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A *Geometrical* Perspective of The Four Colour Theorem[1]

B. S. Anand[2]
Non-institutional research
Mumbai, Maharashtra, India
e-mail: bhup.anand@gmail.com

Abstract

All acknowledged proofs of the Four Colour Theorem (4CT) are computer-dependent. They appeal to the existence, and manual identification, of an 'unavoidable' set containing a sufficient number of explicitly defined configurations—each evidenced only by a computer as 'reducible'—such that at least one of the configurations must occur in any chromatically distinguished, putatively minimal, planar map. For instance, Appel and Haken 'identified' 1,482 such configurations in their 1977, computer-dependent, proof of 4CT; whilst Neil Robertson et al 'identified' 633 configurations as sufficient in their 1997, also computer-dependent, proof of 4CT. However, treating any specific number of 'reducible' configurations in an 'unavoidable' set as sufficient entails a minimum number as necessary and sufficient. We now show that the minimum number of such configurations can only be *the* one corresponding to the 'unavoidable' set of the single, 'reducible', 4-sided configuration identified by Alfred Kempe in his, seemingly fatally flawed, 1879 'proof' of 4CT. We shall further show that although Kempe fallaciously concluded that a 5-sided configuration was also in the 'unavoidable' set, and appealed to unproven properties of 'Kempe' chains in a *graphical* representation to then argue for its 'reducibility', neither flaw in his 'proof' is fatal when the argument is expressed *geometrically*; and that, essentially, Kempe correctly argued that any planar map which admits a chromatic differentiation with a five-sided area C that shares non-zero boundaries with four, all differently coloured, neighbours can be 4-coloured.

Keywords: computer-assisted proof, four colour theorem, Kempe chains, minimal planar map, geometrical proof, unavoidable but reducible configurations.
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1 Introduction

Although the Four Colour Theorem 4CT is considered passé, it would probably be a fair assessment that the mathematical significance of any new 'proof' of the Four

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[2] ORCID: https://orcid.org/0000-0003-4290-9549; CV: https://www.dropbox.com/s/ohrhhinjvg5zj7s/

Colour Theorem 4CT continues to be perceived[3] as lying not in any ensuing theoretical or practical utility of the Theorem per se, but in whether the putative proof can address the lack of mathematical insight[4] as to *why* four colours suffice to chromatically differentiate any set of contiguous, simply connected and bounded, spaces in a planar map.

Now, all acknowledged proofs of 4CT appeal to the existence, and manual 'identification', of a sufficient number of explicitly defined configurations, each evidenced only by a computer as 'reducible' (see [16], Ch.8); and claimed (see §5) to be an 'unavoidable' set of configurations, at least one of which must occur in any chromatically distinguished, minimal, planar map that claims to essentially require five colours.

Thus, Kenneth Appel and Wolfgang Haken claimed to have identified an unavoidable set of 1,482 reducible configurations in their 1977, computer-dependent, proof [3] of 4CT; whilst Neil Robertson et al claimed to have identified an unavoidable set of 633 reducible configurations as sufficient in their 1997, also computer-dependent, proof [9] of 4CT.

Since claiming any specific number of 'reducible' configurations in an 'unavoidable' set as sufficient entails a minimum number as necessary and sufficient—which, moreover, can only be a mathematical constant (such as π or e)—we shall show that:

> The minimum number of 'reducible' configurations in an 'unavoidable' set can only be the mathematical constant 1[5]; and not 2 as fallaciously claimed by Kempe *explicitly* in [23] (see §4), and by Appel and Haken *implicitly* in [3] (see §5).

Although Kempe fallaciously argued that a 5-sided configuration[6]—identified by him as *Plate II, Fig.11 and Fig.12* in [23]—was also in the 'unavoidable' set, and appealed to unproven properties of 'Kempe' chains, in a *graphical* representation, to conclude it was 'reducible', we show in §3 that neither flaw is fatal when the argument is expressed *geometrically.* In other words, Kempe correctly argued that:

> Any planar map which admits a chromatic differentiation with a five-sided area C that shares non-zero boundaries with four, all differently coloured, neighbours can be 4-coloured.

[3]See Appel and Haken: [3]; Appel, Haken and Koch: [4]; Tymoczko: [5]; Swart: [6]; Stewart: [7], *Appendix, pp.503-505*; Robertson et al: [8], *Pre-publication*; Robertson, Sanders, Seymour, and Thomas: [9]; Thomas: [10]; Calude: [11]; Brun: [12], *§1. Introduction (Article for undergraduates)*; Gonthier: [13]; Zeilberger: [14]; Rogers: [15]; Wilson: [16]; Conradie and Goranko: [17], *§7.7.1, Graph Colourings, p.417*; Allo: [18], *Conclusion, p.562*; Nanjwenge: [19], *Chapter 8, Discussion (Student Thesis)*; Najera: [20]; Gardner: [21], *§11.1, Colourings of Planar Maps, pp.6-7 (Lecture notes)*; D'Alessandro and Lehet: [22], *§3.2 The trouble with schemas: objectual understanding is not explanatory understanding*, p.3.

[4]Particularly in currently accepted, computer-dependent, proofs of the Theorem whose validity—as we highlight in §5—is not beyond doubt.

[5]The one corresponding to the 'reducible' 4-sided configuration identified by Alfred Kempe as *Plate II, Fig.9* in his, seemingly fatally flawed, 1879 'proof' [23] of 4CT; and to the 'quadrilateral' identified by Appel and Haken as P_4 in [3].

[6]The 'pentagon' identified by Appel and Haken as P_5 in [3]; see also §4.

Comment: Lemma 3.1 details the *geometrical* argument Kempe needed for validating his 'proof' of 4CT in [23]. Seemingly, Kempe failed to recognise the significance of the *geometrical* argument, since he preferred appeal to considerations of Euler's formula $V + F = E + 2$ in what he mistakenly claimed (see §6) as an 'equivalent', *graphical*, representation; which may have led him to falsely conclude that any *minimal* planar map must contain an 'unavoidable' set with *two* 'reducible' configurations (see §4).

The intuitive truth of Lemma 3.1 is evidenced by the fact that it is trivial to paste (or even merely imagine pasting) a small, grey, piece of paper anywhere on a children's 4-coloured globe to confirm that:

- there is never a need to re-colour the, inherited, 4-coloured areas of the globe in order to maintain chromatic differentiation; *and*,
- if the grey area increases the number of countries by *one* by partitioning an existing country into *two*, none of the *other* countries abutting the grey area *after* partitioning could have shared the same colour as the partitioned country *before* partitioning.

In the *geometrical* proof of 4CT in §3, Theorem 1 we thus proffer elementary, computer-independent, arguments which transparently illustrate *why* four colours suffice to chromatically differentiate any set of contiguous, simply connected and bounded, planar spaces; by essentially arguing that:

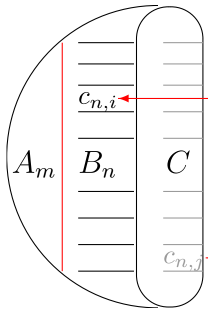
- If there is a *minimal* planar map $\mathcal{H}$ with $(m + n + 1)$ areas that contains an area C which necessarily requires a 5^{th} colour, whilst any planar map with $\leq m+n$ areas can be 4-coloured (defined as needing at most four colours);
- Then shrinking C to a point P_C yields a *sub-minimal* map $\mathcal{M}_C$ that can always be 4-coloured, such that all the areas meeting at the apex P_C require at most 3 colours.
- Recreating C in $\mathcal{M}_C$ would now yield a chromatic differentiation of $\mathcal{H}$ that requires at most only a 4^{th} colour for C, contradicting the putative minimality of $\mathcal{H}$.

2 The Minimal Planar Map $\mathcal{H}$ and the Minimality Hypothesis

Without loss of generality, the surface of the hemisphere in Fig.1 is taken to define a *minimal* planar map $\mathcal{H}$ where:

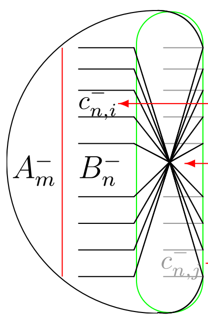
1. A_m denotes a region of m contiguous, simply connected and bounded, surface areas $a_{m,1}, a_{m,2}, \ldots, a_{m,m}$ (of the hemisphere in Fig.1), *none* of which share a non-zero boundary segment with the contiguous, simply connected, surface area C (as indicated by the red barrier which, however, is *not* to be treated as a boundary of the region A_m);

Fig.1: Minimal Planar Map $\mathcal{H}$



2. B_n denotes a region of n contiguous, simply connected and bounded, surface areas $b_{n,1}, b_{n,2}, \ldots, b_{n,n}$, *some* of which, say $c_{n,1}, c_{n,2}, \ldots, c_{n,r}$, share *at least one* non-zero boundary segment with C; where, for each $1 \leq i \leq r$, we have that $c_{n,i} = b_{n,j}$ for some $1 \leq j \leq n$;

Fig.2: Sub-minimal Planar Map $\mathcal{M}_C$ defined uniquely by shrinking C to a point P_C in $\mathcal{H}$



3. C is a single contiguous, simply connected and bounded, area (see Fig.1) constructed *finitarily* by sub-dividing and annexing (compare Kempe [23], Plate II, Fig.14) one or more contiguous, simply connected, portions surrounding a common apex P_C of each area $c^-_{n,i}$ (see Fig.2) in the region B^-_n of some putative sub-minimal map $\mathcal{M}_C$ (see Fig.2), defined uniquely by putatively shrinking C to a point P_C in $\mathcal{H}$.

We define:

Definition 1 (Finitary Constructibility)**.** A single contiguous, simply connected and bounded, area D of a planar map $\mathcal{G}$ is *finitarily constructible* if, and only if, it can be constructed in a finite number of steps by annexing non-zero areas of the planar map $\mathcal{M}_D$ obtained by shrinking D to a point in $\mathcal{G}$.

Lemma 2.1. *Any single contiguous, simply connected and bounded, area D of a planar map $\mathcal{G}$ with n areas is finitarily constructible.*

Proof. If D shares m non-zero boundary segments with abutting areas, then:

- shrinking D to a point (as in Fig.2),
- yields a planar map $\mathcal{M}_D$,
- with at most m areas of $\mathcal{G}$,
- that now meet in $\mathcal{M}_D$,
- at least once at a common apex P_D.

The area D can then be *finitarily* constructed in m steps by annexing m triangular areas of those immediate portions of each area of $\mathcal{M}_D$ that contain P_D. The Lemma follows. $\square$

We next consider the:

Hypothesis 1 (Minimality Hypothesis)**.** Since four colours suffice for all, and are necessary for some, planar maps with fewer than 5 regions, we assume the existence of some m, n, in a putatively *minimal* planar map $\mathcal{H}$, which defines a *specific* configuration of the region $\{A_m + B_n + C\}$ where:

(a) any configuration of p contiguous, simply connected and bounded, areas can be 4-coloured if $p \leq m + n$, where $p, m, n \in \mathbb{N}$, and $m + n \geq 5$;

(b) any chromatically differentiated colouring of $\mathcal{H}$ contains some area C that necessarily requires a 5^{th} colour;

(c) in any such chromatically differentiated colouring, there is a *specific* configuration of $m+n$ contiguous, simply connected and bounded, areas, say $\{A^-_m + B^-_n\}$, of a putatively unique, sub-minimal, 4-colourable planar map, say $\mathcal{M}_C$, where $A_m \subseteq A^-_m$ and $B_n \subseteq B^-_n$;

(d) the area C can be constructed finitarily by sub-dividing and annexing some portions from each area, say $c^-_{n,i}$, of B^-_n in the specific, sub-minimal, planar map $\mathcal{M}_C$;

(e) the region $\{A_m + B_n + C\}$ in the planar map $\mathcal{H}$ is a *specific* chromatic differentiation of the $m + n + 1$ contiguous, simply connected and bounded, areas of $\mathcal{H}$ in which C necessarily requires a 5^{th} colour.

We further define:

Definition 2 (Finitary Definability)**.** A single contiguous, simply connected and bounded, area D of a planar map $\mathcal{G}$ is finitarily definable if, and only if, it can be shrunk in a finite number of steps to a point in $\mathcal{G}$.

We note that:

Lemma 2.2. *The minimal map $\mathcal{H}$ cannot admit two areas, say C and C', both of which necessarily require a 5^{th} colour.*

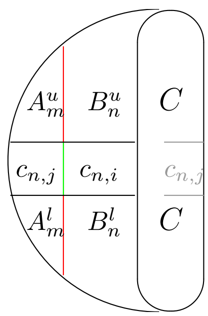
Proof. Shrinking C to a point would reduce $\mathcal{H}$ to a 4-colourable map where C' does not require a 5^{th} colour. Restoring C with a 5^{th} colour establishes the Lemma. $\square$

Since any area D of $\mathcal{H}$ can be shrunk to a point, the argument of Lemma 2.2 immediately entails:

Corollary 1. *The minimal map $\mathcal{H}$ can always be chromatically distinguished so that any specified area D of $\mathcal{H}$ requires a 5^{th} colour.* $\square$

We note next that:

Fig.3: No two, non-adjacent, areas $c_{n,i}$ and $c_{n,j}$ can share a non-zero boundary in $\mathcal{H}$



Lemma 2.3. *No two, non-adjacent, areas of B_n, each sharing a non-zero boundary segment with C in the minimal planar map $\mathcal{H}$ in Hypothesis 1, can also share a non-zero boundary that has no point in common with C.*

Proof. Let two, non-adjacent, areas of B_n, say $c_{n,i}$ and $c_{n,j}$ in Fig.1, *each* of which shares a non-zero boundary with C, *also* share a non-zero boundary with each other that *does not* intersect C (as shown in green in Fig.3). This would divide $\{A_m + B_n - c_{n,i} - c_{n,j}\}$ into two *non-empty* regions $A^u_m + B^u_n$ and $A^l_m + B^l_n$, such that *no* area of the region $A^u_m + B^u_n$ shares a non-zero boundary with *any* area of the region $A^l_m + B^l_n$.

However, it would entail that the areas $c_{n,k}$ $(k \neq i, j)$ which abut C in each of the regions B^u_n and B^l_n would necessarily require 2 additional colours not shared with the areas C, $c_{n,i}$ and $c_{n,j}$; since:

(a) if all such $c_{n,k}$ require only 1 additional colour, then $\mathcal{H}$ would be 4-colourable; which would violate the *minimality* of $\mathcal{H}$;

(b) if all such $c_{n,k}$ in only one of the regions, say B^u_n, require only 1 additional colour,

- then annexing one of the areas of B^l_n, say $c_{n,lower}$, which has this colour, say x, into the area C would again reduce the map $\mathcal{H}$ to a *sub-minimal* map, say $\mathcal{M}'$,

- where any re-colouring of $\mathcal{M}'$ would still require 5 colours, since the merged area $(c_{n,\ lower} + C)$ must continue to necessarily abut areas with four colours if $\mathcal{H}$ was a *minimal* map; thus violating the *minimality* of $\mathcal{H}$;

Consequently, *each* of the regions $\{B^u_n + c_{n,i} + c_{n,j} + C\}$ and $\{B^l_n + c_{n,i} + c_{n,j} + C\}$—when considered as separate planar maps, each with less than $m+n+1$ areas—would necessarily require C to have the 5^{th} colour, thus violating Hypothesis 1. The Lemma follows. □

Corollary 2. *No area $c_{n,i}$ of B_n in the minimal planar map $\mathcal{H}$ can share two, distinctly separated, non-zero boundary segments with C.* □

Lemma 2.4. *Every area D of a minimal planar map $\mathcal{H}$ shares non-zero boundaries with at least four neighbours.*

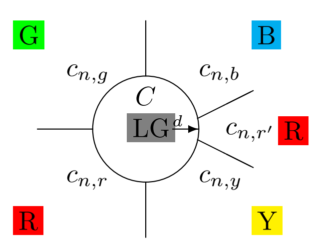
Proof. Shrinking *any* area D of a *minimal* map $\mathcal{H}$ to a point would yield a 4-colourable, *sub-minimal*, map. By Hypothesis 1, restoring D in any 4-colouring of such a *sub-minimal* map must require a 5^{th} colour for D. The Lemma follows. □

3 A *geometrical* proof of the 4-Colour Theorem

We now show how the following Lemma improves upon, and bridges the gap, in Alfred Kempe's—seemingly fatally failed (see §4)—'proof' of the Four Colour Theorem in [23].

Prima facie, Kempe failed to recognise the significance of the *geometrical* argument, since he preferred appeal to considerations of Euler's formula $V + F = E + 2$ in what he mistakenly claimed (see §6) as an 'equivalent', *graphical*, representation of the 4-colour problem; which may have led him to fallaciously conclude that any *minimal* planar map must contain an 'unavoidable' set with *two* 'reducible' configurations (see §4). Lemma 3.1 now shows why the conclusion was false.

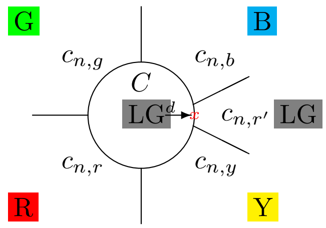
***Fig.4:** C **cannot share a non-zero boundary segment with 5 or more areas in** $\mathcal{H}$*



Lemma 3.1. *The area C in $\mathcal{H}$ can share a non-zero boundary with only four, differently coloured, areas $c_{n,i}$.*

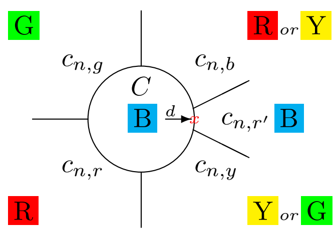
Proof. (i) Without loss of generality, we shall consider only the case where C has five neighbours. If, now, C (Light Gray) shares a non-zero boundary with identically-coloured areas $c_{n,r}$ and $c_{n,r'}$ (see Fig.4)—where $r \neq r'$ by Corollary 2—then either area can be annexed by C without disturbing the chromatic differentiation of $\mathcal{H}$.

Fig.5: Annexing $c_{n,r'}$ into C would then yield a sub-minimal map $\mathcal{M}_{\mathcal{C}}$



(ii) However, if C annexes $c_{n,r'}$ by erasing the boundary d (see Fig.5), that would then yield a *sub-minimal* map $\mathcal{M}_{\mathcal{C}}$.

Fig.6: The sub-minimal map $\mathcal{M}_{\mathcal{C}}$ would be 4-colourable

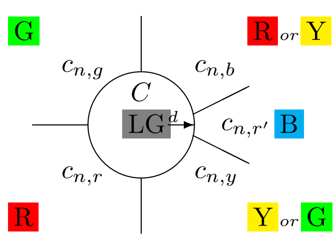


(iii) Without loss of generality, we can keep $c_{n,r}$-Red and $c_{n,g}$-Green, such that the *sub-minimal* map $\mathcal{M}_{\mathcal{C}}$ is now 4-colourable (requiring at most four colours by definition) as shown, for instance, in Fig.6 (where Yellow and Blue are inter-changeable) and:

(a) the areas $c_{n,r}$ and $c_{n,r'}$ are necessarily differently coloured;

(b) neither $c_{n,r}$ nor $c_{n,y}$ share a non-zero boundary with $c_{n,b}$ by the non-sharing Lemma 2.3.

Fig.7: After restoration $c_{n,r}$ and $c_{n,r'}$ cannot share identical colours in $\mathcal{H}$ as postulated



(iv) However, we now have the contradiction that:

- there is *no* 4-colouring of the *sub-minimal* map $\mathcal{M}_{\mathcal{C}}$ which,

- on restoration of the area C as the necessary 5^{th} coloured area in the *putatively minimal* planar map $\mathcal{H}$ (see Fig.7)

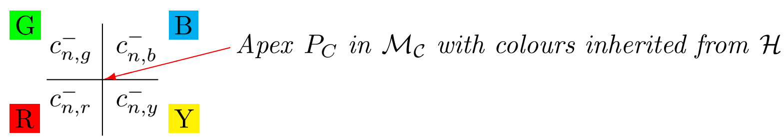
- would admit the identical colouring for $c_{n,r}$ and $c_{n,r'}$ in $\mathcal{H}$,

- as *postulated* in (i) above.

The Lemma follows. $\square$

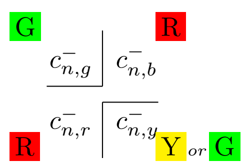
Theorem 1. *No chromatically differentiated planar map needs more than four colours.*

Fig.8: Areas which meet at the apex P_C in $\mathcal{M}_{\mathcal{C}}$ with colours inherited from $\mathcal{H}$



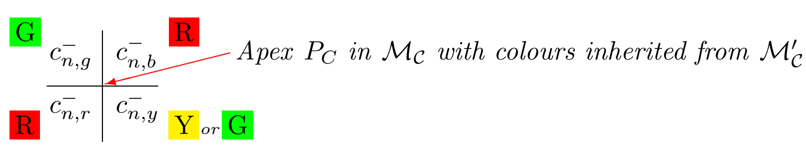
Proof. (1) By Lemma 3.1, only four differently coloured areas meet at the apex P_C (see Fig.8) of the *sub-minimal* map $\mathcal{M}_C$ in any colouring which is inherited from the *putatively minimal* map $\mathcal{H}$.

Fig.9: Merging areas $c^{-}_{n,r}$ and $c^{-}_{n,b}$ at the apex P_C in $\mathcal{M}_{\mathcal{C}}$ and recolouring $\mathcal{M}'_{\mathcal{C}}$



(2) Merging $c^{-}_{n,r}$ with $c^{-}_{n,b}$ at P_C (see Fig.9) now yields another *sub-minimal*, hence 4-colourable, map $\mathcal{M}'_{\mathcal{C}}$.

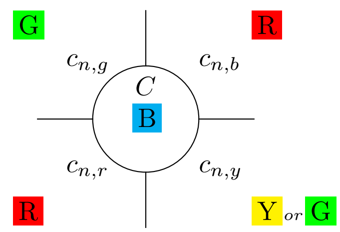
Fig.10: Restoring areas $c^{-}_{n,r}$ and $c^{-}_{n,b}$ at the apex P_C in $\mathcal{M}_{\mathcal{C}}$ with colours inherited from $\mathcal{M}'_{\mathcal{C}}$



(3) Keeping $c_{n,r}$-Red and $c_{n,g}$-Green in $\mathcal{M}'_{\mathcal{C}}$ (see Fig.10), and restoring P_C, further yields a fresh 4-colouring of $\mathcal{M}_C$ in which only 3 colours at most meet at the apex P_C.

(4) Recreating C in $\mathcal{M}_C$ now yields a chromatic differentiation of $\mathcal{H}$ that requires (see Fig.11) at most a 4^{th} colour for C; contradicting the *putative minimality* of $\mathcal{H}$.

Fig.11: Areas in $\mathcal{H}$ with C recreated from $\mathcal{M}_C$ and colours inherited from $\mathcal{M}'_C$



We conclude that Hypothesis 1 is false. The Theorem follows. □

4 The perceived 'flaw' in Kempe's 1879 argument

In their computer-assisted proof of the Four Colour Theorem [3], Appel and Haken review the 'flaw' in Kempe's 1879 'proof' [23]:

> "The first published attempt to prove the Four Color Theorem was made by A. B. Kempe [19] in 1879. Kempe proved that the problem can be restricted to the consideration of "normal planar maps" in which all faces are simply connected polygons, precisely three of which meet at each node. For such maps, he derived from Euler's formula, the equation
>
> (1.1) $$4p_2 + 3p_3 + 2p_4 + p_5 = \sum_{k=7}^{k_{max}} (k-6)p_k + 12$$
>
> where p_i is the number of polygons with precisely i neighbors and k_{max} is the largest value of i which occurs in the map. This equation immediately implies that every normal planar map contains polygons with fewer than six neighbors.
>
> In order to prove the Four Color Theorem by induction on the number p of polygons in the map ($p = \sum p_i$), Kempe assumed that every normal planar map with $p \leq r$ is four colorable and considered a normal planar map M_{r+1} with $r+1$ polygons. He distinguished the four cases that M_{r+1} contained a polygon P_2 with two neighbors, or a triangle P_3, or a quadrilateral P_4, or a pentagon P_5; at least one of these cases must apply by (1.1). In each case he produced a map M_r, with r polygons by erasing from M_{r+1} one edge in the boundary of an appropriate P_k. By the induction hypothesis, M_r admits a four coloring, say c_{r+1}, and Kempe attempted to derive a four coloring c_{r+1} of M_{r+1} from c_r. This task was very easy in the cases of P_2 and P_3. To treat the cases of P_4 and P_5, Kempe invented the method of interchanging the colors in a maximal connected part which was colored by c_r with a certain pair of colors (two-colored chains were later called Kempe chains) to obtain a coloring $c_r{}'$ of M_r from which one can then obtain a four coloring c_{r+1} of M_{r+1}.
>
> While Kempe's argument was correctly applied to the case of P_4, it was incorrectly applied to the case of P_5 as was shown by Heawood [18] in 1890." *. . . Appel and Haken: [3], §1. Introduction, p.429.*

We note, however, that the 'flaw' is not fatal if Kempe's argument is expressed *geometrically*.

Reason: The case Appel and Haken refer to as P_5 corresponds to §3, Lemma 3.1 where:

- We do not appeal—in a *graphical* representation of *minimal* 'normal planar maps'—to a 'method of interchanging the colors' in 'Kempe chains', so as to identify 'reducible' configurations in an 'unavoidable' set.

- Instead, we appeal—in a *geometrical* representation of *minimal* planar maps—to the Minimality Hypothesis 1, and argue that:

  - in any *minimal* planar map such as $\mathcal{H}$ in Fig.1,

  - any area such as P_5 which necessarily requires a 5^{th} colour,

  - cannot share non-zero boundaries with *two*, similarly coloured, neighbours.

This then yields P_4 as the sole configuration in an 'unavoidable' set. Moreover, as Appel and Haken note, P_4 is shown by Kempe to be 'reducible' (corresponding to the proof of the Four Colour Theorem in §3, Theorem 1).

5 Could there be an *un*perceived, inherited, 'flaw' in Appel and Haken's argument?

Unarguably meriting a philosophical discussion of consequences that lie beyond the immediate ambit of this investigation, we merely note here that:

If the 'flaw' in Kempe's 1879 'proof' [23] is perceived as falsely claiming to have proven the argument that:

> (α) Any *minimal* 'normal planar map' admits an *unavoidable* set containing a 'pentagon' that can be shown as *reducible*; where (cf. [16], Ch.8):
>
> (i) An *unavoidable* set is a set of configurations such that every map that satisfies some necessary conditions for being a minimal non-4-colorable triangulation (such as having minimum degree 5) must have at least one configuration from this set.
>
> (ii) A *reducible* configuration is one that cannot occur in a minimal counterexample. If a map contains a reducible configuration, the map can be reduced to a smaller map. This smaller map has the condition that if it can be colored with four colors, this also applies to the original map. This implies that if the original map cannot be colored with four colors the smaller map cannot either and so the original map is not minimal.

then the following remarks suggest that Appel and Haken's computer-dependent 'proof' in [3] (as also Robertson et al's proof in [9]), too could be viewed as 'flawed' (in the sense of being vacuously true, even if logically valid):

> "While Kempe's argument was correctly applied to the case of P_4, it was incorrectly applied to the case of P_5 as was shown by Heawood [18] in 1890. Kempe's argument proved, however, that five colors suffice for coloring planar maps and that a minimal counter-example to the Four Color Conjecture (minimal with respect to the number p of polygons in the map) could not contain any two-sided polygons, triangles, or quadrilaterals. This restricts the Four Color Problem to the consideration of normal planar maps in which each polygon has at least five neighbors. Each such map must contain at least twelve pentagons since in (1.1) we have $p_2 = p_3 = p_4 = 0$ and thus
>
> (1.2) $$p_5 = \sum_{k=7}^{k_{max}} (k-6)p_k + 12.$$
>
> Since 1890 a great many attempts have been made to find a proof of the Four Color Theorem. We distinguish two types of such attempts: (i) attempts to repair the flaw in Kempe's work; and (ii) attempts to find new and different approaches to the problem. Among attempts of type (i) we distinguish two subtypes: (i)(a) attempts to find an essentially stronger chain argument for "reducing the pentagon," i.e., proving that a minimal counter-example to the Four Color Conjecture cannot contain any pentagon, and thus does not exist; and (i)(b) attempts to make more extended use of Kempe's arguments in different directions and, instead of "reducing" the pentagon directly, to replace it by configurations of several polygons. Since the method used in this paper is of type (i)(b) we shall restrict our attention to further developments in this branch."
>
> *. . . Appel and Haken: [3], §1. Introduction, p.430.*

Reason: By Lemma 3.1 (essentially Appel/Haken's 'type (i)(a)') no *minimal* planar map can admit an 'unavoidable' set containing a pentagon.

In other words, both Kempe and Appel/Haken argue that:

(I) 4CT is equivalent to proving that, in any *minimal* 'normal planar map', there is an 'unavoidable' set of *two* configurations, P_4 and P_5, each of which is 'reducible';

(II) Kempe [23] has validly shown that the configuration P_4 is 'reducible'.

Moreover, Appel/Haken restricted their argument to 'type (i)(b)' to further argue that:

(a) Kempe did not prove in [23] that the configuration P_5 is 'reducible'.

(b) If each of the 1,482 configurations—'in which each polygon has at least five neighbors', as *manually* defined in their 'unavoidable' set in [3]—is 'reducible', then P_5 is 'reducible';

(c) A computer-dependent proof validates that each of the 1,482 configurations is 'reducible';

(d) Hence 4CT is proven.

However, Lemma 3.1 shows that argument (I) *implicitly* appeals to an invalid assumption, since no *minimal* planar map can contain a configuration such as P_5; whence (d) would hold *vacuously* as having proven:

> (β) If every *minimal* planar map admits an 'unavoidable' set containing a five-sided figure such as P_5, then P_5 is 'reducible'.

and not that:

> (γ) No *minimal* planar map can admit an 'unavoidable' set containing a five-sided figure such as P_5.

6 Why the *geometrical* proof of 4CT may not be expressible *graphically*

We note that, since classical graph theory (see, for instance, Brun [12], Conradie/Goranko [17], Gardner [21]) represents non-empty areas as points (*vertices*), and a non-zero boundary between two areas as a line (*edge*) joining two points (*vertices*) (see Fig.12), the theory does not immediately *evidence* a graphical proof of Theorem 1.

Fig.12: Graphical representation of $\mathcal{H}$

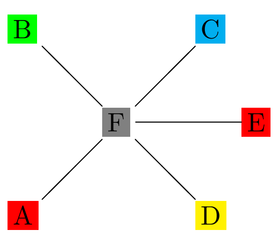
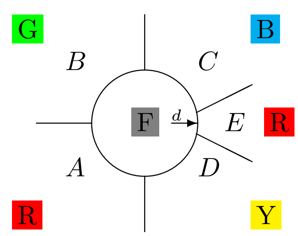


Fig.13: Geometrical representation of $\mathcal{H}$



In other words, the proof of Theorem 1 appeals critically to re-configuring the *geometrical* representation of the, putatively *minimal*, planar map $\mathcal{H}$ in Fig.13 by:

Fig.14: Graphical representation of $\mathcal{M}_{\mathcal{F}}$

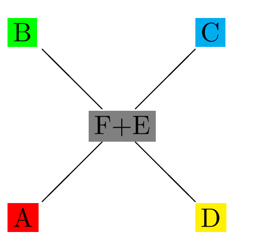
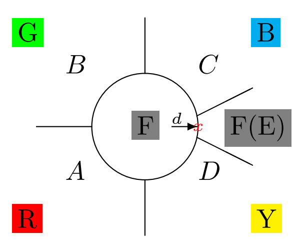


Fig.15: Geometrical representation of $\mathcal{M}_{\mathcal{F}}$



- first *removing* (see Fig.15), and then *restoring* after recolouring (see Fig.17), the non-zero boundary d in Fig.13,

- to merge/de-merge the areas F and E (in Fig.13), in a *geometrically* distinguishable way, that, prima facie, cannot be immediately *evidenced* in the corresponding argument, *when* represented *graphically* by Figs.12, 14 and 16.

Fig.16: Graphical representation of $\mathcal{H}$

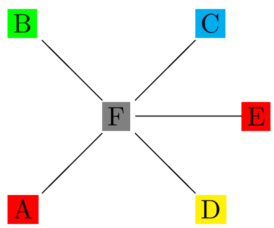
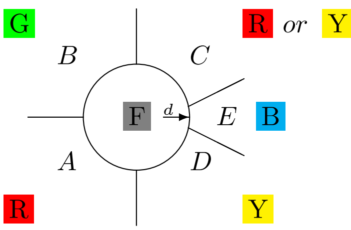


Fig.17: Geometrical representation of $\mathcal{H}$



We thus speculate that the barriers to proving 4CT *graphically* may possibly lie in Alfred Kempe's unsupported *postulation*, that the four-color map problem could be reformulated *equivalently* as a problem involving *linkages* between the 'lettering' of colours at *unspecified* points of a map in a *graph*:

> "If we lay a sheet of tracing paper over a map and mark a point on it over each district and connect the points corresponding to districts which have a common boundary, we have on the tracing paper a diagram of a "linkage," and we have as the exact analogue of the question we have been considering, that of lettering the points in the linkage with as few letters as possible, so that no two directly connected points shall be lettered with the same letter. Following this up, we may ask what are the linkages which can be similarly lettered with not less than n letters?
>
> The classification of linkages according to the value of n is one of considerable importance. I shall not, however, enter here upon this question, as it is one which I propose to consider as part of an investigation upon which I am engaged as to the general theory of linkages. It is for this reason also that I have preferred to treat the question discussed in this paper in the manner I have done, instead of dealing with the analogous linkage." *. . . Kempe: [23], p.200*

In other words, it is conceivable—perhaps even likely—that Kempe was misled by a *pseudo-graphical* representation of $\mathcal{M}_{\mathcal{F}}$ (see Fig.18) into believing that a *graphical* argument *must* follow which entails that a five-sided configuration in $\mathcal{H}$ (see Fig.12) must be 'reducible'.

Fig.18: Pseudo-graphical representation of $\mathcal{M}_{\mathcal{F}}$

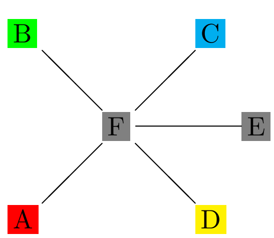
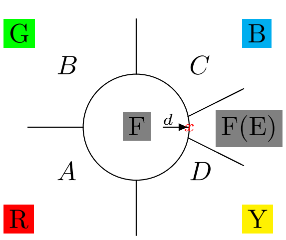


Fig.19: Geometrical representation of $\mathcal{M}_{\mathcal{F}}$



Reason: In a *pseudo-graphical* representation—as shown in Fig.18 just *before* their merger—countries A and E obviously could not have been identically coloured in any 4-colouring of $\mathcal{M}_{\mathcal{F}}$ inherited by, or from, $\mathcal{H}$ (as is evident in the *geometrical* representation in Fig.19).

> **Comment**: The intuitive truth of Lemma 3.1 is evidenced by the fact that it is trivial to paste (or even merely imagine pasting) a small, grey, piece of paper anywhere on a children's 4-coloured globe to confirm that:
>
> - there is never a need to re-colour the, inherited, 4-coloured areas of the globe in order to maintain chromatic differentiation; and,
> - if the grey area increases the number of countries by *one* by partitioning an existing country into *two*, none of the *other* countries abutting the grey area *after* partitioning could have shared the same colour as the partitioned country *before* partitioning;
> - whence we cannot create a *minimal* map from a *sub-minimal* map in which the created area *both* requires a fifth colour *and* abuts two areas which share a common colour.

This could account for Kempe's intuitively 'preferred' alternative in informal explanations vis à vis his explicit assumption that a formal representation by 'linkages' may be viewed 'as the exact analogue' of the four-color map problem; a preference reflected in Robin Wilson's *italicised* remark in [16], wherein he too, seemingly *uncritically*, assumes such an equivalence:

> "Any coloring of the countries of the map gives rise to a lettering of the points in the linkage in which no two directly connected points are lettered the same.
>
> We now refer to such a linkage as a *graph* . . . and to the preceding process as forming the graph (or dual graph) of the map. This reformulation of the four-color problem as a problem involving the lettering of points reappeared briefly in the 1880s (see Chapter 6) and was later reintroduced in the 1930s and used in all subsequent attempts to solve the problem.
>
> *So as not to complicate matters, we shall usually stick to coloring the countries of maps (rather than switching to lettering the points of a graph) throughout the rest of this book.*" *. . . Wilson: [16], p.67.*

It is thus also conceivable that subsequent articulations of 4CT failed to recognise the *geometrical* argument in Lemma 3.1 *only* because Kempe's formal appeal to Euler's formula $V + F = E + 2$ 'seemingly' simplified the problem substantially by entailing that every *minimal* planar map must contain a configuration of fewer than six sides.

'Seemingly', since it is not obvious—unlike the *geometrical* argument of Lemma 3.1 which is immediately evident in Fig.15—that:

- a *graphical* argument *must* follow from Fig.14,
- which admits the possibility that a five-sided figure may *not* be definable in a *minimal* planar map[7].

[7] In which case any proof of 4CT that appeals to the argument that every *minimal* planar map contains a five-sided figure which is *reducible* would be *vacuous*.

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