T_j \models \psi)^+ \equiv T^j_{\beta} \models \psi^+\), where \(\beta < \alpha\). For parts 1–2, the reason we cannot merge these parts into \([i\text{-}\text{Deduction}]\) is because \([i\text{-}\text{Deduction}]\) is not \(i\)-stratified.

Lemma 63. (Compare Lemma 17) For any \(i, j \in \omega\), each of the following families is stratified-r.e.-generic.
1. [i-Stratified deduction].
2. [j-Deduction], if \( j < i \).
3. \([S]_i\) if \( i < j \) where \( S \) is the following schema (\( \phi, \psi \) range over \( \mathcal{L}_{PA}(\omega) \)-formulas):

\[
\text{(Modified j-Deduction)} \quad \text{ucl}(T_j \models (\phi \rightarrow \psi) \rightarrow T_j \models \phi \rightarrow T_j \models (\psi \land \phi)).
\]

**Proof.** Clearly these families are unstratified. Recursive enumerability follows from the fact that \( \prec \) is r.e. In each case below, let \( U = (U_k)_{k \in \omega} \) be a straticlosed r.e. family extending the family in question. For brevity, let \( \mathcal{M} = \mathcal{M}_{\text{Str}(U)} \).

(1) Let \( I(i) \) be any computable stratifier-set above \( i \), we must show \( \mathcal{M}^{I(i)} \models \text{ucl}(T^\omega_i \models (\phi \rightarrow \psi) \rightarrow T^\omega_i \models \phi \rightarrow T^\omega_i \models (\psi \land \phi)) \) assuming this formula is \( i \)-stratified. Let \( s \) be an assignment and assume \( \mathcal{M}^{I(i)} \models T^\omega_i \models (\phi \rightarrow \psi)[s] \) and \( \mathcal{M}^{I(i)} \models T^\omega_i \models \phi[s] \). By Definition \[5\], \( \mathcal{M}^{I(i)} \) and \( \mathcal{M} \) agree on \( T^\omega_i \models \), so \( \mathcal{M} \models T^\omega_i \models (\phi \rightarrow \psi)[s] \). By definition of \( \mathcal{M} = \mathcal{M}_{\text{Str}(U)} \), this means \( U_i \cap \alpha \models (\phi \rightarrow \psi)^s \). Clearly \( (\phi \rightarrow \psi)^s \equiv \phi^s \rightarrow \psi^s \), so \( U_i \cap \alpha \models \phi^s \rightarrow \psi^s \). By similar reasoning, \( U_i \cap \alpha \models \phi^s \). By modus ponens, \( U_i \cap \alpha \models \psi^s \), which means \( \mathcal{M} \models T^\omega_i \models \psi[s] \). Since \( \mathcal{M} \) and \( \mathcal{M}^{I(i)} \) agree on \( T^\omega_i \models \), \( \mathcal{M}^{I(i)} \models T^\omega_i \models \psi[s] \), as desired.

(2) Let \( I(i) \) be any computable stratifier-set above \( i \), we must show \( \mathcal{M}^{I(i)} \models \text{ucl}(T^\omega_j \models (\phi \rightarrow \psi) \rightarrow T^\omega_j \models \phi \rightarrow T^\omega_j \models (\psi \land \phi)) \).

Let \( s \) be an assignment and assume \( \mathcal{M}^{I(i)} \models T^\omega_j \models (\phi \rightarrow \psi)[s] \) and \( \mathcal{M}^{I(i)} \models T^\omega_j \models \phi[s] \). Since \( I(i) \) is above \( i \) and \( j < i \), \( \mathcal{M}^{I(i)} \) and \( \mathcal{M} \) agree on \( T^\omega_j \models \), so \( \mathcal{M} \models T^\omega_j \models (\phi \rightarrow \psi)[s] \) and \( \mathcal{M} \models T^\omega_j \models \phi[s] \). By definition of \( \mathcal{M} \), \( U^-_j \models \phi^s \rightarrow \psi^s \) and \( U^-_j \models \phi^s \), thus \( U^-_j \models \psi^s \), so \( \mathcal{M} \models T^\omega_j \models \psi[s] \) and thus so does \( \mathcal{M}^{I(i)} \).

(3) Let \( I(i) \) be any computable stratifier-set above \( i \), we must show \( \mathcal{M}^{I(i)} \models \text{ucl}(T^\omega_j \models (\phi \rightarrow \psi) \rightarrow T^\omega_j \models \phi \rightarrow T^\omega_j \models (\psi \land \phi)) \).

Let \( s \) be an assignment and assume \( \mathcal{M}^{I(i)} \models T^\omega_j \models (\phi \rightarrow \psi)[s] \) and \( \mathcal{M}^{I(i)} \models T^\omega_j \models \phi[s] \). If \( j \notin \text{Indices}(I(i)) \), then \( \mathcal{M}^{I(i)} \) and \( \mathcal{M} \) agree on \( T^\omega_j \models \), so reason as in (2) above. If not, we can write \( I(i) = I_0 \cup \{\bullet^+\} \) where \( \bullet^+ \) is a computable \( j \)-stratifier, and Lemma \[57\] ensures that \( \mathcal{M}^{I(i)} \) and \( \mathcal{M}^+ \) agree on \( T^\omega_j \models \). By definition of \( \mathcal{M}^+ \), \( \mathcal{M} \models (T^\omega_j \models (\phi \rightarrow \psi))^+ \models [s] \) and \( \mathcal{M} \models (T^\omega_j \models \phi)^+ \models [s] \). Let \( \alpha, \beta \in \omega \cdot \omega \) be such that \( (T^\omega_j \models (\phi \rightarrow \psi))^+ \models T^\omega_j \models (\phi^+ \rightarrow \psi^+) \) and \( (T^\omega_j \models \phi)^+ \models T^\omega_j \models (\phi^+)^+ \). Then \( \mathcal{M} \models T^\omega_j \models (\phi^+ \rightarrow \psi^+)[s] \) and \( \mathcal{M} \models T^\omega_j \models (\phi^+)^+ [s] \). This means \( U_j \cap \alpha \models (\phi^+ \rightarrow \psi^+)^s \) and \( U_j \cap \beta \models (\phi^+)^s \). Since \( \phi \) is a subformula of \( \phi \rightarrow \psi \), it follows \( \beta \leq \alpha \), thus \( U_j \cap \alpha \models (\psi^+ \land \phi^+)^s \). So \( \mathcal{M} \models T^\omega_j \models (\psi^+ \land \phi^+)[s] \). By Definition \[51\]

\[
T^\omega_j \models (\psi^+ \land \phi^+) \equiv (T^\omega_j \models (\psi \land \phi))^+
\]

(this is the reason for the \( \psi \land \phi \) clause) and finally \( \mathcal{M}^+ \models T^\omega_j \models (\psi \land \phi)[s] \). \( \square \)

In Lemma \[18\] we introduced Assigned Validity as a single schema for inclusion in \( T_i \) for any \( i \). In the following lemma, we need to break the stratified version of Assigned Validity into different \( \omega \)-indexed families because the stratified version of Assigned Validity intended for inclusion in \( T_i \) (for any particular \( i \)) needs to be \( i \)-stratified.

**Lemma 64.** (Compare Lemma \[18\]) For any \( i, j \in \omega \), each of the following families is straticlosed-r.e.-generic.

1. \([S]_i\) where \( S \) is: (i-Assigned Strativalidity) the schema \( \phi^s \) (\( \phi \) valid and \( i \)-stratified, \( s \) an assignment).
2. \([i-\text{Assigned Strativalidity}]_i \cup [i-\text{Strativalidity}]_i \).
3. \([i-\text{Assigned Strativalidity}]_i \cup [\text{-Strativalidity}]_j \) (if \( j \neq i \)).

**Proof.** For unstratifiedness, use Corollary \[34\]. Recursive enumerability follows from the fact that \( \prec \) is r.e. In each case below, let \( U = (U_k)_{k \in \omega} \) be a straticlosed r.e. family extending the family in question. For brevity, let \( \mathcal{M} = \mathcal{M}_{\text{Str}(U)} \).

(1) Let \( I(i) \) be any computable stratifier-set above \( i \), let \( \phi \) be any valid \( i \)-stratified formula, and let \( s \) be any assignment. Since \( \phi \) is valid, \( \mathcal{M}^{I(i)} \models s \). By Lemma \[59\], \( \mathcal{M}^{I(i)} \models \phi^s \), as desired.
(2) Let $I(i)$ be any computable stratifier-set above $i$. By (1), $\mathcal{M}^{(i)} \models \phi(i)$, where $\phi$ is any valid $\mathcal{L}_{PA}(\{\ldots\})$-formula and $\alpha < \varepsilon_0 \cdot \omega$ is any ordinal such that $T^\alpha_0 \models \phi$ is i-stratified. Let $s$ be any assignment. Since $U_i$ contains i-Assignment Strativality, in particular $U_i$ contains $\phi^\alpha$. Since $T^\alpha_0 \models \phi$ is i-stratified, $\phi^\alpha$ exceeds all the superscripts in $\phi$ (hence in $\phi^\alpha$), so $U_i \cap \alpha \models \phi^\alpha$. By definition of $\mathcal{M}$, this means $\mathcal{M} \models T^\alpha_0 \models \phi^\alpha[s]$. By Definition 54, $\mathcal{M}$ and $\mathcal{M}^{(i)}$ agree on $T^\alpha_0 \models \phi^\alpha[s]$, as desired.

(3) By (1), $\mathcal{M}^{(i)} \models \phi(i)$-Assigned Strativality for every computable stratifier-set $I(i)$ above $i$. Let $J(j)$ be a computable stratifier-set above $j$, we must show $\mathcal{M}^{(j)} \models \phi$. Let $\phi$ be a valid $\mathcal{L}_{PA}(\omega)$-formula, $s$ an assignment.

Case 1: $i \not\in \text{Indices}(J(j))$. Then $\mathcal{M}^{(j)}$ and $\mathcal{M}$ agree on $T^\alpha_0 \models \phi$. Let $\phi^+$ be an i-stratifier. Since $\phi$ is valid, so is $\phi^+$ (by Lemma 56), so $(\phi^+)^* \in U_i$ (since $[\phi(i)$-Assigned Strativality], is part of line 3). Clearly $((\phi^+)^*)^- \equiv \phi^\omega$, so $\phi^\omega \in U_i^\omega$, thus $\mathcal{M} \models T^\phi_0 \models \phi^\omega[s]$, and so does $\mathcal{M}^{(j)}$.

Case 2: $i \in \text{Indices}(J(j))$. Thus $j < i$ and we can write $J(j) = J_0 \cup \{\phi^+\}$ for some computable i-stratifier $\phi^+$. By Lemma 57, $\mathcal{M}^{(j)}$ and $\mathcal{M}^+$ agree on $T^\alpha_0 \models \phi$. Let $\alpha < \varepsilon_0 \cdot \omega$ be such that $(T^\alpha_0 \models \phi^+) \equiv T^\phi_0 \models \phi^+$. As in Case 1, $(\phi^+)^*$ is an instance of $\phi(i)$-Assigned Strativality, so $(\phi^+)^* \in U_i$ (since $[\phi(i)$-Assigned Strativality] is part of line 3). In fact by choice of $\alpha$, $((\phi^+)^*)^\omega \in U_i \cap \alpha$, so $\mathcal{M} \models T^\phi_0 \models \phi^+$. Thus $\mathcal{M}^{(j)}$ and $\mathcal{M}^+$ agree on $T^\phi_0 \models \phi^+$. By Lemma 55, $\mathcal{M}^{(j)}$ and $\mathcal{M}^+$ agree on $T^\phi_0 \models \phi^+$. Since $\mathcal{M}^{(j)}$ and $\mathcal{M}^+$ agree on $T^\phi_0 \models \phi^+$, as desired.

In Lemma 63 above, we had to modify what $T_i$ says about j-Deduction for $i \not< j$. No such modification is needed in the following lemma. This is interesting because in modal logic, positive introspection is generally considered much more controversial and demanding than basic deduction.

Lemma 65. (Compare Lemma 19) For any $i, j, \omega$ of each of the following families is stratified-closed r.e.-generic.

1. $[\phi(i)$-Assigned Strativality], $[\phi(i)$-Strativality], $[\phi(i)$-Stratival deduction], $[\phi(i)$-Introspection] $j \not< i$.

2. $[\phi(i)$-Stratival deduction], $[\phi(i)$-Stratival deduction], $[\phi(i)$-Stratival deduction], $[\phi(i)$-Introspection] $i, S$ where $S$ is:

\[ (i\text{-Strativalitation}) \] $\mathcal{L}(T^\phi_0 \models \phi \to T^\phi_0 \models \phi)$ whenever this is i-stratified.

Proof. For unistratifiedness, use Corollary 34. Recursive enumerability follows from the fact that < is r.e. In each case below, let $U = (U_k)_{k \in \omega}$ be a stratified-closed r.e. family extending the family in question. For brevity, let $\mathcal{M} = \mathcal{M}_\text{Str}(U)$.

(1) By Lemma 63 (part 1) and Lemma 64 (part 2), $\mathcal{M}^{(i)} \models \phi(i)$-Assigned Strativality, $\phi(i)$-Strativality, and $\phi(i)$-Stratival deduction for every computable stratifier-set $I(i)$ above $i$. Let $J(j)$ be a computable stratifier-set above $j$, we must show $\mathcal{M}^{(j)} \models \phi(i)$-Introspection. In other words, we must show $\mathcal{M}^{(j)} \models \mathcal{L}^\alpha_0 \models \phi$ for any $\mathcal{L}_{PA}(\alpha)$-formula $\phi$. Let $s$ be any assignment and assume $\mathcal{M}^{(j)} \models \mathcal{L}^\alpha_0 \models \phi[s]$. By Definition 1, $\mathcal{M}^{(j)}$ and $\mathcal{M}^+$ agree on $T^\alpha_0 \models \phi$. Let $\alpha$ be such that $(T^\alpha_0 \models \phi) \equiv T^\phi_0 \models \phi^+$, so $\mathcal{M} \models T^\phi_0 \models \phi^+$.

Let $\beta$ be such that $(\phi^+)^\omega \in U_i \cap \alpha$, so $\mathcal{M} \models T^\phi_0 \models \phi^+$. By Part 1 of Theorem 48, $U_i \cap \beta \models T^\phi_0 \models \phi^+$. Thus $\mathcal{M} \models T^\phi_0 \models \phi^+$. By Lemma 56, $\mathcal{M}^{(j)}$ and $\mathcal{M}^{(j)}$ agree on $T^\phi_0 \models \phi^+$, as desired.

Case 2: $\phi(i)$-Stratival deduction. Thus $j < i$ and we can write $J(j) = J_0 \cup \{\phi^+\}$ for some computable i-stratifier $\phi^+$. By Lemma 57, $\mathcal{M}^{(j)}$ and $\mathcal{M}^+$ agree on $T^\phi_0 \models \phi$. Let $\phi^+$ be the i-veristratifier. By Theorem 50, $\mathcal{M} \models (T^\phi_0 \models \phi)^+[s]$. Let $\alpha$ be such that $(T^\phi_0 \models \phi)^+ \equiv T^\phi_0 \models \phi^+$, so $\mathcal{M} \models T^\phi_0 \models \phi^+$. By definition of $\mathcal{M}$, this means $U_i \cap \alpha \models (\phi^+)^*$. Let $\beta$ be such that $(T^\alpha_0 \models \phi)^+ \equiv \mathcal{L}^\alpha_0 \models \phi^+$, so $\beta > \alpha$. By Part 1 of Theorem 48, $U_i \cap \beta \models T^\phi_0 \models \phi^+$. Thus $\mathcal{M} \models T^\phi_0 \models \phi^+$. By Lemma 56, $\mathcal{M}^{(j)}$ and $\mathcal{M}^{(j)}$ agree on $T^\phi_0 \models \phi^+$, as desired.

(2) Let $I(i)$ be any computable stratifier-set above $i$, we must show $\mathcal{M}^{(i)} \models \mathcal{L}(T^\phi_0 \models \phi \to T^\phi_0 \models \phi)$ assuming this is i-stratified (so $\beta > \alpha$). Let $s$ be any assignment and assume $\mathcal{M}^{(i)} \models T^\phi_0 \models \phi$. By Definition 54, $\mathcal{M}^{(i)}$ and $\mathcal{M}^{(i)}$ agree on $T^\phi_0 \models \phi$, so $\mathcal{M} \models T^\phi_0 \models \phi$. By definition of $\mathcal{M}$, this means $U_i \cap \alpha \models \phi^\alpha$. By Part 1 of Theorem 48, $U_i \cap \beta \models T^\phi_0 \models \phi^\alpha$. Thus $\mathcal{M} \models T^\phi_0 \models \phi^\alpha$. In other words, $\mathcal{M} \models (T^\phi_0 \models \phi^\alpha)$, as desired. By Lemma 55, $\mathcal{M}^{(i)}$ and $\mathcal{M}^{(i)}$ agree on $T^\phi_0 \models \phi$, as desired.

□
For the next lemma, note that the proof shows more than is necessary, namely that the structures in question satisfy all the axioms of Peano arithmetic for $\mathcal{L}_{PA}(I)$, not just the $i$-stratified ones. But of course, the full set of Peano axioms for $\mathcal{L}_{PA}(I)$ is not $i$-stratified.

**Lemma 66.** (Compare Lemma 20) For any $i \in \omega$, $[S]$, is straticlosed-r.e.-generic, where $S$ is the set of those axioms of Peano arithmetic for $\mathcal{L}_{PA}(I)$ that are $i$-stratified.

**Proof.** Unistratifiedness and recursive enumerability are clear. In each case below, let $\text{Lemma 68.}$

**Lemma 67.** (Compare Lemma 21) For any $i, j \in \omega$, each of the following families is straticlosed-r.e.-generic.

1. $[j\text{-SMT}]_{i}$ ($j \neq i$).

2. $[S]$, where $S$ is: (i-Strati-SMT) ucl$(\exists \forall x(T_i^\phi \leftrightarrow x \in W_e)$ when this is $i$-stratified, $e \notin FV(\phi)$.

**Proof.** Unistratified and recursive enumerability are clear. In each case below, let $U = (U_k)_{k \in \omega}$ be a straticlosed r.e. family extending $[S]$. By Lemma 59, $\mathcal{M}_{\text{Str}}^{(i)}(U)$ interprets formulas by substitution. By Lemma 10 $\mathcal{M}_{\text{Str}}^{(i)}$ satisfies the axioms of Peano Arithmetic for $\mathcal{L}_{PA}(I)$, as desired.

**Case 1:** $j \notin \text{Indices}(I(i))$. Then $\mathcal{M}_{\text{Str}}^{(i)}$ and $\mathcal{M}$ agree on $T_j^\phi$. Since $U_k^\phi$ is r.e., by the $S$-m-n theorem there is some $n$ such that $W_n = \{m : U_j^\phi \models \phi[x \bar{m}(x_1 \overline{s(x_1)}) \cdots (x_k \overline{s(x_k)})]\}. Since $\phi$ is r.e., by the $S$-m-n theorem there is some $n$ such that $U_n = \{m : U_j \cap \alpha \models \phi \vdash \phi \vdash x \in W_e\}$. Since $\phi$ is r.e., by the $S$-m-n theorem there is some $n$ such that $W_n = \{m : U_j \cap \alpha \models \phi \vdash \phi \vdash x \in W_e\}$. Since $\phi$ is r.e., by the $S$-m-n theorem there is some $n$ such that $W_n = \{m : U_j \cap \alpha \models \phi \vdash \phi \vdash x \in W_e\}$.

**Case 2:** $j \in \text{Indices}(I(i))$. Thus $i \prec j$ and we can write $I(i) = I_0 \cup \{i^+\}$ for some computable $j$-stratifier $i^+$. By Lemma 57, $\mathcal{M}_{\text{Str}}^{(i)}$ and $\mathcal{M}^+$ agree on $T_j^\phi$. Let $\alpha$ be such that $(T_j^\phi)^+ \equiv T_j^\phi$. Since $U_j \cap \alpha$ is r.e., by the $S$-m-n theorem there is some $n$ such that $W_n = \{m : U_j \cap \alpha \models \phi \vdash \phi \vdash x \in W_e\}$. Since $\phi$ is r.e., by the $S$-m-n theorem there is some $n$ such that $W_n = \{m : U_j \cap \alpha \models \phi \vdash \phi \vdash x \in W_e\}$.

If $T = (T_k)_{k \in \omega}$ is straticlosed-r.e.-generic, we cannot simply take an axiom $\phi$ from $T_j$ and insert $T_j^\phi$ into $T_i$ without violating straticlosed-r.e.-genericness, because such a $\phi$ is not necessarily $i$-stratified. Thus, the following lemma has a somewhat more complicated structure than Lemma 22.

**Lemma 68.** (Compare Lemma 22) Let $i, j \in \omega$ and suppose $T = (T_k)_{k \in \omega}$ is straticlosed-r.e.-generic. Then each of the following families is straticlosed-r.e.-generic.

1. $T \cup [S]$, where $S$ is the schema $T_i^\phi \phi$ ($\phi \in T_i$ such that this is $i$-stratified).

2. $T \cup [S]$, where $S$ is the schema $T_j^\phi \phi$ ($\phi \in T_j$, $j \neq i$).

**Proof.** Unistratifiedness and recursive enumerability are clear. In each case below, let $U = (U_k)_{k \in \omega}$ be a straticlosed r.e. family extending the family in question. For brevity, let $\mathcal{M} = \mathcal{M}_{\text{Str}}(U)$. 

18
(1) Since $\mathcal{T}$ is stratified-r.e.-generic and $\mathcal{U} \supseteq \mathcal{T}$ is stratified and r.e., immediately $\mathcal{M}^j \models \mathcal{T}$ (by Definition 61) for all $j \in \omega$ and any computable stratifier-set $J(j)$ above $j$. Let $I(i)$ be any computable stratifier-set above $i$. Suppose $\phi \in T_i$ and $\alpha \in \epsilon_0 \cdot \omega$ are such that $T_\alpha^i \models \phi$ is $i$-stratified, and let $s$ be any assignment. Since $U_i \supseteq T_i$, $\phi \in U_i$, in fact since $T_\alpha^i \models \phi$ is $i$-stratified, it follows that $\phi \in U_i \cap \alpha$. Since $\phi$ is a sentence, $\phi \equiv \phi^s$, and so $U_i \cap \alpha \models \phi^s$, and so $\mathcal{M} \models T_\alpha^i \models \phi[s]$. By Definition 54, $\mathcal{M}^{i(i)}$ agrees with $\mathcal{M}$ on $T_i^e$, so $\mathcal{M}^{i(i)} \models T_i^e \phi[s]$, as desired.

(2) Since $\mathcal{T}$ is stratified-r.e.-generic and $\mathcal{U} \supseteq \mathcal{T}$ is stratified and r.e., immediately $\mathcal{M}^{K(k)} \models \mathcal{T}$ (by Definition 61) for all $k \in \omega$ and any computable stratifier-set $K(k)$ above $k$. Let $I(i)$ be any computable stratifier-set above $i$. Suppose $\phi \in T_j$ where $j < i$. Let $s$ be any assignment. Since $U_j \supseteq T_j$, $\phi \in U_j$. By Lemma 41 there is some very $j$-stratified $\psi \in U_j$ such that $\psi^- \equiv \phi^-$. Clearly since $\phi$ is a sentence, so is $\psi$. By compactness, there is some positive integer multiple $\alpha$ of $\epsilon_0$ such that $U_j \cap \alpha \models \psi$. Since $\psi$ is a sentence, $\psi \equiv \psi^s$, and thus $U_j \cap \alpha \models \psi^s$. Thus, $\mathcal{M} \models T_j^\psi \models \psi^s$. By Theorem 50, $\mathcal{M} \models T_j^\psi \models \psi^s$. Since $I(i)$ is above $i$ and $j \not\prec i$, $\mathcal{M}$ and $\mathcal{M}^{i(i)}$ agree on $T_j^e$, so $\mathcal{M}^{i(i)} \models T_j^e \models \phi^-[s]$, as desired.

7.2 Stratifiable-r.e.-generic Building Blocks

We have established some stratified-r.e.-generic building blocks, but the goal of this paper is to better understand the structure of non-stratified theories—stratification is only a means to an end. Therefore, we introduce a non-stratified building-block notion.

Definition 69. If $T^0 = (T_i^0)$$_{i \in \omega}$ where each $T_i^0$ is an $\mathcal{L}_{PA}(\omega)$-theory, we say $T^0$ is $\prec$-stratifiable-r.e.-generic (or stratifiable-r.e.-generic if $\prec$ is clear from context) if there is some $\prec$-stratified-r.e.-generic family $T = (T_i)_{i \in \omega}$ of $\mathcal{L}_{PA}(J)$-theories such that each $T_i^- = T_i^0$.

Lemma 70. If $T = (T_i)_{i \in \omega}$ is any stratifiable-r.e.-generic family of $\mathcal{L}_{PA}(J)$-theories, then $T^- = (T_i^-)$$_{i \in \omega}$ is a stratifiable-r.e.-generic family of $\mathcal{L}_{PA}(\omega)$-theories.

Proof. Straightforward. □

Corollary 71. (Compare Corollary 23) For all $i, j \in \omega$, each of the following families of $\mathcal{L}_{PA}(\omega)$-theories is stratifiable-r.e.-generic.

1. $[j$-Deduction]$_i$ (if $j \leq i$).
2. $[Modified \ j$-Deduction]$_j$ (if $i \prec j$).
3. $[Assigned \ Validity]_i$.
4. $[Assigned \ Validity]_j \cup [i$-Validity]$_j$.
5. $[Assigned \ Validity]_j \cup [i$-Validity]$_j \cup [i$-Deduction]$_j \cup [i$-Introspection]$_j$.
6. $[S]_i$ where $S$ is the axioms of Peano Arithmetic for $\mathcal{L}_{PA}(\omega)$.
7. $[j$-SMT]$_i$.
8. (If $j \leq i$) $T \cup [S]_i$, for any stratifiable-r.e.-generic $T = (T_k)_{k \in \omega}$, where $S$ is the schema: $T_j \models \phi \ (\phi \in T_j)$.

Proof. By combining Lemma 70 with Lemmas 63 68. For parts involving validity, Lemma 50 can be used to provide valid stratified counterparts of valid non-stratified formulas. □

Comparing the stratifiable-r.e.-generic families we exhibited (Corollary 71) with the closed-r.e.-generic families we exhibited (Corollary 23), we see that the stratifiable-r.e.-generic families are weaker in exactly two ways:

1. They do not allow $T_1$ to state $j$-Deduction for $T_j$ when $i \prec j$, instead allowing what we called Modified $j$-Deduction.
2. Their closure property is more restricted: if \( T^1 = (T^1_k)_{k \in \omega} \) is closed-r.e.-generic and \( T^2 = (T^2_k)_{k \in \omega} \) is stratifiable-r.e.-generic, and if \( S_1 \) is the schema \( T_j \models \phi \ (j \in T^1_1) \), and if \( S_2 \) is the schema \( T_j \models \phi \ (j \in T^2_1) \), then Corollary \( 23 \) says \( T^1 \cup [S_1]_i \) is closed-r.e.-generic with no restrictions on \( j \), whereas Corollary \( 71 \) only says that \( T^2 \cup [S_2]_i \) is stratifiable-r.e.-generic if \( j < i \).

We leave it an open question to what extent Corollary \( 71 \) could be further strengthened. Our primary motivation in choosing building blocks was to facilitate creation of background provability theories at least strong enough to make our own consistency result (Theorem \( 72 \) below) generalize Carlson’s consistency result \( [5] \). If that were our lone motivation, we could restrict Corollary \( 71 \) to only those families where \( i = j \), but a secondary motivation was to provide inter-theory versions of those restricted building blocks.

8 Second Consistency Result: Prioritizing Self-Truth

In this section, we continue to fix an r.e. well-founded partial-order \( \prec \) of \( \omega \). The following theorem will satisfy the second promise from the introduction: it will exhibit true theories \( (T_i)_{i \in \omega} \) such that \( T_i \) expresses a Gödel number of \( T_j \) \( (j \prec i) \) and the truth of \( T_j \) \( (j \preceq i) \). These theories can further be taken so that \( T_i \) expresses the fact that \( T_j \) has some Gödel number (all \( i, j \)), by Lemma \( 67 \).

**Theorem 72.** Let \( T^0 = (T^0_i)_{i \in \omega} \) be any stratifiable-r.e.-generic family of \( \mathcal{L}_{PA}(\omega) \)-theories. For every \( i \in \omega \) and \( n \in \mathbb{N} \), let \( T_i(n) \) be the smallest \( T_i \)-closed \( \mathcal{L}_{PA}(\omega) \)-theory containing the following axioms.

1. The axioms contained in \( T^0_i \).
2. Assigned Validity, \( i \)-Validity and \( i \)-Deduction.
3. \( \text{ucl}(T_j \models \phi \rightarrow \phi) \) whenever \( j \preceq i \).
4. \( \forall x(T_j \models \phi \leftrightarrow (\phi^+|\bar{x}, x) \in W_n) \) whenever \( j \prec i \), \( \text{FV}(\phi) \subseteq \{x\} \).

Let each \( T(n) = (T_i(n))_{i \in \omega} \). There is some \( n \in \mathbb{N} \) such that \( T(n) \) is true.

**Proof.** By the \( S-m-n \) Theorem, there is a total computable \( f : \mathbb{N} \rightarrow \mathbb{N} \) such that \( \forall n \in \mathbb{N} \),

\[
W_{f(n)} = \{ \langle \bar{\phi}, j, m \rangle \in \mathbb{N} : \phi \text{ is an } \mathcal{L}_{PA}(\omega) \text{-formula, } \text{FV}(\phi) \subseteq \{x\}, \text{ and } T_j(n) \models \phi(x|m) \}. 
\]

By the Recursion Theorem, there is an \( n \in \mathbb{N} \) such that \( W_n = W_{f(n)} \). We will show \( T(n) \) is true. For the rest of the proof, we write \( T \) for \( T(n) \), \( T_i \) for \( T_i(n) \).

The structure of the proof is as follows.

- (“Definition of \( U \)” below) First, we will define a certain carefully-chosen family \( U = (U_i)_{i \in \omega} \) of \( \mathcal{L}_{PA}(\mathcal{I}) \)-theories (with each \( U_i^- = T_i \)) and the \( \mathcal{L}_{PA}(\mathcal{I}) \)-structure \( \mathcal{M} = \mathcal{M}_{\text{Str}(U)} \).
- (“Preliminary Result” below) Next, we will show that \( \forall i \in \omega \), \( \mathcal{M} \models U_i \cup T_i \). In order to deal with the difficulty mentioned at the beginning of Section \( 6 \) we will prove more than necessary, to obtain a strong \( \prec \)-induction hypothesis. Namely, we will prove, by \( \prec \)-induction, that \( \forall i \in \omega \), for every computable stratifier-set \( I(i) \) above \( i \), \( \mathcal{M}^{I(i)} \models U_i \cup T_i \).

- (Claim 1 below) In order to prove \( \mathcal{M}^{I(i)} \models U_i \), we will use induction on \( \alpha \) to show that \( \mathcal{M}^{I(i)} \models U_i \cap \alpha \) for all \( \alpha \in \varepsilon_0 \cdot \omega \).

- (Case 3 below) Part of proving \( \mathcal{M}^{I(i)} \models U_i \cap \alpha \) will be proving \( \mathcal{M}^{I(i)} \models \text{ucl}(T_i^{\alpha_0} \models \phi \rightarrow \phi) \) whenever this is \( i \)-stratified, \( \alpha_0 < \alpha \). This is where we will use the \( \alpha \)-induction hypothesis.

- (Case 4 below) Part of proving \( \mathcal{M}^{I(i)} \models U_i \cap \alpha \) will be proving \( \mathcal{M}^{I(i)} \models \text{ucl}(T_j \models \phi \rightarrow \phi^+) \) whenever \( j \prec i \), \( \phi \) is an \( \mathcal{L}_{PA}(\omega) \)-formula, and \( \phi^+ \) is an \( i \)-stratifier. This is where we will take advantage of our strong \( \prec \)-induction hypothesis.

- (Claims 2–3 below) Once we’ve established \( \mathcal{M}^{I(i)} \models U_i \), we will essentially be able to conclude \( \mathcal{M}^{I(i)} \models T_i \) using the Stratification Theorem (Theorem \( 60 \)).
• At the very end of the proof, having established that \( \forall i \in \omega, \mathcal{M} \models U_i \cup T_i \), we will use that to prove that \( \mathcal{M}_T \models T \), i.e., that \( T \) is true.

**Definition of \( U \).** Since \( T^0 \) is stratifiable-r.e.-generic, there is a stratisclosed-r.e-generic family \( V = \{ V_i \}_{i \in \omega} \) of \( \mathcal{L}_{PA}(I) \)-theories such that each \( V_i^- = T_i^0 \). For every \( i \in \mathbb{N} \), let \( U_i \) be the smallest \( i \)-stratified \( \mathcal{L}_{PA}(I) \)-theory such that the following hold.

1. \( U_i \) contains \( V_i \).
2. \( U_i \) contains \( i \)-Assigned Strativalidity, \( i \)-Strativalidity, \( i \)-Strativeduction and \( i \)-Collapse.
3. \( U_i \) contains \( \text{ucl}(T_i^0 \models \phi \rightarrow \phi) \) whenever \( T_i^0 \models \phi \) is \( i \)-stratified.
4. \( U_i \) contains \( \text{ucl}(T_j \models \phi \rightarrow \phi^+ \text{ for every } \mathcal{L}_{PA}(\omega) \text{-formula } \phi, j < i, \text{ and } i \)-stratifier \( \bullet^+ \).
5. \( U_i \) contains \( \forall \phi \ (T_j \models \phi \leftrightarrow (\overline{\phi}, j, x) \in W_{\mathcal{M}}) \) whenever \( j < i \), \( \text{FV}(\phi) \subseteq \{ x \} \) and \( \phi \) is an \( \mathcal{L}_{PA}(\omega) \text{-formula} \).
6. Whenever \( \phi \in U_i \) and \( T_i^0 \models \phi \) is \( i \)-stratified, \( T_i^0 \models \phi \in U_i \).

Let \( U = \{ U_i \}_{i \in \omega} \). Observe that \( U \) is stratisclosed and r.e. (to see \( U \) is r.e., use Theorem \ref{thm:26} part 1); \( U \supseteq V \); and for each \( i \in \omega, U_i^- = T_i \).

Let \( \mathcal{M} = \mathcal{M}_{\text{Str}(U)} \). Recall that \( \text{Str}(U) \) is the \( \mathcal{L}_{PA}(I) \)-family \( \{ S_i \}_{i \in \mathcal{I}} \) where \( \forall i \in \omega \) and \( \alpha \in \epsilon_0 \cdot \omega \), \( S_i = U_i^- \cap T_i \) and \( S_{i, \alpha} = U_i \cap \alpha \). For the reader’s convenience, here is how (by definition) \( \mathcal{M} \) interprets \( T_i^0 \) and \( T_i^0 \models \phi \) for all \( i \in \omega, \alpha \in \epsilon_0 \cdot \omega \):

\[
\mathcal{M} \models T_i^0 \models \phi[s] \text{ iff } T_i \models \phi^s, \\
\mathcal{M} \models T_i^0 \models \phi[s] \text{ iff } U_i \cap \alpha \models \phi^s.
\]

**Preliminary Result.** We would like to prove the following preliminary result: \( \forall i \in \omega, \mathcal{M} \models U_i \cup T_i \). For the sake of a stronger induction hypothesis, we will prove that \( \forall i \in \omega \), for every computable stratifier-set \( I(i) \) above \( i \), \( \mathcal{M}^{I(i)} \models U_i \cup T_i \). This is more than enough because \( \mathcal{M}^{I(i)} \models \mathcal{M} \) when \( I(i) = \emptyset \).

Fix \( i \in \omega \). By \(<\alpha \)-induction, we have the following:

\((*)\) For every \( j \prec i \), for every computable stratifier-set \( J(j) \) above \( j \), \( \mathcal{M}^{J(j)} \models U_j \cup T_j \).

Let \( I(i) \) be any computable stratifier-set above \( i \). We must show \( \mathcal{M}^{I(i)} \models U_i \cup T_i \).

**Claim 1:** \( \forall \alpha \in \epsilon_0 \cdot \omega, \mathcal{M}^{I(i)} \models U_i \cap \alpha \).

By induction on \( \alpha \). Let \( \sigma \in U_i \cap \alpha \).

**Case 1:** \( \sigma \in V_i \). Then \( \mathcal{M}^{I(i)} \models \sigma \) because \( V \) is stratisclosed-r.e-generic and \( U \supseteq V \) is stratisclosed and r.e.

**Case 2:** \( \sigma \) is an instance of \( i \)-Assigned Strativalidity, \( i \)-Strativalidity, or \( i \)-Strativeduction. Then \( \mathcal{M}^{I(i)} \models \sigma \) by Lemma \ref{lem:63} or Lemma \ref{lem:54}.

**Case 3:** \( \sigma \) is \( \text{ucl}(T_i^{\sigma_0} \models \phi \rightarrow \phi) \) for some \( i \)-stratified \( \mathcal{L}_{PA}(I) \)-formula \( \phi \) such that \( T_i^{\sigma_0} \models \phi \) is \( i \)-stratified. Since \( \sigma \in U_i \cap \alpha \), this forces \( \sigma_0 < \alpha \). Let \( s \) be an assignment and assume \( \mathcal{M}^{I(i)} \models T_i^{\sigma_0} \models \phi[s] \), then:

\[
\mathcal{M}^{I(i)} \models T_i^{\sigma_0} \models \phi, \\
\mathcal{M} \models T_i^{\sigma_0} \models \phi[s] \quad \text{(Assumption)}
\]

\[
U_i \cap \sigma_0 \models \phi^s. \\
\mathcal{M}^{I(i)} \models \phi^s. \\
\mathcal{M}^{I(i)} \models \phi[s].
\]

**Case 4:** \( \sigma \) is \( \text{ucl}(T_i \models \phi \rightarrow \phi^+ \text{ for some } \mathcal{L}_{PA}(\omega) \text{-formula } \phi, j \prec i, \text{ and } i \)-stratifier \( \bullet^+ \). By Lemma \ref{lem:52} we may assume \( \bullet^+ \) is computable. Let \( J(j) \) be the computable stratifier-set \( J(j) = I(i) \cup \{ \bullet^+ \} \), which is above \( j \).
since \( I(i) \) is above \( i \) and \( j < i \). Let \( s \) be an assignment and assume \( M^i(i) \models T_j \models \phi[s] \), then:

\[
\begin{align*}
M^i(i) & \models T_j \models \phi[s] & \text{(Assumption)} \\
M & \models T_j \models \phi[s] & \text{(Since } j < i \text{ and } I(i) \text{ is above } i, M^i(i) \text{ and } M \text{ agree on } T_j \models \) \\
T_j & \models \phi^* & \text{(Definition of } M^i(i)) \\
M^i(i)^+ & \models \phi^* & \text{(Since } M^i(i) \models T_j \text{ by (+))} \\
M^i(i)^{+} & \models \phi^* & \text{(Lemma 54)} \\
M^i(i) & \models (\phi^*)^+ & \text{(Clearly } (\phi^*)^+ \equiv (\phi^+)^* \text{)} \\
M^i(i) & \models \phi^+[s]. & \text{(Lemma 58)}
\end{align*}
\]

**Case 5:** \( \sigma \) is \( \forall x(T_i \models \phi \leftrightarrow \langle \langle \phi \rangle, j, x \rangle \in W_T \) for some \( \mathcal{L}_{PA}(\omega) \)-formula \( \phi \) with \( \text{FV}(\phi) \subseteq \{x\} \) and \( j < i \). Let \( s \) be any assignment, say \( s(x) = m \). The following biconditional statements are equivalent:

\[
\begin{align*}
M^i(i) & \models T_j \models \phi \leftrightarrow \langle \langle \phi \rangle, j, x \rangle \in W_T[s] & \text{(} M^i(i) \text{ and } M \text{ agree on the symbols in question)} \\
M & \models T_j \models \phi \leftrightarrow \langle \langle \phi \rangle, j, x \rangle \in W_T[s] & \text{(} M \text{ has standard first-order part)} \\
M & \models T_j \models \phi[s] \text{ iff } M & \models \langle \langle \phi \rangle, j, m \rangle \in W_T & \text{(Lemma 59)} \\
M & \models T_j \models \phi[s] \text{ iff } \langle \langle \phi \rangle, j, m \rangle \in W_N & \text{(Definition of } M \text{)} \\
T_j & \models \phi^*[s] \text{ iff } \langle \langle \phi \rangle, j, m \rangle \in W_N & \text{(Definition of } M \text{)} \\
T_j & \models \phi(x|m) \models \langle \langle \phi \rangle, j, m \rangle \in W_N & \text{(Since FV}(\phi) \subseteq \{x\})
\end{align*}
\]

The latter is true by definition of \( n \).

**Case 6:** \( \sigma \) is an instance \( T_i \models \phi \leftrightarrow \langle \langle \phi \rangle, j, x \rangle \in W_T \) of \( i \)-Collapse (so \( \beta \leq_1 \gamma \) and \( T_i \models \phi \leftrightarrow T_i \models \phi \) is \( i \)-stratified). Let \( s \) be any assignment, since \( M^i(i) \) and \( M \) agree on \( T_i \models \phi \leftrightarrow T_i \models \phi[s] \), we need only show \( M \models T_i \models \phi \leftrightarrow T_i \models \phi[s] \).

In other words, we must show \( U_i \cap \beta \models \phi^* \) if and only if \( U_i \cap \gamma \models \phi^* \). This is by Theorem 38.

**Case 7:** \( \sigma \) is \( T_i \models \phi \) for some \( \mathcal{L}_{PA}(\mathcal{I}) \)-formula \( \phi \) such that \( T_i \models \phi \) is \( i \)-stratified and \( \phi \in U_i \). Since \( T_i \models \phi \) is \( i \)-stratified, \( \text{On}(\phi) \subseteq \alpha_0 \), so \( \phi \in U_i \cap \alpha_0 \). Thus \( M \models T_i \models \phi \), so \( M^i(i) \models T_i \models \phi \) since \( M^i(i) \) and \( M \) agree on \( T_i \models \phi \).

Cases 1–7 establish \( M^i(i) \models U_i \cap \alpha \). By arbitrariness of \( \alpha \), Claim 1 is proved.

**Claim 2:** For any assignment \( s \) and any very \( i \)-stratified \( \mathcal{L}_{PA}(\mathcal{I}) \)-formula \( \phi \), \( M^i(i) \models \phi[s] \) if and only if \( M^i(i) \models \phi^*[s] \).

By Theorem 50, for all such \( s \) and \( \phi \), \( M \models \phi[s] \) if and only if \( M \models \phi^*[s] \). The claim now follows from Lemma 58 (\( i \notin \text{Indices}(i)) \) because \( I(i) \) is above \( i \).

**Claim 3:** \( M^i(i) \models T_j \).

For any \( \sigma \in T_i \), there is some \( \tau \in U_i \) such that \( \tau \equiv \sigma \); since \( U_i \) is \( i \)-unistratified, we may take \( \tau \) to be very \( i \)-stratified (Lemma 11). By Claim 1, \( M^i(i) \models U_i \), so \( M^i(i) \models T_j \). By Claim 2, \( M^i(i) \models \sigma \).

For each \( i \in \omega \), letting \( I(i) = \emptyset \), Claims 1–3 show that \( M \models U_i \cup T_i \). It follows that \( M \models T_i \). Now, for every \( i \in \omega \), \( M_T \) interprets \( T_i \models \) as follows:

\[
M_T \models T_i \models \phi[s] \text{ iff } T_i \models \phi^*.
\]

This is exactly the same way that \( M \) interprets \( T_i \models \). It follows that \( M \) and \( M_T \) agree on \( \mathcal{L}_{PA}(\omega) \)-formulas. Thus, since \( M \models T \), \( M_T \models T \), i.e., \( T \) is true.

9 Well-Foundation and Ill-Foundation

The following is a variation on Kleene’s \( \mathcal{O} \).
Definition 73. Simultaneously define $\mathcal{O} \subseteq \mathbb{N}$ and $\bullet : \mathcal{O} \to \text{Ord}$ so that $\mathcal{O} \subseteq \mathbb{N}$ is the smallest set such that:

1. $0 \in \mathcal{O}$ (it represents the ordinal $|0| = 0$).
2. $\forall n \in \mathcal{O}, 2^n \in \mathcal{O}$ (it represents the ordinal $|2^n| = |n| + 1$).
3. If $\varphi_e$ (the $e$th partial recursive function) is total and range($\varphi_e$) $\subseteq \mathcal{O}$, then $3 \cdot 5^e \in \mathcal{O}$ (it represents the ordinal $|3 \cdot 5^e| = \sup\{|\varphi_e(0)|, |\varphi_e(1)|, \ldots\}$).

To avoid technical complications, we have differed from the usual Kleene’s $\mathcal{O}$ in the following way: in the usual definition, in order for $3 \cdot 5^e$ to lie in $\mathcal{O}$, it is also required that $|\varphi_e(0)| < |\varphi_e(1)| < \ldots$.

Definition 74. $\mathcal{L}^\mathcal{O}_\text{PA}$ is the language of Peano arithmetic extended by a unary predicate $\mathcal{O}$. The following notions are defined by analogy with Section 3.

1. For any assignment $s$ and $\mathcal{L}^\mathcal{O}_\text{PA}(I)$-formula $\phi$ with FV($\phi$) = $\{x_1, \ldots, x_n\}$, $\phi^s \equiv \phi(x_1[s(x_1)] \cdots (x_n[s(x_n)]).

2. If $T = (T_i)_{i \in I}$ is an $I$-indexed family of $\mathcal{L}^\mathcal{O}_\text{PA}(I)$-theories, the intended structure for $T$ is the $\mathcal{L}^\mathcal{O}_\text{PA}(I)$-structure $\mathcal{M}_T$ with universe $\mathbb{N}$, interpreting symbols of PA as usual and interpreting $\mathcal{O}$ as $\mathcal{O}$, and interpreting $T_i \models (i \in I)$ as in Definition 3. For any $\mathcal{L}^\mathcal{O}_\text{PA}(I)$-structure $\mathcal{N}$, we write $\mathcal{N} \models T$ if $\forall i \in I, \mathcal{N} \models T_i$. We say $T$ is true if $\mathcal{M}_T \models T$.

Definition 75. If $I$ is an index set and $T = (T_i)_{i \in I}$ is a family of $\mathcal{L}^\mathcal{O}_\text{PA}(I)$-theories, then for any $i \in I$ such that $\mathcal{M}_T \models T_i$, we define the ordinal $\|T_i\| = \sup\{|m| + 1 : T_i \models \mathcal{O}^\mathcal{M}_T(m)\}$.

The above definition makes sense: since $\mathcal{M}_T \models T_i$ and $\mathcal{O}^\mathcal{M}_T = \mathcal{O}$, the supremands are defined.

Definition 76. The basic axioms of $\mathcal{O}$ are the following $\mathcal{L}^\mathcal{O}_\text{PA}$-axioms.

1. $\mathcal{O}(0)$.
2. $\mathcal{O}(\pi) \to \mathcal{O}(\pi')$, for every $n \in \mathbb{N}$.
3. $\forall x(\varphi_\pi(x) \downarrow \& \mathcal{O}(\varphi_\pi(x))) \to \mathcal{O}(3 \cdot 5^n)$, for every $n \in \mathbb{N}$.

We have written the last two lines using infinitc schemata to strengthen the following result.

Theorem 77. Let $I$ be an index set, $\prec$ a binary relation on $I$. Suppose $\mathbf{T} = (T_i)_{i \in I}$ is a family of $\mathcal{L}^\mathcal{O}_\text{PA}(I)$-theories with the following properties:

1. $\forall i \in I, T_i$ contains the axioms of Peano arithmetic.
2. $\forall i \in I, T_i$ contains the basic axioms of $\mathcal{O}$.
3. $\forall i \in I, \forall j < i, \exists n \in \mathbb{N}$ such that $T_i \models \forall x(T_j \models \mathcal{O}(x) \iff x \in W_\pi)$.
4. $\forall i \in I, \forall j < i, T_i \models \forall x(T_j \models \mathcal{O}(x) \to \mathcal{O}(x))$.

If $\mathcal{M}_T \models T_i \cup T_j$ (in particular if $T$ is true) and $j \prec i$, then $\|T_j\| < \|T_i\|$.

Proof. Assume $\mathcal{M}_T \models T_i \cup T_j$ and $j \prec i$. By hypothesis there is some $n \in \mathbb{N}$ such that $T_i \models \forall x(T_j \models \mathcal{O}(x) \iff x \in W_\pi)$ and $T_i \models \forall x(T_j \models \mathcal{O}(x) \to \mathcal{O}(x))$. From these, $T_i \models \forall x(x \in W_\pi \to \mathcal{O}(x))$.

Since $\mathcal{M}_T \models T_i$, in particular $\mathcal{M}_T \models \forall x(T_j \models \mathcal{O}(x) \iff x \in W_\pi)$. This means $W_n = \{m \in \mathbb{N} : T_j \models \mathcal{O}(\pi_m)\}$. Since $T_j$ includes the axiom $\mathcal{O}(0)$, $W_n \neq \emptyset$.

Since $W_n \neq \emptyset$, by computability theory there is some $k \in \mathbb{N}$ such that

$\mathbf{PA} \models (\text{domain}(\varphi_\pi) = \mathbb{N}) \land (\text{range}(\varphi_\pi) = W_\pi)$.

Since $T_i$ includes PA, $T_i$ also implies as much. Combined with $T_i \models \forall x(x \in W_\pi \to \mathcal{O}(x))$, it follows that $T_i \models \forall x(\mathcal{O}(\varphi_\pi(x)) \downarrow \& \mathcal{O}(\varphi_\pi(x)))$. Since $T_i$ contains the basic axiom $\forall x(\varphi_\pi(x) \downarrow \& \mathcal{O}(\varphi_\pi(x))) \to \mathcal{O}(3 \cdot 5^k)$, $T_i \models \mathcal{O}(3 \cdot 5^k)$.
To finish the proof, calculate
\[ \|T_j\| = \sup\{ |m| + 1 : T_j \models O(m) \} \]
\[ = \sup\{ |m| : T_j \models O(m) \} \quad \text{(Since } T_j \text{ contains } O(\overline{n}) \rightarrow O(\overline{2n}) \text{ for all } n \in \mathbb{N}) \]
\[ = \sup\{ |m| : m \in W_n \} \quad \text{(Since } W_n = \{ m \in \mathbb{N} : T_j \models O(m) \} \} \]
\[ = \sup\{ |\varphi_k(0)|, |\varphi_k(1)|, \ldots \} \quad \text{(By choice of } k \text{)} \]
\[ = |3 \cdot 5^k| \quad \text{(Definition } \ref{78}) \]
\[ < \sup\{ |m| + 1 : T_i \models O(m) \} \quad \text{(Since } T_i \models O(3 \cdot 5^k) \} \]
\[ = \|T_i\|. \]
\[ \square \]

**Corollary 78.** (Well-Foundedness of True Self-Referential Theories) Let \( I, T, \prec \) be as in Theorem \ref{77}. If \( T \) is true then \( \prec \) is well founded, by which we mean there is no infinite descending sequence \( i_0 > i_1 > \cdots \).

In particular Corollary \ref{78} says that if \( I, T, \prec \) are as in Theorem \ref{77} and \( T \) is true then \( \prec \) is strict: there is no \( i \) with \( i \prec i \). This gives a new form (under the additional new assumption of containing/knowing basic rudiments of computable ordinals) of the Lucas–Penrose–Reinhardt argument that a truthful theory (or machine) cannot state (or know) its own truth and its own Gödel number.

We could remove Peano arithmetic from Theorem \ref{77} if we further departed from Kleene and changed line 3 of Definition \ref{73} to read:

3. If \( W_e \subseteq O \), then \( 3 \cdot 5^e \in O \) and \( |3 \cdot 5^e| = \sup\{ |n| : n \in W_e \} \), or \( |3 \cdot 5^e| = 0 \) if \( W_e = \emptyset \)

(and altered Definition \ref{76} accordingly). The previous paragraph would still stand, in fact giving a version of the Lucas–Penrose–Reinhardt argument in which the theory (machine) is not required to contain (know) arithmetic.

We close the paper by showing that Corollary \ref{78} fails without \( O \). Let WF be the set of all r.e. well-founded partial orders on \( \omega \) and let \( Tr \) be the set of all true \( \mathcal{L}_{PA} \)-sentences. It is well-known that WF is computability theoretically \( \Pi^1_1 \)-complete and \( Tr \) is \( \Delta^1_1 \), so WF cannot be defined in \( \mathcal{L}_{PA} \cup \{ Tr \} \).

**Theorem 79.** (Ill-Foundedness of True Self-Referential Theories)

1. There exists an r.e., ill-founded partial order \( \prec \) on \( \omega \) such that for every closed r.e.-generic \( T^0 = (T^0_i)_{i \in \omega} \) there is an \( n \in \mathbb{N} \) such that \( T(n) \) is true, where \( T(n) \) is as in Theorem \ref{24}.

2. There exists an r.e., ill-founded partial order \( \prec \) on \( \omega \) such that for every \( \prec \)-stratifiable r.e.-generic \( T^0 = (T^0_i)_{i \in \omega} \) there is an \( n \in \mathbb{N} \) such that \( T(n) \) is true, where \( T(n) \) is as in Theorem \ref{72}.

**Proof.** We prove (1), (2) is similar. Assume \( \neg (1) \). For each r.e. partial order \( \prec \) on \( \omega \), let \( S(\prec) \) be the statement of Theorem \ref{24} for \( \prec \), minus the requirement that \( \prec \) be well founded. Combining \( \neg (1) \) with Theorem \ref{24} \( \prec \) is well founded if and only if \( S(\prec) \) is true. We will argue that \( S(\prec) \) is expressible in \( \mathcal{L}_{PA} \cup \{ Tr \} \), which is absurd because that would mean it is possible to define WF in \( \mathcal{L}_{PA} \cup \{ Tr \} \).

\( S(\prec) \) is equivalent to the following:

- For any (Gödel number of an) r.e. family \( T^0 = (T^0_i)_{i \in \omega} \) of \( \mathcal{L}_{PA}(\omega) \)-theories, if \( T^0 \) is closed r.e.-generic
  (i.e., if \( \mathcal{M}_U \models T^0 \) for every closed r.e. family \( U \supseteq T^0 \) of \( \mathcal{L}_{PA}(\omega) \)-theories), then there is some \( n \in \mathbb{N} \) such that \( T(n) \) is true (i.e., such that \( \mathcal{M}_{T(n)} \models T(n) \)), where \( T(n) = (T(n)_i)_{i \in \omega} \), where each \( T(n)_i \) is the smallest \( T_j \)-closed theory containing the following:
    1. The axioms in \( T^0_i \).
    2. \( \forall x(T_j \models \phi \rightarrow (\exists \phi, j, x) \in W_T) \) whenever \( j \in \omega \), \( \text{FV}(\phi) \subseteq \{ x \} \).
    3. \( \text{ucl}(T_j \models \phi \rightarrow \phi) \) whenever \( j \prec i \).

This is manifestly expressible in \( \mathcal{L}_{PA} \) except for the clauses \( \mathcal{M}_U \models T^0 \) and \( \mathcal{M}_{T(n)} \models T(n) \). We will show that \( \mathcal{M}_U \models T^0 \) is expressible in \( \mathcal{L}_{PA} \cup \{ Tr \} \); the expressibility of \( \mathcal{M}_{T(n)} \models T(n) \) is similar.

Define an operator \( F_U \) which takes an \( \mathcal{L}_{PA}(\omega) \)-formula \( \phi \) and outputs an \( \mathcal{L}_{PA} \)-formula \( F_U(\phi) \) as follows:
• If $\phi$ is atomic, let $F_U(\phi) \equiv \phi$.

• If $\phi$ is $\neg \phi_0$, $\phi_1 \rightarrow \phi_2$, or $\forall x \phi_0$, let $F_U(\phi)$ be $\neg F_U(\phi_0)$, $F_U(\phi_1) \rightarrow F_U(\phi_2)$, or $\forall x F_U(\phi_0)$, respectively.

• Suppose $\phi$ is $\mathcal{T}_i \models \psi$ and $FV(\psi) = \{x_1, \ldots, x_k\}$. Let $f : \mathbb{N}^k \rightarrow \mathbb{N}$ be the computable function such that for all $m_1, \ldots, m_k \in \mathbb{N}$, $f(m_1, \ldots, m_k) = \psi(x_1 \upharpoonright m_1) \cdots (x_k \upharpoonright m_k)$). Let $F_U(\phi)$ be: “$U_i$ proves the sentence with Gödel number $f(x_1, \ldots, x_k)$” (so $FV(F_U(\phi)) = \{x_1, \ldots, x_k\}$).

It is easy to check that for every $\mathcal{L}_{PA}(\omega)$-formula $\phi$ and assignment $s$, $\mathcal{M}_U \models \phi[s]$ if and only if $\mathbb{N} \models F_U(\phi)[s]$. In particular, for every $\mathcal{L}_{PA}(\omega)$-sentence $\phi$, $\mathcal{M}_U \models \phi$ if and only if $\mathbb{N} \models F_U(\phi)$. Thus, the clause $\mathcal{M}_U \models T^0$ can be expressed in $\mathcal{L}_{PA} \cup \{\text{Tr}\}$ as follows: $\forall \forall x \in T^0 \rightarrow \text{Tr}(\upharpoonright F_U(\phi))$.

The way we prove Theorem 79 by referring to the computability theoretical complexity of WF is similar to a recent argument by Kripke [7].

References