THE LANGUAGE OF MATHEMATICS

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Mohan Ganesalingam. ***The Language of Mathematics: A Linguistic and Philosophical Investigation***. FoLLI Publications on Logic, Language and Information. Springer, 2013. ISBN: 978-3-642-37011-3 (hbk); 978-3-642-37012-0 (ebook). Pp. xx + 260.

This ambitious book sets out to provide a linguistic analysis of the language used in written mathematics, both textual and symbolic. It is a revised version of the author's Cambridge Ph.D. thesis, a worthy recipient of FoLLI's E. W. Beth dissertation award for 2011. Mohan Ganesalingam is a linguist with a Ph.D. in computer science, and his work combines insights from these disciplines with a substantial grasp of mathematics. However, there is much in the book that should interest philosophers of mathematics. Firstly, Ganesalingam's project leads him to confront some significant issues in the foundations of mathematics, for which he proposes a response that is, in part, novel. Secondly, and perhaps more importantly, he demonstrates something which is often discussed but seldom attempted: he shows how his account of mathematics can be applied to a significant body of actual mathematical practice.

The book is very clearly structured. Chapter 1 begins with a defence of Ganesalingam's methodological presuppositions. Critically, and in distinction from earlier projects of more modest scope, notably the work of Aarne Ranta (Ranta, 1997), he insists on sufficient breadth to encompass all of pure mathematics and on what he calls 'full adaptivity', that any mathematical content be extracted from the text under analysis, and not baked into the analytic system (p. 3). The latter constraint prevents him from, for example, building set theory into his linguistic model. Although his account is intended to provide an analysis of the content of 'rigorous, careful textbooks' he confines it to what he calls the 'formal mode' of the language found therein: the statements exclusively concerning mathematical objects and mathematical properties (p. 7). Characteristically, such textbooks also contain much that is in an informal mode—remarks about the context, or value, or interest, of the mathematical results, say—but, as Ganesalingam notes, analysis of these comments would require a full analysis of natural language (p. 8). Conversely, one of the attractions mathematics in the formal mode holds for the linguist is its comparative simplicity.

In Chapter 2, Ganesalingam surveys the problem that he has set himself, identifying some of the distinctive linguistic features of mathematics that his analysis, or any comparable rival, should account for. These include the interdependency of text and symbols, the extensive use of stipulative definition to expand the language as it is being used, and some idiomatic features of symbolic notation that lack counterparts in natural language. In particular, Ganesalingam observes that the syntax of mathematics is type-dependent: for example, an expression such as $\alpha \to (\beta)^m_n$ may be syntactically well-defined only if α and β are ordinals and

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m and n natural numbers, say (p. 28). Although mathematical language (at least in the formal mode) lacks the great diversity of rhetorical features that bedevil linguistic analysis of natural language, Ganesalingam does draw attention to two rhetorical features it possesses that natural language lacks: 'blocks', explicitly labelled theorems, proofs, and the like, and the overt introduction of assumptions or variables. On Ganesalingam's own estimation, the most distinctive feature of this picture is what he calls 'reanalysis', a term borrowed from philology, by which he describes the process whereby a mathematician comes to revise the sense of his terminology as he learns more mathematics (p. 36). For example, a mathematician's understanding of expressions of the form x^n becomes successively more sophisticated upon learning that the exponent n is not just restricted to values such as 2 or 3, but can range over the natural numbers and, indeed, the integers, the rationals, and so forth.

Chapter 3 sets out the tools with which Ganesalingam proposes to tackle the problems identified in Chapter 2. The type-dependency of syntax adverted to in Chapter 2 leads Ganesalingam to augment the context-free grammar employed by Ranta by a system of types, discussed in greater detail in later chapters. Problems of anaphoric reference lead Ganesalingam to ground his semantics in Discourse Representation Theory (DRT) (Kamp et al., 2011). As he observes, DRT is also the basis for the semantics of the automated proof-checking software NaProChe, although that project is substantially different in intent (p. 82).

Chapters 4–6, which comprise the formal heart of the book, tackle the problem of ambiguity in mathematical writing. Chapter 4 surveys the different forms of ambiguity that can arise in mathematical symbols, text, and, crucially, combinations of the two. Ganesalingam concludes that the textual and the symbolic are inextricably intertwined, and that ambiguities arising in the latter, and therefore ambiguities in general, can only be resolved if the types of the entities involved are known (p. 111). In Chapter 5, Ganesalingam presents his system of types. This comprises types of three different kinds: *fundamental* types, such as NUMBER or GROUP, that are associated with mathematical objects; *relational* types, such as ELEMENT OF A GROUP, that are associated with positions in structures; and *inferential* types, such as SET OF NUMBERS that support inferences about the types of other objects (p. 142). This system resists paradox by allowing some types, including SET OF NUMBERS, to be *non-extensional*: that is, they function as 'tags' but not as properties (p. 123). Ganesalingam distinguishes his treatment of types from more orthodox varieties of type theory in terms of its 'irreducible notion of time: when the declaration of a structure type is encountered, a new type is created' (p. 156). This emphasis on the internal chronology of mathematical understanding arose earlier in his discussion of reanalysis and is the subject of a more protracted defence in Chapter 7. Chapter 6 proposes a parsing procedure for the disambiguation of mathematics by means of Ganesalingam's system of types.

Chapter 7 contains the most overtly philosophical themes in Ganesalingam's book. He begins by positing an asymmetry between foundational and non-foundational mathematics: 'definitions in the foundations are post hoc rationalisations', whereas in 'sufficiently advanced mathematics ... the definitions are the final arbiter of truth' (p. 177). Hence the purely descriptive methodology that he has applied to mathematical practice in general will be insufficient for the foundations. He regards a more revisionary tack as warranted, not least by the disconnect between much mathematical practice and foundational assertions. For example, although most mathematicians would accept intellectually that the natural numbers and the real numbers are disjoint classes of set, in practice they treat natural numbers as a special sort of real number and treat numbers and sets quite differently (p. 183). These reflections on foundational mathematics lead Ganesalingam to three philosophical problems which his account must confront (p. 200). Firstly, it must reconcile the ontology and epistemology of mathematics, reflecting the epistemic practice of mathematicians without positing a novel ontology at odds with the axioms they accept. Secondly, he needs an account of cross-sortal identification which can specify when abstract objects introduced as components of different structures are to be treated as the same object. The '2' which is an element of the natural numbers and the '2' which is an element of the reals are usually identified, despite being defined quite differently in the foundations of mathematics. Lastly, he is concerned to accommodate the way that mathematics develops over time, not just as a discipline, but in the career of the individual mathematician. Here he employs a familiar biological metaphor, distinguishing *phylogeny* from *ontogeny*, respectively.

It is in Ganesalingam's engagement with these questions that philosophers of mathematics may expect to find the most relevance to their own work. However, for better or worse, his response is disconnected from recent work in the philosophy of mathematics. For example, he notes that one possible solution to the identification problem, treating relationships such as that between $\mathbb{N}$ and $\mathbb{R}$ as isomorphic embeddings, runs up against a problem first noted by John Burgess as a difficulty for Stewart Shapiro's ante rem structuralism: there are non-trivial automorphisms of $\mathbb{C}$ (and many other mathematical structures) (Burgess, 1999, p. 288). That is, mapping every complex number to its complex conjugate preserves the structure of the complex numbers, but is obviously not an identification in the desired sense, since mathematicians consider i and $-i$ distinct. Although Ganesalingam cite's Burgess's review (p. 184), he does not cite Shapiro's proposed solution to Burgess's problem (Shapiro, 2008), or anyone else's (such as Keränen, 2001; Ladyman, 2005).[1] This omission makes sense: defences of ante rem structuralism from Burgess's problem are not necessarily any help to Ganesalingam, since his concerns are not Shapiro's. However, the broader issue of cross-sortal identification has drawn attention from perspectives besides structuralism (Cook and Ebert, 2005), yet Ganesalingam does not engage with this work either. His own solution to the problem turns on two new kinds of block, *systems* and *models* (pp. 203 f.). Within a system a mathematician is at liberty to posit new kinds of object, providing that the system is followed by a model which cashes these new objects out in terms of existing objects. (So a model can secure the object's place within the 'official' ontology, however mathematical practice may treat it.) Objects from two different systems may then be identified precisely when each system models the other, subject to the somewhat ad hoc constraint that objects may only be mapped to themselves by identity functions (p. 213).

In his focus on the chronology of mathematics, Ganesalingam is in philosophically less well-travelled territory (although see Dutilh Novaes, 2013, for an independent account of mathematical phylogeny and ontogeny). As he observes, philosophers of mathematical practice have paid substantial attention to the phylogeny of mathematics, most memorably in (Lakatos, 1976), but much less attention to its ontogeny (p. 186). An important insight that Ganesalingam derives from his focus on time is what he calls the principle of 'isochrony', that no stage in the ontogenetic development of mathematics should be privileged over any other (p. 190). That is, the linguistic analysis of a particular stage in a mathematician's development ought not to appeal to mathematics as yet unknown to that mathematician—no peeking into the ontogenetic future, as it were. However,

[1] Even more recent work includes (Heathcote, 2014; Kouri, 2016).

Ganesalingam notes that, particularly in more elementary work, mathematical concepts are often acquired informally in a quite different sequence from how they are studied formally: informally, we learn about numbers before we learn about sets; formally, sets come first (p. 188). Hence he distinguishes between *formal* ontogeny and *psychological* ontogeny, and stresses that his concern is with formal ontogeny alone (although for sufficiently advanced mathematics the two should coincide). This suggests that what Ganesalingam is actually concerned with is conceptual priority, rather than any sort of chronological sequence. If that is correct, then the notion of time may not be an irreducible feature of his account after all.

The book closes with two short chapters: an outline of ways in which Ganesalingam's system might be extended and a concise summary by way of conclusion. While I have criticized aspects of Ganesalingam's work, I should reiterate my admiration for his project and my broad sympathy with many of his conclusions. If his project would have benefitted from a greater engagement with prior philosophical work, it is equally true that future philosophical work should benefit from a greater engagement with his project.

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