Why the Perceived *Flaw* in Kempe's 1879 *Graphical* 'Proof' of the Four Colour Theorem is *Not* Fatal When Expressed *Geometrically*

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Abstract. All accepted proofs of the Four Colour Theorem (4CT) are *computer-dependent*; and appeal to the existence, and *manual* identification, of an 'unavoidable' set containing a *sufficient* number of explicitly defined configurations—each evidenced *only* by a *computer* as 'reducible'—such that at least one of the configurations *must* occur in any chromatically distinguished, *minimal*, planar map. For instance, Appel and Haken 'identified' 1,482 such configurations in their 1977, computer-dependent, proof of 4CT; whilst Neil Robertson et al 'identified' 633 configurations as sufficient in their 1997, also computer-dependent, proof of 4CT. However, treating any *specific* number of 'reducible' configurations in an 'unavoidable' set as *sufficient* entails a *minimum* number as *necessary* and *sufficient*. We now show that the *minimum* number of such configurations can only be the one corresponding to the 'unavoidable' set of a 'reducible', 4-sided configuration identified by Alfred Kempe in his, seemingly fatally flawed, 1879 'proof' of 4CT. Although Kempe appealed to putative properties of 'Kempe' chains in a *graphical* representation to fallaciously argue that a 5-sided configuration was also in the 'unavoidable' set, and 'reducible', we shall show why the flaw in his 'proof' is not fatal when the argument is expressed *geometrically*; and that, essentially, Kempe correctly argued that *any* planar map which admits a chromatic differentiation with a five-sided area C that shares non-zero boundaries with four, all differently coloured, neighbours can be 4-coloured.

Keywords. computer-assisted proof, four colour theorem, Kempe chains, minimal planar map, geometrical proof, unavoidable but reducible configurations.

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1. Introduction

Although the Four Colour Theorem 4CT is considered passé, it would probably be a fair assessment that the mathematical significance of any new proof of the Four Colour Theorem 4CT continues to be perceived[1] as lying not in any ensuing theoretical or practical utility of the Theorem per se, but in whether the proof can address the lack of mathematical insight—in currently accepted, *computer-dependent*, proofs of the Theorem—as to *why* four colours suffice to chromatically differentiate any set of contiguous, simply connected and bounded, spaces in a planar map.

All accepted proofs of 4CT appeal to the existence, and *manual* 'identification', of a *sufficient* number of explicitly defined configurations, each evidenced *only* by a *computer* as 'reducible' (see [14], Ch.8); and *claimed* (see §3., Appendix A) to be an 'unavoidable' set of configurations, at least one of which *must* occur in any chromatically distinguished planar map which claims to *essentially* require five colours.

Thus, Kenneth Appel and Wolfgang Haken claimed to have identified an *unavoidable* set of 1,482 *reducible* configurations in their 1977, computer-dependent, proof [1] of 4CT; whilst Neil Robertson et al claimed to have identified an *unavoidable* set of 633 *reducible* configurations as sufficient in their 1997, also computer-dependent, proof [7] of 4CT.

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[1]See Appel and Haken: [1]; Appel, Haken and Koch: [2]; Tymoczko: [3]; Swart: [4]; Stewart: [5], *Appendix, pp.503-505*; Robertson et al: [6], *Pre-publication*; Robertson, Sanders, Seymour, and Thomas: [7]; Thomas: [8]; Calude: [9]; Brun: [10], *§1. Introduction (Article for undergraduates)*; Gonthier: [11]; Zeilberger: [12]; Rogers: [13]; Wilson: [14]; Conradie and Goranko: [15], *§7.7.1, Graph Colourings, p.417*; Allo: [16], *Conclusion, p.562*; Nanjwenge: [17], *Chapter 8, Discussion (Student Thesis)*; Najera: [18]; Gardner: [19], *§11.1, Colourings of Planar Maps, pp.6-7 (Lecture notes).*

Since claiming any *specific* number of 'reducible' configurations in an 'unavoidable' set as *sufficient* entails a *minimum* number as *necessary* and *sufficient*, we shall show that:

> The *minimum* number of 'reducible' configurations in an 'unavoidable' set can *only* be the one corresponding to the 'reducible' 4-sided configuration (the 'quadrilateral' P_4 in [1]; see §3., Appendix A), identified by Alfred Kempe as *Plate II, Fig.9* in his, seemingly fatally flawed, 1879 'proof' [20] of 4CT.

Although Kempe appealed to putative properties of 'Kempe' chains, in a *graphical* representation, to fallaciously argue that a 5-sided configuration (the 'pentagon' P_5 in [1]; see §3., Appendix A), identified by him as *Plate II, Fig.11 and Fig.12* in [20], was also in the 'unavoidable' set, and 'reducible', we shall show the flaw is not fatal when the argument is expressed *geometrically*; and that, essentially, Kempe correctly concluded that *any* planar map which admits a chromatic differentiation with a five-sided area C that shares non-zero boundaries with four, all differently coloured, neighbours can be 4-coloured.

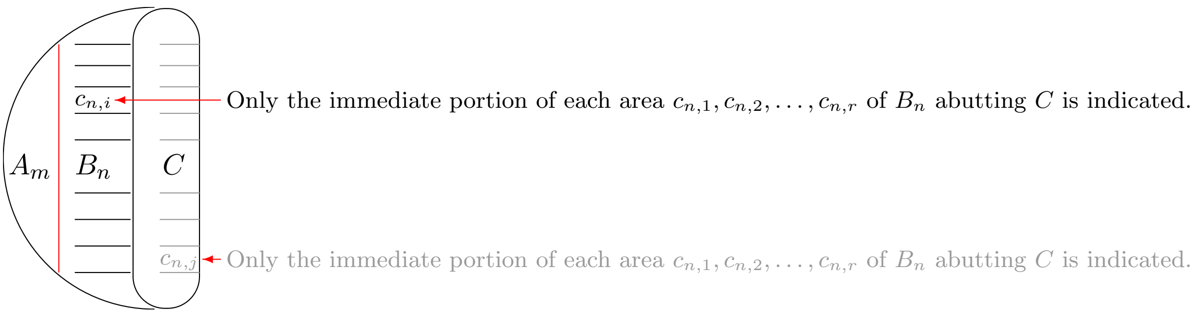
> **Comment**: Lemma 2.7 details the *geometrical* argument Kempe needed for validating his 'proof' of 4CT in [20]. Seemingly, Kempe failed to recognise the *geometrical* argument since he preferred appeal to Euler's formula $V + F = E + 2$, in what he claimed as an 'equivalent', *graphical*, representation (see §5., Appendix C), to falsely conclude that any *minimal* planar map *must* contain an 'unavoidable' set with *two* 'reducible' configurations (see §3., Appendix A).

In the *geometrical* proof of 4CT in Theorem 2.8[2] we thus seek a *computer-independent* argument which *transparently* illustrates *why* four colours suffice to chromatically differentiate any set of contiguous, simply connected and bounded, planar spaces by arguing that (compare with Alfred Kempe's summary of his argument in §4., Appendix B):

- If there is a *minimal* planar map $\mathcal{H}$ with $(m + n + 1)$ areas that contains an area C which *necessarily* requires a 5^{th} colour, whilst *any* planar map with $\leq m + n$ areas *can* be 4-coloured;
- Then shrinking C to a point P_C yields a *sub-minimal* map $\mathcal{M}_C$ that can always be 4-coloured, such that all the areas meeting at the apex P_C require only 3 colours.
- Recreating C in $\mathcal{M}_C$ would now yield a chromatic differentiation of $\mathcal{H}$ that requires only a 4^{th} colour for C, contradicting the *putative minimality* of $\mathcal{H}$.

2. A *geometrical* 'proof' of the 4-Colour Theorem

Fig.1: Minimal Planar Map $\mathcal{H}$



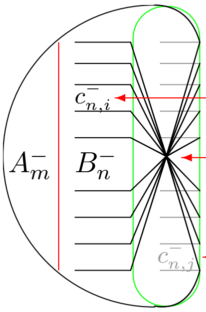
Without loss of generality, the hemisphere in Fig.1 is taken to define a *minimal* planar map $\mathcal{H}$ where:

[2]Essentially a formalisation of the *pre-formal* proof of 4CT in [21]. See also [22], §1.G: *Evidence-based (pictorial), pre-formal, proofs of the Four Colour Theorem.* The need for distinguishing between *belief-based* 'informal', and *evidence-based* 'pre-formal', reasoning is addressed by philosopher Markus Pantsar in [23].

1. A_m denotes a region of m contiguous, simply connected and bounded, surface areas $a_{m,1}, a_{m,2}, \ldots, a_{m,m}$ (of the hemisphere in Fig.1), *none* of which share a non-zero boundary segment with the contiguous, simply connected, surface area C (as indicated by the red barrier which, however, is *not* to be treated as a boundary of the region A_m);

2. B_n denotes a region of n contiguous, simply connected and bounded, surface areas $b_{n,1}, b_{n,2}, \ldots, b_{n,n}$, *some* of which, say $c_{n,1}, c_{n,2}, \ldots, c_{n,r}$, share *at least one* non-zero boundary segment of $c_{n,i}$ with C; where, for each $1 \leq i \leq r$, we have that $c_{n,i} = b_{n,j}$ for some $1 \leq j \leq n$;

Fig.2: Sub-minimal Planar Map $\mathcal{M}_C$ defined uniquely by shrinking C to a point P_C in $\mathcal{H}$



3. C is a single contiguous, simply connected and bounded, area (see Fig.1) constructed *finitarily* by sub-dividing and annexing (compare Kempe [20], Plate II, Fig.14) one or more contiguous, simply connected, portions surrounding a common apex P_C of each area $c^-_{n,i}$ (see Fig.2) in the region B^-_n of some *putative* sub-minimal map $\mathcal{M}_C$ (see Fig.2), defined uniquely by shrinking C to a point P_C in $\mathcal{H}$.

We define:

Definition 1. (Finitary Constructibility) *A single contiguous, simply connected and bounded, area D of a planar map $\mathcal{G}$ is finitarily constructible if, and only if, it can be constructed in a finite number of steps by annexing non-zero areas of the planar map $\mathcal{M}_D$ obtained by shrinking D to a point in $\mathcal{G}$.*

Lemma 2.1. *Any single contiguous, simply connected and bounded, area D of a planar map $\mathcal{G}$ with n areas is finitarily constructible.*

Proof. If D shares m non-zero boundary segments with abutting areas, then shrinking D to a point (as in Fig.2) yields a planar map $\mathcal{M}_D$ with at most m areas of $\mathcal{G}$ that now meet in $\mathcal{M}_D$ at least once at a common apex P_D. The area D can then be *finitarily* constructed in m steps by annexing m triangular areas of those immediate portions of each area of $\mathcal{M}_D$ that contain P_D. The Lemma follows. □

We next consider the:

Hypothesis 1. (Minimality Hypothesis) *Since four colours suffice for all, and are necessary for some, planar maps with fewer than 5 regions, we assume the existence of some m, n, in a putatively minimal planar map $\mathcal{H}$, which defines a specific configuration of the region $\{A_m + B_n + C\}$ where:*

(a) any configuration of p contiguous, simply connected and bounded, areas can be 4-coloured if $p \leq m + n$, where $p, m, n \in \mathbb{N}$, and $m + n \geq 5$;

(b) any chromatically differentiated colouring of $\mathcal{H}$ contains some area C that necessarily requires a 5^{th} colour;

(c) *in any such chromatically differentiated colouring, there is a specific configuration of $m + n$ contiguous, simply connected and bounded, areas, say $\{A_m^- + B_n^-\}$, of a putatively unique, sub-minimal, 4-colourable planar map, say $\mathcal{M}_C$, where $A_m \subseteq A_m^-$ and $B_n \subseteq B_n^-$;*

(d) *the area C can be constructed finitarily by sub-dividing and annexing some portions from each area, say $c_{n,i}^-$, of B_n^- in the specific, sub-minimal, planar map $\mathcal{M}_C$;*

(e) *the region $\{A_m + B_n + C\}$ in the planar map $\mathcal{H}$ is a specific chromatic differentiation of the $m+n+1$ contiguous, simply connected and bounded, areas of $\mathcal{H}$ in which C necessarily requires a 5^{th} colour.*

where we define:

Definition 2. (Finitary Definability) *A single contiguous, simply connected and bounded, area D of a planar map $\mathcal{G}$ is finitarily definable if, and only if, it can be shrunk in a finite number of steps to a point in $\mathcal{G}$.*

We note that:

Lemma 2.2. *The minimal map $\mathcal{H}$ cannot admit two areas, say C and C', both of which necessarily require a 5^{th} colour.*

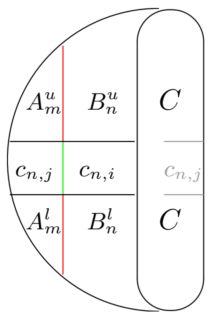
Proof. Shrinking C to a point would reduce $\mathcal{H}$ to a 4-colourable map where C' does not require a 5^{th} colour. Restoring C with a 5^{th} colour establishes the Lemma. □

Since any area D of $\mathcal{H}$ can be shrunk to a point, the argument of Lemma 2.2 immediately entails:

Corollary 2.3. *The minimal map $\mathcal{H}$ can always be chromatically distinguished so that any specified area D of $\mathcal{H}$ requires a 5^{th} colour.* □

We note next that:

Fig.3: No two, non-adjacent, areas $c_{n,i}$ and $c_{n,j}$ can share a non-zero boundary in $\mathcal{H}$



Lemma 2.4. *No two, non-adjacent, areas of B_n, each sharing a non-zero boundary segment with C in the minimal planar map $\mathcal{H}$ in Hypothesis 1, can also share a non-zero boundary that has no point in common with C.*

Proof. Let two, non-adjacent, areas of B_n, say $c_{n,i}$ and $c_{n,j}$ in Fig.1, *each* of which shares a non-zero boundary with C, *also* share a non-zero boundary with each other that *does not* intersect C (as shown in green in Fig.3). This would divide $\{A_m + B_n - c_{n,i} - c_{n,j}\}$ into two *non-empty* regions $A_m^u + B_n^u$ and $A_m^l + B_n^l$, such that *no* area of the region $A_m^u + B_n^u$ shares a non-zero boundary with *any* area of the region $A_m^l + B_n^l$.

However, it would entail that the areas $c_{n,k}$ ($k \neq i, j$) which abut C in each of the regions B_n^u and B_n^l would necessarily require 2 additional colours not shared with the areas C, $c_{n,i}$ and $c_{n,j}$; since:

— if all such $c_{n,k}$ require only 1 additional colour, $\mathcal{H}$ would be 4-colourable, and violate *minimality*;

— if all such $c_{n,k}$ in only one of the regions, say B_n^u, require only 1 additional colour,

- then annexing one of the areas of B_n^l, say $c_{n,lower}$, which has this colour, say x, into the area C would again reduce the map $\mathcal{H}$ to a *sub-minimal* map, say $\mathcal{C}'$,

- where $\mathcal{C}'$ still requires 5 colours, since the merged area $(c_{n,\ lower} + C)$ would now abut areas with all the four colours of the map $\mathcal{C}'$, thus violating the *minimality* of $\mathcal{H}$;

Consequently, *each* of the regions $\{B_n^u+c_{n,i}+c_{n,j}+C\}$ and $\{B_n^l+c_{n,i}+c_{n,j}+C\}$—when considered as separate planar maps, each with less than $m+n+1$ areas—would necessarily then require C to have the 5^{th} colour, thus violating Hypothesis 1. The Lemma follows. □

Corollary 2.5. *No area $c_{n,i}$ of B_n in the minimal planar map $\mathcal{H}$ can share two, distinctly separated, non-zero boundary segments with C.* □

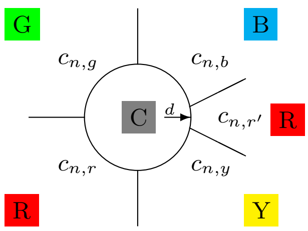
We now show how Lemma 2.7 improves upon, and bridges the gap, in Alfred Kempe's—seemingly fatally failed (see §3.)—'proof' of the Four Colour Theorem in [20].

Lemma 2.6. *Every area D of a minimal planar map $\mathcal{H}$ shares non-zero boundaries with at least four neighbours.*

Proof. Shrinking *any* area D of a *minimal* map $\mathcal{H}$ to a point would yield a 4-colourable, *sub-minimal*, map. By Hypothesis 1, restoring D in any 4-colouring of such a *sub-minimal* map must require a 5^{th} colour for D. The Lemma follows. □

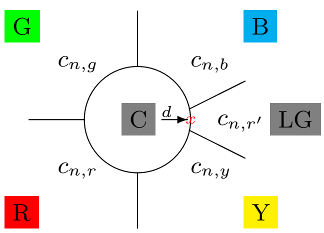
Lemma 2.7. *The area C in $\mathcal{H}$ can share a non-zero boundary with only four, differently coloured, areas $c_{n,i}$.*

***Fig.4:** C **cannot share a non-zero boundary segment with 5 areas in the minimal planar map** $\mathcal{H}$*



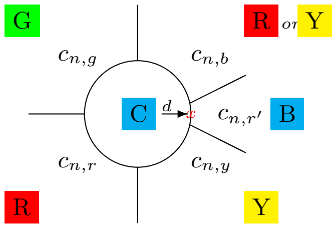
Proof. (i) If C shares a non-zero boundary with identically-coloured areas $c_{n,r}$ and $c_{n,r'}$ (see Fig.4)—where $r \neq r'$ by Corollary 2.5—then either area can be annexed by C without disturbing the chromatic differentiation of $\mathcal{H}$.

***Fig.5:** **Annexing** $c_{n,r'}$ **into** C **would then yield a sub-minimal map** $\mathcal{M}_\mathcal{C}$*



(ii) However, if C annexes $c_{n,r'}$ by erasing the boundary d (see Fig.5), that would then yield a *sub-minimal* map $\mathcal{M}_\mathcal{C}$.

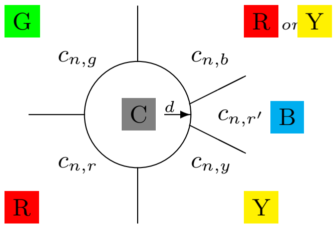
Fig.6: The sub-minimal map $\mathcal{M}_C$ would be 4-colourable



(iii) By definition, the *sub-minimal* map $\mathcal{M}_C$ is now 4-colourable as shown in Fig.6, where:

- the areas $c_{n,r}$ and $c_{n,r'}$ are necessarily differently coloured; and
- neither $c_{n,r}$ nor $c_{n,y}$ share a non-zero boundary with $c_{n,b}$ by the non-sharing Lemma 2.4.

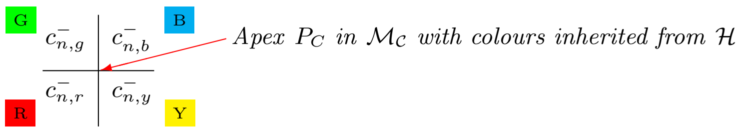
Fig.7: After restoration $c_{n,r}$ and $c_{n,r'}$ cannot share identical colours in $\mathcal{H}$ as postulated



(iv) However, we now have the contradiction that there is *no* 4-colouring of the *sub-minimal* map $\mathcal{M}_C$ which—on restoration of the area C as the necessary 5^{th} coloured area in the *putatively minimal* planar map $\mathcal{H}$ (see Fig.7)—would admit the identical colouring for $c_{n,r}$ and $c_{n,r'}$ in $\mathcal{H}$, as *postulated* in (i) above. The Lemma follows. □

Theorem 2.8. *No chromatically differentiated planar map needs more than four colours.*

Fig.8: Areas which meet at the apex P_C in the sub-minimal planar map $\mathcal{M}_C$ with colours inherited from $\mathcal{H}$

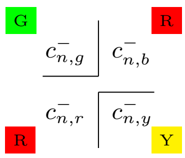


Proof. (1) By Lemma 2.7, only four differently coloured areas meet at the apex P_C (see Fig.8) of the *sub-minimal* map $\mathcal{M}_C$ in any colouring which is inherited from the *putatively minimal* map $\mathcal{H}$.

(2) By Lemma 2.4, *no* two areas $c_{n,i}$, $c_{n,j}$ of B_n in the *minimal* planar map $\mathcal{H}$ in Fig.3 can share a non-zero boundary segment that has *no* point in common with C.

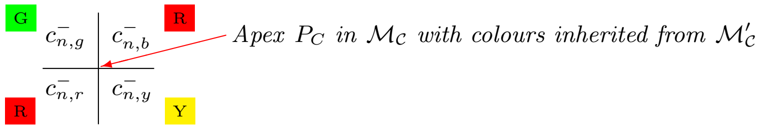
(3) Merging any $c^-_{n,i}$ with a differently coloured $c^-_{n,j}$ at P_C (see Fig.9), where both $c^-_{n,i}$ and $c^-_{n,j}$ do not share a non-zero boundary, and are abutted by areas that do not share an inherited colour with either of them, thus yields another *sub-minimal*, hence 4-colourable, map $\mathcal{M}'_C$.

Fig.9: Merging areas $c^-_{n,r}$ and $c^-_{n,b}$ at the apex P_C in $\mathcal{M}_C$ and recolouring the sub-minimal planar map $\mathcal{M}'_C$



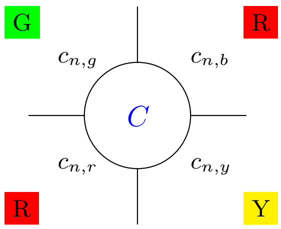
(4) Restoring P_C now yields a fresh 4-colouring of $\mathcal{M}_C$ in which only 3 colours at most (see Fig.10) meet at the apex P_C.

Fig.10: Restoring areas $c^-_{n,r}$ and $c^-_{n,b}$ at the apex P_C in $\mathcal{M}_C$ with colours inherited from $\mathcal{M}'_C$



(5) By Lemma 2.1, recreating C in $\mathcal{M}_C$ would now yield a chromatic differentiation of $\mathcal{H}$ that requires (see Fig.11) only a 4^{th} colour for C, thus contradicting the *putative minimality* of $\mathcal{H}$.

Fig.11: Areas in the minimal planar map $\mathcal{H}$ with C recreated from $\mathcal{M}_C$ and colours inherited from $\mathcal{M}'_C$



We conclude that Hypothesis 1 is false. The Theorem follows. □

3. Appendix A: The perceived 'flaw' in Kempe's 1879 argument

In their *computer-assisted* proof of the Four Colour Theorem [1], Appel and Haken review the 'flaw' in Kempe's 1879 'proof' [20]:

> "The first published attempt to prove the Four Color Theorem was made by A. B. Kempe [19] in 1879. Kempe proved that the problem can be restricted to the consideration of "normal planar maps" in which all faces are simply connected polygons, precisely three of which meet at each node. For such maps, he derived from Euler's formula, the equation
>
> $$4p_2 + 3p_3 + 2p_4 + p_5 = \sum_{k=7}^{k_{max}} (k-6)p_k + 12 \qquad (1.1)$$
>
> where p_i is the number of polygons with precisely i neighbors and k_{max} is the largest value of i which occurs in the map. This equation immediately implies that every normal planar map contains polygons with fewer than six neighbors.
>
> In order to prove the Four Color Theorem by induction on the number p of polygons in the map ($p = \sum p_i$), Kempe assumed that every normal planar map with $p \leq r$ is four colorable and considered a normal planar map M_{r+1} with $r+1$ polygons. He distinguished the four cases that M_{r+1} contained a polygon P_2 with two neighbors, or a triangle P_3, or a quadrilateral P_4, or a pentagon P_5; at least one of these cases must apply by (1.1). In each case he produced a map M_r, with r polygons by erasing from M_{r+1} one edge in the boundary of an appropriate P_k. By the induction hypothesis, M_r admits a four coloring, say c_{r+1}, and Kempe attempted to derive a four coloring c_{r+1} of M_{r+1} from c_r. This task was very easy in the cases of P_2 and P_3. To treat the cases of P_4 and P_5, Kempe invented the method of interchanging the colors in a maximal connected part which was colored by c_r with a certain pair of colors (two-colored chains were later called Kempe chains) to obtain a coloring ${c_r}'$ of M_r from which one can then obtain a four coloring c_{r+1} of M_{r+1}.
>
> While Kempe's argument was correctly applied to the case of P_4, it was incorrectly applied to the case of P_5 as was shown by Heawood [18] in 1890."
>
> *Appel and Haken: [1], §1. Introduction, p.429.*

We note, however, that the 'flaw' is not fatal if Kempe's argument is expressed *geometrically*.

Reason: The case Appel and Haken refer to as P_5 corresponds to §2., Lemma 2.7 where:

- We do not appeal—in a *graphical* representation of *minimal* 'normal planar maps'—to a 'method of interchanging the colors' in 'Kempe chains', so as to identify 'reducible' configurations in an 'unavoidable' set.

- Instead, we appeal—in a *geometrical* representation of *minimal* planar maps—to the Minimality Hypothesis 1, and argue that:

  - in any *minimal* planar map such as $\mathcal{H}$ in Fig.1,
  - any area such as P_5 which necessarily requires a 5^{th} colour,
  - cannot share non-zero boundaries with *two*, similarly coloured, neighbours.

This then yields P_4 as the sole configuration in an 'unavoidable' set. Moreover, as Appel and Haken note, P_4 is shown by Kempe to be 'reducible' (corresponding to the proof of the Four Colour Theorem in §2., Theorem 2.8).

3.A. Could there be an *un*perceived, inherited, 'flaw' in Appel and Haken's argument?

Unarguably meriting a philosophical discussion of consequences that lie beyond the immediate ambit of this investigation, we merely note here that if the 'flaw' in Kempe's 1879 'proof' [20] is perceived as falsely claiming to have proven the argument that:

> Any *minimal* 'normal planar map' admits an *unavoidable* set containing a 'pentagon' that can be shown as *reducible*;

where (cf. [14], Ch.8):

(i) An *unavoidable* set is a set of configurations such that every map that satisfies some necessary conditions for being a minimal non-4-colorable triangulation (such as having minimum degree 5) must have at least one configuration from this set.

(ii) A *reducible* configuration is one that cannot occur in a minimal counterexample. If a map contains a reducible configuration, the map can be reduced to a smaller map. This smaller map has the condition that if it can be colored with four colors, this also applies to the original map. This implies that if the original map cannot be colored with four colors the smaller map cannot either and so the original map is not minimal.

then the following remarks suggest that Appel and Haken's computer-dependent 'proof' [1] (as also Robertson et al's [7]), too could be viewed as 'flawed' (in the sense of being vacuously true, even if logically valid):

> "While Kempe's argument was correctly applied to the case of P_4, it was incorrectly applied to the case of P_5 as was shown by Heawood [18] in 1890. Kempe's argument proved, however, that five colors suffice for coloring planar maps and that a minimal counter-example to the Four Color Conjecture (minimal with respect to the number p of polygons in the map) could not contain any two-sided polygons, triangles, or quadrilaterals. This restricts the Four Color Problem to the consideration of normal planar maps in which each polygon has at least five neighbors. Each such map must contain at least twelve pentagons since in (1.1) we have $p_2 = p_3 = p_4 = 0$ and thus
>
> (1.2) $$p_5 = \sum_{k=7}^{k_{max}} (k-6)p_k + 12.$$

> Since 1890 a great many attempts have been made to find a proof of the Four Color Theorem. We distinguish two types of such attempts: (i) attempts to repair the flaw in Kempe's work; and (ii) attempts to find new and different approaches to the problem. Among attempts of type (i) we distinguish two subtypes: (i)(a) attempts to find an essentially stronger chain argument for "reducing the pentagon," i.e., proving that a minimal counter-example to the Four Color Conjecture cannot contain any pentagon, and thus does not exist; and (i)(b) attempts to make more extended use of Kempe's arguments in different directions and, instead of "reducing" the pentagon directly, to replace it by configurations of several polygons. Since the method used in this paper is of type (i)(b) we shall restrict our attention to further developments in this branch."
>
> *. . . Appel and Haken: [1], §1. Introduction, p.430.*

Reason: By Lemma 2.7 (essentially type (1)(a)) no *minimal* planar map can admit an 'unavoidable' set containing a pentagon.

In other words, both Kempe and Appel/Haken argue that:

(I) 4CT is equivalent to proving that, in any *minimal* 'normal planar map', there is an 'unavoidable' set of *two* configurations, P_4 and P_5, each of which is 'reducible';

(II) Kempe [20] has validly shown that the configuration P_4 is 'reducible'.

Seemingly, Appel/Haken further argue that:

(a) Kempe did not prove in [20] that the configuration P_5 is 'reducible'.

(b) If each of the 1,482 configurations, as *manually* defined in their 'unavoidable' set in [1], is 'reducible', then P_5 is 'reducible';

(c) A *computer-dependent* proof validates that each of the 1,482 configurations is 'reducible';

(d) Hence 4CT is proven.

However, Lemma 2.7 shows that (I) admits an invalid *implicit* assumption, since no *minimal* planar map can contain a configuration such as P_5; whence (d) would hold *vacuously* as having proven:

> If every *minimal* planar map admits an 'unavoidable' set containing a five-sided figure such as P_5, then P_5 is 'reducible'.

and not that:

> No *minimal* planar map can admit an 'unavoidable' set containing a five-sided figure such as P_5.

4. Appendix B: Kempe's concluding argument in his 1879 'proof'

Notwithstanding the critical difference, as highlighted in §3., between:

- Kempe's 'flawed' appeal in [20] to 'the method of interchanging the colors' in 'Kempe chains'; and
- The pictorially *transparent* appeal to the Minimality Hypothesis 1 in the proof of Theorem 2.8,

the two arguments can be seen to share a similar structure.

For instance, Kempe's concluding argument in his 1879 'proof' of 4CT by finite induction:

> "Returning to the question of colour, if the map at any stage of its development, can be coloured with four colours, we can arrange the colours so that, at the point of concourse on the patch next to be taken off, where less than six boundaries meet, only three colours shall appear, and, therefore, when the patch is stripped off, only three colours surround the disclosed district, which can. therefore, be coloured with the fourth colour, *i. e.* the map can be coloured at the next stage. But, at the first stage, one colour suffices, therefore, four suffice at all stages, and therefore, at the last. This proves the theorem and shows how the map may be coloured."
>
> *. . . Kempe: [20], p.199.*

can be viewed as faithfully mirrored in the *language* of Definition 1, and the *intent* of Theorem 2.8, as follows:

> Returning to the question of colour, if the map at any stage of its [finitary creation by annexation], can be coloured with four colours, we can arrange the colours so that, at [an apex where the next area is to be created by annexation], where less than six boundaries meet, only three colours shall appear, and, therefore, when the [new area is created therein by annexation], only three colours surround the [newly created area], which can. therefore, be coloured with the fourth colour, *i. e.* the map can be coloured at the next stage. But, at the first stage, one colour suffices, therefore, four suffice at all stages, and therefore, at the last. This proves the theorem and shows how the map may be coloured.

if we correspond [finitary creation by annexation at an apex] to Kempe's 'disclosed district' at the 'point of concourse on the patch next to be taken off' (as illustrated by Kempe's Fig.14 in [20], Plate II).

5. Appendix C: Why the *geometrical* proof of 4CT may not be expressible *graphically*

Fig.12: Graphical representation of $\mathcal{H}$

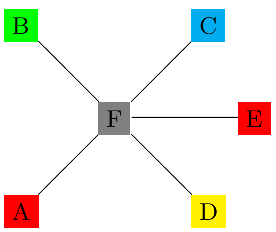
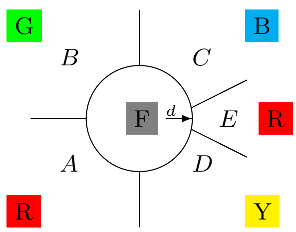


Fig.13: Geometrical representation of $\mathcal{H}$



We note that, since classical graph theory (see, for instance, Brun [10], Conradie/Goranko [15], Gardner [19]) represents non-empty areas as points (*vertices*), and a non-zero boundary between two areas as a line (*edge*) joining two points (*vertices*), the theory does not immediately *evidence* a graphical proof of Theorem 2.8.

Fig.14: Graphical representation of $\mathcal{M}_{\mathcal{F}}$

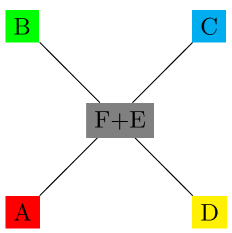
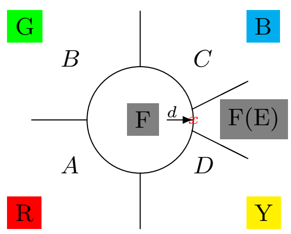


Fig.15: Geometrical representation of $\mathcal{M}_{\mathcal{F}}$



In other words, the proof of Theorem 2.8 appeals critically to re-configuring the *geometrical* representation of the, putatively *minimal*, planar map $\mathcal{H}$ in Fig.13; by first *removing* (see Fig.15), and then *restoring* after recolouring (see Fig.17), the non-zero boundary d in Fig.13 to merge/de-merge the areas F and E in a *geometrically* distinguishable way that, prima facie, cannot be immediately *evidenced* in the corresponding argument *when* represented *graphically* by Figs.12, 14 and 16.

Fig.16: Graphical representation of $\mathcal{H}$

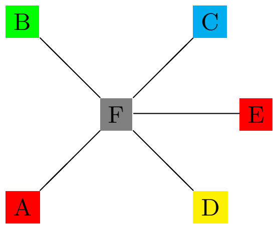
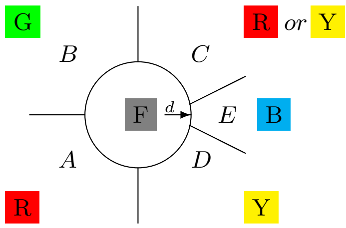


Fig.17: Geometrical representation of $\mathcal{H}$



We thus speculate that the barriers to proving 4CT *graphically* may possibly lie in Alfred Kempe's unsupported *postulation*, that the four-color map problem could be reformulated *equivalently* as a problem involving *linkages* between the 'lettering' of colours at *unspecified* points of a map in a *graph*:

> "If we lay a sheet of tracing paper over a map and mark a point on it over each district and connect the points corresponding to districts which have a common boundary, we have on the tracing paper a diagram of a "linkage," and we have as the exact analogue of the question we have been considering, that of lettering the points in the linkage with as few letters as possible, so that no two directly connected points shall be lettered with the same letter. Following this up, we may ask what are the linkages which can be similarly lettered with not less than n letters?
>
> The classification of linkages according to the value of n is one of considerable importance. I shall not, however, enter here upon this question, as it is one which I propose to consider as part of an investigation upon which I am engaged as to the general theory of linkages. It is for this reason also that I have preferred to treat the question discussed in this paper in the manner I have done, instead of dealing with the analogous linkage."
>
> *...Kempe: [20], p.200*

Fig.18: Pseudo-graphical representation of $\mathcal{M}_{\mathcal{F}}$

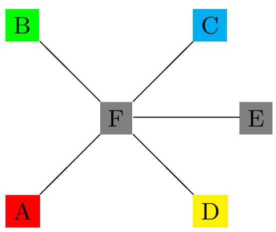
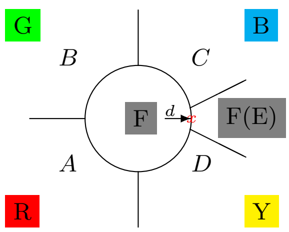


Fig.19: Geometrical representation of $\mathcal{M}_{\mathcal{F}}$



In other words, it is conceivable—perhaps even likely—that Kempe was misled by a *pseudo-graphical* representation of $\mathcal{M}_{\mathcal{F}}$ (see Fig.18) into believing that a *graphical* argument *must* follow which entails that a five-sided configuration in $\mathcal{H}$ (see Fig.12) must be 'reducible'.

Reason: In the above *pseudo-graphical* representation—as shown in Fig.18 just *before* their merger—countries A and E *obviously* could not have been identically coloured in $\mathcal{H}$.

This could account for Kempe's intuitively 'preferred' alternative in informal explanations vis à vis his explicit assumption that a formal representation by 'linkages' may be viewed 'as the exact analogue' of the four-color map problem; a preference reflected in Robin Wilson's *italicised* remark in [14], wherein he too, seemingly *uncritically*, assumes such an equivalence:

> "Any coloring of the countries of the map gives rise to a lettering of the points in the linkage in which no two directly connected points are lettered the same.
>
> We now refer to such a linkage as a *graph* ... and to the preceding process as forming the graph (or dual graph) of the map. This reformulation of the four-color problem as a problem involving the lettering of

points reappeared briefly in the 1880s (see Chapter 6) and was later reintroduced in the 1930s and used in all subsequent attempts to solve the problem.

So as not to complicate matters, we shall usually stick to coloring the countries of maps (rather than switching to lettering the points of a graph) throughout the rest of this book."

. . . Wilson: [14], p.67.

It is thus also conceivable that subsequent articulations of 4CT failed to recognise the *geometrical* argument in Lemma 2.7 *only* because Kempe's formal appeal to Euler's formula $V + F = E + 2$ 'seemingly' simplified the problem substantially by entailing that every *minimal* planar map must contain a configuration of fewer than six sides.

'Seemingly', since it is not obvious whether—unlike the *geometrical* argument of Lemma 2.7 which is immediately evident in Fig.15—a *graphical* argument *must* follow, from Fig.14, which admits the possibility that a five-sided figure may *not* be definable in a *minimal* planar map[3].

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