This sentence does not contain the symbol X

Samuel A. Alexander*
Department of Mathematics, the Ohio State University

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In order to formalize the Liar's Paradox, one approach is as follows. Work in the language of Peano arithmetic extended by a unary predicate symbol T, and use Gödel's diagonal lemma to produce a sentence λ such that Peano arithmetic proves $\lambda \leftrightarrow \neg T(\ulcorner \lambda \urcorner)$. One then refers to λ as a liar sentence, glossing it as "This sentence is not true."

A suprise may occur if we use a similar strategy to formalize

This sentence does not contain the symbol X.

Work in the language $\mathscr{L}$ of Peano arithmetic extended by a new symbol X (for example, X can be a constant symbol, this is unimportant). Let $\phi \mapsto \ulcorner \phi \urcorner$ be an effective Gödel numbering of the $\mathscr{L}$-formulas, such that (for convenience) every $n \in \mathbb{N}$ is a Gödel number of some $\mathscr{L}$-formula. By the Church-Turing thesis, there is a total computable function $h : \mathbb{N} \to \mathbb{N}$ such that for every $\mathscr{L}$-formula ϕ, $h(\ulcorner \phi \urcorner) = 1$ if and only if X occurs in ϕ. It follows that there is a formula ψ with one free variable x, in the language of Peano arithmetic without X, such that $\mathbb{N} \models \psi(\ulcorner \phi \urcorner)$ precisely when X occurs in ϕ. By Gödel's diagonal lemma, there is a sentence λ, not containing X, such that Peano arithmetic proves $\lambda \leftrightarrow \neg \psi(\ulcorner \lambda \urcorner)$. Following the liar's precedent, we feel tempted to gloss λ as "This sentence does not contain the symbol X." The main difference is that unlike the liar's sentence, the sentence we've just constructed is entirely syntactical, not depending on the semantics of X.

The surprise is that this λ we have constructed is, in fact, true (at least if Peano arithmetic is true). This is surprising because the English sentence, "This sentence does not contain the symbol X," certainly does appear to contain the symbol X.

*Email: alexander@math.ohio-state.edu