First-Order Quantifiers

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Quantifiers as second-level concepts

In §21 of *Grundgesetze der Arithmetik* asks us to consider the forms:

$$\vdash\!\!\overset{a}{\smile}\!\!\vdash a^2 = 4 \quad \text{and} \quad \vdash\!\!\overset{a}{\smile}\!\!\vdash a > 0$$

and notices that they can be obtained from $\vdash\!\!\overset{a}{\smile}\!\!\vdash \phi(a)$ by replacing the function-name placeholder $\phi(\xi)$ by names for the functions $\xi^2 = 4$ and $\xi > 0$ (and the placeholder cannot be replaced by names of objects or of functions of 2 arguments).

So the above forms can be regarded as values of the same function for different arguments, which are, in turn, themselves functions.

Since the values of the *second-level function* $\vdash\!\!\overset{a}{\smile}\!\!\vdash \phi(a)$ are always truth values, the functions is a *second-level concept*.

Generalized Quantifiers

The modern study of generalized quantifiers begins with Mostowski [1957] and continues with Montague [1974]. The work on generalized quantifiers spans linguistics and mathematical logic, the linguists focusing on quantifiers as tools for natural language semantics, and the logicians focusing on the expressive power and properties such as axiomatizability, decidability etc.

A quantifier Q over a domain D is just a collection of subsets of D: $\mathsf{Q} \subseteq \mathscr{P}(D)$.

For instance:

- $\forall = \{D\}$;
- $\exists = \{X \subseteq D : X \neq \emptyset\}$;
- $\exists!^k = \{X \subseteq D : |X| = k\}$;
- $\mathsf{John} = \{X \subseteq D : \text{John} \in X\}$.

Binary, ternary quantifiers

Some quantifiers are best viewed as n-ary *relations* over $\mathscr{P}(D)$. Here are some examples:

- All A are B: $\mathsf{All} = \{(A,B) : A \subseteq B\}$
- Some A are B: $\mathsf{Some} = \{(A,B) : A \cap B \neq \emptyset\}$
- Most A are B: $\mathsf{Most} = \{(A,B) : |A \cap B| > |A - B|\}$;
- Twice as many A as B are C:

$$\mathsf{Twice} = \{(A,B,C) : |A \cap C| = 2 \cdot |B \cap C|\}.$$

Some binary quantifiers can be represented by means of the corresponding unary quantifier applied to a Boolean combination of their arguments:

$$\mathsf{All}[A,B] = \forall[-A \cup B)],$$

but for instance Most cannot be so represented.

Classification of quantifiers

Quantifiers are distinguished by their *arity* as well as their *adicity*, where the former is the number of arguments (formulas) they take, and the latter is the number of variables of such formulas.

All(A,B) is a binary monadic quantifier, whereas $\mathsf{Q}(R) = 1 \Leftrightarrow \forall x \forall y Rxy$ is a unary dyadic quantifier.

In general, the type of a quantifier Q can be represented by a type $\langle n_1, \ldots, n_k \rangle$, where k represents the arity, and n_i the adicity of the i-th argument: $\mathsf{Q} \in \langle n_1, \ldots, n_k \rangle \Longleftrightarrow \mathsf{Q} \subseteq \prod_{i=1}^{k} \mathscr{P}(D^{n_i})$.

Second order quantifiers are just collections (relations) of first-order quantifiers. Consider the sentence $\exists P \phi(P)$: the quantifier has type $\langle \langle 1 \rangle \rangle$ and can be identified with $\{X \in \mathscr{P}^2(D) : X \neq \emptyset\}$.

The distinction is semantical, not merely notational.

Properties of quantifiers

Several important properties have been singled out for binary quantifiers $\mathsf{Q}(A, B)$:

- Conservativity: $\mathsf{Q}(A, B) = \mathsf{Q}(A, A \cap B)$;
- Right monotony: $\mathsf{Q}(A, B)$ and $B \subseteq C$ implies $\mathsf{Q}(A, C)$ (all, most);
- Left monotony: $\mathsf{Q}(A, B)$ and $A \subseteq C$ implies $\mathsf{Q}(C, B)$;
- Right anti-monotony: $\mathsf{Q}(A, B)$ and $C \subseteq B$ implies $\mathsf{Q}(A, C)$ (no, few);
- Left anti-monotony: $\mathsf{Q}(A, B)$ and $C \subseteq A$ implies $\mathsf{Q}(C, B)$;
- Permutation invariance: if π is a permutation of D, then $\mathsf{Q}(A, B)$ holds iff $\mathsf{Q}(\pi[A], \pi[B])$ holds.

New quantifiers from old ones

Several constructions have been singled out to combine quantifiers to obtain new ones.

ITERATION: Noun phrases such as John can be viewed as *arity-reducing* (projection) operators. If R is a $k+1$-ary relation, then:

$$\mathsf{John}(R) = R_{\mathsf{John}},$$

where $R_a = \{(x_1, \ldots, x_k) : R(x_1 \ldots, x_k, a)\}$.

This subsumes the monadic case, viewing `true`, `false` as 0-ary relations. Then:

$$\text{John kissed Mary} = \mathsf{John}(\mathsf{Mary}(\mathit{kissed}))$$

New quantifiers from old ones, cont'd

In general, define:

- $\mathbf{R}_0 = \{\mathtt{true}, \mathtt{false}\}$
- $\mathbf{R}_{n+1} = \mathscr{P}(D^{n+1})$.

Then:

$$AR^k = \{\mathsf{Q} : R \in \mathbf{R}_{n+k} \Rightarrow \mathsf{Q}(R) \in \mathbf{R}_n\}.$$

COMPOSITION: Quantifiers in AR^k are closed under composition: if $\mathsf{Q}_1 \in AR^n$ and $\mathsf{Q}_2 \in AR^k$ then $\mathsf{Q}_1\,\mathsf{Q}_2 \in AR^{n+k}$

Note that composition is *associative*, so we can write $\mathsf{Q}_1\,\mathsf{Q}_2 \ldots \mathsf{Q}_n$.

First-Order Definability

A (binary) quantifier $\mathsf{Q}(A, B)$ is *first-order definable* over D iff there is a formula $\phi \in \mathscr{L}(P, Q)$ such that

$$\langle D, A, B \rangle \models \phi.$$

For instance, `at least two` is first order definable.

A quantifier $\mathsf{Q}(A, B)$ is *proportional* (over finite D) iff

$$\exists m, n \in \mathbb{N} \left[\mathsf{Q}(A, B) \Leftrightarrow \frac{|A \cap B|}{|A|} \geq \frac{m}{n} \right]$$

The quantifiers most, at least half, more than 10%, ... are all proportional.

Thorem: Proportional quantifiers are not first-order definable.

Cardinality Quantifiers

We single out two closely related quantifiers that deal with cardinality restrictions:

- The *Härtig quantifier*: $\mathsf{I}(A, B) \Leftrightarrow |A| = |B|$;
- the *Rescher quantifier*: $\mathsf{R}(A, B) \Leftrightarrow |A| > |B|$. These quantifiers, first introduced by Rescher [1962] and Härtig [1965], have been extensively studied from a mathematical point of view.
- Härtig's quantifier is definable from Rescher's (using choice) but not vice-versa: $\mathsf{I}(A, B) \Leftrightarrow \neg\, \mathsf{R}(B, A) \wedge \neg\, \mathsf{R}(A, B)$.
- Neither quantifier is conservative: the following both *fail*
  - ◇ $\mathsf{I}(A, B) \Leftrightarrow \mathsf{I}(A, A \cap B)$;
  - ◇ $\mathsf{R}(A, B) \Leftrightarrow \mathsf{R}(A, A \cap B)$.
- Both quantifiers are obviously permutation-invariant.
- Both quantifiers have type $\langle 1, 1 \rangle$ and hence are first-order.

Detour: the neo-logicist program

With the recent neo-logicist work of Hale and Wright [2001] the logical/epistemological status of cardinality notions (equinumerosity etc.) has come into focus.

The neo-logicists focus on the so-called *Hume's Principle* specifying identity conditions for numerical terms:

The number of F = the number of G iff $F \approx G$,

where $F \approx G$ is the second-order statement that there is a 1-1 correspondence of the F onto the G.

Hume's principle has a privileged epistemological status (analytic, explicative, constitutive of the notion of number).

The *number of . . .* operator maps concepts into the objects, in such a way that equinumerous concepts are mapped into the same object. The right-hand-side of Hume's principle is usually taken to be logically innocent (and the innocence seeps into the left-hand-side)

Cardinality quantifiers and the neo-logicist program

The neologicists have claimed (at times) that Hume's principle is (close to) a logical truth, and that therefore, by implication, that only logical notions are mentioned in it.

If the notion of *equinumerosity* is so logically innocent, it is interesting then, to see what happens when we take the idea that equinumerosity is a logical notion seriously.

In order to get an unobstructed view we consider this notion in conjunction with only the barest of logical apparatus.

The Frege quantifier F

We consider a *first-order* language $\mathscr{L}$ with formulas built up from individual, predicate, and function constants by means of Boolean connectives ($\wedge$, $\vee$, $\neg$, and $\rightarrow$, say) and the quantifier $\mathsf{F}\,x$ satisfying the clause:

> if ϕ, ψ are formulas and x a variable, then $\mathsf{F}\,x(\phi, \psi)$ is a formula.

So $\mathsf{F}\,x$ is a binary quantifier (like All) with the intuitive interpretation (similar to that of the Rescher quantifier R) that there is an injection of the ϕ's into the ψ's.

As for the Rescher quantifier, we abbreviate $\neg Fx(\phi, \psi) \wedge \neg\,\mathsf{F}\,x(\psi, \phi)$ by $\mathsf{I}\,x(\phi, \psi)$.

Standard semantics for $\mathscr{L}_{\mathsf{F}}$

A model $\mathfrak{M}$ with non-empty domain D provides an interpretation for the non-logical constants of $\mathscr{L}$ in the usual way (e.g., n-place predicates are mapped onto relations $\subseteq D^n$, etc.)

Given a formula $\phi(\overline{x})$ and a function s assigning objects from D to the variables of $\mathscr{L}$, satisfaction $\mathfrak{M} \models \phi[s]$ is defined in the usual way, with the clause:

$$\mathfrak{M} \models \mathsf{F}\,x(\phi, \psi)[s] \Leftrightarrow \exists f : \{s(x) : \mathfrak{M} \models \phi[s]\} \xrightarrow{1-1} \{s(x) : \mathfrak{M} \models \psi[s]\}.$$

Alternatively, if $s\frac{\overline{a}}{\overline{x}}$ is just like s, except "shifted" to assign $\overline{a} = a_1, \ldots, a_k$ to $\overline{x} = x_1, \ldots, x_k$ (respectively), we can define:

$$[\![\phi]\!]_s^{\overline{x}} = \{\overline{a} : \mathfrak{M} \models \phi[s\tfrac{\overline{a}}{\overline{x}}]\};$$

then the above clause becomes:

$$\mathfrak{M} \models \mathsf{F}\,x(\phi, \psi)[s] \Leftrightarrow \exists f : [\![\phi]\!]_s^x \xrightarrow{1-1} [\![\psi]\!]_s^x.$$

Expressibility in $\mathscr{L}_{\mathsf{F}}$

- The standard first order quantifiers are expressible in $\mathscr{L}_{\mathsf{F}}$:
  - ◇ $\forall x\phi(x) = \mathsf{F}\, x(\neg\phi(x), x \neq x)$;
  - ◇ $\exists x\phi(x) = \neg\, \mathsf{F}\, x(\phi(x), x \neq x)$.
- There is an *axiom of infinity* in the pure identity fragment of $\mathscr{L}_{\mathsf{F}}$:

  AxInf: $\exists y\, \mathsf{F}\, x(x = x, x \neq y)$.

- AxInf is true in *all and only* the infinite models, and therefore, its negation is true in all and only the finite models.
- As a consequence, *compactness fails* in $\mathscr{L}_{\mathsf{F}}$.
- Let us abbreviate by $\mathsf{Fin}\, x(\phi(x))$ the statement that $\{x : \phi(x)\}$ is Dedekind finite: $\forall y \neg\, \mathsf{F}\, x(\phi(x), \phi(x) \wedge x \neq y)$.

Characterizing $\mathbb{N}$

It's easy to see that there is a sentence ϕ of $\mathscr{L}_{\mathsf{F}}(<)$ which is true if and only if $<$ has order type $\leq \omega$. Such a sentence says:

- $<$ is a strict transitive linear order; and
- $\exists x \forall y(y \neq x \to x < y)$; and
- $\forall x\, \mathsf{Fin}\, y(y < x)$.

Then $\phi \wedge \mathsf{AxInf}$ is true precisely if $<$ is a countably infinite linear order, which can be conjoined with a set of arithmetical axioms (such as PA minus induction) to characterize the standard model $(\mathbb{N}, +, \times)$ up to isomorphism.

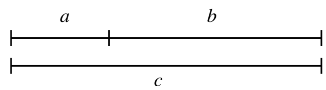
As further consequences we have that:

- the set of $\mathscr{L}_{\mathsf{F}}(+, \times)$ validities is not recursively axiomatizable (although $\mathscr{L}_{\mathsf{I}}(<)$ validities are decidable);
- The Löwenheim-Skolem theorem for $\mathscr{L}_{\mathsf{F}}$ (and $\mathscr{L}_{\mathsf{I}}$) fails vary badly.

Further properties of F

It is well known that addition is not definable in first-order logic over the structure $(\mathbb{N}, <)$, the following defines in $\mathscr{L}_{\mathsf{F}}$:

$$a + b = c \Leftrightarrow (\mathbb{N}, <) \models \mathsf{I}\, x(x < b, a \leq x < c) :$$



Further evidence of how badly Löwenheim-Skolem fails: there are sentences ϕ_1 and ϕ_2 of $\mathscr{L}_{\mathsf{F}}$ such that ϕ_1 is true in all and only the successor cardinalities and ϕ_2 is true in all and only the limit cardinalities.

Since $\mathbb{N}$ is categorically definable in $\mathscr{L}_{\mathsf{F}}$, we can define a Gödel numbering of finite sequences and finite sets, and hence *implicitly* define satisfaction. Since by Tarski's theorem satisfaction is not *explicitly* definable we have: the *Beth definability* property fails in $\mathscr{L}_{\mathsf{F}}$.

Non-standard semantics for F

A *general model* $\mathfrak{M}$ for $\mathscr{L}_{\mathsf{F}}$ provides a non-empty domain D, interpretations for the non-logical constants, and a collection $\mathcal{F}$ of 1-1 functions $f : A \to B$ with $\mathrm{dom}(f) = A$, and $\mathrm{rng}(f) \subseteq B$, for $A, B \in \mathscr{P}(D)$.

The satisfaction clause for the quantifier F then becomes:

$$\mathfrak{M} \models \mathsf{F}\, x(\phi, \psi)[s] \Leftrightarrow (\exists f \in \mathcal{F}) f : [\![\phi]\!]_s^x \xrightarrow{1-1} [\![\psi]\!]_s^x.$$

In practice, we expect $\mathcal{F}$ to satisfy certain *closure conditions*, such as, for instance:

- For each A, the identity map on A belongs to $\mathcal{F}$ (including the empty map on $\emptyset$);
- if $f \in \mathcal{F}$ and $f : A \to B$ and $x \notin A$ and $y \notin B$, then there is a $g \in \mathcal{F}$ such that $g : A \cup \{x\} \to B \cup \{y\}$.

A system of axioms for F

Consider the following set of axioms for the general interpretation of $\mathscr{L}_{\mathsf{F}}$:

1. $\mathsf{F}\,x(\phi(x), x \neq x) \rightarrow \neg\phi(t)$ provided t is free for x in ϕ (universal instantiation);
2. $\mathsf{F}\,x(\phi, \psi) \wedge \mathsf{F}\,x(\psi, \theta) \rightarrow \mathsf{F}\,x(\phi, \theta)$ (transitivity);
3. $\forall x(\phi \rightarrow \psi) \rightarrow \mathsf{F}\,x(\phi, \psi)$ (if $A \subseteq B$ then $|A| < |B|$);
4. $(\mathsf{F}\,z(\phi, \psi) \wedge \neg\phi(x) \wedge \neg\psi(y)) \rightarrow \mathsf{F}\,z(\phi(z) \vee z = x, \psi(z) \vee z = y)$ (extension of injections).

These axioms are valid in every standard model. Axioms 1 and 2 are valid in *every* non-standard model, regardless of closure conditions; Axiom 3 holds as long as the identity map belongs to $\mathcal{F}$; and axiom 4 holds in models whose maps are closed under finite unions.

Claim: The axioms are complete for the class of non-standard models satisfying the closure conditions.

The existential quantifier $\exists$

- Just like F can be given a non-stantard or general interpretation, so can $\exists$.
- Define a general first-order model $\mathfrak{M}$ as providing a non-empty domain D along with a collection $\mathcal{E}$ of *non-empty* subsets $A \subseteq D$.
- The satisfaction clause for the quantifier then becomes:

$$\mathfrak{M} \models \exists x\phi[s] \iff [\![\phi]\!]^x_s \in \mathcal{E}.$$

- Notice that this quantifier is *not permutation invariant*, and hence its logical nature can be questioned.
- **Question:** What is the logic of the non standard $\exists$?

The logic of the generalized ∃

A general model $\mathfrak{M}_{\mathcal{E}}$ provides a class $\mathcal{E} \subseteq \mathscr{P}(D) \setminus \{\emptyset\}$.

An alternative is given by *outer-inner domain* models, where such a model supplies a non-empty domain D as well as a (possibly empty) subset D' of D (the "inner domain"), with the clause:

$$\mathfrak{M} \models \exists x\phi[s] \Leftrightarrow \exists d \in D' : \mathfrak{M} \models \phi[s(d/x)].$$

We can go back and forth between the two kinds of models by putting $\mathcal{E} = \{X \subseteq D : X \cap D' \neq \emptyset\}$, and conversely $D' = \bigcup \mathcal{E}$. The two maps preserve satisfaction and hence the two kinds of models give rise to the same set of validities.

but the logic of outer-domain models is well known: it is the *free logic* axiomatized by K. Lambert by dropping universal instantiation and replacing it with the axiom $\exists x(x = t) \rightarrow (\forall x\phi(x) \rightarrow \phi(t))$ or, better: $\forall y(\forall x\phi(x) \rightarrow \phi(y))$.

Concluding questions

- The expressive power of F under the standard interpretation raises the question whether it really is *first-order*. But being first-order is a genuine semantical property, not a mere notational convenience.
- The same expressive power (even in the absence of an explicit notion of number as contained in Hume's principle) leads to the question of whether cardinality really is a logical notion, and the implications this has for the neo-logicist program.
- The expressive power of a genuine first-order quantifier further leads to the conclusion that there is more to first-order quantifiers than ∃ and ∀.
- As to the general (non-standard) interpretation of ∃, we observed that the quantifier is *not invariant*. Of course, we knew that the outer domain semantics is also *not invariant*, but that semantics has been regarded as artificial. In contrast, we now have a completely *natural* route to an equivalent semantics.
- The non invariance of ∃ under the natural generalization leads to the question of whether free logic really is *logic*.

References

J. Barwise and S. Feferman, *Model theoretic logics*, Springer Verlag, 1985.

J. van Benthem, Questions about quantifiers, *Journal of Symbolic Logic*, 49:443–66, 1984.

R. Hale and C. Wright, *The Reason's Proper Study. Essays toward a Neo-Fregean Philosophy of Mathematics*, Oxford-Clarendon Press, 2001.

H. Härtig, Über einen Quantifikator mit zwei Wirkungsbereichen, in L. Kalmár, editor, *Colloquium on the foundations of mathematics, mathematical machines and their applications*, pages 31–36, Akadémiai Kiadó, Budapest, 1965.

H. Herre, M. Krynicki, A. Pinus, and J. Väänänen, The Härtig quantifier: a survey, *Journal of Symbolic Logic*, 56(4):1153–83, 1991.

E.L. Keenan and D. Westerstål, Generalized quantifiers, in J. van Bentham and A. ter Meulen, editors, *Handbook of Logic and Language*, pages 837–93, MIT Press, 1997.

P. Lindström, First-order predicate logic with generalized quantifiers, *Theoria*, 35:186–95, 1966.

R. May, Interpreting logical form, *Linguistics and Philosophy*, 12(4):387–435, 1989.

R. Montague, English as a formal language, in R.H. Thomason, editor, *Formal Philosophy*, Yale University Press, 1974, originally published 1969.

A. Mostowski, On a generalization of quantifiers, *Fundamenta Mathematicæ*, 44:12–36, 1957.

N. Rescher, Plurality quantification, *Journal of Symolic Logic*, 27:373–47, 1962.