PURITY IN ARITHMETIC: SOME FORMAL AND INFORMAL ISSUES

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1. INTRODUCTION

There has been since antiquity a tradition in mathematics of preferring solutions to problems / proofs of theorems that are restricted to considerations "close" or "intrinsic" to what is being solved/proved. A classical example of such preference is the desire for "synthetic" geometrical solutions to geometrical problems rather than "analytic" solutions to those problems that have struck many mathematicians as "rather far" from the problems at hand. Arithmetic in particular has been the locus of much concern over purity. For instance, regarding Hadamard and de la Vallée Poussin's 1896 proof of the prime number theorem using complex analysis, the distinguished number theorist A.E. Ingham remarked that it "may be held to be unsatisfactory in that it introduces ideas very remote from the original problem, and it is natural to ask for a proof of the prime number theorem not depending on the theory of a complex variable" (cf. [Ing32], pp. 5–6).

Investigation of purity in mathematical practice reveals that there are several different strains of purity differentiated by how they measure what is "close" or "intrinsic" to what is being solved/proved. In [DA11] we settled on one such strain as the one most central to purity in historical practice, which we called the "topical" strain.[1] Very roughly, a solution to a problem is "topically pure" if it draws only on what is "contained" in (the content of) that problem, where what is "contained" in a problem is what grounds its understanding and is what we call that problem's "topic". If a solution to a problem draws on something extrinsic to that problem's topic, then it is topically impure.

The chief aim of this essay is to shed further light on topical purity by examining two cases from arithmetic: the infinitude of primes, and Gödel's incompleteness theorems. The discussion of the infinitude of primes will extend the discussion in [DA11] of the familiar Euclidean proof. After a brief recapitulation of that work's discussion of the

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[1]Cf. [Ara08], [Ara09] and [Ara11a] for discussions of other important strains of purity.

topical impurity of Furstenberg's topological proof of the infinitude of primes, stressing different notions of content that arise in reflections on this proof, attention will shift to Gödel's work. The main question to be addressed is whether this work shows that there are some arithmetic theorems for which no topically pure proof is possible.

2. Topical purity

The insight behind the analysis of topical purity in [DA11] was expressed well by Hilbert in his 1898/1899 lectures on geometry:

> In modern mathematics such criticism is raised very often, where the aim is to preserve *the purity of method* [*die Reinheit der Methode*], i.e. to prove theorems if possible using means that are suggested by [*nahe gelegt*] the content of the theorem.[2]

What is critical for a proof's being pure or not, then, is whether the means it draws upon are "suggested by the content of the theorem" being proved.

Since what it is for an element of a proof to be "suggested" by its content is not particularly clear, a chief task of [DA11] was to clarify this. Call the commitments that together determine the understanding of a given problem (for a particular investigator α) the "topic" of that problem (for α). Among these commitments are definitions, axioms, inferences, etc. These together are constitutive of α's understanding of the problem, and hence of the identity of the problem (to α).[3] A solution to a problem is "topically pure" (for α) if it draws only on what belongs to that problem's topic. In other words topically pure solutions to problems draw only on what is constitutive of the identity of that problem.

The heart of the account in [DA11] of the epistemic value of topical purity is the following counterfactual: if a component of a topically pure solution to a problem were retracted by an investigator, then that investigator's understanding of that problem would change. This is because every component of a topically pure solution to a problem belongs to the topic of that problem, and hence is partly determinative of the understanding of that problem. This is not the case for topically impure solutions to problems, since some of their components do not belong to their problems' topics.

The epistemic significance of this counterfactual is as follows. A topically pure solution to a problem remains a solution to that problem even when some component of that solution is retracted, for such retraction "dissolves" that problem, by changing its understanding and hence its identity. While dissolving a problem is not typically taken to count as

[2] Cf. [HM04], pp. 315–6. The original reads, "In der modernen Mathematik wird solche Kritik sehr häufig geübt, wobei das Bestreben ist, *die Reinheit der Methode* zu wahren, d.h. beim Beweise eines Satzes wo möglich nur solche Hülfsmittel zu benutzen, die durch den Inhalt des Satzes nahe gelegt sind."

[3] These relativizations to a particular investigator are needed because how a problem is understood may differ from investigator to investigator. In practice there is not too much local variation in this.

solving it, we argue that it should, since the aim of problem solution is the relief of rational ignorance, and we cannot (rationally) be ignorant about dissolved problems. Hence topically pure solutions persist as solutions even when one of their components are retracted. Thus the relief of ignorance provided by topically pure solutions to problems is quite "stable" with respect to changes in attitude regarding their components. The same cannot be said of topically impure solutions to problems, however. That is because some components of topically impure solutions may be retracted without dissolving their problems and hence the relief of ignorance they provide is not as "stable" to changes in attitude regarding their components as is the relief provided by topically pure solutions. The epistemic value of topical purity is thus that topically pure solutions are more resilient to retraction of their components than are topically impure solutions.

This characterization of topical purity and its value is only as clear as the notion of topic it uses. In the next section we thus turn to an example from arithmetic, the infinitude of primes, that sheds further light on this notion. This characterization also leaves open the question of whether every theorem has a topically pure proof. We will turn to this question in Section 4.

3. The infinitude of primes

Section 4 of [DA11] raised as an example for topical purity the problem of determining whether there are infinitely many primes. Two positive solutions to that problem were considered: the classical Euclidean solution, and Furstenberg's topological solution. In that section the case was made that the former solution is topically pure, while the latter is topically impure. In this section the purity of the Euclidean solution will be discussed in further detail, and the impurity of the topological solution will be discussed again with an eye toward seting up the discussion of Gödel's work later in this article.

3.1. **The Euclidean solution.** Consider a contemporary investigator α who has a typical contemporary understanding of arithmetic. Suppose α formulates the question concerning the infinitude of primes (henceforth called IP) as follows: for every natural number, is there a greater natural number that is prime? A solution to IP for α is a proof of the result that for all natural numbers a, there exists a natural number $b > a$ such that b is prime. A topically pure solution to IP for α may draw on what belongs to the topic of IP, that is, on the commitments that determine the content of the problem as she has formulated it. To determine whether a given solution to IP is topically pure, then, more must be said about what commitments constitute the topic of IP.

The topic of IP must at least include definitions and axioms for natural number, an ordering on the natural numbers, and primality. The natural numbers are typically understood to begin with a first number 1, followed by its successor $S(1)$, and continuing with the successors of each number already reached. Hence axioms for successor would seem to be included, as would induction axioms for making precise the view that the natural numbers "start" with 1 and "continue" onward thereafter. Definitions and axioms for an ordering on the natural numbers would also be needed for IP's topic, and following typical practice these would specify a linear discrete ordering. Additionally, a definition of primality is also needed, and since a natural number a is ordinarily defined as prime if and only if $a \neq 1$ and the only numbers dividing a are 1 and a, a definition of divisibility (written $a|b$) also belongs to IP's topic.

In light of this preliminary specification of IP's topic, consider the well-known Euclidean proof from *Elements* IX.20. If $a = 1$, then since $2 = S(1)$ is prime, we know that there is a prime greater than $a = 1$. So suppose that $a > 1$. Let $p_1, p_2, \ldots, p_n$ be all the primes less than or equal to a, and let $Q = S(p_1 \cdot p_2 \cdots p_n)$. Note that Q has a prime divisor b. For each i, $b \neq p_i$; if not, then $b|(p_1 \cdot p_2 \cdots p_n)$ and $b|S(p_1 \cdot p_2 \cdots p_n)$, and so $b = 1$, contradicting the primality of b. Finally, either $b > a$, or $b \leq a$, but since $b \leq a$ contradicts that the p_i were all of the primes less than or equal to a, we may conclude that $b > a$.

This proof has several steps that themselves require proof. Examples are the step that consists in the assertion that if $b|(p_1 \cdot p_2 \cdots p_n)$ and $b|S(p_1 \cdot p_2 \cdots p_n)$, then $b = 1$, or the step in which it is asserted that if $a|b$ and $a|S(b)$, then $a = 1$. Whether or not the Euclidean solution is topically pure thus depends on whether or not the main proof, and all of the subproofs needed to establish the steps of the main proof, draw only what belongs to the topic of IP.

On the face of it, there is nothing unarithmetic about these proofs, and so a favored initial diagnosis is that the Euclidean solution is pure. However, there are good reasons to think this too quick. A first reason for concern about the purity of this solution is that it appeals to multiplication in generating Q, though multiplication was not included in the preliminary specification of IP's topic. A second reason for concern is that the subproofs have not been given fully, and so appeal to elements foreign to IP's topic cannot yet be ruled out. A reply to both concerns would be to note that the main proof and each of the needed subproofs can be carried out from the first-order Peano axioms (PA), as can be (tediously) checked. Provided that the axioms of PA (augmented by definitions of the appropriate ordering and primality) belong to the topic of IP, its sufficiency for expressing the main proof and each subproof answers the second concern, and its inclusion of a definition of multiplication answers the first concern. Thus, provided that the Peano axioms belong to the topic of IP, and that the proofs when carried out in PA remain faithful to the Euclidean proof and subproofs, the Euclidean proof is topically pure.

This is not an especially convincing reply, however, because it begs both questions. The sufficiency of PA for the Euclidean solution was not in question; this is indicative of PA's being widely considered an adequate axiomatization of elementary arithmetic. What is in question is the *topicality* of the commitments engendered in accepting PA for IP. The reply simply asserts that these commitments are topical for IP, but that is exactly what being questioned. What is needed is a more fine-grained analysis of the topic of IP, in particular investigation of what *operations* (divisibility? multiplication? addition?) and *modes of inference* (classical logic? how much induction?) belong to IP's topic. To this the essay now shifts.

3.1.1. *The topicality of arithmetic operations for IP.* The second concern just raised is that the fully spelled-out Euclidean solution may contain elements that do not belong to IP's topic. When spelling out the solution in PA this is the case, since addition is used in establishing the needed properties of multiplication, while addition is not explicitly mentioned in the problem as formulated. But this is not merely an issue with PA. Any proof that uses multiplication must either take to belong to the definition of multiplication the properties of multiplication that it needs, or prove them on some other basis. If the latter, proof via addition is the obvious choice since multiplication is ordinarily defined as iterated addition (as in PA, for instance). If the former, then some plausible non-additive definition of multiplication is needed; and moreover some answer will be needed for the reply that multiplication is also not mentioned explicitly in IP. We will return to the issue concerning multiplication; let us for now discuss the additive case.

Concerning the use of addition in the Euclidean solution, one response would be to defend the use of addition as topically pure for IP. One could do so on the grounds that addition (and multiplication) are "basic" to understanding the natural numbers, because we are talking about a discretely ordered ring in usual practice, that is, as a structure with both an addition and a multiplication operator. But this seems wrong: Presburger and Skolem arithmetic (with just addition and multiplication, respectively) are just as "basic" as Peano arithmetic. Indeed, children seem to grasp the natural numbers before they understand the concepts of addition and multiplication. The sequence starting with 1 and generated by successors is more plausibly basic (though not necessarily the most basic).

Another response to this objection concerning the use of addition in the Euclidean solution notes that the only need for addition in the Euclidean solution is to establish properties of multiplication such as commutativity and associativity. We may thus isolate these properties of multiplication and find a proof directly from them, without adverting to addition.

The following seventeen assumptions are an attempt to do this. They include assumptions regarding successor and the ordering in addition to multiplicative assumptions, in order to yield a set of assumptions sufficient for solving IP without using addition.

Assumption 1. For all x, there exists y such that $y = S(x)$.

Assumption 2. For all x, y, $x = y$ if and only if $S(x) = S(y)$.

Assumption 3. For all x, y, there exists z such that $z = x \cdot y$.

Assumption 4. For all x, $x \cdot 1 = x$.

Assumption 5. For all x, y, z, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

Assumption 6. For all x, y, $x \cdot y = y \cdot x$.

Assumption 7. For every sequence of primes $p_1, \ldots, p_n$, there exists z such that $z = p_1 \cdot p_2 \cdots p_n$.

Assumption 8. For all x, y, z, if $x < y$ and $y < z$, then $x < z$.

Assumption 9. For all x, $x \not< x$.

Assumption 10. For all x, y, either $x < y$, $x = y$, or $y < x$.

These three assumptions together imply that if $x < y$, then $y \not< x$, and hence that the trichotomy asserted in Assumption 10 is exclusive, i.e. for each x, y, exactly one of $x < y$, $x = y$, and $y < x$ holds.

Assumption 11. For all x, $1 \leq x$.

Assumption 12. For all x, y, $x < y$ if and only if $S(x) < S(y)$.

Assumption 13. For all x, $x < S(x)$.

Assumption 14. For all x, y, if $x < y$ then $S(x) \leq y$.

Assumption 15. For all x, y, z, $x < y$ if and only if $xz < yz$.

Assumption 16. For all $y \neq 1$ and all x, $S(yx) < y \cdot S(x)$.

Assumption 17. For each formula $\varphi(x, \overline{y})$, where x is a free variable and the $\overline{y}$ are terms, if $\varphi(1, \overline{y})$ and if for all a and all $b < a$, $\varphi(b, \overline{y})$ implies that $\varphi(a, \overline{y})$, then for all a, $\varphi(a, \overline{y})$.

These assumptions may be grouped as follows: Assumptions 1 and 2 concern successor, 3–7 concern multiplication, 8–10 concern the ordering, 11–16 concern how successor and multiplication respect the ordering, and 17 is an induction schema.[4]

Next we will give a non-additive solution to IP using these assumptions. To simplify the structure of the main proof, we will separate from the main proof the following three lemmas, and prove them separately.

Lemma 3.1. *$S(1)$ is prime.*

[4] We are not claiming that these assumptions are mutually independent of each other.

Lemma 3.2. *Every natural number $a \neq 1$ has a prime divisor $p \leq a$.*

Lemma 3.3. *For all a, b, if $a|b$ and $a|S(b)$, then $a = 1$.*

Using these lemmas, the main result, that for all a, there exists $b > a$ such that b is prime, can be proved as follows, with the assumptions referred to therein listed afterwards.

(1) Either $a = 1$ or $a > 1$. [Assumption 11]
(2) (a) Say $a = 1$.
  (b) By Lemma 3.1, $S(1)$ is prime. [Assumption 1]
  (c) $S(1) > 1$. [Assumption 13]
(3) (a) Say $a < 1$.
  (b) Let $p_1, p_2, \ldots, p_n$ be all the primes less than or equal to a.
  (c) Let $Q = S(p_1 \cdot p_2 \cdots p_n)$. [Assumptions 1, 7]
  (d) By Lemma 3.2, Q has a prime divisor b.
  (e) (i) Suppose $b = p_i$.
    (ii) Then $b|(p_1 \cdot p_2 \cdots p_n)$. [Assumptions 5, 6]
    (iii) By Lemma 3.3, $b = 1$, contradicting the primality of b.
  (f) Thus for each i, $b \neq p_i$.
  (g) Either $a < b$, or $b \leq a$. [Assumption 10]
  (h) $b \leq a$ contradicts that the p_i were all the primes less than or equal to a.
  (i) Thus $a < b$.

Proof of Lemma 3.1, $S(1)$ is prime:

(1) For all n, $n < S(1)$, $n = S(1)$, or $S(1) < n$. [Assumptions 1 and 10]
(2) (a) Suppose $n = S(1)$.
  (b) $S(1)|S(1)$. [Assumption 4]
(3) (a) Suppose $n < S(1)$.
  (b) $S(n) \leq S(1)$. [Assumptions 1, 14]
  (c) If $S(n) < S(1)$, then $n < 1$, a contradiction. [Assumptions 11 and 12; and 8–10 which imply that only one of the cases of trichotomy of $<$ obtains]
  (d) If $S(n) = S(1)$, then $n = 1$. [Assumption 2]
  (e) $n = 1$.
  (f) $1|S(1)$. [Assumptions 4, 6]
(4) (a) Suppose $S(1) < n$.
  (b) (i) Suppose $n|S(1)$.
    (ii) There exists x such that $nx = S(1)$.
    (iii) $S(1) \cdot x < nx$. [Assumption 15]
    (iv) $S(1) \cdot x < S(1) \cdot 1$. [Assumption 4]
    (v) $x < 1$, a contradiction. [Assumptions 11, 15]

(c) Thus, if $S(1) < n$, then $n \not| S(1)$.

(5) So the only numbers dividing $S(1)$ are 1 and $S(1)$, and so $S(1)$ is prime.

Proof of Lemma 3.2, every natural number $a \neq 1$ has a prime divisor $p \leq a$:

(1) We proceed by strong induction on a.
(2) Base case: $S(1)$ is prime by Lemma 3.1.
(3) Inductive case:
  (a) Suppose that for all $y < a$, $y \neq 1$ has a prime divisor $p \leq y$.
  (b) Either a is prime or composite.
  (c) If a is prime, we are finished.
  (d) So suppose a is composite, i.e. that there is some b such that $1 < b < a$ and $b|a$.
  (e) By the inductive hypothesis, b has a prime divisor $p \leq b$.
  (f) Since $p|b$ and $b|a$, $p|a$. [Assumption 5]
  (g) Since $p \leq b$ and $b < a$, $p \leq a$. [Assumption 8]
(4) So for all $a \neq 1$, a has a prime divisor $p \leq a$. [Assumption 17]

Proof of Lemma 3.3, for all a, b, if $a|b$ and $a|S(b)$, then $a = 1$:

(1) Suppose $a|b$ and $a|S(b)$.
(2) Then there exist x and y such that $ax = b$ and $ay = S(b)$.
(3) $ax = b < S(b) = ay$. [Assumption 13]
(4) $x < y$. [Assumptions 6 and 15]
(5) (a) Suppose $a \neq 1$.
  (b) $S(ax) < a \cdot S(x)$. [Assumptions 1, 3, and 16]
  (c) $S(x) \leq y$. [Assumptions 1 and 14]
  (d) $a \cdot S(x) \leq ay$. [Assumptions 3, 6 and 15]
  (e) $S(ax) < ay$ [Assumption 8]
  (f) $S(b) < S(b)$, a contradiction [Assumption 9]
(6) $a = 1$.

To argue that this solution is pure, we would need to argue that each of the seventeen assumptions belong to IP's topic, that is, that each assumption partly determines the content of IP as formulated (for an ordinary investigator). If we are willing to grant PA as topical for IP, then this is trivial, since each of these assumptions may be derived in PA. Otherwise, this is a difficult task, because it is hard to say definitively whether an assumption is determinative of the content of a problem formulation, even for an ordinary investigator. Indeed it is not even clear what the standards are for making such a determination. Toward this, we note in particular that Assumption 7 is provable by induction, that is, from Assumption 17; and that Assumption 16 asserts that multiplication grows faster than successor, which seems essential to the typical contemporary understanding of the relation of these two functions.

While the issue concerning the topicality of addition for IP remains of interest, let us expand the discussion by considering the topicality of *multiplication* for IP as well. We raised two issues for said topicality earlier: firstly, some plausible non-additive definition of multiplication would be needed if multiplication is to belong "natively" to IP's topic; and secondly, some answer would be needed to the point that like addition, multiplication is also not mentioned explicitly in IP; rather, only division is, in the definition of prime number. Both points may be met by introducing work in mathematical logic. To that work we now turn.

We first point out that since primality is defined in terms of divisibility, a definition of divisibility uncontroversially belongs to IP's topic. This leaves open precisely *which* such definition is included. Divisibility is often defined in terms of multiplication—a divides b if and only if there exists x such that $a \cdot x = b$—but this is not required. Informally, a divides b if a collection of b many objects can be divided into a groups with none left over. While this can be expressed in terms of multiplication, we have just shown that it need not be. The question then is what definitions and axioms ground (our understanding of) divisibility. There has been some logical work on axiomatizations of the arithmetic of divisibility, in which divisibility is taken as a primitive, notably by Cegielski (cf. [Ceg84], [CMR96]), but this work takes the infinitude of primes as an axiom and so is not fine-grained enough for the question in focus here.

We turn instead to work of Julia Robinson which offers a more promising direction for our investigation. Robinson showed how to define addition (and multiplication) for the natural numbers in terms of just successor and divisibility, both of which are explicitly referenced in the problem's formulation (cf. [Rob49], pp. 100–2). She firstly showed that addition is definable in terms of successor and multiplication as follows: $a + b = c$ if and only if $S(a \cdot c) \cdot S(b \cdot c) = S[(c \cdot c) \cdot S(a \cdot b)]$. She next showed both that two numbers being relatively prime, and that a number being the least common multiple of other numbers, can be defined in terms of successor and divisibility, without appeal to addition. She lastly showed how to define multiplication using successor, relative primality, and least common multiple.

Using these explicit definitions, the Euclidean solution to IP (as carried out in PA) may be translated into a language with just 1, S, $|$, and $<$. In particular, all the axioms, definitions and propositions used in proving the Euclidean solution could be translated into this language. By avoiding explicit reference to addition and multiplication, this translated solution would answer worries concerning the topicality of both addition and multiplication, by showing that neither operation is needed for this translated version of the Euclidean solution, and so that a topically pure solution to IP has been identified.

One might resist this response by noting, as Robinson does, that this "mechanical" translation, as Robinson describes it, results in axioms that are "complicated and artificial" (cf. [Rob49], pp. 102–103). They are *syntactically* more complex than the ordinary axioms of PA, in particular *longer* than those axioms, and not identifiable to the "naked eye" as equivalent to PA. Robinson thus set out to find "a simple and elegant axiom system" for arithmetic in this restricted language. She identified a candidate for such an axiom system, and demonstrated that it proves the same theorems as PA, and thus proves the Euclidean solution to IP. This would seem to answer the objection. However, the "simple and elegant" axiom system Robinson identifies includes second-order induction. She could not prove that this axiom system proves the same theorems as PA when including just first-order induction. As far as we know, it remains open whether this can be accomplished.

It is not clear, however, what is wrong with "complicated and artificial" axioms, from the perspective of topical purity. It is not clear that the "simplicity" and "elegance" of a definition or axiom, construed for instance in terms of syntactic complexity, is relevant to whether a definition or axiom belongs to a problem's topic. There could be problems whose topics contain irreducibly complex elements. Such problems would not be simple to understand, at least when considering all the commitments required to ground its understanding, but there is no *a priori* reason to think that every problem, even every commonly-studied problem, is simple to understand in this respect. If that is correct, then Robinson's "mechanical" approach remains a viable response to the objection to the topical purity of the Euclidean solution on account of its use of addition and multiplication.

The objection that the Euclidean solution's use of addition and multiplication entails its topical impurity can thus be met in several ways. In closing this section, we observe that these responses raise interesting issues concerning "implicit" understanding/commitment that we can only raise here. For instance, we can ask whether in understanding the "translated" Euclidean solution we are implicitly committed to the ordinary additive properties used in the untranslated version. We might think this if we thought there was something "basic" about those ordinary additive properties, so that these properties can be expressed in superficially non-additive ways while remaining additive in content. The Robinson definability approach points toward a way to make this sense of "basicness" somewhat sharper: if the axioms resulting from the translation are too "complex", then the original axioms are more basic. (We return to such issues in Sections 3.2 and 4.) The notion of complexity to be used here awaits further clarification, though. Such a suggestion would be worth pursuing. If indeed the "translated" non-additive Euclidean solution turns out to be essentially additive in content, then to avoid concluding that the Euclidean solution is topically impure a reasonable strategy would be to press the point that commitment to additive properties is fundamental to an understanding of the natural numbers.

3.1.2. *The topicality of arithmetic inferences for IP.* Let's now turn to the question of whether the Euclidean solution to IP remains pure when we vary the *modes of inference* belonging to IP's topic. For instance, note that the Euclidean proof would still be topically pure if IP's logic were taken to be intuitionistic, since it uses excluded middle only for effectively decidable predicates such as "is prime", and while it uses reductio it does not use double negation. A fuller investigation would check whether some uses of substitution are topical for IP but not others. In this section we focus on *inductive* modes of inference. For instance, if the logic of IP were taken to be second-order, then the Euclidean solution would still qualify as topically pure by a trivial modification to use second-order induction rather than the first-order induction schema.

Let's consider a different variant of IP. If the logic of IP were taken to be finitary rather than classical, then the Euclidean solution is also plausibly pure. This is because it is straightforward to check that our demonstration uses the induction schema for arithmetical formulas no more complex than Σ^0_1—that is, it can be carried out in the weak fragment of PA known as $I\Sigma_1$. If we follow Tait [Tai81] in taking finitary arithmetic to be Primitive Recursive Arithmetic (PRA), and follow Hájek and Pudlák [HP98] in taking PRA to be equivalent to $I\Sigma_1$, then a finitary solution to IP is one that can be done in $I\Sigma_1$. Hence the Euclidean solution is finitist (on this construal of finitary arithmetic), and thus would still qualify as topically pure.

Next, let's take the logic of IP to be feasible rather than classical. If we follow Parikh [Par71] and take feasible arithmetic to be $I\Delta_0$ (where this is PA with the induction schema restricted to formulas with just bounded quantifiers), then it is an open problem whether IP can be solved purely. It is known, though, that the Euclidean solution is not pure for this formulation of IP. The problem is proving the existence of $Q = (p_1 \cdot p_2 \cdots p_n) + 1$ as is done in the Euclidean solution. This product has exponential growth (by a result of Chebyshev), but Parikh showed [Par71] that every Δ_0-definable function that is provably total in $I\Delta_0$ has polynomial growth.[5] Hence in $I\Delta_0$ it is unprovable that every product of primes exists (cf. [D'A92], p. 13).

However, the existence of Q can be proved using bounded induction provided that we add another axiom asserting the totality of the exponential relation, resulting in a theory called $I\Delta_0(exp)$ (cf. [D'A05], p. 153). $I\Delta_0(exp)$ has been well-studied (many call it EFA, for Elementary Function Arithmetic).[6] For this solution of IP carried out in EFA to be pure

[5]Cf. [D'A05] pp. 164–7 for more on what is known concerning the rate of growth of the function yielding products of primes in $I\Delta_0$.

[6]It is claimed that every result in elementary number theory (for instance, every result in Hardy and Wright's canonical [HW79]) can be proved in EFA (cf. [D'A05], p. 149n1). Indeed, Harvey Friedman has gone further with his "grand conjecture" that (in Avigad's words, "Every theorem published in the *Annals of Mathematics* whose statement involves only finitary mathematical objects (i.e., what logicians call an arithmetical statement) can be proved in elementary arithmetic." (cf. [Avi03] , p. 258)

when IP's logic is construed as feasible, the axiom asserting the totality of exponentiation must be part of the topic of IP so construed. By a result of Gödel [Göd31] we know that the exponential relation is definable in $(\mathbb{N}, 1, S, +, \cdot)$ (for a reasonably explicit definition of this type, cf. [End01], pp. 276–9). It is unclear, though, whether an axiom asserting the *totality* of this definable relation is part of the topic of IP with its logic construed feasibly.

Let's add one more data point to this discussion of feasibility. In his dissertation, Alan Woods [Woo81] was able to solve IP in $I\Delta_0$ augmented by a weak version of the pigeonhole principle. Rather than giving a version of the Euclidean solution, though, Woods gave a version of a solution due to Sylvester. Woods' theory, called $I\Delta_0 + PHP$, is logically weaker than $I\Delta_0(exp)$, in that $I\Delta_0(exp)$ proves $I\Delta_0 + PHP$ but not vice-versa; and indeed Paris, Wilkie, and Woods were able to give a solution to IP using an even weaker version of the pigeonhole principle (cf. [PWW88], and [D'A05], pp. 162–4). Again, we may wonder whether one of these pigeonhole-principle-like axioms belong to IP's topic with its logic construed feasibly. A positive answer would imply that IP has a topically pure solution when its logic is understood feasibly. As far as we are concerned, this remains open.

We have thus surveyed some variations of IP based on what induction is taken to belong to its topic. These variations correspond to foundationally familiar arithmetic theories. As we have seen, the purity of the Euclidean solution to IP, and indeed whether there is any known pure solution to IP, depends on what inferences are licensed by its topic.

3.2. **The topological solution to IP.** We will next consider a solution to IP that we have argued (in Section 5 of [DA11]) is topically *impure*. This is a topological solution due to Harry Furstenberg that goes as follows (cf. [Fur55], p. 353). We begin by putting a topology on the integers, by taking the arithmetic progressions

$$B_{a,b} = \{a + bn : n \in \mathbb{Z}\} \text{ with } a, b \in \mathbb{Z}, b > 0$$

as the basic open sets. The following can then be shown (but we omit the details here): these sets $B_{a,b}$ together form a basis for a topology on the integers; and each $B_{a,b}$ is closed as well as open. By the latter it follows that the union of finitely many $B_{a,b}$ is closed since in a topological space, unions of finitely many closed sets are closed.

We now consider the set $A = \bigcup_p B_{0,p}$ for $p \geq 2$ prime. Since every integer besides ± 1 has a prime factor (by the Fundamental Theorem of Arithmetic), every integer besides ± 1 is contained in some $B_{0,p}$. Thus, $A = \mathbb{Z} - \{-1, 1\}$. If A were a union of finitely many $B_{0,p}$, then it would be a closed set in our topology. Then $\{-1, 1\}$, being the complement of a closed set, would be open. But this is impossible, since the basic open sets $B_{a,b}$ are all infinite, and by the definition of basis, each open set is a superset of some basic open set. Thus A is not a union of finitely many $B_{0,p}$. Hence there are infinitely many primes.

Our argument that Furstenberg's solution to IP is topically impure simply observes that its commitments to definitions of topological space, topological basis, and open and closed sets in a topology do not belong to IP's topic. Each could be retracted without a corresponding change in our understanding of IP.

One might object that some set-theoretic commitments *are* necessary for understanding IP. For instance, one might reply that the "proper" definition of natural number is set-theoretic, as in the original second-order Dedekind-Peano axioms, or in Frege or Russell's work. This makes it clear how difficult it is to say definitively what belongs to a problem's topic, for a full response to this objector would be an argument against this understanding of number, a significant philosophical achievement in its own right. Rather than offer such a response, we observe that it is open whether what is defined by set-theoretic definitions of natural number is the same as what is defined by purely first-order definitions. It is consistent with what is presently accepted that we are discussing (at least) two different problems, one with set-theoretic commitments in its topic, one without.[7] In that case, the topical purity of Furstenberg's proof, with respect to its set-theoretic commitments, comes down to which of these problems is the IP being considered.

Another objection of this type would be that while set-theoretic commitments may not be necessary to understanding natural number, they are necessary for understanding the arithmetic *functions* appealed to by IP. In reply we point out that arithmetic functions can be understood algorithmically, without appeal to set theory. We see no good reason why a set-theoretic understanding of function take precedence, particularly in the case of IP where the functions are merely used for computations.

Note also that the topology used in Furstenberg's proof, when carried out in set theory, is quite weak, i.e. it can be carried out in a fragment of set theory that uses just boolean operations on "simple" sets of natural numbers. On this point, D. Cass and G. Wildenberg have shown that Furstenberg's proof can be reformulated in terms of periodic functions on integers, "avoiding the language of topology" (cf. [CW03], p. 203). In reply, we observe that the issue again is whether *any* set-theoretic commitments are engendered by understanding arithmetic problems. Whether or not these set-theoretic commitments are "weak" is only relevant inasmuch as it bears on whether those commitments belong to IP's topic, and we see no reason to think that they do.

In Section 5 of [DA11] we surveyed a further objection to the topical impurity of Furstenberg's solution, articulated by Colin McLarty in correspondence. McLarty's view is that to have a full understanding of IP, one must include not merely set-theoretic commitments but indeed topological commitments of the type appealed to in Furstenberg's proof. Hence

[7] We say "at least" because it is conceivable that IP's topic may not contain all of the commitments needed to permit Furstenberg's proof, while containing other set-theoretic commitments.

Furstenberg's proof should not therefore be regarded as impure simply because it appeals to topological principles. We want to revisit this objection here in order to clarify further McLarty's point, and to pinpoint how it suggests a notion of content that differs from the one at play in topical purity, in which the understanding of ordinary practitioners is foundational.

In taking this view, McLarty is aligning himself with the *Bourbakiste* tradition of arithmetic research, a tradition to which Furstenberg's work also belongs. McLarty suggests that Furstenberg developed his proof from then-current work of Claude Chevalley in class field theory, work that was considered cutting-edge *arithmetic* despite the central role of topology in it. In [Che40] Chevalley took himself to have made progress in realizing a purist ideal; as he remarked to open the paper, "Class field theory is presented a little more simply today than a few years ago, in particular because of the elimination of "transcendental means" [*La théorie du corps de classes se présente un peu plus simplement aujourd'hui qu'il y a quelques années, notamment du fait de l'élimination des "moyens transcendants"*] (cf. [Che40], p. 394). The "transcendental means" in question are ζ-functions; as Olga Taussky-Todd remarks in her review of this paper, a striking achievement of this paper was "the exclusion of analytical methods. . . the theory of the ζ-functions which for so long a time seemed necessary can now be omitted."[8] Chevalley thus sought and achieved the elimination of complex analysis from what he considered arithmetic, though he had an expansive view of what counts as arithmetic; as Taussky-Todd puts it, "topological methods play an important part in this new presentation of class field theory."

From this standpoint, Furstenberg's proof is a way to illustrate these sophisticated methods to non-experts, and by providing a simple solution to a classical arithmetical problem, a demonstration that they are arithmetic. This approach yielded new results on the cutting-edge of arithmetic, but Furstenberg's solution to IP shows that the approach also yielded new solutions to elementary arithmetic problems. Furthermore, the topological means it draws upon are topological only in an axiomatic, lattice-theoretic sense, rather than in the sense typical of the Poincaré-Lefschetz topology according to which essential use is made of continua such as the real or complex lines. Chevalley judged topology in the latter sense to be non-arithmetic, but in the former sense to be arithmetic.

[8]Cf. *Math. Reviews* MR0002357 (2, 38c). The Riemann zeta function $\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$ is a well-known ζ-function used heavily in analytic number theory (cf. [Gra08]). The general notion of ζ-function, or L-functions as they are sometimes known, arose from work of Euler and Dirichlet, and have been used heavily in analytic and algebraic number theory (cf. [Maz08], [Buz08]). H. Weber had used ζ-functions in class field theory (cf. [Web08]); it was this use of analysis that Chevalley's work was an attempt to purify. For a historical discussion of ζ-functions in Chevalley's context, cf. [Cog07]; and for a detailed technical discussion of ζ-functions in this context, cf. [IR90], Chapter 11.

This, then, is the view that McLarty offers in objection to our determination that Furstenberg's solution to IP is topically impure. He claims that the topological elements in Furstenberg's solution belong to IP's topic, and as a result Furstenberg's solution should not be judged topically impure on the basis of its use of these elements.

McLarty's *Bourbakiste* point is an important one. Work like Chevalley's and Furstenberg's shows that IP is not merely a problem of significance for arithmetic, but of topology as well. It shows that there are "deep" connections between arithmetic and topology, connections that were hidden to previous investigators. It brings to light the fusion of what were once thought separate domains. McLarty's point is that solutions to problems that draw on commitments concerning domains that are "deeply" connected to the topic of a problem are of special epistemic importance. Their importance seems to be twofold: firstly, they improve our knowledge of the connections between domains by showing how one domain can be used to solve problems in another; and secondly, through this gain of knowledge of connections, they afford the investigator "considerable economy of thought" [*économie de pensée considérable*] by providing her with results applicable to multiple domains of investigation rather than to just a single one (cf. [Bou48], §5). Such solutions help combat the splintering of mathematics into autonomous disciplines with different methods and aims (cf. [Bou48], §1).

We have argued (in Section 5 of [DA11]) against McLarty's view, on the grounds that a topological solution of IP provides a "deep" solution but not its most "basic" solution, where "basicness" reflects the conceptual resources corresponding most closely to those which are needed to grasp the problem. Our diagnosis rests on McLarty's claim that the Chevalley-inspired reading properly articulates IP's topic, even though an ordinary understanding of IP would seem to contain no topological commitments. There are thus two competing notions of problem understanding that might be thought to be determinative of topics and hence of the content of problems, what could be called "basic" and "deep" understanding.[9] On the latter, *Bourbakiste* notion suggested by McLarty, Furstenberg's solution qualifies as topically pure. While this notion of deep understanding is important and deserves further investigation, the view here is that this notion should not replace the "basic" sense of understanding in topic determination. This is, in short, because McLarty and Bourbaki make clear that they see deep understanding as articulating connections between the domain of the problem being investigated and other domains, rather than articulating just the content of the problem being investigation. Commitments of the latter type are the ones relevant to topical purity, however, since a topically pure solution of

[9]There is a discussion of a related distinction in [AMng] between "informal" or "intuitive" content of a statement, by which is meant what someone with a casual understanding of geometry would (be able to) grasp, and "formal" or "axiomatic" content, by which is meant the inferential role of that statement in an axiomatic system.

a problem is best thought of as a solution of precisely *that* problem, not some different problem—even if there are good reasons to pursue the solution of that different problem, as the McLarty/Bourbaki view argues.

4. Incompleteness and the possibility of purity

In an article [Kre80], Georg Kreisel explained the consequences of Gödel's work for purity as follows:

> Gdel's paper [Göd30] established that logical purity can be achieved in principle, and [1931] that arithmetic purity cannot be achieved; in fact, the result in [Göd31] is so general that it is quite insensitive to any genuine ambiguities in the notion of purity of method. (pp. 163–4)

The idea seems to be that Gödel sentences are arithmetical sentences, and that a pure proof of an arithmetical sentence must draw only upon arithmetical means. But since Gödel sentences are unprovable by just arithmetical means, they do not admit of pure proof. Such, at any rate, seems to have been Kreisel's view.

Daniel Isaacson has articulated a view concerning the content of Gödel sentences that can be used to argue against Kreisel that Gödel sentences can be proved purely. In [Isa96] Isaacson argued that sentences in the first-order language $\mathcal{L}_{\mathrm{PA}}$ of arithmetic may have purely arithmetical content, or may have in addition "higher-order", i.e. infinitary, non-arithmetical content. Since the ordinary understanding of sentences in $\mathcal{L}_{\mathrm{PA}}$ involves only arithmetical content, he says that their higher-order content, if any, is only "implicit" or "hidden". This follows from his view that the content of arithmetical sentences is determined by what is necessary and sufficient for "perceiving" that that sentence is true, where said "perception" amounts either to "articulating" our grasp of the structure of the natural numbers, as he claims yields the axioms of PA, or to the recognition of a proof of that sentence.

Since we have that Gödel sentences are PA-provably equivalent to sentences expressing by coding metamathematical properties of arithmetic (such as unprovability or consistency), it follows that these sentences are provably unprovable in PA but provable by higher-order means. Such equivalences reveal "the implicit (hidden) higher-order content" of truths in the language of arithmetic, Isaacson writes. He holds that "the understanding of these sentences rests crucially on understanding this coding and our grasp of the situation being coded." Hence, he concludes, Gödel sentences are not arithmetical sentences, but rather have higher-order content. If correct, this would seem to imply that pure proofs of Gödel sentences could draw on non-arithmetical resources and hence are available, contra Kreisel.

In reply, we point out that Isaacson's view seems muddled in the following respect: on the one hand, the non-arithmetical nature of Gödel sentences is the result of their provable unprovability in PA, and on the other hand, of their having coded metamathematical content. In identifying these two, is Isaacson's view committed to maintaining that every arithmetical sentence independent of PA has coded metamathematical content?

The first criterion seems to embody the view that the content of a sentence is determined by the inferential role it plays within an axiomatic theory (in this case a theory in which the metamathematics of PA can be carried out, for instance ZFC). This view does not permit obviously arithmetic sentences like the Goldbach conjecture to be judged as arithmetical at present, since there is at present no reason to believe its (plus-minus) truth is "directly perceivable" from our grasp of the structure of the natural numbers, nor from any other truths, arithmetic or not. This tells against the first criterion as a compelling view concerning the content of sentences in the language of arithmetic.

The second criterion, that Gödel sentences are higher-order rather than arithmetical in virtue of having coded metamathematical content, is more promising. However, it suffers from the following problem. While we cannot see that Gödel sentences are Gödel sentences without grasping their coded metamathematical content, we can grasp them the way we do ordinary universally quantified sentences in the language of arithmetic without seeing that they are Gödel sentences. For instance, we could reasonably *try* to prove such sentences while only accepting the axioms of PA. It is true that our interest in Gödel sentences stems from their metamathematical content, generally speaking, but whether a sentence is arithmetical should be independent of our reasons for interest in it. We *could* encounter Gödel sentences in mainstream number-theoretic work, without knowing beforehand that these sentences are equivalent to metamathematical sentences, and could in that case grasp these sentences without grasping any higher-order content (which is not to say we could *prove* them without such grasp).

A defender of Isaacson's view could draw on the distinction between "basic" and "deep" content made in the previous section. While the basic content of Gödel sentences would seem to be arithmetical, their "deep" content would seem to be metamathematical or higher-order. On this view, the basic content of any sentence expressible in the language of arithmetic is arithmetical, while its deep content depends on other theoretical factors such as its inferential role in axiomatic arithmetic. However, for evaluating Kreisel's claim that Gödel sentences cannot be proved purely, it is basic rather than deep content that is relevant, at least if the type of purity at issue is topical. As we have explained, grasp of their deep content is unnecessary for grasping these sentences in the ordinary way sufficient for attempting their proof, for instance. But it is precisely the latter type of grasp that determines what belongs to a problem's topic, and hence what may be drawn upon in a topically pure proof. Consequently Isaacson's observations indicate in another way the two types of content to which we have already drawn attention, but do not pose a convincing argument against Kreisel's claim that Gödel sentences have no topically pure proofs.

5. Closing thoughts

The case of the infinitude of primes is valuable because it highlights several key issues important for getting clearer on topical purity. How topics of problems are determined awaits further systematic study. Case studies like the one presented here are necessary and important preludes to this type of investigation. This particular case study highlights the difficulty of determining exactly what belongs to the topic of even a quite elementary problem. While addition does not explicitly appear in the problem's formulation, it is natural to think that addition belongs to the topic of every arithmetic problem, in virtue of the natural numbers's identity as an additive structure. The discussion of the Euclidean solution here was meant to show how to argue for its topical purity without just granting this point about the additive identity of the natural numbers. The discussion of Furstenberg's topological solution illustrates two competing notions of problem content that might be thought to be determinative of topics, what could be called "basic" and "deep" content. We argue that what belongs to the deep content of a problem may not necessarily be drawn upon by a topically pure solution of that problem, and so in particular that Furstenberg's solution of IP is not topically pure. We then consider Kreisel's claim that Gödel sentences have no pure proofs and observe that Isaacson's point that these sentences have hidden higher-order content does not contradict this claim, since this hidden content is again deep rather than basic and so does not bear on the purity or impurity of proofs drawing on these higher-order means.

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