Defectiveness of formal concepts

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January 2024 - SUBMITTED DRAFT

Abstract

It is often assumed that concepts from the formal sciences, such as mathematics and logic, have to be treated differently from concepts from non-formal sciences. This is especially relevant in cases of concept defectiveness, as in the empirical sciences defectiveness is an essential component of lager disruptive or transformative processes such as concept change or concept fragmentation. However, it is still unclear what role defectiveness plays for concepts in the formal sciences. On the one hand, a common view sees formal concepts to be protected against defects because of their precise and stable nature. On the other, studies going back as far as Lakatos (1963) showcase the changeability of such concepts.

In this paper, I will investigate if and how defectiveness based on the occurrence of inconsistencies can appear with formal concepts. To make the case as strong as possible, I employ a strict notion of formal concept that assumes the concept to have a fixed and definite extension. I will show that there are indeed certain types of defectiveness that cannot occur with such concepts; but that there are other types of defectiveness that do occur. This means that while formal concepts have to be treated differently than non-formal concepts, questions about defectiveness—as raised in the conceptual engineering debate and philosophy of science—can still be applied to them. I will highlight this point by showing how formal sciences have special strategies available to them that allow them to resolve defectiveness of their concepts in a flexible and informative manner.

1 Introduction

Concepts from the formal sciences, such as mathematics and logic, have a special status in the philosophical debate. In contrast to concepts from the empirical sciences, they are seen as being especially stable, abstract and rigorous. When discussing questions like conceptual change, concept pluralism and concept defectiveness, it is generally assumed that these questions require special treatment or even that they are completely exempt from these discussions. One example is the debate about the possibility of Kuhnian revolutions for formal sciences. Serious doubts exist about the applicability of this phenomenon to mathematics (for a collection of standpoints on this see Gillies, 1992). The belief that formal concepts are somehow exempt from problems that befall other concepts can also be found in recent debates about conceptual engineering. As Tanswell (2018, 882) puts it for mathematical concepts: "There is a temptation to view mathematics in an idealised way as a safe-haven for concepts, where rigor and formality protect against the defects, imprecision, sloppiness and inconsistencies of everyday concepts."

Tanswell (2018) argues against this idealization, pointing out that mathematical concepts can be fluid and changeable (relating back to the seminal Lakatos, 1963). And indeed there are concepts fundamental to mathematics that are notoriously problematic, such as the concept of set (see e.g. Incurvati, 2020). So, the assumption that all concepts from the formal sciences are protected against defects seems to be inaccurate. But it seems equally inaccurate to state that all concepts of the formal sciences are imprecise and changeable as this does not conform with the way exact definitions for concepts are used, for example in mathematical proofs. The question remains if there is a special status formal concepts can have and if this status makes them impervious to defects discussed regarding concepts in the empirical sciences.

I will approach these questions pursuing a different strategy. Instead of showing like Tanswell (2018) that formal concepts can be defective because they are not precise, I will show that *even if* they are rigorous and precise, defectiveness can still occur in certain contexts. This will allow us to get a more detailed picture of the way formal concepts and defectiveness are related: One the one hand, we will be able to make exact the claim that there are certain types of defectiveness that formal concepts are indeed impervious to. But, on the other hand, we will see that are still other types of defectiveness that occur with formal concepts nonetheless. So, while the idea that formal concepts are a safe haven against defects can be explicated to a certain degree, it has to be rejected in general. In particular, defectiveness is not necessarily dependent on the imprecision of the concept, but can also occur with precise concepts.

To argue for this view, I will employ a rather strong notion of formal concept, according to which the concept is given by a clearly delimited extension[1]. This excludes most of the well-known problematic concepts, such as the concept of set or other concept given by implicit definitions. Instead, we will concentrate on concepts that are stable and clearly given. For those, problems that rely on an inaccuracy regarding their extensions do not apply.[2] But, I will argue, they can still exhibit a strong form of defectiveness, namely one relying on the occurrence of an inconsistency. However, they do so in a manner that is markedly different to the way in which defectiveness via inconsistency of non-formal concepts occurs.

Such a difference can also be found in resolution strategies for defectiveness of formal concepts. Because such concepts are always embedded in specific formal frameworks (logics, axiomatizations, models etc.) it is often much easier to accommodate changes in these frameworks. This flexibility is a powerful feature of formal sciences and we will see that it provides resolution strategies that are usually not available for non-formal sciences.

One conclusion of this paper will therefore be that formal sciences and their concepts are *sui generis* and have to be handled differently than those in the empirical sciences. However, this does not mean that they are impervious to problems sim-

[1]Note that the view I develop here is not dependent on a particular theory about the structure of general concepts. In particular, I do not claim that all concepts are given by such extensions or that the so-called classical theory of concepts holds (see Margolis and Laurence, 2023). I also do not rely on a specific theory about ontology of concept and I will therefore not address such questions further in this article.

[2]One example are inaccuracies such as as spelled out in the notion of *open-texture*. Open-texture, such as discussed by Waismann (1947), expresses the general idea that there may be objects where it is unclear if they are part of the concept or not; see also Tanswell (2018).

ply because they are rigorous and stable in the idealized manner addressed in the quote of Tanswell (2018) given above. The position developed here is therefore an intermediate position between the view that formal sciences are unaffected by the problems that befall other sciences and the view that entirely rejects such a special status of formal sciences. Such an intermediate position provides a fruitful basis on which considerations about general concepts can be transferred and applied to formal concepts while at the same time taking into account the way in which formal concepts differ from general ones.

I develop this intermediate position in the following manner: First, I focus on concepts that exhibit rigor and formality to a very high degree. For this purpose I develop a stringency criterion for formal concepts that is based on their use in the formal sciences as well as the way in which they are defined (Section 2). I will then make the case that such concepts can still be defective, although in a different way than is usually discussed in the literature on conceptual engineering: formal concepts cannot be inconsistent concepts in the sense of Scharp (2020) (Section 3). But they can be defective via inconsistencies that arise relative to specific settings (Section 4). I will discuss two ways in which such defectiveness can occur: One is related to the possibility of having two definitions for the same concept that are formally equivalent in one setting but not in another (Section 4.1). The other is based on an inconsistency that comes up between the definition of a concept and the formal framework it is considered in (Section 4.2). I will provide arguments as to why both cases are not instances of inconsistent concepts while at the same time still giving rise to an inconsistency and therefore defectiveness of the concept involved. Lastly, I will discuss resolution strategies for this kind of defectiveness and point out how the special way in which formal sciences depend on their formal frameworks makes them very flexible in encountering and defusing defectiveness of formal concepts (Section 5). In Section 6 I will then conclude the article by providing a short outlook of how the position developed here provides a good way of unifying the competing views on the special status of formal concepts.

2 Formal concepts

There are different notions about concepts in the formal sciences. Often they are seen as rigid and precisely given by definitions that fully characterize them. However, some philosophers, such as Lakatos (1963), have focused on the way concepts form and change over time and characterize them as fluid and changeable. As mentioned in Section 1, these characterizations open up different possibilities to account for defectiveness of formal concepts. For examples, the way in which Tanswell (2018) transfers discussions from the conceptual engineering debate to mathematical concepts rests on the a fluid notion of mathematical concepts. Under such a picture, approaches to defectiveness developed for general concepts can be applied to formal concepts as well. Note that Tanswell (2018) does not claim that *all* mathematical concepts are fluid, he only claims that some of them are and therefore that the realm of mathematical concepts in general is not immune from such defects. However, this might not convince staunch supporters of the view that formal concepts cannot be defective. They can explain away the fluidity of formal concepts and the resulting defectiveness by maintaining that they only occur in specific contexts where concepts are not fully formed yet.[3] But, they argue, when concepts are fully formed and stable, for example when used in proofs, these imprecisions have been eradicated and therefore defectiveness does not play a role any longer.

For a stronger argument, it is necessary to show that defectiveness can also occur with formal concepts that are stable and rigorous. I will therefore provide a notion of formal concepts that is especially strong in having stable and well-delimited extensions. It is important to note that I don't see such a notion as an idealization of what formal concepts can be. Instead, concepts with such extensions are frequently used in the practice of formal sciences. Considering them is, then, also a necessary part of providing a full picture of which concepts are used in the formal sciences. Let us therefore start with a characterization of what I mean what talking about formal concepts.

[3]This can be embedded in the broader context of distinctions such as the context of discovery and context of justifications as well as specific versions of it for formal sciences such as mathematics (see e.g. Maddy, 2019).

Formal concepts, as treated here, are concepts that satisfy two criteria: first, they are concepts from the formal sciences; and second, they are fixed and definite. The first criterion can be used to differentiate formal concepts from general abstract concepts that include e.g. “justice” or “friendship” by restricting the focus to abstract concepts that are part of the formal sciences. Standard examples for formal sciences are mathematics and logic, but one can also count (parts of) theoretical computer science, theoretical linguistics and theoretical philosophy among them.[4] It is, however, not essential to my investigation to draw sharp boundaries here and definitely include or exclude specific disciplines. Neither do I claim that all concepts from these sciences are what I take to be formal, nor do I claim that these are the only areas where formal concepts exists.[5] Instead, I will take examples mostly from disciplines that uncontroversially may count as formal, such as mathematics and logic, and will leave it to further case studies to provide examples from other (parts of) sciences. Furthermore, the criterion of being part of a formal science will mostly be used to distinguish concepts that are actually used in the formal sciences from made-up concepts that can be formally formulated but are of no importance to the sciences (see Section 4.

The second criterion, that of being fixed and definite, provides us with a specific notion of formal concepts that encompasses the idea of being stable and well-delimited. It is modeled on a distinction made by Schlimm (2012) for mathematical concepts. Schlimm distinguishes between two notions of mathematical concepts, one called Lakatosian and the other Fregean. He characterizes the Lakatosian notion as embracing the “elasticity and inexactness of concepts” (Schlimm, 2012, 133) that is especially apparent during concept formation and the development of new theories, theorems, definitions etc. In contrast,

> Frege promoted a view of mathematics according to which its subject matter is regarded as static, representable by fixed and definite concepts.

[4] As a current acknowledgment of their status, these disciplines are listed as formal sciences in the introduction to the “Foundations of the Formal Sciences” Series (see Löwe, 2002).

[5] Take, for example, concepts specific to theoretical physics for which no empirical evidence exists.

> Frege's understanding of a concept as being fixed can be interpreted to mean that its extension does not change over time. If an object a falls under a concept P at some point in time, then it always falls under it, eternally. That a concept is definite means that it is determined for every object, whether it falls under the concept or not. (Schlimm, 2012, 128)

With these stipulations at hand, some often discussed cases of concept defectiveness are excluded: First of all, conceptual change during concept formation, as detailed in Lakatos (1963), is here not seen as the extension of one concept changing over time. Instead the concept is fixed, i.e. it is stipulated that for times t and s, a falls under P at time s if and only if it falls under P for all t as well (Schlimm, 2012, 129). Second, defectiveness resting on the vagueness of the extension is excluded by the criterion of definitiveness. Schlimm (2012, 129) makes this precise when elaborating that for each object it either falls under P or it does not (for each point in time t). So we do not only get a clearly delimited extension but also a clearly delimited complement of the extension (a point that will be significant when discussing defectiveness in Section 4). Note that assuming a Fregean notion does not mean that one cannot account for historical examples of concept change; one simply interprets such a change as giving rise to different concepts, such as the natural numbers excluding 1 give rise to a different concept than the natural numbers including 1 (cf. Schlimm, 2012, 139).

The distinction between a Lakatosian and Fregean notion provides a fitting framework for this papers approach: The Fregean notion spells out a common view on mathematical and formal concepts, which is often seen to be impervious against defectiveness. When defectiveness of such a concept nonetheless occurs, it can either be argued that the Fregean notion is faulty and that we should accept that mathematical concepts should be considered under a Lakatosian one; or it can be argued that such defects only occur with concepts that don't fit the strong requirements of the Fregean notion and therefore are not really mathematical concepts (yet). I will take neither of these two options. Instead, I will take the Fregean notion to be a good explication of the often assumed special status of mathematical and formal concepts and show that defectiveness *still* occurs under this notion.

The characterization of "formal concepts" that I will focus on for the rest of this paper can therefore be characterized as follows: A concept is a formal concept if

1. it is a concept from the formal sciences; and
2. the extension given by its definition does not change over time (i.e. it is *fixed*) and it can always be determined if either some object is contained in the extension or if it is not (i.e. the concept is *definite*).

Note that in the second clause of this characterization we deviate slightly from the way in which the Fregean notion is spelled out in Schlimm (2012) in that we refer to definitions.[6] This is necessary because definitions act as the ultimate "extension-providers" for concepts in the formal sciences. We will therefore consider definitions that give rise to different extensions as being definitions for different concepts. This works just like in the case of natural numbers, where one extension includes 1 and the other does not and thus give rise to two different concepts of natural numbers. It is not relevant for our present purpose to specify which kind of definition is employed, i.e. if it is stipulative, explicative, etc. We simply require that the definition provides us with a fixed and definite extension of the concept under consideration.

This characterization explicates the precision and stringency that is often seen as the defining feature of concepts in the formal sciences. Dropping either fixedness or definiteness would weaken the characterization too much. It would leave the extension of a concept not clearly delimited and would therefore essentially collapse it to the Lakatosian notion. As we aim for a strong notion of formal concepts, it is unproblematic that this characterization may exclude certain concepts, for example ones given by implicit definitions. With this I do not claim that all concepts in the formal sciences satisfy the Fregean notion.[7] But I start from the assumption that

[6] However, Schlimm (2012, 130) still shows that, for Frege, definitions are needed to be able to formulate the requirements of fixedness and definitiveness, as the definitions allow us to clearly judge what is (and is not) part of the extension of the concept.

[7] In fact, I agree with Schlimm (2012) who argues for a pluralistic view of the notion of mathematical concepts by pointing out that both notions are relevant and occur in the practice.

there are (a substantial amount of) "Fregean concepts" in the formal sciences.[8] In this sense, I take a stance on the question of the special status of formal sciences: Formal sciences are indeed *sui generis* in that they are made up to a substantial degree by what I characterized here as formal concepts. When investigating questions of defectiveness, they have to be treated differently than concepts from the empirical sciences. But—and that is the main point of this paper— these questions can be investigated here, too, as defectiveness still appears with formal concepts. Therefore, endorsing a Fregean notion does not automatically mean that mathematics, or other formal sciences, are a "save heaven" against all forms of defectiveness.

3 Formal concepts are not inconsistent concepts

Concepts can be said to be defective in various ways. For philosophical concepts, Plunkett and Cappelen (2020, 3) list

> cognitive defects (that undermine our ability to reason properly), moral or political defects (that undermine moral or political values of various sorts), theoretical defects (that undermine progress within some theoretical field), or semantic defects (where the semantic value is incoherent, incomplete, or missing).

This is a huge range, so in this article I will focus on one specific type of defectiveness, namely defectiveness via inconsistency.

Considering defectiveness via inconsistency is a promising approach for formal concepts. It is a very strong type of defectiveness: if we can show compatibility of this type with formal concepts, we can derive a strong version of our claim.[9] Furthermore, inconsistencies are generally accepted by the scientific communities as

[8]Indeed, one could argue that this is an essential part of what makes a science a formal one.

[9]There are areas in the formal sciences, such as paraconsistent logics, where this kind of problem does not occur, because explosion is rejected. So it may be the case that tying defectiveness to inconsistency is not appropriate for formal concepts within such areas. Note, however, that it does not restrict our overall task of defining defectiveness for formal concepts too much, because in the vast majority of formal sciences classical logic is assumed as the unstated background logic.

a warning sign that something is amiss. When inconsistencies come up, they often lead to broader discussions about the objects, methods and foundation of the theory. Take for example the inconsistency arising from the use of infinitesimals in 17th century calculus, or the inconsistency arising from Russell's Paradox and similar problems in the early development of set theory. Inconsistencies don't always have to have such far-reaching consequences, but they usually point to a problem in the underlying theory that has to be resolved. In the following I will consider cases, where this problem is seen to indicate a defectiveness in the related concept(s).[10].

Considering defectiveness via inconsistency is an important part of the conceptual engineering debate. In (Scharp, 2013) inconsistent concepts are introduced as a PRIME[11] example of defective concepts. Here, a concept is inconsistent if an inconsistency arises from its constitutive principles. Scharp's toy example is the concept RABLE whose definition is given by two constitutive principles: The first principle stipulates that RABLE applies to x if x is a table, and the second that RABLE disapplies to x if x is a red thing (where Scharp uses "disapplies" as an antonym of "applies", cf. 2013, 36). The inconsistency occurring here comes from the fact that by way of the first constitutive principle a red table is rable, whereas a red table is *not* rable by employing the second principle. Let's spell this out in more detail: Scharp (2013, 40) calls the domain of objects to which a concept applies the range of applicability that is then divided into the application set, consisting of all objects to which the concept applies, and the disapplication set, consisting of all object to which the concept does not apply. A concept is called application-inconsistent, if its application and disapplication set are not disjoint.[12] RABLE is application-inconsistent

[10] There is another type of defectiveness that is discussed for the formal sciences, namely the problem of not defining the concept in the "right" manner. What this "right" manner comprises of is often unclear as it can include several aspects like naturalness, fruitfulness or purity. But most would agree that a minimum requirement is that the definition is consistent. I will therefore concentrate on defectiveness that is connected to inconsistencies and not investigate these other types of defectiveness any further.

[11] In the following, I will use the convention of referring to concepts by using uppercase letters, whereas the use as adjectives is marked by lowercase letters.

[12] He also introduces the range of inapplicability that contains all objects which fall outside of the application of the concept, and the respective notion of range-inconsistent concepts. This mainly concerns partial concepts and we can set this aside for the investigation attempted here.

because a red table is both part of its application and its disapplication set and they are therefore not disjoint.

This approach can be applied to concepts from the formal sciences as well. Scharp and Shapiro (2017) discuss the logical conception of set which is characterized by an axiomatization containing the Axiom of Unrestricted Comprehension. Here, each of the axioms can be seen as providing one constitutive principle. One can show that taking such a collection of constitutive principles results in an application-inconsistent concept. According to Unrestricted Comprehension, the collection of all sets is again a set. But by the Power Set Axiom, the collection of all sets cannot be a set, because then we run into inconsistencies like Cantor's paradox.[13] So the collection of all sets is both in the application and in the disapplication set of SET given by such constitutive principles.[14]

So Scharp (2013)'s analysis provides a good starting point for our investigation into formal concepts. However, some changes have to be made to account for the special setting in the formal sciences. The major change consist in taking definitions instead of constitutive principles as the basis for showing inconsistencies. This is a non-trivial change, as it is not as straightforward as it might seem to form definitions from constitutive principles (and vice versa). One example, provided by Scharp, shows that constitutive principles give rise to very different definitions depending on how they are joined together. In the case of RABLE, a consistent concept can be derived by joining the two constitutive principles by using an "and" clause and obtaining the extension that contains all non-red tables (Scharp, 2013, 39).[15]

Choosing definitions as starting points for investigating inconsistencies also accords with the picture of formal concepts developed in Section 2. There we assumed that definitions provide the extensions of concepts. Concepts are usually only used

[13] If it were a set, it would have a power set and due to a theorem by Cantor the power set of a set is strictly larger than the set itself. So the power set is not a subset of the set of all sets, producing a logical inconsistency.

[14] Note, however, that SET is not a formal concept according to the characterization given in Section 2.

[15] Also, it might well be that there are definitions that are not made up (solely) of constitutive principles. This can be the case if some clauses in the definition are only there to make it work (e.g. by excluding some problematic cases).

in the formal sciences if a definition exists for them, therefore not relying only on constitutive principles, but also on the way the latter are joined together to form a definition. This becomes particularly important when studying phenomena that are, perhaps, more prevalent in the formal sciences than in other disciplines. One of these is the existence of different, formally equivalent definitions for the same concept. This is a prominent feature, for instance, in mathematics, where there are concepts that have several, formally equivalent definitions.[16] Such concepts are usually seen to be particularly informative, especially if these definitions stem from different fields and therefore unify different areas of research. As we will see in Section 4, concepts with different equivalent definitions also play an important role when investigating defectiveness. To analyze these equivalences however, we have to consider the complete definitions, and not only components of them. We therefore have to work with definitions instead of constitutive principles.

Beside considering definitions, I will make two further adaptations: For one, I will narrow the focus to logical inconsistencies. For Scharp (2013, 36), this is not necessarily the case as he allows for other inconsistencies as well. For example, the inconsistency of RABLE consists in the red table. But in the case of formal concepts, logical inconsistencies are a strong and reliable notion. Formulating a logical inconsistency makes sure that there really is an inconsistency. This excludes cases that might seem inconsistent but turn out not to be when they are appropriately spelled out. This is for example the case with Skolem's Paradox, which is not logically inconsistent when its (meta)-mathematical setting is made exact (cf. Bays, 2007). Also, narrower focus excludes cases where something is in the (dis)application set because of the definition of a concept, but it could be argued that it should not be because it does not match with what is non-formally understood under this concept. Such situations might certainly arise in the formal sciences and a mismatch between intended meaning and extension of the concept can give rise to a form of defectiveness. But it is not one of inconsistency in a logical sense and therefore not the target of

[16]For example, the mathematical concept of a topological space can be defined via neighborhoods or via open sets; the real numbers can be defined via Dedekind cuts or via Chauchy sequences; in set theory the natural numbers can be defined as the finite von Neumann ordinals, or the finite Zermelo ordinals etc.

our investigation.

The last adaption will be to consider the formal setting in which the inconsistency occurs. For Sharp (2013)'s general setting it is of "no practical importance" if an inconsistency occurs "relative to some set of claims" (36). But this is of significant import to formal concepts. Defectiveness can occur in the formal sciences when inconsistencies appear relative to the formal framework the concepts are considered in. We will make this exact in Section 4.[17]

Considering all of these adaptations, I formulate the following definition: A formal concept is (application-)inconsistent if the application set and disapplication set of the definition are not disjoint and this is witnessed by a logical inconsistency.

When comparing this definition to the characterization of formal concepts given above, it becomes immediately clear that *formal concepts cannot be application-inconsistent*. Formal concepts are concepts from the formal sciences that are fixed and definite. The latter means that an object is either in the application set of the concept or in the disapplication set and that this does not change over time.[18] So, for formal concepts it can never happen that the application and disapplication set are not disjoint and that this is witnessed by a logical inconsistency. Therefore formal concepts cannot be (application-)inconsistent.

This is not a trivial result: Recall that we did not develop the notions of formal concept and application inconsistency with direct reference to one another. Instead, we started with two different goals: the first was to explicate the special status often ascribed to formal concepts. Second, and independently from the characterization of formal concept, we transferred a common notion of defectiveness from the conceptual engineering debate to the formal sciences. The outcome given above, then, means that the notions resulting from these two approaches are incompatible. It provides an exact explication of the belief that formal concepts are not subject to defectiveness because their special form makes them resilient against it.

[17] One area where we do not have to make any adaptions is how to deal with inconsistent concepts. Scharp (2013) engages in detail with the question if inconsistent concepts can exist and how they can be possessed and employed. As we will see at the end of this section, we will not have to deal with such questions for formal concepts.

[18] Note again, that we do not consider the inapplication set of concepts here.

However, it also shows the boundaries of such a belief. First, it does not mean that all concepts of the formal sciences are generally not application-inconsistent, just as we never assumed that all concepts from the formal sciences are formal concepts in the sense of Section 2. For mathematics, Schlimm (2012) describes such non-formal concepts as concepts under the Lakatosian notion. Alternatively, Tanswell (2018) discusses such concepts under the viewpoint of open-texture. For formal sciences, such as parts of theoretical philosophy, this is even more obvious. Indeed, the whole point of Scharp (2013) is to show that TRUTH is such an application-inconsistent concept—and, of course, then it is not a formal concept in our sense.

Second, even if all concepts from the formal sciences were formal concepts, our investigation only states that they cannot be application-inconsistent. I already mentioned that there are other forms of defectiveness that can arise in the formal sciences. In the following I will take a second look at defectiveness via inconsistency and show that there is indeed a way of making this idea workable in the setting of formal concepts.

4 Formal concepts can be defective via relative inconsistency

In this section I develop an account of defectiveness of formal concepts that is still focused on inconsistency, but not on application-inconsistency as defined above. Instead, we will adapt it to two situations that arise in the formal sciences. First, we will study inconsistencies that come up when two formally equivalent definitions for the same concept are not equivalent any longer due to changes in the formal setting. And, second, we will investigate inconsistencies that occur directly between a definition and the formal setting it is considered in.

4.1 Inconsistency from different definitions

In the formal sciences it is quite common to have equivalent definitions for the same concept. The existence of such definitions is considered to be highly informative.

They contribute to unification when occurring in different sub-fields, or to understanding when providing different constructions. For example, consider the different ways the real numbers can be defined in mathematics, either by axiomatic approaches (such as Tarski and Tarski, 1994) or the construction via Dedekind Cuts. In theoretical computer science a prominent example is the various formal definitions of the concept of computability (for an overview, see Soare, 1999).

In the following, we will develop an account of defectiveness via inconsistencies that arise from the case of equivalent definitions. Much of the setting stays the same: We will continue to use the definition of formal concepts given in Section 2 and to consider concepts to be given by a definition and the corresponding extension. To have two equivalent definition, then, means that both definition give rise to the same extension and, therefore, to the same concept. As we pointed out in Section 3, it would not do working with constitutive principles instead of definitions here. When considering different constitutive principles for the same concept, the concept will only be given through all of them (otherwise they would not be constitutive). But for definitions it is enough to have one of them to fully determine the concept, despite the additional insight that equivalent definitions might offer.

Having two (or more) equivalent definitions is of course not problematic by itself. But it can give rise to a form of defectiveness if it happens that the two definitions are considered in a context where they are no longer equivalent. Let us consider the case of the concept PRIME, referring to prime numbers in mathematics. This concept can be defined in two ways:[19]

Definition$_\mathbf{1}$. A natural number greater than 1 is prime if it is evenly divisible only by 1 and itself.

Definition$_\mathbf{2}$. A natural number greater than 1 is prime if whenever it evenly divides some product bc it divides either b or c.

[19]Note, again, that each of these are complete definitions and do not have to be combined to provide the concept PRIME.

These definitions are equivalent if considered over the natural numbers, meaning that they give rise to the same set of number $\{2, 3, 5, 7, \ldots\}$. However they differ in their extension when considered over quadratic integer rings such as $\mathbb{Z}[\sqrt{-5}]$. Here 2 is prime via Definition$_1$ but not prime via Definition$_2$, therefore producing two different extensions for PRIME.[20] So, we have one concept of prime number given by two equivalent definitions. However, once we extend the domain in which we consider these definitions, we see that they are not equivalent in general.[21]

To analyze this situation, first note that it gives rise to a logical inconsistency. In the set-up described above the sentence "In $\mathbb{Z}[\sqrt{-5}]$, 2 is a prime number and 2 is not a prime number." can be proven. At the same time, this is not an application-inconsistency. Each of the definitions given above provide us with disjoint application and disapplication sets. Definition$_1$ has 2 in its application set and not in its disapplication set; Definition$_2$ has 2 in its disapplication set, but not in its application set. The inconsistency arises because the application set of Definition$_1$ and the disapplication set of Definition$_2$ are not disjoint and because both definitions are assumed to provide an extension for PRIME.

This type of inconsistency does not stand in contrast to the definition of a formal concept, as it was in the case of application-inconsistency. First, PRIME is a concept from a formal science, i.e. mathematics. More importantly, as we are working with definitions and not with constitutive principles, we can consider them independently from one another. So, when checking if PRIME is a formal concept, we check if the extension given by a definition, say Definition$_1$, is fixed and definite. Because this is the case, PRIME is a formal concept. Alternatively, we could have checked if the extension given by Definition$_2$ is fixed and definite and we would have come to the same conclusion. This is also why we would not say that PRIME is inconsistent the way Scharp (2013) posits in the case of application-inconsistent concepts. Each of its definitions taken by itself does not give rise to an inconsistency. But PRIME is still defective, as it gives rise to a *relative inconsistency*[22]. This inconsistency

[20]For more details see for example Tappenden (2008, 267–268).

[21]Historically, considering prime numbers over the natural numbers was standard for a long time, while the consideration of quadratic integer rings came much later. See also Schlimm (2012, 10–11).

[22]One might object that this situation is one of *two* concepts that stand in contract to each

occurs when considering one definition relative to the other, as well as relative to the theory of numbers in which both definitions are considered (remember that it does not occur when considering the definitions over the natural numbers). I will therefore call such concepts *defective via relative inconsistency*. Before discussing this notion of defectiveness more generally, let us consider another version of it that also occurs in the formal sciences.

4.2 Inconsistency from formal background

There is another type of relative inconsistency that can be the cause of defectiveness of formal concepts. It occurs between the definition of a concept and the theory in which it is considered. We will elaborate this idea using the example of Reinhardt cardinals from set theory.

Reinhardt cardinals are a specific type of large cardinal, i.e. uncountable limit cardinal numbers that are "too big" to be reachable by ordinary means. An example of a "small" large cardinal are *inaccessible cardinals* where κ is inaccessible if one cannot use operations such as cardinal arithmetic or power set on cardinals strictly less then κ and get something equal or bigger than κ.[23] Large cardinals are of great importance in set theory, not least because they provide a very nice structure of the uncountable.[24] One notable features of large cardinals is that there is a uniform way of defining many different large cardinal notions, namely, via elementary embeddings of a set-theoretic model V in another model M.[25] The idea is that the first model V represents the "real" or "intended" model of the axioms ZFC of set theory and M is an elementary submodel of V. The embedding will map ordinal numbers to ordinal numbers and will be strictly increasing, meaning that eventually there will

other rather than one concept being defective. I disagree because I interpret the formation of two distinct concepts as a reaction towards the defectiveness of PRIME. I provide an arguments to that end in Section 5.

[23]A cardinal κ is (strongly) inaccessible if and only if for every $\lambda < \kappa$, it holds that $2^\lambda < \kappa$, and κ is regular. An infinite cardinal κ is regular if $cf\kappa = \kappa$, where $cf\kappa$ is the least limit ordinal α such that there is an increasing α sequence $\langle \kappa_\eta : \eta < \alpha \rangle$ with $lim_{\eta \to \alpha} \kappa_\eta = \kappa$.

[24]For an introduction and thorough discussion see Kanamori (2003).

[25]An embedding $j : V \to M$ is elementary, if M is an elementary submodel of V, meaning that the models agree on all formulas with parameters from M.

be one ordinal that does not get mapped onto itself by j (so as to assure that V is embedded in M). This ordinal is called the critical point of the embedding and this is then identified as the large cardinal in question. This type of definition can give rise to different large cardinal notions depending on how closely the model M resembles V. The closer this resemblance is, the stronger the resulting large cardinal notion. For example for *measurable cardinals*, the simple existence of such an embedding is enough, whereas for the stronger notion of *strong cardinals* the additional requirement has to be satisfied that V is contained in M for every cardinal bigger than κ.[26]

Reinhardt cardinals derive out of a natural generalization of these definitions for large cardinals (cf. Reinhardt, 1967). They express the situation in which M resembles V as closely as possible: they are the critical point of an elementary embedding j from V to V (in a non-trivial manner so that that j is not the identity). As natural as this definition is, it turns out that the existence of Reinhardt cardinals is inconsistent with the Axiom of Choice and therefore with the standard system of set theory, ZFC. This result was proven by Kunen (1971) and is often called the Kunen inconsistency.[27] The proof of the Kunen inconsistency[28] shows that there cannot be a nontrivial elementary embedding from V into V, so no embedding exists that could give rise to an instance of the concept of Reinhardt cardinal. That means that the extensions of REINHARDT is empty in ZFC.

In the case of REINHARDT we see a logical inconsistency, namely the inconsistency between the existence of the respective elementary embedding and the Axiom of Choice. However, this concept is not application-inconsistent. The application and disapplication set of REINHARDT are disjoint for the trivial reason that the application set is empty. Scharp (2013) calls such concepts *unsatisfiable concept*.[29] and excludes them from his investigation as they are not inconsistent themselves.

[26]A cardinal is *strong* if for every $\lambda > \kappa$, there is an elementary embedding $j : V \to M$ where κ is the critical point of j and $V_\lambda \subseteq M$.

[27]Actually, Kunen proved this in von Neumann-Bernays-Gödel class theory and later Suzuki (1999) proved it in ZFC.

[28](See, for example Jech, 2003, 290)

[29](See Scharp, 2013, 39).

However, as the focus on our investigation is *defectiveness* which is witnessed by an inconsistency, the case of REINHARDT is relevant. First, observe that this is indeed an example of a formal concept leading to an inconsistency. The extension of REINHARDT is fixed and definite as it is always empty. One might object that having an empty extension simply means that there is no such concept and that it therefore also cannot serve as an example for a defective formal concept. We will set this aside for now and first focus on Clause 1 of the characterization of formal concepts, namely that they are concepts from the formal sciences.

In general, there are potentially a great number of unsatisfiable concepts that can be invented. Scharp (2013, 39) provides one of these to illustrate what an unsatisfiable concept looks like: x is a squircle if and only if x is a square and x is a circle. As is the case of REINHARDT, SQUIRCLE is definite and fixed because its application set is empty. Also, the definition of SQUIRCLE makes use of two concepts of a formal science, namely SQUARE and CIRCLE. Still, it does not constitute a formal concept in our sense as it is itself not a concept that is used in a formal science and therefore fails to fulfill the first clause of the characterization of formal concepts. So, SQUIRCLE and other artificially constructed concepts are not examples for defective formal concepts. And they shouldn't be as the target of this investigation are concepts that are genuinely used in the formal sciences and considering made-up concepts will only muddy the waters.

In contrast, REINHARDT is not an artificial concept: it was developed and used in set theory and, most interestingly, is still used despite knowing that it is inconsistent with the Axiom of Choice. Reinhardt cardinals were introduced by Reinhardt (1967) in his doctoral dissertation. He had been working on strengthening large cardinals defined by elementary embeddings and this was a natural continuation of this work. After Kunen's proof of the inconsistency, the concept of Reinhardt cardinals continued to used, namely as a benchmark for when inconsistency occurs, as it serves as a natural limit for large cardinal strength: Different proofs of Kunen's inconsistency were found connecting the phenomenon back to different areas of set theory (cf. Kanamori, 2003, 318–324); generalizations of the inconsistency have been studied (Hamkins et al., 2012) and other large cardinals notions were developed that are weaker than Reinhardt cardinals but close enough to them to make it questionable whether they were consistent themselves (cf. Kanamori, 2003, Ch. 24). To this day it remains an open question whether Reinhardt cardinals are merely inconsistent with the Axiom of Choice or also with ZF itself.

In the last decade, a further application of Reinhardt cardinals has gained importance. They are studied not only as a limit for inconsistency, but the large cardinal concepts itself is an object of investigation, despite its inconsistency with ZFC. Indeed, Reinhardt cardinals are the starting point for a whole class of large cardinals, called the *choiceless large cardinals*. In the last years, work has been done on generalizations of Reinhardt cardinals (Koellner, 2014; Cutolo, 2018; Schlutzenberg, 2020); Reinhardt cardinals are used to investigate fundamental properties of models of set theory, such as the HOD conjecture (Bagaria et al., 2019); general set theoretic tools, like forcing (Schlutzenberg, 2022); the cumulative hierarchy (Goldberg and Schlutzenberg, 2023); and consistency results between Reinhardt cardinals and their generalizations are considered (Goldberg, 2021). We can conclude that Reinhardt cardinals are a genuine concept of set theory and so the first clause of the characterization for formal concepts is satisfied. We can conclude that REINHARDT is a formal concept.

Let us return to the objection that there cannot be a concept REINHARDT because its extension is empty. Such an objection would maintain that the definition given for REINHARDT does not denote anything and that REINHARDT can therefore not serve as an example for a defective formal concept. But stating that such a concept does not exists stands in contrast to the fact that the concept is used in a formal science. This use is not based on the fact that set theorists are not aware of the inconsistency arising from REINHARDT.[30] Instead, its use is warranted by two reasons: One is that the case of REINHARDT is highly informative. As we have seen, proofs and generalizations of the Kunen inconsistency were studied as well as cardinal notions developed that narrowly avoid the inconsistency. In general, the concept provides insight into where the barrier to the strength of large cardinal

[30] For this reason the argument given by Scharp (2013, 37) to counter a similar objection against inconsistent concepts does not hold here.

notions lies, i.e. how "high" in the large cardinal hierarchy one can go before the resulting notions become inconsistent. The interest in this concept is, however, not only due to the simple fact that it is inconsistent.[31] REINHARDT is also used in set theory because it is such a natural generalization for large cardinal notions. This leads to the second reason of why its use is warranted and this reason is specific to the setting of formal theories.

A prominent feature of formal sciences is their flexibility as to the context in which their concepts are considered. This context can be the choice of an appropriate logic, such as classical logic, modal logic, etc.; it can the choice of an the axiomatization or the restriction to a certain sub-field or class of objects under consideration. The fact that formal sciences are generally free in these choices is due to their relative independence from natural phenomena in the empirical world. This flexibility is highly relevant for the question of defectiveness of concepts and it will be discussed in more detail in Section 5. In the case of REINHARDT, it provides a reason as to why and how it can be used despite the Kunen inconsistency. When studying choiceless large cardinals, set theorists simply switch context: as they cannot work in ZFC, they work in ZF, leaving out the Axiom of Choice. As no know inconsistency arises from ZF alone, REINHARDT is a satisfiable concept in this context. This is also the axiomatization in which the above mentioned work on generalizations of the concept takes place and gives rise to the hierarchy of choiceless large cardinals.

So, the use of REINHARDT in set theory is warranted both by the insights the concept offers and because there is a formally exact way of working with the concept without involving an inconsistency. This means that REINHARDT only has empty extension in a specific context, namely, when it is considered in the axiomatization of ZFC. But it would be much too strong to claim that this implies that the concept does not exist; rather, it means that the concept is defective with respect to an inconsistency that arises relative to a certain context in which the concept

[31] Arbitrary inconsistencies could be constructed at will by introducing artificial concepts. For example one could introduce the concept NOPOWER that is defined to be all sets that have no power set. This is an unsatisfiable concept as, according to the Power Set Axiom, all sets have a power set, giving rise to the corresponding inconsistency. However, neither the concept nor the corresponding inconsistency are studied in set theory, as they offer no further insight.

is considered. This is why I propose that REINHARDT is another example for defectiveness via relative inconsistency.

In general, we can conclude that formal concepts can be defective via relative inconsistency, meaning that a concept entails a logical inconsistency within a formal context of some formal science. As in the case of artificial concepts and arbitrary inconsistencies, let me add that these contexts should not be made-up or randomly chosen. Allowing made-up contexts would make every formal concept defective for trivial reasons, because for every concept one can arguably produce a context in which it is unsatisfiable. We therefore require that, for a formal concept to be defective in a formal science, these contexts should come up in and be relevant to the science in question. As we have seen, this is the case for REINHARDT, as it is inconsistent with the standard axiomatization of set theory, ZFC, and considered in ZF, which is an often used, relatively mild weakening of ZFC. This also holds for PRIME, because it was given by the two equivalent definitions in the context of the natural numbers and the inconsistency arose in the context of generalizations to quadratic integer rings that are of genuine mathematical interest as well.

5 Revision strategies in the formal sciences

The notion of defectiveness for formal concepts developed in the last section is a notion of relative defectiveness. It addresses a situation in which a formal concept comes up in a relevant context and in which its defectiveness is assessed relative to this context. The requirement that the context ought to be relevant is important here, as non-relevant (artificial, made-up) contexts can always be given in which an inconsistency can arise. In the following, we will consider what the existence of such a type of defectiveness entails for formal sciences. I will illustrate this point by regarding an objection to relative defectiveness as developed above. The objection aims to interpret the situation investigated above in a manner that seems to avoid the conclusion that formal concepts can be defective via relative inconsistency. The case of PRIME, the objection goes, is not an example of a defective concept, but an example of a case of two non-defective concepts, one given by Definition$_1$ and the other given by Definition$_2$.

The alternative interpretation of the case of PRIME introduces two concepts. One, called PRIME$_1$ is given by Definition$_1$ and the other, PRIME$_2$, is given by Definition$_2$. The extension of these concepts are the same over the natural numbers, but they differ over quadratic integer rings. This is not problematic at all: Concept can coincide over certain restricted contexts. For example the concept of odd number and prime number has the same extension when considered over the interval of natural numbers from 3 to 8. Still, nobody would claim that they are the same concept. This interpretation seems plausible also when taking into account the way in which the situation arising from two prime number definitions was resolved in mathematics. The notion of prime number was generalized to *prime element* and the definition based on Definition$_2$.[32] A generalization of Definition$_1$, instead, resulted in the concept of *irreducible element*.[33] Over the quadratic integer ring $\mathbb{Z}[\sqrt{-5}]$, then, 2 is irreducible but not prime. So, there are two concepts, PRIME$_2$ and IRREDUCIBLE, that are both not defective as they do not lead to an inconsistency. I claim that, despite the way in which these concept are defined today, this is not the right interpretation of the situation first described in Section 4.1. Rather, it is a description of the reaction of the mathematical community towards this situation. Before the problem arising with quadratic integer rings was discovered, the concept of prime number was that of PRIME and not that of PRIME$_1$ and PRIME$_2$. Mathematician did not consider there to be two concepts of prime number that differed by two definitions. Instead, there was one concept with two equivalent definitions, exactly what we described by PRIME above. The separation of these two definitions, which served as basis for generalizations producing two separate concepts, is a way of resolving this defectiveness. But a reaction to a defect requires the defect to be present in the first place—and that is what was described in Section 4.1.

[32] An element p (that is not zero and not a unit) of a commutative ring R is prime if whenever p divides bc for some b and c in R, it divides either b or c. An element u of a ring R is a unit if there exists v in R such that $vu = uv = 1$, where 1 is the identity respective to the multiplication operation on R.

[33] An element r (that is not zero and not a unit) of a ring R is irreducible, it it is only divisible by units or products of units with r (i.e. ua for a unit u).

Still, the alternative interpretation raises the general question of resolution strategies for defectiveness in the formal sciences. It would go beyond the scope of this paper to discuss resolution strategies in general. But we will study the cases of PRIME and REINHARDT with respect to the question of how relative defectiveness of concepts can be resolved and what type of resolution strategies this involves. This will allow us to draw some preliminary conclusions for resolutions in the case of formal concepts. Consider the following account from Scharp and Shapiro (2017):

> Let C be an inconsistent term or phrase. Define a replacement for C to be a term, or a batch of terms, that is, or are, consistent (at least as is known) and which can play at least some of the roles played by C in our linguistic and intellectual lives. (Scharp and Shapiro, 2017, 264)

As we determined in Section 3, we are not dealing with inconsistent concepts in the case of formal concepts. Still, we can define a replacement in the above manner for a concept C that is defective via relative inconsistency. The replacement should then still be made up of consistent concepts, but it should play (at least some of) the roles played by C in the respective formal science. Scharp and Shapiro (2017) gives guidelines for initiating such a replacement. First, they require that C comes up in a "valuable project" and is not a just made-up, like the concept RABLE (Scharp and Shapiro, 2017, 264). We already have such a requirement built into our account for formal concepts, as we only want to consider concepts that come up in a relevant manner in a formal science.[34] The relevant project is then the possibility to pursue the formal science without running into inconsistencies. This also matches with Scharp and Shapiro (2017) e.g. when they state that it "counts as a valuable project on its own" to be a branch of mathematics (Scharp and Shapiro, 2017, 265).

The second condition for replacement requires the defective concept C to "inhibit the pursuit of the project" (Scharp and Shapiro, 2017, 267). This is the case for PRIME, when the project is to have a concept of prime number that works

[34] Compare also the discussion in Section 4 about the relevance of the inconsistency and the contexts the formal concept appears in.

both over the natural numbers as well as in the case of certain rings. Here the inconsistency comes up and in order to remove this inconsistency, PRIME has to be replaced. Notice that it does not necessarily inhibit the project of defining a concept of prime number over the natural numbers alone. Even nowadays, where the distinction between prime and irreducible elements is well known, when talking about prime numbers over the natural numbers Definition_1 is sometimes still used in textbooks. While being possibly misleading, it is not completely wrong, as the two definitions are extensionally equivalent. As long as the project does not require the switch to the more general context (e.g. when working with prime numbers in cryptography), the relative inconsistency of PRIME will not inhibit the respective project and therefore a replacement is not necessarily required. For the REINHARDT case, consider two projects that are valuable in set theory. The first is to understand the large cardinal hierarchy in the axiomatization that is considered to be standard for mathematics, ZFC. Here the inconsistency appears, but it does not inhibit the project. It simply adds a precise boundary for large cardinal notions in showing that an elementary embedding from V to V cannot exist. The second project is to investigate (natural) generalizations of large cardinals via elementary embeddings. Here, the inconsistency inhibits the project because it does not allow a consideration of the natural generalization of Reinhardt cardinals. Both cases therefore show that relative defectiveness inhibits projects in the formal sciences to different degrees so that it might call for a resolution in some cases and not in others. In general, this is not a problem, as in the formal sciences we have precise tools to delimit these cases and make sure that it is always clear when a resolution is called for and employed and when it is not (for example, making sure that PRIME is only used when operating within the natural number system).

Scharp and Shapiro (2017) call for two other considerations when deciding if a concept should be replaced. These are connected to the questions whether the inconsistency connected to the concept C can be avoided (Scharp and Shapiro, 2017, 670) and if there even is a replacement available that is suitable for taking over the role of C (Scharp and Shapiro, 2017, 671). For PRIME, the answer to the latter is "yes" and the answer to the former is "not really". Yes, there is a replacement provided by the concepts of prime element and irreducible element. Other than using this replacement the inconsistency can not really be avoided, except in the local case where one only considered natural numbers. And even then it would be considered to be more desirable to use Definition$_2$ for PRIME as it corresponds better with the general setting. So, PRIME seems to be a straightforward candidate for a resolution via replacement. The case or REINHARDT is much different. Here, there is no (batch of) replacement concept that can take over the role that Reinhardt cardinals play in the axiomatization of ZFC, as there is no sensible way of changing Reinhardt cardinals to make them consistent with ZFC.[35] However, REINHARDT offers a clear and precise manner in which the inconsistency can be avoided, namely, by changing the axiomatization in which it is considered. This is a way out very specific to formal sciences. They can accommodate concepts that would be inconsistent with certain frameworks and therefore provide a place where these concepts can be fruitfully studied and employed. So, in set theory we are actually able to pursue both projects mentioned above, namely, to understand the large cardinal hierarchy in ZFC and to investigate natural generalizations of large cardinal notions, despite the relative inconsistency of REINHARDT. Each project simply has to be pursued in a different formal framework: ZFC for the first and ZF for the second.

6 Conclusion

In this article I have investigated the idea that formal concepts are impervious against defects commonly discussed with respect to concepts from philosophy and the empirical sciences. To spell out the core of this idea, I have developed a very strict definition of formal concept. It focuses on the extension of a concept and fixes this extension completely. This excludes often discussed cases of concept defectiveness based on unclear extensions and therefore makes it as hard as possible to apply usual

[35] One possible change would be to weaken the models in the embedding; however, the result by Kunen also holds for embeddings from $V_{\lambda+2}$ to $V_{\lambda+2}$ for a limit ordinal λ. Another change could consist in weakening only the second model in the embedding; however, then we would not talk about Reinhardt cardinals any longer as we would give up the essential feature that $V = M$.

types of defectiveness to concept from the formal sciences. Additionally, I restricted the type of defectiveness under consideration to ones that can be spelled out clearly within the formal science by requiring a logical inconsistency to occur.

In this setting, the idea that formal concepts so understood are impervious to defects can be made exact and, in general, rejected. It turns out that formal concepts cannot be inconsistent concepts. This exempts them from discussions about defectiveness of inconsistent concepts arising in the conceptual engineering debate (e.g. by Scharp, 2013). However, it can also be shown that a variation of defectiveness via inconsistency still applies to them. Formal concepts can lead to inconsistencies that occur relative to specific settings, which can be made formally exact. So, even when formal concepts are spelled out under very strong assumptions regarding their exactness, they can still be defective. When a defective concept is discovered, different ways of responding to this situation are possible. For the examples discussed in this paper, we presented cases in which the defective concept is replaced, at least in the formal contexts the inconsistency occurs in. But we also presented a strategy of avoiding the defectiveness by switching to an unproblematic formal framework—a strategy that seems to be specifically available in formal sciences.

This leads to an intermediate position about the special status of formal concepts: On the one hand, there is a way to show that formal concepts are not defective as they cannot be inconsistent. On the other hand, a variation of defectiveness via inconsistency still applies to them—a very strong result, considering the strictness of our definition of formal concepts. The conclusion is that current discussions about defectiveness of concepts and related topics such as resolution strategies can be applied to formal concepts, too. But the special character of formal sciences and the concepts employed therein still set them apart, for instance with respect to which resolution strategies are available and what they entail. This means it is neither the case that problems such as defectiveness cannot occur for formal concepts, nor is it the case that formal concepts can be treated in exactly the same way as concepts from empirical sciences with regards to these problems.

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