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The Truth Assignments That Differentiate Human Reasoning From Mechanistic Reasoning

The Evidence-Based Argument for Lucas' Gödelian Thesis

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Abstract. We consider the argument that Tarski's classic definitions permit an intelligence—whether human or mechanistic—to admit *finitary* evidence-based definitions of the satisfaction and truth of the *atomic* formulas of the first-order Peano Arithmetic PA over the domain $\mathbb{N}$ of the natural numbers in two, hitherto unsuspected and essentially different, ways: (1) in terms of *classical* algorithmic verifiabilty; and (2) in terms of *finitary* algorithmic computability. We then show that the two definitions correspond to two distinctly different assignments of satisfaction and truth to the *compound* formulas of PA over $\mathbb{N}$—$\mathcal{I}_{PA(\mathbb{N},\ SV)}$ and $\mathcal{I}_{PA(\mathbb{N},\ SC)}$. We further show that the PA axioms are true over $\mathbb{N}$, and that the PA rules of inference preserve truth over $\mathbb{N}$, under both $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ and $\mathcal{I}_{PA(\mathbb{N},\ SC)}$. We then show: (a) that if we assume the satisfaction and truth of the *compound* formulas of PA are always *non-finitarily* decidable under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$, then this assignment corresponds to the classical *non-finitary* putative standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ S)}$ of PA over the domain $\mathbb{N}$; and (b) that the satisfaction and truth of the *compound* formulas of PA are always *finitarily* decidable under the assignment $\mathcal{I}_{PA(\mathbb{N},\ SC)}$, from which we may *finitarily* conclude that PA is consistent. We further conclude that the appropriate inference to be drawn from Gödel's 1931 paper on undecidable arithmetical propositions is that we can define PA formulas which—under interpretation—are algorithmically verifiable as always true over $\mathbb{N}$, but not algorithmically computable as always true over $\mathbb{N}$. We conclude from this that Lucas' Gödelian argument is validated if the assignment $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ can be treated as circumscribing the ambit of human reasoning about 'true' arithmetical propositions, and the assignment $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ as circumscribing the ambit of mechanistic reasoning about 'true' arithmetical propositions.

Keywords. Algorithmic computability, Algorithmic verifiability, Arithmetical satisfaction, Arithmetical truth, Arithmetical provability, Classical arithmetical reasoning, Cauchy sequence, Consistency of Arithmetic, Evidence-based interpretation, Finitary arithmetical reasoning, Gödel's β-function, Gödelian undecidability, Human Reasoning, Lucas' Gödelian argument, Hilbert's First Problem, Mechanistic Reasoning, Peano Arithmetic PA, Poincaré-Hilbert debate, Standard interpretation, Tarski's definitions, Uncomputability.

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Contents

1. Introduction

We briefly consider a philosophical challenge that arises when an intelligence—whether human or mechanistic—accepts arithmetical propositions as true under an interpretation—either axiomatically or on the basis of subjective self-evidence—*without* any specified methodology for evidencing such acceptance[1].

For instance conventional wisdom, whilst accepting Alfred Tarski's classical definitions of the satisfiability and truth of the formulas of a formal language under an interpretation as adequate to the intended purpose, *postulates* that under the classical putative standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ Standard,\ Classical)}$[2] of the first-order Peano Arithmetic PA[3] over the domain $\mathbb{N}$ of the natural numbers:

(i) The satisfiability/truth of the atomic formulas of PA can be assumed as *uniquely* decidable under $\mathcal{I}_{PA(\mathbb{N},\ S)}$;

(ii) The PA axioms can be assumed to *uniquely* interpret as satisfied/true under $\mathcal{I}_{PA(\mathbb{N},\ S)}$;

(iii) The PA rules of inference—Generalisation and Modus Ponens—can be assumed to *uniquely* preserve such satisfaction/truth under $\mathcal{I}_{PA(\mathbb{N},\ S)}$;

(iv) Aristotle's particularisation[4] can be assumed to hold under $\mathcal{I}_{PA(\mathbb{N},\ S)}$.

We shall argue that the seemingly innocent and self-evident assumptions of *uniqueness* in (i) to (iii)—as also the seemingly innocent assumption in (iv) which, despite being obviously *non-finitary*, is unquestioningly accepted in classical literature[5] as equally self-evident under any logically unexceptionable interpretation of the classical first-order logic FOL—conceal an ambiguity with far-reaching consequences.

The ambiguity is revealed if we note[6] that Tarski's classic definitions permit both human and mechanistic intelligences to admit *finitary*[7] evidence-based definitions of the satisfaction and truth of the *atomic* formulas of PA over the domain $\mathbb{N}$ of the natural numbers in two, hitherto unsuspected and essentially different, ways:

(1a) In terms of *classical* algorithmic verifiabilty; and

(1b) In terms of *finitary* algorithmic computability.

We shall show[8] that:

(2a) The two definitions correspond to two distinctly different assignments of satisfaction and truth to the *compound* formulas of PA over $\mathbb{N}$—say $\mathcal{I}_{PA(\mathbb{N},\ Standard,\ Verifiable)}$ and $\mathcal{I}_{PA(\mathbb{N},\ Standard,\ Computable)}$[9]; where

[1] For a brief recent review of such challenges, see [Fe06], [Fe08]; also [An04] and Rodrigo Freire's informal essay on '*Interpretation and Truth in Cantorian Set Theory*'.

[2] See Section 9., Appendix A. We shall refer to this henceforth as $\mathcal{I}_{PA(\mathbb{N},\ S)}$.

[3] We take this to be the first-order theory S defined in any standard text such as [Me64], p.102.

[4] See Section 9., Appendix A. Informally, Aristotle's particularisation is the *non-finitary* assumption that an assertion such as, 'There exists an x such that $F(x)$ holds'—usually denoted symbolically by '$(\exists x)F(x)$'—can always be validly inferred in the classical logic of predicates from the assertion, 'It is not the case that: for any given x, $F(x)$ does not hold'—usually denoted symbolically by '$\neg(\forall x)\neg F(x)$' ([HA28], pp.58-59).

[5] See Section 9., Appendix A.

[6] See [An12] and [An15].

[7] We mean 'finitary' in the sense that "...there should be an algorithm for deciding the truth or falsity of any mathematical statement" ... http://en.wikipedia.org/wiki/Hilbert's_program. For a brief review of 'finitism' and 'constructivity' in the context of this paper see [Fe08].

[8] cf. [An12] and [An15].

[9] We shall refer to these henceforth as $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ and $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ respectively.

(2b) The PA axioms are true over $\mathbb{N}$, and the PA rules of inference preserve truth over $\mathbb{N}$, under both $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ (Section 5.A.) and $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ (Section 6.A.).

We shall then show that[10]:

(3a) If we assume the satisfaction and truth of the compound formulas of PA are always *non-finitarily* decidable under the assignment $\mathcal{I}_{PA(\mathbb{N},\ SV)}$, then this assignment defines a *non-finitary* interpretation of PA in which Aristotle's particularisation always holds over $\mathbb{N}$; and which may be taken to correspond to the intended (putative) classical *non-finitary* standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ S)}$ of PA over the domain $\mathbb{N}$—from which only a human intelligence may *non-finitarily* conclude that PA is consistent; whilst

(3b) The satisfaction and truth of the compound formulas of PA are always *finitarily* decidable under the assignment $\mathcal{I}_{PA(\mathbb{N},\ SC)}$, which thus defines a *finitary* interpretation of PA—from which both intelligences may *finitarily* conclude that PA is consistent[11].

We shall show further that both intelligences would logically conclude that:

(4a) The assignment $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ defines a subset of PA formulas that are algorithmically computable as true under the putative standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ S)}$ if, and only if, the formulas are PA provable;

(4b) PA is not ω-consistent[12]; and

(4c) PA is categorical with respect to algorithmic computability.

Both intelligences would also logically conclude that:

(5a) Since PA is not ω-consistent, Gödel's argument in [Go31] (p.28(2))—that "$Neg(17Gen\ r)$ is not κ-PROVABLE"[13]—does not yield a 'formally undecidable proposition' in PA[14];

(5b) The appropriate conclusion to be drawn from Gödel's argument in [Go31] (p.27(1))—that "$17Gen\ r$ is not κ-PROVABLE"—is that his 'undecidable arithmetical proposition' is an instantiation of the argument[15] that we can define number-theoretic formulas which are algorithmically verifiable as always true, but not algorithmically computable as always true.

We shall finally conclude from this that:

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10 cf. [An12] and [An15].

11 As sought by David Hilbert for the second of the twenty three problems that he highlighted at the International Congress of Mathematicians in Paris in 1900.

12 See Section 9., Appendix A.

13 The reason we prefer to consider Gödel's original argument (rather than any of its subsequent avatars) is that, for a purist, Gödel's remarkably self-contained 1931 paper—it neither contained, nor needed, any formal citations—remains unsurpassed in mathematical literature for thoroughness, clarity, transparency and soundness of exposition, *from first principles* (thus avoiding any implicit mathematical or philosophical assumptions), of his notion of arithmetical 'undecidability' as based on his Theorems VI and XI and their logical consequences.

14 We note that if PA is not ω-consistent, then Aristotle's particularisation does not hold in any finitary interpretation of PA over $\mathbb{N}$. Now, J. Barkeley Rosser's 'undecidable' arithmetical proposition in [Ro36] is of the form $[(\forall y)(Q(h, y) \rightarrow (\exists z)(z \leq y \wedge S(h, z)))]$. Thus his 'extension' of Gödel's proof of undecidability too does not yield a 'formally undecidable proposition' in PA, since it assumes that Aristotle's particularisation holds when interpreting $[(\forall y)(Q(h, y) \rightarrow (\exists z)(z \leq y \wedge S(h, z)))]$ under a finitary interpretation over $\mathbb{N}$ ([Ro36], Theorem II, pp.233-234; [Kl52], Theorem 29, pp.208-209; [Me64], Proposition3.32, pp.145-146).

15 Corresponding to Cantor's diagonal argument and Turing's halting argument as reflected in Theorem 2.1.

Lucas' Gödelian argument[16] is validated if the assignment $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ can be treated as circumscribing the ambit of human reasoning about 'true' arithmetical propositions, and the assignment $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ can be treated as circumscribing the ambit of mechanistic reasoning about 'true' arithmetical p[ropositions.

2. Defining algorithmic verifiability and algorithmic computability

We begin by introducing the following two concepts:

Definition 1. ***Algorithmic verifiability:***

A number-theoretical relation $F(x)$ is algorithmically verifiable if, and only if, for any given natural number n, there is an algorithm $AL_{(F,\ n)}$ which can provide objective evidence[17] for deciding the truth/falsity of each proposition in the finite sequence $\{F(1), F(2), \ldots, F(n)\}$.

Definition 2. ***Algorithmic computability:***

A number theoretical relation $F(x)$ is algorithmically computable if, and only if, there is an algorithm AL_F that can provide objective evidence for deciding the truth/falsity of each proposition in the denumerable sequence $\{F(1), F(2), \ldots\}$.

We note that algorithmic computability implies the existence of an algorithm that can finitarily decide the truth/falsity of each proposition in a well-defined denumerable sequence of propositions[18], whereas algorithmic verifiability does not imply the existence of an algorithm that can finitarily decide the truth/falsity of each proposition in a well-defined denumerable sequence of propositions.

The following argument shows that although every algorithmically computable relation is algorithmically verifiable, the converse is not true.

Theorem 2.1. *There are number theoretic functions that are algorithmically verifiable but not algorithmically computable.*

Proof: (a) Since any real number R is mathematically definable as the unique limit of a correspondingly unique Cauchy sequence $\{\Sigma_{i=0}^{n} r(i).2^{-i} : n = 0, 1, \ldots\}$ of rational numbers:

- Let $r(n)$ denote the n^{th} digit in the decimal expression of the real number $R = Lt_{n \rightarrow \infty} \Sigma_{i=0}^{n} r(i).2^{-i}$ in binary notation.

- Then, for any given natural number n, Gödel's β-function[19] defines an algorithm $AL_{(R,\ n)}$ that can verify the truth/falsity of each proposition in the finite sequence:

[16] Which Lucas advanced in [Lu61].

[17] cf. [Mu91]: "It is by now folklore ... that one can view the *values* of a simple functional language as specifying *evidence* for propositions in a constructive logic ...".

[18] We note that the concept of 'algorithmic computability' is essentially an expression of the more rigorously defined concept of 'realizability' in [Kl52], p.503.

[19] In Theorem VII of his 1931 paper Gödel defined ([Go31], p.31, Lemma 1; see also [Me64], p.131, Proposition 3.21) a primitive recursive function—Gödel's β-function—as:

$\beta(x_1, x_2, x_3) = rm(1 + (x_3 + 1) \star x_2, x_1)$

where $rm(x_1, x_2)$ denotes the remainder obtained on dividing x_2 by x_1. Gödel then showed that, for any non-terminating sequence of values $f(x_1, 0), f(x_1, 1), \ldots$, we can construct natural numbers b, c such that:

(i) $j = max(n, f(x_1, 0), f(x_1, 1), \ldots, f(x_1, n))$;

(ii) $c = j!$;

(iii) $\beta(b, c, i) = f(x_1, i)$ for $0 \leq i \leq n$.

and that $\beta(x_1, x_2, x_3)$ is strongly represented in PA by $[Bt(x_1, x_2, x_3, x_4)]$, which is defined as follows:

$[(\exists w)(x_1 = ((1 + (x_3 + 1) \star x_2) \star w + x_4) \wedge (x_4 < 1 + (x_3 + 1) \star x_2))]$.

$\{r(0) = 0, r(1) = 0, \ldots, r(n) = 0\}$.

- Hence, for any real number R, the relation $r(x) = 0$ is algorithmically verifiable trivially.

(b) Since it follows from Alan Turing's Halting argument[20] that there are algorithmically uncomputable real numbers:

- Let $r(n)$ denote the n^{th} digit in the decimal expression of an algorithmically *uncomputable* real number R in binary notation.

- By (a), the relation $r(x) = 0$ is algorithmically verifiable trivially.

- However, by definition there is no algorithm AL_R that can decide the truth/falsity of each proposition in the denumerable sequence:

  $\{r(0) = 0, r(1) = 0, \ldots\}$.

- Hence the relation $r(x) = 0$ is algorithmically verifiable but not algorithmically computable. □

3. Reviewing Tarski's inductive assignment of truth-values under an interpretation

We shall essentially follow standard expositions[21] of Tarski's inductive definitions on the 'satisfiability' and 'truth' of the formulas of a formal language under an interpretation where:

Definition 3. *If $[A]$ is an atomic formula $[A(x_1, x_2, \ldots, x_n)]$*[22] *of a formal language S, then the denumerable sequence $(a_1, a_2, \ldots)$ in the domain $\mathbb{D}$ of an interpretation $\mathcal{I}_{S(\mathbb{D})}$ of S satisfies $[A]$ if, and only if:*

> *(i) $[A(x_1, x_2, \ldots, x_n)]$ interprets under $\mathcal{I}_{S(\mathbb{D})}$ as a unique relation $A^*(x_1, x_2, \ldots, x_n)$ in $\mathbb{D}$ for any witness $\mathcal{W}_{\mathbb{D}}$ of $\mathbb{D}$;*
>
> *(ii) there is a Satisfaction Method that provides objective evidence*[23] *by which any witness $\mathcal{W}_{\mathbb{D}}$ of $\mathbb{D}$ can objectively* ***define*** *for any atomic formula $[A(x_1, x_2, \ldots, x_n)]$ of S, and any given denumerable sequence $(b_1, b_2, \ldots)$ of $\mathbb{D}$, whether the proposition $A^*(b_1, b_2, \ldots, b_n)$ holds or not in $\mathbb{D}$;*
>
> *(iii) $A^*(a_1, a_2, \ldots, a_n)$ holds in $\mathbb{D}$ for any $\mathcal{W}_{\mathbb{D}}$.*
>
> **Witness:** From a constructive perspective, the existence of a 'witness' as in (i) above is implicit in the usual expositions of Tarski's definitions.
>
> **Satisfaction Method:** From a constructive perspective, the existence of a Satisfaction Method as in (ii) above is also implicit in the usual expositions of Tarski's definitions.
>
> **A constructive perspective:** We highlight the word '***define***' in (ii) above to emphasise the constructive perspective underlying this paper; which is that the concepts of 'satisfaction' and 'truth' under an interpretation are to be explicitly viewed as objective assignments by a convention that is witness-independent. A Platonist perspective would substitute 'decide' for 'define', thus implicitly suggesting that these concepts can 'exist', in the sense of needing to be discovered by some witness-dependent means—eerily akin to a 'revelation'—if the domain $\mathbb{D}$ is $\mathbb{N}$.

[20] [Tu36], p.132, §8.

[21] See Section 9., Appendix A.

[22] We shall use square brackets to indicate that the contents represent a symbol or a formula of a formal theory, generally assumed to be well-formed unless otherwise indicated by the context.

[23] In the sense of [Mu91].

We further define the truth values of 'satisfaction', 'truth', and 'falsity' for the compound formulas of a first-order theory S under the interpretation $\mathcal{I}_{S(\mathbb{D})}$ in terms of *only* the satisfiability of the atomic formulas of S over $\mathbb{D}$ as usual[24].

We now show how Tarski's definitions yield two distinctly different 'standard' interpretations of the first-order Peano Arithmetic PA.

4. The ambiguity in the classical putative standard interpretation of PA over the domain $\mathbb{N}$ of the natural numbers

The classical putative standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ S)}$ of PA over the domain $\mathbb{N}$ of the natural numbers is obtained if, in $\mathcal{I}_{S(\mathbb{D})}$:

(a) we define S as PA with standard first-order predicate calculus as the underlying logic[25];

(b) we define $\mathbb{D}$ as the set $\mathbb{N}$ of natural numbers;

(c) we assume for any atomic formula $[A(x_1, x_2, \ldots, x_n)]$ of PA, and any given sequence $(b_1^*, b_2^*, \ldots, b_n^*)$ of $\mathbb{N}$, that the proposition $A^*(b_1^*, b_2^*, \ldots, b_n^*)$ is decidable in $\mathbb{N}$;

(d) we define the witness $\mathcal{W}_{(\mathbb{N},\ Standard,\ Classical)}$ informally as the 'mathematical intuition' of a human intelligence for whom, classically, (c) has been *implicitly* accepted as *objectively* 'decidable' in $\mathbb{N}$.

(e) we postulate that Aristotle's particularisation holds over $\mathbb{N}$[26].

Clearly, (e) does not form any part of Tarski's inductive definitions of the satisfaction, and truth, of the formulas of PA under the above interpretation. Moreover, its inclusion makes $\mathcal{I}_{PA(\mathbb{N},\ S)}$ extraneously non-finitary[27].

We shall show that the implicit acceptance in (d) conceals an ambiguity that needs to be made explicit since:

Lemma 4.1. *$A^*(x_1, x_2, \ldots, x_n)$ is both algorithmically verifiable and algorithmically computable in $\mathbb{N}$ by $\mathcal{W}_{(\mathbb{N},\ Standard,\ Classical)}$.*

Proof (i) It follows from the argument in Theorem 5.1 (below) that $A^*(x_1, x_2, \ldots, x_n)$ is algorithmically verifiable in $\mathbb{N}$ by $\mathcal{W}_{(\mathbb{N},\ Standard,\ Classical)}$.

(ii) It follows from the argument in Theorem 6.1 (below) that $A^*(x_1, x_2,\ \ldots, x_n)$ is algorithmically computable in $\mathbb{N}$ by $\mathcal{W}_{(\mathbb{N},\ Standard,\ Classical)}$. The lemma follows. □

We note without proof that[28] (i) is consistent with, whilst (ii) is inconsistent with, the assumption of Aristotle's particularisation.

[24] See Section 9., Appendix A.

[25] Where the string $[(\exists \ldots)]$ is defined as—and is to be treated as an abbreviation for—the PA formula $[\neg(\forall \ldots)\neg]$. We do not consider the case where the underlying logic is Hilbert's formalisation of Aristotle's logic of predicates in terms of his ϵ-operator ([Hi27], pp.465-466).

[26] This postulates that a PA formula such as $[(\exists x)F(x)]$ can always be taken to interpret under $\mathcal{I}_{PA(\mathbb{N},\ S)}$ as 'There is some natural number n such that $F(n)$ holds in $\mathbb{N}$.

[27] As argued by Brouwer in [Br08].

[28] For a more detailed argument see [An12] and [An15].

5. The standard verifiable interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA over $\mathbb{N}$

We now consider a standard verifiable interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA, under which we define:

Definition 4. *An atomic formula $[A]$ of PA is satisfiable under the interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ if, and only if, $[A]$ is algorithmically verifiable under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.*

We note that:

Theorem 5.1. *The atomic formulas of PA are algorithmically verifiable as true or false under the standard verifiable interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.*

Proof It follows from Gödel's definition of the primitive recursive relation xBy[29]—where x is the Gödel number of a proof sequence in PA whose last term is the PA formula with Gödel-number y—that, if $[A]$ is an atomic formula of PA, we can algorithmically verify which one of the PA formulas $[A]$ and $[\neg A]$ is necessarily PA-provable and, ipso facto, true under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$. $\Box$

We note that the interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ cannot claim to be finitary[30].

Reason: It follows from Theorem 2.1 that we cannot conclude finitarily from Tarski's Definitions 3 to 10 whether or not a quantified PA formula $[(\forall x_i)R]$ is algorithmically verifiable as true under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ if $[R]$ is algorithmically verifiable but not algorithmically computable under the interpretation[31].

5.A. The PA axioms are algorithmically verifiable as true under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$

The significance of defining satisfaction in terms of algorithmic verifiability under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ is that:

Lemma 5.2. *The PA axioms PA_1 to PA_8 are algorithmically verifiable as true over $\mathbb{N}$ under the interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.*

Proof Since $[x+y]$, $[x\star y]$, $[x=y]$, $[x']$ are defined recursively[32], the PA axioms PA_1 to PA_8 interpret as recursive relations that do not involve any quantification. The lemma follows straightforwardly from Theorem 5.1 and Tarski's Definitions 3 to 10. $\Box$

Lemma 5.3. *For any given PA formula $[F(x)]$, the Induction axiom schema $[F(0) \rightarrow (((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x))]$ interprets as an algorithmically verifiable true formula under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.*

Proof

(a) If $[F(0)]$ interprets as an algorithmically verifiable false formula under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ the lemma is proved.

> *Reason*: Since $[F(0) \rightarrow (((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x))]$ interprets as an algorithmically verifiable true formula under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ if, and only if, either $[F(0)]$ interprets as an algorithmically verifiable false formula or $[((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x)]$ interprets as an algorithmically verifiable true formula under under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.

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[29] [Go31], p. 22(45).

[30] See [An12] and [An15] for a proof that $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ is non-finitary, since it defines a model of PA if, and only if, PA is ω-consistent and so we may always non-finitarily conclude from $[(\exists x)R(x)]$ the existence of some numeral $[n]$ such that $[R(n)]$.

[31] Although a proof that such a PA formula exists is not obvious, we shall show that Gödel's 'undecidable' arithmetical formula $[R(x)]$ is algorithmically verifiable but not algorithmically computable under the interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.

[32] cf. [Go31], p.17.

(b) If $[F(0)]$ interprets as an algorithmically verifiable true formula, and $[(\forall x)(F(x) \rightarrow F(x'))]$ interprets as an algorithmically verifiable false formula, under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$, the lemma is proved.

(c) If $[F(0)]$ and $[(\forall x)(F(x) \rightarrow F(x'))]$ both interpret as algorithmically verifiable true formulas under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ then, for any natural number n, there is an algorithm which (by Definition 1) will evidence that $[F(n) \rightarrow F(n')]$ is an algorithmically verifiable true formula under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.

(d) Since $[F(0)]$ interprets as an algorithmically verifiable true formula under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$, it follows for any natural number n that there is an algorithm which will evidence that each of the formulas in the finite sequence $\{[F(0),\ F(1),\ \ldots,\ F(n)\}]$ is an algorithmically verifiable true formula under the interpretation.

(e) Hence $[(\forall x)F(x)]$ is an algorithmically verifiable true formula under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.

Since the above cases are exhaustive, the lemma follows. □

We note that if $[F(0)]$ and $[(\forall x)(F(x) \rightarrow F(x'))]$ both interpret as algorithmically verifiable true formulas under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$, then we can only conclude that, for any natural number n, there is an algorithm which will give evidence for any $m \leq n$ that the formula $[F(m)]$ is true under $\mathcal{I}_{PA(\mathbb{N},\ S)}$.

We cannot conclude that there is an algorithm which, for any natural number n, will give evidence that the formula $[F(n)]$ is true under $\mathcal{I}_{PA(\mathbb{N},\ S)}$.

Lemma 5.4. *Generalisation preserves algorithmically verifiable truth under* $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.

Proof The two meta-assertions:

'$[F(x)]$ interprets as an algorithmically verifiable true formula under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$[33]'

and

'$[(\forall x)F(x)]$ interprets as an algorithmically verifiable true formula under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$'

both mean:

$[F(x)]$ is algorithmically verifiable as always true under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$. □

It is also straightforward to see that:

Lemma 5.5. *Modus Ponens preserves algorithmically verifiable truth under* $\mathcal{I}_{PA(\mathbb{N},\ SV)}$. □

We thus have that:

Theorem 5.6. *The axioms of PA are always algorithmically verifiable as true under the interpretation* $\mathcal{I}_{PA(\mathbb{N},\ SV)}$, *and the rules of inference of PA preserve the properties of algorithmically verifiable satisfaction/truth under* $\mathcal{I}_{PA(\mathbb{N},\ SV)}$. □

By Theorem 5.1 we conclude that:

Theorem 5.7. *If the PA formulas are algorithmically verifiable as true or false under* $\mathcal{I}_{PA(\mathbb{N},\ SV)}$, *then PA is consistent.* □

We note that, like Gentzen's argument, such a proof of consistency would be debatably 'finitary', since we cannot conclude from Theorem 5.1 that the quantified formulas of PA are 'finitarily' decidable as true or false under the interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.

[33] See Definition 9

6. The standard computable interpretation $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ of PA over $\mathbb{N}$

We next consider a standard computable interpretation $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ of PA, under which we define:

Definition 5. *An atomic formula $[A]$ of PA is satisfiable under the interpretation $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ if, and only if, $[A]$ is algorithmically computable under $\mathcal{I}_{PA(\mathbb{N},\ SC)}$.*

We note that:

Theorem 6.1. *The atomic formulas of PA are algorithmically computable as true or as false under the standard computable interpretation $\mathcal{I}_{PA(\mathbb{N},\ SC)}$.*

Proof If $[A(x_1, x_2, \ldots, x_n)]$ is an atomic formula of PA then, for any given sequence of numerals $[b_1, b_2, \ldots, b_n]$, the PA formula $[A(b_1, b_2, \ldots, b_n)]$ is an atomic formula of the form $[c = d]$, where $[c]$ and $[d]$ are atomic PA formulas that denote PA numerals. Since $[c]$ and $[d]$ are recursively defined formulas in the language of PA, it follows from a standard result[34] that $[c = d]$ is algorithmically computable as either true or false in $\mathbb{N}$ since there is an algorithm that, for any given sequence of numerals $[b_1, b_2, \ldots, b_n]$, will give evidence whether $[A(b_1, b_2, \ldots, b_n)]$ interprets as true or false in $\mathbb{N}$. The lemma follows. □

We note that the interpretation $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ is finitary since:

Lemma 6.2. *The formulas of PA are algorithmically computable finitarily as true or as false under $\mathcal{I}_{PA(\mathbb{N},\ SC)}$.*

Proof The Lemma follows by finite induction from Definition 2, Tarski's Definitions 3 to 10, and Theorem 6.1. □

6.A. The PA axioms are algorithmically computable as true under $\mathcal{I}_{PA(\mathbb{N},\ SC)}$

The significance of defining satisfaction in terms of algorithmic computability under $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ as above is that:

Lemma 6.3. *The PA axioms PA_1 to PA_8 are algorithmically computable as true under the interpretation $\mathcal{I}_{PA(\mathbb{N},\ SC)}$.*

Proof Since $[x+y]$, $[x \star y]$, $[x = y]$, $[x']$ are defined recursively[35], the PA axioms PA_1 to PA_8 interpret as recursive relations that do not involve any quantification. The lemma follows straightforwardly from Definitions 3 to 10 in Section 3. and Theorem 5.1. □

Lemma 6.4. *For any given PA formula $[F(x)]$, the Induction axiom schema $[F(0) \rightarrow (((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x))]$ interprets as an algorithmically computable true formula under $\mathcal{I}_{PA(\mathbb{N},\ SC)}$.*

Proof By Definitions 3 to 10:

(a) If $[F(0)]$ interprets as an algorithmically computable false formula under $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ the lemma is proved.

[34] For any natural numbers $m,\ n$, if $m \neq n$, then PA proves $[\neg(m = n)]$ ([Me64], p.110, Proposition 3.6). The converse is obviously true.

[35] cf. [Go31], p.17.

Since $[F(0) \rightarrow (((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x))]$ interprets as an algorithmically computable true formula if, and only if, either $[F(0)]$ interprets as an algorithmically computable false formula, or $[((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x)]$ interprets as an algorithmically computable true formula, under $\mathcal{I}_{PA(\mathbb{N},\ SC)}$.

(b) If $[F(0)]$ interprets as an algorithmically computable true formula, and $[(\forall x)(F(x) \rightarrow F(x'))]$ interprets as an algorithmically computable false formula, under $\mathcal{I}_{PA(\mathbb{N},\ SC)}$, the lemma is proved.

(c) If $[F(0)]$ and $[(\forall x)(F(x) \rightarrow F(x'))]$ both interpret as algorithmically computable true formulas under $\mathcal{I}_{PA(\mathbb{N},\ SC)}$, then by Definition 2 there is an algorithm which, for any natural number n, will give evidence that the formula $[F(n) \rightarrow F(n')]$ is an algorithmically computable true formula under $\mathcal{I}_{PA(\mathbb{N},\ SC)}$.

(d) Since $[F(0)]$ interprets as an algorithmically computable true formula under $\mathcal{I}_{PA(\mathbb{N},\ SC)}$, it follows that there is an algorithm which, for any natural number n, will give evidence that $[F(n)]$ is an algorithmically computable true formula under the interpretation.

(e) Hence $[(\forall x)F(x)]$ is an algorithmically computable true formula under $\mathcal{I}_{PA(\mathbb{N},\ SC)}$.

Since the above cases are exhaustive, the lemma follows. □

The Poincaré-Hilbert debate: We note that Lemma 6.4 appears to dissolve the Poincaré-Hilbert debate[36] since: (i) the algorithmically verifiable non-finitary interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA validates Poincaré's argument that the PA Axiom Schema of Finite Induction could not be justified finitarily with respect to algorithmic verifiability under the classical putative standard interpretation of arithmetic, as any such argument would necessarily need to appeal to some form of infinite induction[37]; whilst (ii) the algorithmically computable finitary interpretation $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ of PA validates Hilbert's belief that a finitary justification of the Axiom Schema was possible under some finitary interpretation of an arithmetic such as PA.

Lemma 6.5. *Generalisation preserves algorithmically computable truth under* $\mathcal{I}_{PA(\mathbb{N},\ SC)}$.

Proof The two meta-assertions:

'$[F(x)]$ interprets as an algorithmically computable true formula under $\mathcal{I}_{PA(\mathbb{N},\ SC)}$[38]'

and

'$[(\forall x)F(x)]$ interprets as an algorithmically computable true formula under $\mathcal{I}_{PA(\mathbb{N},\ SC)}$'

both mean:

$[F(x)]$ is algorithmically computable as always true under $\mathcal{I}_{PA(\mathbb{N},\ S)}$. □

It is also straightforward to see that:

Lemma 6.6. *Modus Ponens preserves algorithmically computable truth under* $\mathcal{I}_{PA(\mathbb{N},\ SC)}$. □

We thus have that[39]:

[36] See [Hi27], p.472; also [Br13], p.59; [We27], p.482; [Pa71], p.502-503.

[37] cf. Gerhard Gentzen's *non-finitary* proof of consistency for PA, which involves a *non-finitary* Rule of Infinite Induction that admits appeal to the well-ordering property of transfinite ordinals.

[38] See Definition 9

[39] Without appeal, moreover, to Aristotle's particularisation.

Theorem 6.7. *The axioms of PA are always algorithmically computable as true under the interpretation $\mathcal{I}_{PA(\mathbb{N},\ SC)}$, and the rules of inference of PA preserve the properties of algorithmically computable satisfaction/truth under $\mathcal{I}_{PA(\mathbb{N},\ SC)}$.* □

We thus have a finitary proof that:

Theorem 6.8. *PA is consistent.* □

7. Bridging PA Provability and Computability

We now show that PA can have no non-standard model, since it is 'computably' complete in the sense that:

Theorem 7.1. *(Provability Theorem for PA) A PA formula $[F(x)]$ is PA-provable if, and only if, $[F(x)]$ is algorithmically computable as always true in $\mathbb{N}$.*

Proof We have by definition that $[(\forall x)F(x)]$ interprets as true under the interpretation $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ if, and only if, $[F(x)]$ is algorithmically computable as always true in $\mathbb{N}$.

By Theorem 6.7, $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ defines a finitary model of PA over $\mathbb{N}$ such that:

> If $[(\forall x)F(x)]$ is PA-provable, then $[F(x)]$ is algorithmically computable as always true in $\mathbb{N}$;
>
> If $[\neg(\forall x)F(x)]$ is PA-provable, then it is not the case that $[F(x)]$ is algorithmically computable as always true in $\mathbb{N}$.

Now, we cannot have that both $[(\forall x)F(x)]$ and $[\neg(\forall x)F(x)]$ are PA-unprovable for some PA formula $[F(x)]$, as this would yield the contradiction:

> (i) There is a finitary model—say $\mathcal{I}'_{PA(\mathbb{N},\ SC)}$—of PA+$[(\forall x)F(x)]$ in which $[F(x)]$ is algorithmically computable as always true in $\mathbb{N}$.
>
> (ii) There is a finitary model—say $\mathcal{I}''_{PA(\mathbb{N},\ SC)}$—of PA+$[\neg(\forall x)F(x)]$ in which it is not the case that $[F(x)]$ is algorithmically computable as always true in $\mathbb{N}$.

The lemma follows. □

Corollary 7.2. *PA is categorical with respect to algorithmic computability.*

8. An evidence-based perspective of Lucas' Gödelian argument

We finally note that:

Lemma 8.1. *If $\mathcal{I}_{PA(\mathbb{N},\ M)}$ defines a model of PA over $\mathbb{N}$, then there is a PA formula $[F]$ which is algorithmically verifiable as always true over $\mathbb{N}$ under $\mathcal{I}_{PA(\mathbb{N},\ M)}$ even though $[F]$ is not PA-provable.*

Proof Gödel has shown how to construct an arithmetical formula with a single variable—say $[R(x)]$[40]—such that $[R(x)]$ is not PA-provable[41], but $[R(n)]$ is instantiationally PA-provable for any given PA

[40] Gödel refers to the formula $[R(x)]$ only by its Gödel number r ([Go31], p.25(12)).

[41] Gödel's aim in [Go31] was to show that $[(\forall x)R(x)]$ is not P-provable; by Generalisation it follows, however, that $[R(x)]$ is also not P-provable.

numeral $[n]$. Hence, for any given numeral $[n]$, Gödel's primitive recursive relation $xB\lceil[R(n)]\rceil$[42] must hold for some x. The lemma follows. □

By the argument in Theorem 7.1 it follows that:

Corollary 8.2. *The PA formula $[\neg(\forall x)R(x)]$ defined in Lemma 8.1 is PA-provable.* □

Corollary 8.3. *In any model of PA, Gödel's arithmetical formula $[R(x)]$ interprets as an algorithmically verifiable, but not algorithmically computable, tautology over $\mathbb{N}$.*

Proof Gödel has shown that $[R(x)]$[43] always interprets as an algorithmically verifiable tautology over $\mathbb{N}$[44]. By Corollary 8.2 $[R(x)]$ is not algorithmically computable as always true in $\mathbb{N}$. □

Corollary 8.4. *PA is not ω-consistent.*[45]

Proof Gödel has shown that if PA is consistent, then $[R(n)]$ is PA-provable for any given PA numeral $[n]$[46]. By Corollary 8.2 and the definition of ω-consistency, if PA is consistent then it is *not* ω-consistent. □

Corollary 8.5. *The putative standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ S)}$ of PA does not define a model of PA*[47].

Proof If PA is consistent but not ω-consistent, then Aristotle's particularisation does not hold over $\mathbb{N}$. Since the putative standard interpretation of PA appeals to Aristotle's particularisation, the lemma follows. □

We conclude from this that Lucas' Gödelian argument[48] can validly claim that:

Thesis 1. *There can be no mechanist model of human reasoning if the assignment $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ can be treated as circumscribing the ambit of human reasoning about 'true' arithmetical propositions*[49]*, and the assignment $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ can be treated as circumscribing the ambit of mechanistic reasoning about 'true' arithmetical propositions.*

[42] Where $\lceil[R(n)]\rceil$ denotes the Gödel-number of the PA formula $[R(n)]$.

[43] Gödel refers to the formula $[R(x)]$ only by its Gödel number r; [Go31], p.25, eqn.12.

[44] [Go31], p.26(2): "$(n)\neg(nB_\kappa(17Gen\ r))$ holds"

[45] This conclusion is contrary to accepted dogma. See, for instance, Davis' remarks in [Da82], p.129(iii) that "...there is no equivocation. Either an adequate arithmetical logic is ω-inconsistent (in which case it is possible to prove false statements within it) or it has an unsolvable decision problem and is subject to the limitations of Gödel's incompleteness theorem".

[46] [Go31], p.26(2).

[47] I note that finitists of all hues—ranging from Brouwer [Br08], to Wittgenstein [Wi78], to Alexander Yessenin-Volpin [He04]—have persistently questioned the assumption that the putative standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ S)}$ can be treated as well-defining a model of PA; see also [Brm07].

[48] Although Lucas' original thesis ([Lu61] deserves consideration that lies beyond the immediate scope of this investigation, we draw attention to his informal defence of it from a philosophical perspective in The Gödelian Argument: Turn Over the Page, where he concludes with the argument that: "Thus, though the Gödelian formula is not a very interesting formula to enunciate, the Gödelian argument argues strongly for creativity, first in ruling out any reductionist account of the mind that would show us to be, au fond, necessarily unoriginal automata, and secondly by proving that the conceptual space exists in which it intelligible to speak of someone's being creative, without having to hold that he must be either acting at random or else in accordance with an antecedently specifiable rule".

[49] Such a thesis can be justified by the argument in [An13] and [An15a] that: (i) the assignment $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ can be viewed as corresponding to the way human intelligence conceptualises, symbolically represents, and logically reasons about, those sensory perceptions that are triggered by physical processes which can be treated as representable—not necessarily *finitarily*—by algorithmically verifiable formulas, where a physical process is effectively computable if, and only if, it is algorithmically verifiable; whilst: (ii) the assignment $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ can be viewed as corresponding to the way human intelligence conceptualises, symbolically represents, and logically reasons about, only those sensory perceptions that are triggered by physical processes which can be treated as representable—*finitarily*—by algorithmically computable formulas, where a physical process is effectively computable if, and only if, it is algorithmically computable. We suggest how such a perspective offers a resolution to the *EPR* paradox.

Argument: Gödel has shown how to construct an arithmetical formula with a single variable—say $[R(x)]$[50]—such that $[R(x)]$ is not PA-provable, but $[R(n)]$ is instantiationally PA-provable for any given PA numeral $[n]$. Hence, for any given numeral $[n]$, Gödel's primitive recursive relation $xB\lceil[R(n)]\rceil$[51] must hold for some natural number m.

If we assume that any mechanical witness can only reason *finitarily* then although, for any given numeral $[n]$, a mechanical witness can give evidence under the assignment $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ that the PA formula $[R(n)]$ holds in $\mathbb{N}$, no mechanical witness can conclude *finitarily* under the assignment $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ that, for any given numeral $[n]$, the PA formula $[R(n)]$ holds in $\mathbb{N}$.

However, if we assume that a human witness can also reason *non-finitarily*, then a human witness *can* conclude under the assignment $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ that, for any given numeral $[n]$, the PA formula $[R(n)]$ holds in $\mathbb{N}$.

9. Appendix A

Aristotle's particularisation: Aristotle's particularisation is the implicit *non-finitary* assumption that the classical first-order logic FOL is ω-consistent, and so we may always interpret the formal expression '$[(\exists x)\ F(x)]$' of a formal language under an interpretation as 'There exists an object s in the domain of the interpretation such that $F(s)$'.

We note that Aristotle's particularisation is a *non-finitary* but fundamental tenet of classical logic unrestrictedly adopted as *intuitively obvious* by standard literature[52].

However, L. E. J. Brouwer had noted in his seminal 1908 paper on the unreliability of logical principles[53] that the commonly accepted interpretation of this formula is ambiguous if interpretation is intended over an infinite domain.

Brouwer essentially argued that:

(i) Even supposing the formula '$[P(x)]$' of a formal Arithmetical language interprets as an arithmetical relation denoted by '$P^*(x)$'; and

(ii) the formula '$[\neg(\forall x)\neg P(x)]$' interprets as the arithmetical proposition denoted by '$\neg(\forall x)\neg P^*(x)$';

(iii) the formula '$[(\exists x)P(x)]$'—which is formally defined as '$[\neg(\forall x)\neg P^*(x)]$'—need not interpret as the arithmetical proposition denoted by the usual abbreviation '$(\exists x)P^*(x)$'; and

(iv) that such postulation is invalid as a general logical principle in the absence of a means for constructing some putative object a for which the proposition $P^*(a)$ holds in the domain of the interpretation.

Hence we shall follow the convention that the assumption that '$(\exists x)P^*(x)$' is the intended interpretation of the formula '$[(\exists x)P(x)]$'—which is essentially the assumption that Aristotle's particularisation holds over the domain of the interpretation—must always be explicit.

[50] Gödel refers to this formula only by its Gödel number r ([Go31], p.25(12)).

[51] Where xBy denotes Gödel's primitive recursive relation 'x is the Gödel-number of a proof sequence in PA whose last term is the PA formula with Gödel-number y' ([Go31], p. 22(45)); and $\lceil[R(n)]\rceil$ denotes the Gödel-number of the PA formula $[R(n)]$.

[52] See [Hi25], p.382; [HA28], p.48; [Sk28], p.515; [Go31], p.32.; [Kl52], p.169; [Ro53], p.90; [BF58], p.46; [Be59], pp.178 & 218; [Su60], p.3; [Wa63], p.314-315; [Qu63], pp.12-13; [Kn63], p.60; [Co66], p.4; [Me64], p.52(ii); [Nv64], p.92; [Li64], p.33; [Sh67], p.13; [Da82], p.xxv; [Rg87], p.xvii; [EC89], p.174; [Mu91]; [Sm92], p.18, Ex.3; [BBJ03], p.102; [Cr05], p.6.

[53] [Br08].

ω-consistency: A formal system S is ω-consistent if, and only if, there is no S-formula $[F(x)]$ for which, first, $[\neg(\forall x)F(x)]$ is S-provable and, second, $[F(a)]$ is S-provable for any given S-term $[a]$.

In order to avoid intuitionistic objections to his reasoning in his seminal 1931 paper on formally undecidable arithmetical propositions[54], Gödel did not assume that the classical putative standard assignment $\mathcal{I}_{PA(N,\ S)}$ of PA yields a model of PA. Instead, Gödel introduced the syntactic property of ω-consistency as an explicit assumption in his formal reasoning[55]. Gödel explained at some length[56] that his reasons for introducing ω-consistency as an explicit assumption in his formal reasoning was to avoid appealing to the semantic concept of classical arithmetical truth—a concept which is implicitly based on an intuitionistically objectionable logic that assumes Aristotle's particularisation is valid over $\mathbb{N}$.

However, we note that if we assume the classical putative standard assignment $\mathcal{I}_{PA(N,\ S)}$ of PA yields a model of PA, then PA is consistent if, and only if, it is ω-consistent. It can thus be argued that Gödel's Platonism was perhaps rooted (justifiably within the context of the implicit *non-finitary* assumption of Aristotle's particularisation in classical theory) in his implicitly held[57] *non-finitary* belief that any first-order axiomatic theory of arithmetic or set theory is ω-consistent.

Standard interpretation of PA: The classical putative standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ S)}$ of PA over the domain $\mathbb{N}$ of the natural numbers is the one in which the logical constants have their 'usual' interpretations[58] in Aristotle's logic of predications[59] (which subsumes Aristotle's particularisation), and[60]:

(a) The set of non-negative integers is the domain;
(b) The symbol $[0]$ interprets as the integer 0;
(c) The symbol $[']$ interprets as the successor operation (addition of 1);
(d) The symbols $[+]$ and $[\star]$ interpret as ordinary addition and multiplication;
(e) The symbol $[=]$ interprets as the identity relation.

The axioms of first-order Peano Arithmetic (PA)

PA$_1$ $[(x_1 = x_2) \rightarrow ((x_1 = x_3) \rightarrow (x_2 = x_3))]$;
PA$_2$ $[(x_1 = x_2) \rightarrow (x_1' = x_2')]$;
PA$_3$ $[0 \neq x_1']$;
PA$_4$ $[(x_1' = x_2') \rightarrow (x_1 = x_2)]$;
PA$_5$ $[(x_1 + 0) = x_1]$;
PA$_6$ $[(x_1 + x_2') = (x_1 + x_2)']$;
PA$_7$ $[(x_1 \star 0) = 0]$;
PA$_8$ $[(x_1 \star x_2') = ((x_1 \star x_2) + x_1)]$;
PA$_9$ For any well-formed formula $[F(x)]$ of PA:
$[F(0) \rightarrow (((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x))]$.

Generalisation in PA If $[A]$ is PA-provable, then so is $[(\forall x)A]$.

Modus Ponens in PA If $[A]$ and $[A \rightarrow B]$ are PA-provable, then so is $[B]$.

[54][Go31].
[55][Go31], p.23 and p.28.
[56]In his introduction on p.9 of [Go31].
[57][Go31], p.28.
[58]We essentially follow the definitions in [Me64], p.49.
[59]See http://plato.stanford.edu/entries/aristotle-logic/, §4.3.
[60]cf. [Me64], p.107.

Hilbert's Second Problem: "When we are engaged in investigating the foundations of a science, we must set up a system of axioms which contains an exact and complete description of the relations subsisting between the elementary ideas of that science. ... But above all I wish to designate the following as the most important among the numerous questions which can be asked with regard to the axioms: To prove that they are not contradictory, that is, that a definite number of logical steps based upon them can never lead to contradictory results. In geometry, the proof of the compatibility of the axioms can be effected by constructing a suitable field of numbers, such that analogous relations between the numbers of this field correspond to the geometrical axioms. ... On the other hand a direct method is needed for the proof of the compatibility of the arithmetical axioms."[61]

In this paper, we treat Hilbert's intent[62] behind the enunciation of his Second Problem as essentially seeking a finitary proof for the consistency of arithmetic when formalised in a language such as the first order Peano Arithmetic PA.

Tarski's inductive definitions: We shall assume that truth values of 'satisfaction', 'truth', and 'falsity' are assignable inductively—whether *finitarily* or *non-finitarily*—to the compound formulas of a first-order theory S under the interpretation $\mathcal{I}_{S(\mathbb{D})}$ in terms of *only* the satisfiability of the atomic formulas of S over $\mathbb{D}$ as usual[63]:

Definition 6. *A denumerable sequence s of $\mathbb{D}$ satisfies $[\neg A]$ under $\mathcal{I}_{S(\mathbb{D})}$ if, and only if, s does not satisfy $[A]$;*

Definition 7. *A denumerable sequence s of $\mathbb{D}$ satisfies $[A \rightarrow B]$ under $\mathcal{I}_{S(\mathbb{D})}$ if, and only if, either it is not the case that s satisfies $[A]$, or s satisfies $[B]$;*

Definition 8. *A denumerable sequence s of $\mathbb{D}$ satisfies $[(\forall x_i)A]$ under $\mathcal{I}_{S(\mathbb{D})}$ if, and only if, given any denumerable sequence t of $\mathbb{D}$ which differs from s in at most the i'th component, t satisfies $[A]$;*

Definition 9. *A well-formed formula $[A]$ of $\mathbb{D}$ is true under $\mathcal{I}_{S(\mathbb{D})}$ if, and only if, given any denumerable sequence t of $\mathbb{D}$, t satisfies $[A]$;*

Definition 10. *A well-formed formula $[A]$ of $\mathbb{D}$ is false under $\mathcal{I}_{S(\mathbb{D})}$ if, and only if, it is not the case that $[A]$ is true under $\mathcal{I}_{S(\mathbb{D})}$.*

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[61] Excerpted from Maby Winton Newson's English translation [Nw02] of David Hilbert's address [Hi00] at the International Congress of Mathematicians in Paris in 1900.

[62] Compare Curtis Franks' thesis in [Fr09] that Hilbert's intent behind the enunciation of his Second Problem was essentially to establish the autonomy of arithmetical truth without appeal to any debatable philosophical considerations.

[63] See [Me64], p.51; [Mu91].

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