In the Light of Logic
by Solomon Feferman
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Poincaré famously compared the logician's understanding of mathematics to the understanding we would have of chess if we were only to know its rules. "To understand the game," Poincaré wrote, "is wholly another matter; it is to know why the player moves this piece rather than that other which he could have moved without breaking the rules of the game. It is to perceive the inward reason which makes of this series of moves a sort of organized whole." [P, pp. 217-218] The Dutch mathematician L.E.J. Brouwer took a position similar to Poincaré's: genuinely mathematical reasoning is not simply a matter of logical inference. It is, as Poincaré put it, a matter of mathematical *insight*.

Despite those views concerning logic, Poincaré and Brouwer believed that the foundations of mathematics ought to be studied, and indeed carried out fundamental work in this area. This might strike contemporary ears as a bit odd, but it is a consistent view. Mathematical logic and the foundations of mathematics are frequently lumped together, as though they are the same. They are not. Mathematical logic is a mature mathematical subdiscipline, with its own problems generated by reflecting on what is known from other logic problems and solution attempts. Like any mature mathematical subdiscipline, what counts as a good problem is largely determined by factors 'internal' to the subdiscipline, such as how the problem contributes to other work in progress and to what is already known. Foundations of mathematics, on the other hand, has a different standard. It raises questions about the objects and structures of mathematics: what are they, and how do we know anything about them? It raises questions about mathematical statements: how should we go about discovering and justifying them? It raises questions about mathematical proofs: what is a proof, what kinds of proofs do we prefer, and for what reasons? Foundations of mathematics is therefore not a mathematical subdiscipline at all, but rather a body of *reflections* on mathematics itself.

A striking insight reached by David Hilbert and others in the early twentieth century was that the foundations of mathematics could be studied by the application of mathematical logic. By taking mathematical objects and structures to be described by axioms in formal languages, these axioms and their consequences could be studied using mathematical logic. In this way, *contra* Poincaré and Brouwer, logic could be used to shed light on the foundations of mathematics, the light of logic to which the title of Feferman's excellent book refers.

Of those who have shed light on the foundations of mathematics using logic, there is one figure whose influence and views tower over the rest: Kurt Gödel. His incompleteness theorems both answered existing questions and raised many new ones, thereby deepening considerably the study of the foundations of mathematics. On account of that, his specter haunts almost every page of Feferman's book.

Feferman classifies the essays (all previously published) of the book into five parts based on their topics, and for each topic, Gödel's work and views are of utmost importance. In Part I, Feferman raises as a problem the role of transfinite set theory in mathematics. Since transfinite sets are supposed to be infinite objects about which facts are true independently of our abilities to verify them, it seems that these abstract entities must exist independently of human thoughts or constructions. This family of beliefs about sets is frequently called *platonism*. Feferman finds platonism philosophically unsatisfying, and thus presents three projects aimed at avoiding platonism: L.E.J. Brouwer's 'intuitionism', David Hilbert's 'finitism', and Hermann Weyl's 'predicativism'. Feferman characterizes Brouwer's solution as excessively radical, leaving Hilbert's and Weyl's as acceptable options. Feferman believes that Gödel's incompleteness theorems cast doubt on the viability of Hilbert's project, as is commonly (but not universally) thought. This leaves Weyl's predicativism as Feferman's preferred alternative to platonism. I will return to predicativism shortly.

The discussion in Part I sets the agenda for the rest of the book. Finding an acceptable alternative to platonism emerges as one central theme. Another central theme that emerges is the question of whether there is any justification for new axioms for set theory. These two themes are tied together by Gödel's view that platonism could be used to justify new axioms for set theory. These new axioms assert the existence of sets which Gödel thought the platonist had every reason to believe in, on account of their uniformity with sets already believed to exist, and on account of a sense-perception-like faculty he thought we possess for experiencing mathematical objects. In addition, he supported new axioms for set theory because he thought they would eventually be used to solve open mathematical problems, just as they can be used to prove the arithmetically unprovable sentences that he had studied in his work on the incompleteness theorems. I think that we may justly view Feferman's book as a wrestling match with Gödel, the arch-platonist. It is unsurprising, therefore, that Feferman dedicates one of the book's five parts—part III—to essays on Gödel's life and work.

Though these central themes are explored in every part of the book, Feferman returns in Part V to his preferred alternative to platonism, predicativism. Here Feferman argues first against attempts to show that transfinite set theory is *necessary* for ordinary finite mathematics. Responding to arguments of Gödel and Harvey Friedman, Feferman concludes that "the case remains to be established that any use of the Cantorian transfinite beyond $\aleph_0$ is necessary for the mathematics of the finite in the everyday sense of the word" (p. 243). Instead, he supports a much more restricted view on the transfinite, maintaining that only *predicatively definable* sets should be admitted. A set is predicatively definable if it is defined by way of the system of natural numbers, or by way of predicatively definable sets that have already been defined. Sets defined by way of a collection of sets that includes the set to be defined are thereby excluded, such as the 'set' of all sets that do not contain themselves, as used in Russell's paradox. Feferman explains how he used methods from modern logic to develop Weyl's predicative set theory, yielding a system in which, he argues, all "scientifically applicable mathematics" can be proved. This system is up to such a task, he argues, because analysis, both classical and modern, can be formalized within it. Yet any (first-order) truth that can be proved in this system can be proved from the (first-order) Peano Arithmetic axioms, which formalize elementary number theory. Feferman argues that this vindicates his view that the predicativist need not admit any transfinite sets beyond the countably infinite, since, he maintains, commitment to Peano Arithmetic only entails commitment to the countably infinite.

In parts II and IV of the book, Feferman discusses how logic can be used to shed light on aspects of mathematical practice besides that part already formalized within set theory. He critically examines Imre Lakatos' views on mathematical discovery, comparing it with George Pólya's views on discovery. He explains how logic can help clarify vague mathematical concepts such as *construction*, *infinitesimal*, and *natural well-ordering*. In particular, Feferman uses his expertise in proof theory, a branch of mathematical logic, to emphasize its utility for understanding mathematics. As he explains, proof theory can be used to clarify what parts of mathematics can be reduced to other parts, and in what ways. Feferman's moral is that logic is useful for more than just the systematic organization of preexisting, well-understood bodies of mathematics—though it is useful for that too.

Feferman has tried to show in this book that the tools of mathematical logic are useful for understanding aspects of mathematical practice that must be accounted for by any reasonable foundation of mathematics. Part of accounting for mathematical practice is saying how we are justified in admitting the objects we seem to need to do mathematics in specific areas like analysis. Frequently this is done by saying that the objects of, e.g., analysis, are just sets, but this requires that we justify our use of sets. Feferman is critical of platonist attempts to justify set theory, and offers instead a predicativist view of how to do so. I think there are four main reasons for why Feferman thinks that predicatively definable sets are justifiable, as follows. (1) Consider Grelling's paradox. Suppose we define a word as being *heterological* if it does not describe itself. The word "heterological" is heterological if and only if it is not heterological. This definition is unhappy, since it does not determine whether or not "heterological" is heterological. Predicative definitions avoid these vicious circles, as follows. We typically define sets as consisting of all objects satisfying some condition. In predicative definitions, the satisfaction of this condition for all objects is determined independently of the set being defined. Hence, there are no vicious circles. (2) Our commitment to predicatively definable sets entails commitment to whatever is needed for Peano Arithmetic, presumably just countably infinite sets. (3) Predicatively definable sets suffice for doing all scientifically applicable mathematics, so working with just them is adequate for the applicability of mathematics. (4) Predicatively definable sets suffice for doing all ordinary finite mathematics, perhaps the minimum part of mathematics for which any reasonable foundation must account.

I will comment briefly on these four reasons. (1) The avoidance of vicious circle paradoxes does not ensure the consistency of predicative mathematics. Predicative mathematics may be more secure than impredicative mathematics, but that does not mean that it is perfectly secure. Indeed, as Feferman showed in 1964, the consistency of predicative analysis cannot be proved predicatively, though it can be proved impredicatively [F, pp. 1–30]. Furthermore, this characterization of the value of predicativity leaves it open whether predicative definitions have any other value. One reason to be worried about this is that there are many sets that can be defined predicatively, but whose impredicative definitions mathematicians find more 'natural'. For instance, the closure of a set in a topological space is naturally defined as the intersection of all closed sets containing the set, but this is impredicative. Mathematicians typically find this definition unproblematic because the existence of the sets involved follows from set-theoretic axioms such as ZFC. Predicativists reject existence-in-ZFC as sufficient for set existence, demanding instead a description (in some weaker axiomatic system, perhaps) of how a set may be generated from other sets already known to exist. Consider also the following example: given a homeomorphism on a compact space, there is always a "minimal" nonempty closed invariant subset. The standard proof uses Zorn's lemma and intersections, and is thus impredicative. This can be proved predicatively, but it is more involved than the standard proof [BHS, p. 152]. (Thanks to Jeremy Avigad for pointing out this example to me.) Predicativity thus exacts a toll, in that it costs us natural definitions and proofs—leaving what is natural unspecified but, I take it, uncontroversial in the cases under consideration. We must weigh the apparent security purchased by requiring predicative definitions, against the burden of having to abandon in many cases what we, as mathematicians, consider natural definitions.

(2) It is unclear exactly what objects we are committed to when we are committed to Peano Arithmetic. There are plenty of problems in number theory whose proofs use analytic means, for instance. Does commitment to Peano Arithmetic entail commitment to whatever objects are needed for these proofs? More generally, does commitment to a mathematical theory mean commitment to any objects needed for solving problems of that theory? If so, then Gödel's incompleteness theorems suggest that it is open what objects commitment to Peano Arithmetic entails.

(3) As Feferman admits, it is unclear how to account predicatively for some mathematics used in currently accepted scientific practice, for instance in quantum mechanics. In addition, I think that Feferman would not want to make the stronger claim that *all future* scientifically applicable mathematics will be accountable for by predicative means. However, the claim that *currently* scientifically applicable mathematics can be accounted for predicatively seems too time-bound to play an important role in a foundation of mathematics. Though it is impossible to predict all future scientific advances, it is reasonable to aim at a foundation of mathematics that has the potential to support these advances. Whether or not predicativity is such a foundation should be studied critically.

(4) Whether the use of impredicative sets, and the uncountable more generally, is needed for ordinary finite mathematics, depends on whether by

'ordinary' we mean 'current'. If so, then this is subject to the same worry I raised for (3). It also depends on where we draw the line on what counts as finite mathematics. If, for instance, Goldbach's conjecture counts as finite mathematics, then we have a statement of finite mathematics for which it is completely open whether it can be proved predicatively or not.

In emphasizing the degree to which concerns about predicativism shape this book, I should not overemphasize it. There is much to be interested in besides predicativism in this book, as I have tried to indicate. In fact, Feferman advises that we not read his predicativism too strongly. In the preface, he describes his interest in predicativity as concerned with seeing how far in mathematics we can get without resorting to the higher infinite, whose justification he thinks can only be platonic. It may turn out that uncountable sets are needed for doing valuable mathematics, such as solving currently unsolved problems. In that case, Feferman writes, we "should look to see where it is necessary to use them and what we can say about what it is we know when we do use them" (p. ix).

Nevertheless, Feferman's committed anti-platonism is a crucial influence on the book. For mathematics right now, Feferman thinks, "a little bit goes a long way", as one of the essay titles puts it. The full universe of sets admitted by the platonist is unnecessary, he thinks, for doing the mathematics for which we must currently account. Time will tell if future developments will support that view, or whether, like Brouwer's view, it will require the alteration or outright rejection of too much mathematics to be viable. Feferman's book shows that, far from being over, work on the foundations of mathematics is vibrant and continuing, perched deliciously but precariously between mathematics and philosophy.

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