ON WOODRUFF'S CONSTRUCTIVE NONSENSE LOGIC

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ABSTRACT. Sören Halldén's logic of nonsense is one of the most well-known many-valued logics available in the literature. In this paper, we discuss Peter Woodruff's as yet rather unexplored attempt to advance a version of such a logic built on the top of a constructive logical basis. We start by recalling the basics of Woodruff's system and by bringing to light some of its notable features. We then go on to elaborate on some of the difficulties attached to it; on our way to offer a possible solution to such difficulties, we discuss the relation between Woodruff's system and two-dimensional semantics for many-valued logics, as developed by Hans Herzberger.

Keywords: Peter Woodruff, logic of nonsense, constructive logic, Hans Herzberger, two-dimensional semantics.

1. INTRODUCTION

1.1. **Background and aim.** In [24], Peter Woodruff devised a constructive version of Sören Halldén's logic of nonsense, presented in [14], with an additional connective introduced by Lennart Åqvist in [1], and explored further by Krister Segerberg in [22]. Woodruff's project, which is actually pursued having more abstract aims as its goals (cf. [24, p.195]), is extremely rich and, although it has not been fully explored yet, promises to be very fruitful on a variety of different fronts. Part of our aim in this paper is to bring to light many interesting features of Woodruff's proposal.

Another major concern of ours has more philosophical tones, and it is related to the fact that the semantics advanced by Woodruff seems to be not without problems, when seen in the lights of the original motivations set forth by Halldén in the intended semantics for his logic of nonsense. In a nutshell, the problem is that Woodruff's semantics is presented by employing *two dimensions*, one accounting for the behavior of the two usual truth values, and another one accounting for the sense/non-sense distinction. As per the original suggestion by Halldén, these dimensions are not independent: typically, a senseless sentence should have no truth-value, and sentences having a truth value are meaningful; however, given the relatively independent operation of the two dimensions in Woodruff's semantics, this is not what happens in his (Woodruff's) semantics. Discussing

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these tensions will shed an interesting light on the workings of the two-dimensional semantics and how it can be related to an appropriately *constructive* understanding of the semantic notions of truth and falsity, in opposition to a more classical reading of such concepts.

Our aim in this paper, then, is to bring Woodruff's major contributions in [24] to the table again, to shed some light on some of its very fruitful ideas, and also to point to some further directions in which they can be elaborated. After presenting the basic definitions of Woodruff's constructive logic of nonsense, we discuss some of its similarities with a semantics originally presented by Hans Herzberger in [15]. This connection shall guide some of our suggestions of how the main idea found in Woodruff's paper may be further extended and explored. We discuss in particular how these variations on the Herzberger-Woodruff themes may be understood in terms of their informal meaning. This is particularly important because, prima facie, as we also point in the paper, Woodruff's semantics and his system in general seem not to be quite appropriate for a logic of nonsense (just as Herzberger's is not), given the original motivations Halldén advanced for such systems. An alternative reading for such constructive system is then presented, which seems to be more suitable for what is actually found in Woodruff's system.

This paper is structured as follows. In section 2 we revisit the basics of Woodruff's semantics advanced in [24], presenting also the motivations behind his developments. In section 3 we observe and comment on two notable features of the semantics by Woodruff: first, its similarities with an alternative semantics for the logic of nonsense (i.e. Halldén's logic), as well as the fact that it is a clear instance of a mixed consequence relation that was advanced in the literature much before such consequences became fashionable. Having the parallel with Herzberger semantics in our hands, in section 4 we discuss two problems that can be highlighted on such a basis: the question of whether Woodruff's system is paraconsistent (such as Halldén's) and the problem of making sense of nonsense in a bivalent context, such as the one presented in the semantics advanced by Woodruff (and here the similarities with the semantics advanced by Herzberger are of great help in drawing important philosophical distinctions). Finally, in section 5 we present what we consider to be an appropriately constructive reading of Woodruff's semantics that could be useful for answering to some of the questions raised in the previous section. It will result that the combination of a constructive notion of truth and falsity with the sense/nonsense distinction behaves quite distinctively from the classical case. We conclude the paper in section 6 by summarizing our claims and presenting some additional directions that future work based on Woodruff's paper could take.

1.2. **Preliminaries.** The languages $\mathcal{L}$ and $\mathcal{L}_w$ (the subscript 'w' for Woodruff) consist of sets $\{\neg, \wedge, \vee\}$ and $\{\bot, +, \mathsf{T}, \wedge, \vee, \rightarrow\}$ of propositional connectives, respectively, and a countable set Prop of propositional variables which we denote by p, q, etc. We denote by Form and Form_w, the sets of formulas defined as usual in $\mathcal{L}$ and $\mathcal{L}_w$, respectively, and denote a formula of the languages by A, B, C, etc. and a subset of the set of formulas by Γ, Δ, Σ, etc.

Before moving ahead, let us also recall the three-valued semantics for the weak Kleene Logic, as well as Åqvist's expansion of Halldén's logic.

Definition 1. A *weak Kleene interpretation* of $\mathcal{L}$ is a function $v_3 : \mathsf{Prop} \to \{\mathbf{t}, \mathbf{u}, \mathbf{f}\}$. Given a weak Kleene interpretation v_3, this is extended to a function I_3 that assigns every formula a truth value by the following truth functions:

	$\neg$
t	**f**
u	**u**
f	**t**

$\wedge$	**t**	**u**	**f**
t	**t**	**u**	**f**
u	**u**	**u**	**u**
f	**f**	**u**	**f**

$\vee$	**t**	**u**	**f**
t	**t**	**u**	**t**
u	**u**	**u**	**u**
f	**t**	**u**	**f**

Definition 2. A *Åqvist-Segerberg interpretation* of $\mathcal{L}_w$ is a function $v_3 : \mathsf{Prop} \to \{\mathbf{t}, \mathbf{u}, \mathbf{f}\}$. Given a Åqvist-Segerberg interpretation v_3, this is extended to a function I_3 that assigns every formula a truth value by the following truth functions:

	$\neg$	T	$+$
t	**f**	**t**	**t**
u	**u**	**f**	**f**
f	**t**	**f**	**t**

$\wedge$	**t**	**u**	**f**
t	**t**	**u**	**f**
u	**u**	**u**	**u**
f	**f**	**u**	**f**

$\vee$	**t**	**u**	**f**
t	**t**	**u**	**t**
u	**u**	**u**	**u**
f	**t**	**u**	**f**

$\to$	**t**	**u**	**f**
t	**t**	**u**	**f**
u	**u**	**u**	**u**
f	**t**	**u**	**t**

Remark 3. As observed in [22, p.202], it will suffice to take $\neg$, T, and $\wedge$ as the only primitive connectives. Moreover, to be completely precise, the language $\mathcal{L}_w$ contains $\bot$ as primitive, not $\neg$, but these choices are equivalent in the present context of three-valued semantics. Finally, note that in the latter semantics, one can define a very strong *classical* conditional. For details, see [17].

2. Woodruff's semantics, revisited

We first revisit the semantics originally proposed by Woodruff in [24]. One of the most notable features of the apparatus advanced by Woodruff is that it is constituted by two valuations, being a sort of two-dimensional semantics. One of the dimensions is concerned with truth and falsity in a constructive setting, the other one is concerned with the 'meaningful/nonsense' distinction. As we have already anticipated, the intuitive plan is that meaningful sentences coincide with those having a specific truth value, while meaningless sentences have no truth value (our proposal for an appropriate understanding of the formalism is advanced in §5).

Definition 4 (Woodruff)**.** A *Woodruff model* for $\mathcal{L}_w$ is a structure $\mathcal{M} = \langle I, \leq, v_t, v_+ \rangle$, where

- I is a non-empty set of states,

- $\langle I, \leq \rangle$ is a pre-order, and
- v_t and v_+ are valuation functions from $I \times \mathsf{Prop}$ to $\{\mathbf{t}, \mathbf{f}\}$ and $\{1, 0\}$, respectively, such that they are persistent i.e. for every $p \in \mathsf{Prop}$, and all states x, x':
  - if $x \leq x'$ and $v_t(x, p) = \mathbf{t}$, then $v_t(x', p) = \mathbf{t}$ and
  - if $x \leq x'$ and $v_+(x, p) = 1$, then $v_+(x', p) = 1$.

  Moreover, it is required that for all $p \in \mathsf{Prop}$ and all $x \in I$, if $v_t(x, p) = \mathbf{t}$ then $v_+(x, p) = 1$.

For $x \in I$ the relations $\mathcal{M}, x \models^{\mathbf{t}} A$ and $\mathcal{M}, x \models^{+} A$ are inductively defined as follows:

- $\mathcal{M}, x \models^{\mathbf{t}} p$ iff $v_t(x, p) = \mathbf{t}$
- $\mathcal{M}, x \models^{+} p$ iff $v_+(x, p) = 1$
- $\mathcal{M}, x \not\models^{\mathbf{t}} \perp$
- $\mathcal{M}, x \models^{+} \perp$
- $\mathcal{M}, x \models^{\mathbf{t}} +A$ iff $\mathcal{M}, x \models^{+} A$
- $\mathcal{M}, x \models^{+} +A$
- $\mathcal{M}, x \models^{\mathbf{t}} \mathsf{T}A$ iff $\mathcal{M}, x \models^{\mathbf{t}} A$
- $\mathcal{M}, x \models^{+} \mathsf{T}A$
- $\mathcal{M}, x \models^{\mathbf{t}} A \wedge B$ iff $\mathcal{M}, x \models^{\mathbf{t}} A$ and $\mathcal{M}, x \models^{\mathbf{t}} B$
- $\mathcal{M}, x \models^{+} A \wedge B$ iff $\mathcal{M}, x \models^{+} A$ and $\mathcal{M}, x \models^{+} B$
- $\mathcal{M}, x \models^{\mathbf{t}} A \vee B$ iff $\mathcal{M}, x \models^{\mathbf{t}} A$ or $\mathcal{M}, x \models^{\mathbf{t}} B$
- $\mathcal{M}, x \models^{+} A \vee B$ iff $\mathcal{M}, x \models^{+} A$ and $\mathcal{M}, x \models^{+} B$
- $\mathcal{M}, x \models^{\mathbf{t}} A \to B$ iff $\mathcal{M}, x \models^{+} A$ and $\mathcal{M}, x \models^{+} B$ and for every $x' \geq x : \mathcal{M}, x' \not\models^{\mathbf{t}} A$ or $\mathcal{M}, x' \models^{\mathbf{t}} B$
- $\mathcal{M}, x \models^{+} A \to B$ iff $\mathcal{M}, x \models^{+} A$ and $\mathcal{M}, x \models^{+} B$

Remark 5. Note that negation in $\mathcal{L}_w$ is defined as in intuitionistic logic. Namely, we define $\neg A$ as $A \to \perp$. Therefore, we obtain the following.

- $\mathcal{M}, x \models^{\mathbf{t}} \neg A$ iff $\mathcal{M}, x \models^{+} A$ and for every $x' \geq x : \mathcal{M}, x' \not\models^{\mathbf{t}} A$
- $\mathcal{M}, x \models^{+} \neg A$ iff $\mathcal{M}, x \models^{+} A$

Based on these, Woodruff defined the semantic consequence relation as follows.

Definition 6 (Woodruff)**.** For $\Gamma \cup \{A\} \subseteq \mathsf{Form}_w$, $\Gamma \models A$ iff for all Woodruff models $\mathcal{M}$, and for all x in $\mathcal{M}$, if $\mathcal{M}, x \models^{\mathbf{t}} B$ for all $B \in \Gamma$ and $\mathcal{M}, x \models^{+} A$, then $\mathcal{M}, x \models^{\mathbf{t}} A$.

Given these definitions, let us now pause for a moment to recall Woodruff's major aim with this system: to present a constructive version of Halldén's logic of nonsense. The plan is quite simple, and suggests that just as Halldén's logic of nonsense is thought to be elaborated on the top of classical logic, with a third category of truth values added and with the behavior of the connectives remaining classical when only classical truth values are involved, Woodruff's logic is an attempt to bring the nonsensical dimension on the top of a *constructive* basis. The job, however, is not done in the same way as in the classical case. The idea, as we have just seen, is implemented by the use of two orthogonal components of the semantics; while one of the dimensions deals with the alethic element, the second component deals with the sense/nonsense distinction. The heredity condition, along with the demand that true sentences are always meaningful (i.e. they have sense) connects truth and meaningfulness. On the other hand, false sentences may be divided in terms of meaningful and nonsensical, depending on the value they receive from the second valuation (more on this later).

As mentioned in the introduction, this specific construction seems to generate some trouble for the very idea underlying the sense/nonsense division as formulated in the original motivations provided by Halldén himself. The trouble, in a nutshell, is that in Woodruff's system truth and falsity operate in such a way that every sentence receives exactly one of such truth values in each valuation, and this means that if one is to understand 'nonsense' as 'non truth-evaluable', as Halldén suggested, one ends up having every sentence as being actually truth evaluable, which clashes with the very idea of the original sense/nonsense distinction. Of course, true sentences are required to be always meaningful, as we commented, but false sentences are open for a sense/nonsense additional classification which seems to be in tension with the very idea of nonsense as advanced by Halldén.

In the next section we introduce and briefly discuss a closely related approach to formal semantics, advanced by Hans Herzberger in [15], which also incorporates two dimensions: one for (classical) truth and falsity, and another for the sense/nonsense dimension. This will allow us to elaborate on the nature of the difficulties highlighted here for Woodruff's system, and will also contribute to illuminate some possible alternatives for understanding the system advanced by Woodruff. Interestingly, the similarities of both semantics seem to point to similar problems. However, as we shall see, the distinction constructive/classical approaches to truth shall also play an important role.

3. Observations

In this section, we turn to discuss some consequences of some of the features of the semantics proposed by Woodruff. We begin by pointing to some remarkable similarities between the Woodruff semantics and Herzberger semantics.[1]

3.1. **Herzberger.** Let us first draw a connection to the semantics presented by Hans Herzberger in [15], which goes as follows.

Definition 7. A *Herzberger interpretation* of $\mathcal{L}$ is a pair $\langle v_t, v_h \rangle$, where $v_t : \mathsf{Prop} \to \{\mathbf{t}, \mathbf{f}\}$ and $v_h : \mathsf{Prop} \to \{0, 1\}$. Valuations v_t and v_h are then extended to interpretations I_t and I_h by the following conditions.

[1]It is interesting to notice that both papers by Herzberger and Woodruff appeared in the same year, 1973.

$I_t(p)=\mathbf{t}$ iff $v_t(p)=\mathbf{t}$	$I_h(p)=1$ iff $v_h(p)=1$
$I_t(\neg A)=\mathbf{t}$ iff $I_t(A)=\mathbf{f}$	$I_h(\neg A)=1$ iff $I_h(A)=1$
$I_t(A\wedge B)=\mathbf{t}$ iff $I_t(A)=\mathbf{t}$ & $I_t(B)=\mathbf{t}$	$I_h(A\wedge B)=1$ iff $I_h(A)=1$ & $I_h(B)=1$
$I_t(A\vee B)=\mathbf{t}$ iff $I_t(A)=\mathbf{t}$ or $I_t(B)=\mathbf{t}$	$I_h(A\vee B)=1$ iff $I_h(A)=1$ & $I_h(B)=1$

Intuitively, the first component v_t represents a truth component and v_h represents an additional dimension on the top of truth. The exact understanding of this additional dimension depends very much on the application being made of the semantics, but a very appealing reading could be made in epistemic terms.[2] In this sense, while I_t attributes one of the classical truth values, I_h attributes a value that can be read as 'known by agent A' for 1 and 'not known by agent A' for 0. Then, a sentence would receive **t** and 1 if it is actually true, and this truth is known to an agent. A similar reading can be adapted to take care of the other three combinations. For the sake of simplicity, we shall denote such combinations as **t**1, **t**0, **f**1 and **f**0. This epistemic reading of the second component allows one, for instance, to make completely precise a claim by Susan Haack, in [13, chap.11], that some many-valued systems do not really require that we abandon bivalence or two-valuedness; it is all a matter of adding an additional layer of epistemic nature on the top of classical truth values.[3]

Now that there are four combinations for elements of Prop, we may easily turn the above two-valued semantics into a four-valued semantics, as also observed by Herzberger.

Definition 8. A *four-valued interpretation* of $\mathcal{L}$ is a function $v_4 : \mathsf{Prop} \to \{\mathbf{t}1, \mathbf{t}0, \mathbf{f}0, \mathbf{f}1\}$. Given a four-valued interpretation v_4, this is extended to a function I_4 that assigns every formula a truth value by the following truth functions:

A	$\neg A$
t1	**f**1
t0	**f**0
f0	**t**0
f1	**t**1

$A\wedge B$	**t**1	**t**0	**f**0	**f**1
t1	**t**1	**t**0	**f**0	**f**1
t0	**t**0	**t**0	**f**0	**f**0
f0	**f**0	**f**0	**f**0	**f**0
f1	**f**1	**f**0	**f**0	**f**1

$A\vee B$	**t**1	**t**0	**f**0	**f**1
t1	**t**1	**t**0	**t**0	**t**1
t0	**t**0	**t**0	**t**0	**t**0
f0	**t**0	**t**0	**f**0	**f**0
f1	**t**1	**t**0	**f**0	**f**1

Remark 9. I_4 is just the direct product of I_t and I_h.

Given a many-valued interpretation of the language under consideration, we need to specify the set of designated values to define the semantic consequence relation. To this end, we introduce three different sets of designated values as follows:

- $\mathcal{D}_1 := \{\mathbf{t}1\}$;
- $\mathcal{D}_2 := \{\mathbf{t}1, \mathbf{t}0\}$;

[2] Herzberger's motivation, as well as application, in [15] concerned the issues related to presuppositions and presuppositions failure.

[3] See [18] for discussions on this point, building the themes from Haack with the help of Herzberger semantics, and see also [20, 19] for further applications of the Herzberger semantics.

- $\mathcal{D}_3 := \{\mathbf{t}1, \mathbf{t}0, \mathbf{f}0\}$.

Based on these sets of designated values, we define three consequence relations in the usual way, understood as preservation of designated values from the premises to the conclusion.[4]

Definition 10. For $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, and for $i \in \{1, 2, 3\}$, $\Gamma \models_i A$ iff for all four-valued interpretations v_4, $I_4(A) \in \mathcal{D}_i$ if $I_4(B) \in \mathcal{D}_i$ for all $B \in \Gamma$.

Remark 11. It is interesting to remark that $\models_1$ is weak Kleene **WK**, $\models_2$ is classical logic **CL** and $\models_3$ is paraconsistent weak Kleene **PWK** (Halldén's logic), as observed by Herzberger in [15] (see also [23]).

The similarities between the semantics offered by Herzberger and the semantics offered by Woodruff should now be clear. In both cases, what we find is the same technique of having two separate valuations, one accounting for the behavior of the truth values, and an additional one whose interpretation may vary, so that distinct understandings are possible ('known', 'known and analytic', 'meaningful', are some examples, see again [18, 19] for additional discussion). The benefits of using the Herzberger semantics for such many-valued logics concerns a gain in understanding: one may provide for a clear picture of how to attribute meaning to sentences in such systems without abandoning bivalence (on a purely classical setting). At the same time, this same move also seems to lead us to the problems we have pointed out before to the semantics advanced by Woodruff: given that the truth conditions require that every sentence does have a truth value, when we plug to that setting the reading of the second component as involving 'nonsense', understood as non truth-evaluable, the result is quite problematic for the Herzberger semantics too. So, in a sense, although the Herzberger semantics is a two dimensional semantics for Halldén's logic, it does not seem appropriate to capture the intended sense/nonsense reading, given that it does endow every sentence with a classical truth value. That seems to suggest that approaching a logic of nonsense by using such two dimensional semantics is a bad idea to begin with, given the kind of limitation it imposes on the reading of the second component. We shall come back to this topic below.

Now, despite the similarities, there are also at least three crucial differences: Woodruff adds a condition on the relation between the two valuations — which, under his intended interpretation implies that every true sentence must be meaningful —, and moreover, the truth condition for the conditional relies on the Kripke semantic framework. The third difference concerns the non-Tarskian definition of consequence relation, but we shall look into that difference in details in the next subsection.

[4] In principle, we may consider an even wider number of consequence relations, including the cases for mixed consequence relations, but given our aims here, that topic will be left for another paper.

Before we go on to the next topic, we need to address a relevant technical worry that may have occurred to the reader while reading this subsection (as it occurred to a referee of this journal, to whom we are grateful). The concern is related to the following well-known fact: the algebra behind the above four-valued valuation may be obtained by a Płonka sum of two two-element Boolean algebras over which **CL**, **WK** and **PWK** may be defined in terms of a *single* valuation (see [3, 4] for technical details). The worry, then, is that given that such a technical tool is available, and that we may arrive at the same results with single valuations, we may be overemphasizing the role of a two dimensional semantics in terms of truth *and* an additional —epistemic or semantic— ingredient. However, we do not consider that this fact should mean that an approach by Płonka sums and single valuations is to be preferred in the absence of further reasons. There are, indeed, some important advantages of the Herzbergerian two-dimensional approach over the Płonka sums approach that are related to our major purposes in the paper. In order to appreciate such advantages, it is important to recall that we are adopting and endorsing here Susan Haack's above mentioned demand to enlighten a system through its semantics; when seen as attempts to address that concern, the differences between the two semantic approaches (i.e., by Płonka sums and by the Herzberger) are abyssal. The two-dimensional semantics have a quite clear and intuitive reading in terms of qualifications added on the top of simple truth and falsity, so that Haack's demand for explanations of the working of the target systems in terms that are already well understood are met. On the other hand, Płonka sums, as far as we can see, have no such simple reading available. That is, given the epistemological task we are setting ourselves to address on what concern the systems being studied here, a Herzberger-style of semantics seems more suitable.

3.2. **Mixed consequence relation.** The second observation we would like to explore in this section concerns the very definition of the consequence relation given by Woodruff.[5] In brief, Woodruff's definition can be seen as a generalization of the so-called *p-consequence relation* (cf. [11]), or in a more recent terminology, the *ST-consequence relation* (strict-tolerant consequence relation, cf. [6, 7]). In order to see the connection more clearly, let us consider the Woodruff models in which there is only one element for the set I. Then, what we obtain is the following truth table.

A	$\bot$	$\top A$	$+A$
t1	**f**1	**t**1	**t**1
f0	**f**1	**f**1	**f**1
f1	**f**1	**f**1	**t**1

$A \wedge B$	**t**1	**f**0	**f**1
t1	**t**1	**f**0	**f**1
f0	**f**0	**f**0	**f**0
f1	**f**1	**f**0	**f**1

$A \vee B$	**t**1	**f**0	**f**1
t1	**t**1	**t**0	**t**1
f0	**t**0	**f**0	**f**0
f1	**t**1	**f**0	**f**1

$A \rightarrow B$	**t**1	**f**0	**f**1
t1	**t**1	**f**0	**f**1
f0	**f**0	**f**0	**f**0
f1	**t**1	**f**0	**t**1

Note also that given the definition of negation, we obtain the following truth table for negation.

[5] We would like to thank Paul Égré for directing our attention to the main point of the observation below.

A	$\neg A$
$\mathbf{t}1$	$\mathbf{f}1$
$\mathbf{f}0$	$\mathbf{f}0$
$\mathbf{f}1$	$\mathbf{t}1$

In other words, we obtain the three-valued semantics devised by Åqvist, as we recalled earlier (but in a slightly different notation).

Then, it is clear that the definition deployed by Woodruff is indeed a generalization since it does not require the mere preservation of one or two values, but rather excludes the cases in which premises are strictly true (i.e. receiving the value $\mathbf{t}1$) and the conclusion is not tolerantly true (i.e. receiving the value $\mathbf{f}1$). It is also quite remarkable that Woodruff is well ahead of time, even ahead of Jean-Yves Girard's [12], and that to the best of our knowledge, there is so far no ST-relation explicitly defined for systems with worlds.

4. Two problems

In this section, we point to two difficulties engendered by the semantics offered by Woodruff in connection with his stated motivations of offering a logic of nonsense.

4.1. **Is it paraconsistent?**

Given that Halldén's logic is paraconsistent, we may naturally expect that constructive generalization of Halldén's logic is also paraconsistent.[6]

First, one may easily observe that the rule form of *Ex Contradictione Quodlibet* (ECQ hereafter) *holds* in Woodruff's system, as follows. Recall again here that we define negation as arrow-falsum, as in intuitionistic logic.

Proposition 1. *For all* $A, B \in \mathsf{Form}_w$, $A, \neg A \models B$.

Proof. It suffices to observe that for all Woodruff models $\mathcal{M}$, for all x in $\mathcal{M}$, and for all $A \in \mathsf{Form}_w$ we have $\mathcal{M}, x \not\models^{\mathbf{t}} A$ or $\mathcal{M}, x \not\models^{\mathbf{t}} \neg A$. □

Note that sometimes, paraconsistency is understood in terms of ECQ as a formula, not as a rule.[7] Note further that although the seemingly more widely accepted working definition of paraconsistency is with the rule form, the formula form is still used, for example, in the literature of discussive logic. So, this may raise the question if Woodruff's system is actually paraconsistent in the sense of invalidating ECQ in the formula form. The answer is that Woodruff's system also contains ECQ in the formula form, since the deduction theorem holds:

Proposition 2. *For all* $\Gamma \cup \{A, B\} \subseteq \mathsf{Form}_w$, $\Gamma, A \models B$ *iff* $\Gamma \models A \to B$.

[6] Note that the combination of constructivity and paraconsistency is not strange, but rather well known and well explored, thanks to the systems related to **N4** (cf. [16] for a systematic investigation into **N4**).

[7] For example, the expansion of intuitionistic logic by empirical negation (cf. [8]) follows this pattern in the sense that the rule form ECQ holds, but not the formula form (cf. [8, §4]).

Proof. For the left to the right direction, assume that $\Gamma, A \models B$ and that $\Gamma \not\models A \to B$. Then by the latter, for some Woodruff model, $\mathcal{M}_0$ and for some w_0 in $\mathcal{M}_0$,

C1: $\mathcal{M}_0, w_0 \models^{\mathbf{t}} C$ for all $C \in \Gamma$,
C2: $\mathcal{M}_0, w_0 \models^{+} A \to B$ and
C3: $\mathcal{M}_0, w_0 \not\models^{\mathbf{t}} A \to B$.

In view of **C2**, **C3** and the truth condition for $\to$, we obtain that for some $w_1 \geq w_0$,

C4: $\mathcal{M}_0, w_1 \models^{\mathbf{t}} A$ and
C5: $\mathcal{M}_0, w_1 \not\models^{\mathbf{t}} B$.

Note also that in view of **C2**, the meaningful condition for $\to$, and the heredity condition for the meaningfulness, we obtain that

C6: $\mathcal{M}_0, w_1 \models^{+} B$.

Moreover, by **C1** and the heredity condition for truth we obtain

C7: $\mathcal{M}_0, w_1 \models^{\mathbf{t}} C$ for all $C \in \Gamma$,

By combining **C7**, **C4**, **C5** and **C6**, we obtain that $\Gamma, A \not\models B$, a contradiction.

For the other direction, assume that $\Gamma \models A \to B$ and that $\Gamma, A \not\models B$. Then by the latter, for some Woodruff model, $\mathcal{M}_0$ and for some w_0 in $\mathcal{M}_0$,

C1: $\mathcal{M}_0, w_0 \models^{\mathbf{t}} C$ for all $C \in \Gamma$,
C2: $\mathcal{M}_0, w_0 \models^{\mathbf{t}} A$ and
C3: $\mathcal{M}_0, w_0 \models^{+} B$ and
C4: $\mathcal{M}_0, w_0 \not\models^{\mathbf{t}} B$.

By **C2** and the condition for the relation between truth and meaningfullness, we obtain

C5: $\mathcal{M}_0, w_0 \models^{+} A$.

Therefore, in view of **C2–C5**, we obtain that

C6: $\mathcal{M}_0, w_0 \models^{+} A \to B$ and
C7: $\mathcal{M}_0, w_0 \not\models^{\mathbf{t}} A \to B$.

By combining **C1**, **C6** and **C7**, we obtain that $\Gamma \not\models A \to B$, a contradiction. □

Remark 12. Note that Woodruff also observes the direction from the left to the right (cf. [24, Lemma 4.1, (m)]). However, Woodruff did not observe the other direction, but instead, observes different forms of Modus Ponens (cf. [24, Lemma 4.1]).

Therefore, as a corollary of the above two propositions, we obtain the following.

Corollary 1. *For all* $A, B \in \mathsf{Form}_w$, $\models (A \land \neg A) \to B$.

Moreover, note that we have the following, i.e. that the law of excluded middle (LEM) fails in Woodruff's system.

Proposition 3. $\not\models p \lor \neg p$.

Proof. The result can be established the following model:

- $I_0 = \{x_0, x_1\}$,
- $\leq_0 = \{\langle x_0, x_0\rangle, \langle x_0, x_1\rangle, \langle x_1, x_1\rangle\}$,
- $v_t(x_0, p) = \mathbf{f}$, $v_t(x_1, p) = \mathbf{t}$, and $v_+(x_0, p) = v_+(x_1, p) = 1$.

Then, we obtain that $v_t(x_0, p \vee \neg p) = \mathbf{f}$, as desired. □

Note, however, that in the classical extension, in the sense of limiting the cardinality of I to be one, we *do* have LEM. This is because the situation in Woodruff's logic is completely parallel to the case of intuitionistic logic.

In view of these observations, on the one hand, Woodruff's logic is *not* a generalization of Halldén's logic of nonsense, if one follows the working definition of paraconsistency in terms of the rule form ECQ (because of the general validity of ECQ), nor of Bochvar's logic of nonsense (because of LEM holding in the classical extension). Instead, Woodruff's logic should be seen as an ST-consequence relation building on matrices expanding the Weak Kleene matrices.[8]

4.2. **Are we still talking about nonsense?** Now, we proceed to discuss the second major difficulty associated with Woodruff's system. Let us begin by recalling the similarities between the Herzberger semantics and the Woodruff semantics. In both cases, we have a two dimensional semantics where truth and falsity are characterized independently of the second component. The fact that the second component is defined on the top of such an alethic basis that is seen as holding for every sentence is precisely what creates difficulties for it in being interpreted as encoding a sense/nonsense distinction. One of the differences we have pointed to, however, very briefly, is related to the fact that in the Herzberger case, the notions of truth and falsity are completely classical. By having classical truth or falsity, along with Halldén's characterization of nonsense, each proposition is required to have meaning, to be completely meaningful. We have suggested that, given the striking similarities between the two semantics, it could well be the case that the same difficulties plague Woodruff's system. Does it? It is to this point that we now turn.

As we have already hinted at in the introduction, Woodruff's semantics is not, *prima facie*, at least, without problems. There seems to be a kind of tension between the two dimensions of his semantics and his explicit motivation of providing a logic of nonsense along the lines of Halldén. This is the time to check this with some details.

Originally, Halldén's understanding of meaningfulness and meaninglessness required that meaningful sentences are precisely those that can have one of the two classical truth values, true or false, attributed to them, while meaningless sentences are those not truth

[8]Note also that Woodruff's logic can be seen as an expansion of the system explored by Melvin Fitting in [10].

evaluable. As Woodruff explicitly remarks, for Halldén, "[a] sentence is meaningful iff it is either true or false" [24, p.193]. That distinction can be quite explicitly represented in a three-valued semantics, with the classical truth and falsity, on the one side, and with a third truth value on the other. In this approach, 'nonsense' is at the same level as 'meaningful', and they exclude each other. Things are not so straightforward in the context of a two dimensional bivalent semantics, as the one presented by Woodruff (and, for the same reason, by Herzberger): given that in Woodruff's framework *every sentence* is either true or false, every sentence should be meaningful in the required sense of Halldén.[9]

Now, while Halldén's requirement of meaningfulness seems to pull us into one direction, the constructive aspect of Woodruff's system pulls us in another, different direction: given the typical anti-realist character of meaning in such settings, meaningfulness needs to be acknowledged by language users; in other words, there are no meaningful sentences that are outside of our grasp. When it comes to explaining how such basic principle is to be represented in his semantics, Woodruff claims:

> The principle substantive semantical assumption of this paper is the following: to be meaningful is to be known meaningful, and hence meaningfulness should be preserved by the accessibility relation for all wffs. [24, p.196]

In the context of a constructive approach to meaning, this is a quite sensible demand, but unless some care is taken, it generates some tensions when conjugated with bivalence, as it happens in the framework presented. The problem appears to be the following: bivalence, along with the just mentioned semantic principle, seems to imply that every sentence is meaningful, robbing us of meaningless/nonsensical, sentences. On the other hand, requiring that meaning always be acknowledged, there must be some room for sentences whose meaning is not known. How to accommodate such apparently incompatible demands?

The root of the difficulties seems to be found in the fact that in Halldén's original semantics, meaningful and nonsense are exhaustive and exclusive categories, operating on the same level. On Woodruff's approach, however, both pairs of categories 'truth and falsity' and 'meaningful and nonsense' are by themselves exhaustive and exclusive, with the second operating on the top of the former. We now go on to suggest how such

[9] A referee kindly remarked that another way out of the difficulties we are pointing to here could involve a slight modification of Woodruff's original approach by making v_t depend on v_+; that is, we could have v_+ defined for all propositions, distinguishing those that are meaningful from those that are nonsense, and after that v_+ would be defined only for those that count as meaningful. In that sense, meaningful propositions could be completely separated between true or false, but nonsense propositions would not receive a truth value. We would like to thank the referee for the suggestion, although we do not explore it here. In a sense, this could be employed for the purposes of having another constructive approach to nonsense, but it conflicts with our intentions here of preserving bivalence even in a context involving nonsense. See also our remarks in the final paragraph of §3.1.

difficulties may be overcome when an appropriate background on the notions of truth and falsity is presented.

5. An alternative reading with Dummettian flavour

Recall that Woodruff endorses a kind of constructive approach to meaning: meaning is, in fact, recognized meaning, one must exhibit knowledge of meaning. This is a feature of the approach to meaning advanced, among others, by Michael Dummett. As it is widely known, Dummett argues that adopting certain constraints on meaning results in a constructive logic being adopted as the appropriate one to account for the behavior of logical vocabulary. As Dummett [9] famously requires, grasp of meaning must be acknowledged by manifestation of such understanding:[10]

> grasp of the meaning of a mathematical statement must, in general, consist of a capacity to use that statement in a certain way, or to respond in a certain way to its use by others.[11]

Can we make sense of this requirement along with the bivalent setting presented by Woodruff? Although there seems to be some difficult obstacles to make sense of the different requirements, we shall now suggest one possible way out. In fact, there seems to be some good Dummettian motivation for adopting the kind of move advanced by Woodruff. Basically, following Halldén and allowing that some sentences do not receive one of the truth values would conflict with the constructive motivation of the project too. As Dummett explains, given the validity of the double negation of the law of excluded middle in intuitionistic logic, "it is inconsistent to assert of any statement that it is neither true nor false" ([9, p.115]). So, while it is not the case that each sentence is true or false, one cannot literally have sentences having no truth value. How can that be?

The key for a possible solution lies in the consideration that falsity (taken as a possible truth value in the formal apparatus) could be understood as representing the wider category of 'not true'. That is, although falsity is appearing in the model proposed here as a prima facie classical opposite to the truth, falsity is not to be read in a symmetric way to truth, as it is done in a classical setting. As Woodruff [24, p.195] also suggests, such asymmetry allows that falsity represents either something like 'known meaningful established falsity', or else 'not true due to meaninglessness'. Then, non-truth bears an open status, meaning either 'established meaningful falsity' or 'not true because not meaningful'. This makes perfect sense with the requirements above, and also resonates

[10] This goes very closely to the previous quotation by Woodruff, although Woodruff uses the connection between meaning and its knowledge to justify the heredity condition for meaningfulness, not to provide a more elaborate discussion on the motivation for his semantics.

[11] This is a clear heritage of Brouwer's [5, p.90] claim that intuitionism is the view holding that, in particular, "there are no non-experienced truths". Just as truth cannot be hidden from us in a kind of metaphysical realm, meaning cannot be available without a manifest comprehension by us.

quite well with the original semantics for the logic of nonsense, where untruth also has a kind of ambiguous status between meaningful falsity and nonsensical. In Woodruff's semantics, given that every truth is meaningful, and known to be so, those sentences that are not true may be such that they are not true, and meaningful, or not true, and not meaningful (which is reflected by the behavior of the second valuation). The latter, let us emphasize again, would be a sort of non-truth attribution: such sentences are not true, not only because they fail to be true, but also because they fail to be meaningful. That means that untruth covers more ground than falsity. So, although the semantics looks bivalent, it can only be held to be so when the intended meaning of the notion of truth and falsity are properly understood, with untruth playing double duty.

Now, if that is correct, then there is another important difference between the two dimensional semantics provided by Herzberger and by Woodruff, given that the former is classical and the latter constructive. The difference lies in the fact that the universal distribution of truth and falsity in a constructive approach may be compatible with sense/nonsense distinctions being plugged on the top of them, differently from the classical case. So, the underlying alethic basis with which we begin is relevant. This is certainly connected to the fact that on the kind of anti-realist theory of meaning advanced on a constructive basis, nonsense or lack of meaning are typical ingredients, while on a classical setting one has to dissociate truth and falsity from our knowledge of meaning. Perhaps, the very idea of nonsense when the basis is constructive is different from the one in classical case.

6. Concluding remarks

Our aim in this paper, recall, was to highlight some quite interesting contributions by Woodruff to the development of a constructive logic of nonsense. We did that by recalling Woodruff's definitions and the motivations behind his system. Interesting facts about the resulting system, and that we have emphasized, concern the status of the consequence relation obtained and the fact that it is an early instance of a mixed consequence relation; also, we have pointed to the fact that differently from the system that inspired Woodruff, the system that results is a logic of constructive nonsense that is not paraconsistent in the sense of violating the rule form of ECQ, although it violates the formula form of ECQ.[12] We have pointed to those features, and also to some similarities between Woodruff's two dimensional semantics and another prominent two dimensional semantics for the logic of nonsense: Herzberger's semantics. By having such similar semantics side by side, we could discuss a major problem that seems to affect them, viz. by incorporating bivalence, they seem unable to account for the sense/nonsense distinction.

[12] Again, whether violating the formula form of ECQ is enough to make a system paraconsistent is an issue we shall not delve into here.

We have discussed how, using a Dummetian account of meaning one can make sense of the constructive nonsense logic as an account of nonsense, while the same is not the case for the classical, Herzberger semantics.[13] This points to very interesting features of the connection between the approach to truth and meaning in logic, which may have very different outcomes depending on the notion of truth and meaning one is using.

For instance, in discussing the three-valued truth tables for **WK** and **PWK**, Susan Haack [13, p.211] emphatically argues that the usual understandings of the third truth value in terms of 'paradoxical', or 'nonsense', are philosophically unsatisfactory. 'Paradoxical', as a logical tool to accommodate semantic paradoxes is an obvious prey to well-known revenge paradoxes. 'Nonsense' is not in better shape: it just fails in contributing to the original idea of a logical project of systematizing valid inferences; premises and conclusions of an argument are supposed to be meaningful sentences of natural languages. Haack suggests that abandoning bivalence, or two valuedness, then, only promotes obscurity in logic (see again [18] for additional discussion). However, what the above discussions show is that one can keep company with Haack, when it comes to preserve bivalence, and still add on the top of it a sense/nonsense distinction. It all depends very much on the notion of truth that is at stake, and how 'nonsense' is understood. As we have suggested, a constructive approach, following Dummett, when borrowed by Woodruff, may make the Haackianly undesirable 'nonsense' to the well behaved world of two values, without disturbing the goals of logic.

Nota bene: we are neither saying that Woodruff has given the final word on these topics, nor that his system provides a definitive answer to calm the worries of those like Haack, who would prefer bivalence and the corresponding intelligibility that comes with it in place of more wild distinctions such as sense/nonsense. What we claim is that interesting proposals may be available on that front once Woodruff's system is on the table, and one may enjoy the benefits of intelligibility brought by the presence of bivalence and still have the sense/nonsense distinction on the top of it. Maybe Haack is thinking too close to a classical notion of truth and falsity? These are issues we leave unexplored by now, but which certainly deal with the very idea of what a logic is and the basic purposes of developing such systems.

Finally, let us conclude the paper by briefly discussing two directions for future investigations. First, recall that there is another interpretation of nonsense, by building on the framework of the plurivalent semantics, as developed by Graham Priest in [21] (see [18] for a comparison of the interpretations of the infectious value). Then, it will be interesting to explore how that framework can be combined with constructive semantics, and compare it with Woodruff's proposal. Second, recall yet another interpretation of **WK** suggested by Jc Beall in [2] in terms of off-topic. This then seems to invite us to

[13]For further discussions of the classical aspects of Herzberger semantics, see [19, 20].

explore an application of Beall's idea within the context of constructive reasoning. The resulting system may be formulated closer to the style of Herzberger, in the sense that we do not require the relations between two (or more) forcing relations, nor we require the additional clause in the truth condition for the conditional, but formulating one such system and comparing the resulting system with Woodruff's logic seems to be another interesting, as well as promising direction.

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