Tractatus versus Quantum Mechanics

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Abstract This paper is divided in four parts. In the first part we introduce the method of internal critique of philosophical theories by examination of their external consistency with scientific theories. In the second part two metaphysical and one epistemological postulate of Wittgenstein's Tractatus are made explicit and formally expressed. In the third part we examine whether Tractarian metaphysical and epistemological postulates (the independence of simple states of affairs, the unique mode of their composition, possibility of complete empirical knowledge) are externally consistent with the theory of quantum mechanics. The result of the inquiry is negative: Tractarian postulates ought to be be revised. Relying on the result we approach the question of the empirical character of logic in the fourth part. The description of theoretical transformations of the notion of disjunction, in its ontological, epistemological, and logical sense, is a common element of in all parts of the text. The conjecture on the existence of different types of disjunctive connectives in the language of quantum mechanics concludes the paper.

Keywords: external consistency, disjunction, logical pluralism, quantum logic, superposition, *Tractatus logico-philosophicus*

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External consistency as an internal critique of philosophy

It is a well-known fact that quantum theory posses numerous challenges to philosophy. Weizsäcker (2007) uses a metaphor in which the dialogue between physics and philosophy is compared to a trial. With the advent of quantum mechanics the roles in the trial have changed.[1] Traditional philosophy ceases to be the judge and is put in the defendant's chair instead, while the former defendant, physics, now becomes the witness. Weizsäcker's metaphor emphasizes the changed expectations about what contemporary philosophy should do. The common ground is lost and, consequently, the philosophical critique is replaced by a need for dialogue between philosophy and science. Weizsäcker's metaphor implies that there is no external position of certainty to be found in philosophy but rather its postulates of logic, metaphysics and epistemology − as conditions of possibility of scientific knowledge − ought to be (re)constructed in continuous dialogue with science.

The four presuppositions of philosophical tradition embedded in Tractatus

According to our reading, Wittgenstein's *Tractatus logico-philosophicus* ([1921] 1974), which in his own words is a "critique of language", exemplifies the transcendental type of critique: the conditions of possibility of scientific knowledge are delineated from a philosophical position external to science. The core of Tractarian theory is depicted in Figure 1.

The world		**Natural science**
↓ *divides into*		↓ *consists of*
Fact		**True proposition**
↓ *is composed of*		↓ *is composed of*
Atomic fact		**Elementary proposition**
↓ *consists of*		↓ *consist of*
Relation and objects		**Predicate and names**
	← Picture relation →	

Figure 1: Tractarian theory.

Wittgenstein's theory exposed in *Tractatus logico-philosophicus* is beautiful in its apparent simplicity and closeness to the common-sense. Some of its main postulates are listed below and the ones that will be discussed here are marked with an asterisk.

- **Ontological postulates**: The world is that which is the case. The world divides into facts. A fact is an existent state of affairs. (*) States of affairs are mutually independent.[2] (*) States of affairs can be combined only in a "conjunctive way".

- **Epistemological postulates**: A simple sentence is true iff it is a picture of (iff it represents)

1 Weizsäcker writes: "It is understandable that quantum theory provoked a debate over its interpretation. Not only is it incompatible with the world view of classical physics, but also with certain positions of classical metaphysics. [...] one must decide whether to consider this incompatibility as philosophical progress or a weakness of the theory. The present book is based on the conviction that we are dealing with fundamental philosophical progress. According to this conviction, it is not quantum theory that must defend itself before the court of traditional philosophies but those philosophies themselves must stand trial - in itself a philosophical process - with quantum theory in the witness stand" (Weizsäcker, 2007: 243).

2 Tractatus ([1921] 1974): "2.062 From the existence or non-existence of one state of affairs it is impossible to infer the existence or non-existence of another."

an existing fact. (*) It is possible to obtain a complete description of reality.

- **Logical postulates**: The meaning of a sentence is defined by its truth-conditions. (*) A complex sentence is a truth-function of its constituent parts.

In the next section postulates marked with an asterisk will be examined for their external consistency with quantum mechanics. In line with Weizsäcker's metaphor of the trial process, we will consider quantum mechanics to be the witness, that is, a theory to which the Tractarian theory ought to be adapted and not vice versa. This method can be understood as a kind of internal (or deconstructive) critique of philosophy by exposing an external inconsistency of one of its most influential theories.

We will confront the Tractarian postulates with the notion of quantum superposition as admitted by quantum mechanics. A brief exposition of the notion is given below.

> The mathematical representation of an initial quantum state uses a symbol such as $|\varphi\rangle$, and an expansion (summed list) over a series of final quantum states as follows: $|\varphi\rangle = a_1|A\rangle + a_2|B\rangle + a_3|C\rangle + \ldots$ The amplitudes are a collection of complex numbers related to the probability that the initial state $|\varphi\rangle$ will change into *one* of $|n\rangle$ final states as the result of a measurement. (Allday, 2009: 70)
>
> Quantum states can be formed by combining other states in quantum superposition. These combinations can be formed from states that would, classically, be impossible simultaneously. (Allday, 2009: 539)

The quotes contain the following presuppositions: there is a representation relation between the language of mathematical formulas and quantum states; superposition is admitted by the theory although it is a "classically impossible" type of combination of simpler states into complex ones.

Examination of external consistency as a Socratic method

There are methods of science and there are methods of philosophy. Let us call the method of building a theory through dialogue the "Socratic method'. This method requires at least two interlocutors, one of which is a philosopher while the other is an expert in the field that is relevant for the philosophical question under investigation. It is well known that many questions are of common interest both for the scientist and the philosopher. For example, the theories of time and space are presupposed by any empirical theory and yet it was physics that first made explicit the theory of time and space. So, if a philosopher wants to investigate the questions of time and space using the Socratic method, then a physicist is needed for the dialogue to take place. *A theory ought to be consistent* is the first requirement to which theory building is subordinated. *The consistency of a theory ought to be proved* is a closely related requirement and it shows that logical research is necessary part of any theory building. Somewhat less known requirement is: *A theory ought to be externally consistent* or, equivalently, *The union of theories ought to be consistent*.[3] When a philosophical theory is confronted with a scientific theory and examined for its external consistency, then a variant of Socratic method is at work and a dialogue between philosophy and science takes place.

Suppose a philosophical theory is confronted with a scientific theory and external inconsistency has been discovered. Then a revision must take place. Typically, the revision will involve a contraction whereby some of old postulates will be abandoned, and an expansion in which some new ones will be introduced. Like in any revision, the learning occurs and probably to the benefit of both sides.

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3 The notion of external consistency has been introduced by Bunge (1998).

The gain for philosophy is an acquisition of a new theory, superior to the old one with respect to external consistency. The gain for the science is that its philosophical presuppositions will become explicit and thus the implicit general knowledge (which is active in public communication, technical application of science and science education) can be revised and improved.

Tractarian postulates

In this section a formalization for Tractarian picture-relation will be proposed and used in formulation of the independence postulate in metaphysics.

Independence postulate

Definition 1 $W=\{s_1, s_2, \ldots, s_n\}$ is the set of all possible simple states of affairs. $el(L)=\{p_1, p_2, \ldots, p_n\}$ is the set of elementary propositions of the language L .

Postulate 1 *Any* $w \subseteq W$ *is a world.*

Definition 2 *Picture relation* P is a one-to-one relation between the set of atomic propositions and the set of all possible states of affairs. Formally, P is a relation such that conditions (1)-(4) hold.

$$el(L)=\{x : \exists y \quad xPy\} \tag{1}$$

$$W=\{y : \exists x \quad xPy\} \tag{2}$$

$$\forall x \forall y \forall z ((xPz \wedge yPz) \rightarrow x=y) \tag{3}$$

$$\forall x \forall y \forall z ((xPy \wedge xPz) \rightarrow y=z) \tag{4}$$

Each simple state of affairs is pictured by exactly one elementary proposition and, conversely, each elementary proposition pictures exactly one simple state of affairs.

Definition 3 Reality is the function $real : el(L) \times 2^W \rightarrow \{true, false\}$ which determines the truth-value of each elementary proposition $p \in el(L)$ in any world $w \in 2^W$: $real(p, w)=true$ if pPs for some $s \in w$, and $real(p, w)=false$ otherwise.

An immediate consequence of Postulate 1 is Theorem 1 on absence of mutually exclusive simple states of affairs.

Theorem 1 *No simple state of affairs excludes no simple state of affairs.*

Proof. Suppose that states of affairs s_1 and s_2 are mutually exclusive. Then no world contains both state of affairs s_1 and state of affairs s_2 . This is not possible since any subset of W is a world and so $\{s_1, s_2\}$ is a world.

Unique mode of composition of states of affairs into complexes

Although there is no explicit statement in *Tractatus* on how states of affairs can be combined, it is at least consistent to assume that *Tractatus* metaphysics accepts the principle of unique mode of composition of states of affairs into complex facts. Explicit formulations of the principle on the unique mode of composition can be found in the literature. E.g., philosopher Armstrong (1997) accepts conjunctive mode and rejects negative and disjunctive modes of composition of states of affairs:

> We require [...] a distinction between atomic and molecular states of affairs. Because negative and disjunctive states of affairs will be rejected, molecular states of affairs are all of them mere conjunctions of atomic states of affairs. (Armstrong, 1997: 241)

Postulate 2 *The unique mode of composition of states of affairs is the conjunctive mode.*

The mereology of complete knowledge

The third postulate concerns language and ascribes unlimited expressive power to it.

Postulate 3 *The complete description of reality is possible.*

The notion of complete description corresponds to the notion of reality. Complete description of a world w can be equated with a set of elementary propositions and their negations, i.e. with a subset of literals of L.

Definition 4 A complete description $d(w)$ of a world w is the smallest set of literals such that $p \in d(w)$ if pPs for some $s \in w$ and $\neg p \in d(w)$ otherwise.

Complete description is consistent since it cannot contain both an atom and its negation. Complete description determines the truth-value of any proposition. Symbolically, for any sentence $\varphi \in L$ it holds that either $\varphi \in Cn(d(w))$ or $\neg\varphi \in Cn(d(w))$, where $Cn(d(w))$ denotes the set of logical consequences of $d(w)$. Any complete description $d(w)$ is a minimal set, that is, there is no proper subset of it which is complete.

Remark 2 It should be noted that if a complete description $d(w)$ contains a disjunction, it also contains at least one of its disjuncts. The proof for this well-known fact will be omitted here.

One can safely assume that the implicit Tractarian linguistic postulate on the possibility of complete description is a necessary condition for the tacit Tractarian epistemic postulate on possibility of complete knowledge. If the world described by the complete description is the real one, then its complete description is called "natural science". The relevant Tractarian thesis is:

> 4.11 The totality of true propositions is the whole of natural science (or the whole corpus of the natural sciences).
> (Wittgenstein [1921] 1974)

The tacit Tractarian epistemic thesis is made explicit in Postulate 4.

Postulate 4 *The complete description of the world can become known.*

Proposition 1 *The complete knowledge of the world is possible.*

Proof. Suppose that the complete description of the world is known. For any $\varphi \in L$ either $\varphi \in Cn(d(w))$ or $\neg\varphi \in Cn(d(w))$ since $d(w)$ is complete. By epistemic necessitation and epistemic distribution (K axiom) the consequences of known truths are known truths.[4]

4 The notion of knowledge used here is not an empirical notion in which computational properties, such as available memory space and computation time, must be taken into account. To avoid misunderstanding the term "knowledge" can be replaced by "semantic information" (van Benthem, 2011).

Proposition 2 *If knowledge is complete, then a disjunction being known implies a disjunct being known.*

Proof. Using epistemic modal operator K the sentence "It is known that φ " can be written down as $\mathrm{K}\varphi$. Proposition 2 is symbolically noted as (5).

$$\text{If } \mathrm{K}p \vee \mathrm{K}\neg p \text{ for all } p \in el(L) \text{, then } \mathrm{K}(\varphi \vee \psi) \text{ implies } \mathrm{K}\varphi \vee \mathrm{K}\psi \text{ for all } \varphi, \psi \in L . \tag{5}$$

The proof for the proposition relies on the deductive closure property of knowledge understood as semantic information, namely, it depends on K axiom by which the consequences of known truths are known truths.

Remark 3 The idea on existence of implication from disjunction to at least one of its disjuncts is not restricted to the realm of complete knowledge but also appears in dialogue rules and in mathematical intuitionistic logic. Dialogical reconstruction of logical intuitions began with Paul Lorenzen. Lorenzen's logic of argumentation results in same theorems as intuitionistic logic does (Krabbe, 2006). Dialogical disjunction is the right of the opponent to seek from the proponent the justification for one disjunct chosen by the proponent. Such disjunction obviously corresponds to "effective disjunction" of intuitionistic logic and its constructive requirement that the proof of disjunction requires the proof of at least one disjunct .

External inconsistency of Tractarian theory with quantum theory

Neither of the here discussed Tractarian postulates withstands the confrontation with the theory of quantum mechanics. If the whole consisting of scientific and philosophical knowledge is subordinated to the requirement of the external consistency of its parts, as our method assumes, then revision either of Tractarian or quantum theory is needed, and, by the method accepted in this paper, it is the former that ought to be revised.

Are states of affairs mutually independent?

Let us identify some state of affairs with the descriptum of the "basic physical propositions". Basic physical propositions have the form $m(e)=r$ with the meaning "the magnitude m has the value r for the object e ". Here we partially follow Putnam's terminology who writes "Statements of the form $m(e)=r$ *- the magnitude* m *has the value* r *in the system* E *-* are the sorts of statements we shall call basic physical propositions..." (Putnam, 1975:178). The descriptum of basic physical proposition is some atomic, non-conjunctive state of affairs. Basic physical propositions are measurement statements and therefore they exploit some abstract measuring structure, for example, a numerical structure. In the measurement, some relations of the measuring structure are used for the description of the measured empirical structure (Suppes and Zinnes, 1963). Therefore, measurement statements cannot be treated as atomic in the proper sense.

The basic physical proposition $m(e)=r$ is not about a relation in the world, it does not describe an atomic state of affairs consisting of a functional relation m that connects a physical object e to an element r of the measuring structure. Birkhoff and von Neumann made the similar point: "how absurd it would be to call an *experimental proposition*, the assertion that the angular momentum ... of the earth around the sun was at a particular instant a rational number" (Birkhoff and von Neumann, 1936: 825). An atomic state of affairs is not a relation between elements of the two structures, empirical and abstract, but the state thus described.

Let us denote by $des(m(e)=r)$ the atomic state of affairs that is the *descriptum* of (i.e., that which is described by) a basic physical proposition $m(e)=r$. Let $Pr(m,e)$ be the set of all propositions describing possible atomic states of affairs with respect to magnitude m and object e.

Examples. Assuming that position is a primitive magnitude, the set $Pr(position,e)$ could be understood as the set of all propositions having the form $position(e)=\langle r_1,r_2,r_3\rangle$ where the set of triples of real numbers performs the role of the measuring structure. Assuming that having vertical polarization is a primitive magnitude, the set $Pr(vertical\, polarisation,e)$, if non-empty, will contain exactly two propositions, e.g., $vertical\, polarisation(e)=1$ and $vertical\, polarisation(e)=0$.

In order to restate the independence postulate in terms of *descripta* of physical propositions, some additional technical terms will have to be introduced. First, let choice function ch be the function that picks exactly one proposition from any non-empty set $Pr(m,e)$ for each m and e. Second, denote by c the set consisting of at most one proposition for each magnitude and each object, i.e., $c=\{ch(Pr(m,e)): Pr(m,e)\neq\emptyset$ for some m and $e\}$. Third, enrich the language with non-primitive or derivative magnitudes f which take the values of primitive magnitude functions as their arguments, thus obtaining non-basic propositions of the form $f(m_1(e),\ldots,m_n(e))=r$. Denote by Fc the set of non-basic propositions built over c so that if f is an *n*-place derivative magnitude and $m_1(e)=r_1$,..., $m_n(e)=r_n$ are propositions in c, then $f(m_1(e),\ldots,m_n(e))=r$ is a proposition in Fc. Finally, the union of the chosen basic propositions and their related non-basic propositions $c\cup Fc$ gives the description of a "magnitude world".

Now the independence postulate (Postulate 1) can be restated as the claim about magnitude world. Denote by $des(c\cup Fc)$ the set of descripta of propositions in $c\cup Fc$. The alleged postulate would be: *Any* $des(c\cup Fc)$ *is a world*. Surprisingly, by the uncertainty principle, the alleged postulate fails since there can be no world having exactly one *descriptum* of a position proposition and exactly one *descriptum* of a momentum proposition.

Postulate 5. For any two *descripta* of a position proposition $des(position(e)=r_1)$ and a momentum proposition $des(momentum(e)=r_2)$ there is no world that contains them.

Postulate 5 shows that Heisenberg's uncertainty principle allows only for those possible worlds that are either *incomplete* (gappy worlds, lacking at least one *descriptum* for propositions from $Pr(m,e)$ for some m and e) or *overfull* (glutty worlds, having either at least two *descripta* for propositions from $Pr(m,e)$ or a new type of *descriptum* corresponding to at least two propositions from $Pr(m,e)$ for some m and e. On the other hand, application of Postulate 1 to magnitude world requires exactly one descriptum for each magnitude of an object. Therefore, there is an external inconsistency between Tractarian explicit metaphysics and Heisenbergian implicit metaphysics.

Is there a unique mode of composition?

Suppose one wants to save representation relation (picture relation) between the language and the world in the face of the fact that quantum disjunction is not a truth-function of its disjuncts.[5] If the

5 See the last section of this paper for the discussion on quantum disjunction.

quantum world is gappy, there will be no state of affairs to represent. So, one must turn to glutty worlds in order to find the *representandum* of the quantum disjunction. Denote by $des(p \vee_Q p')$ the disjunct state of affairs described by quantum disjunction of propositions p and p' describing mutually exclusive or orthogonal states. It should be noted here that orthogonality does not mean alternation ("... just in case that not ...") since for some magnitudes the number of orthogonal states of the same object can be infinite, while in classical logic mutual exclusiveness comes in pairs.[6] According to quantum mechanics disjunctive states of affairs are possible: they are either tacitly permitted as in "orthodox interpretation", or at least constitute the ontological commitments of the theory in the realistic interpretation of mathematical representations used by the quantum mechanics.[7]

The disjunctive mode of composition of states of affairs is forbidden by Tractarian postulate of unique mode of composition (Postulate 2). Quite the opposite holds true in the quantum mechanics: it allows mutually exclusive states of affairs involving the same object and the same magnitude to co-exist.[8] If the co-existence of orthogonal states is admitted, then a different kind of picture relation must be introduced to account for possibility of representation of a glutty world. Since complete worlds of the type $des(c \cup Fc)$ are impossible by uncertainty principle and glutty worlds are required in the realistic interpretation of quantum superposition, the concepts of the world and of the reality function must be accordingly redefined. In the set-theoretic modeling used in this paper, this means that for some set $Pr(m,e)$ of propositions representing possible magnitude values of an object, a possible world will have one and the same *descriptum* for a set of at least two propositions. Skipping the technical details, assume as a simple modeling solution that a glutty world w_Q contains some state of affairs s which is an at least two-membered subset of *descripta* of magnitude propositions, namely, $s \subseteq \{des(p): p \in Pr(m,e)\}$ and $2 \leq |s|$. Consequently the reality function must be appropriately redefined, see Definition 5.

Example. The two propositions $spin(e)=up$ and $spin(e)=down$ about the spin of an electron e describe mutually exclusive states of affairs. The *descriptum* of quantum disjunctive proposition will be denoted here by $des(spin(e)=up \vee_Q spin(e)=down)$ and given the fact that disjuncts describe orthogonal state it will represent a *disjunctive state of affairs* or a superposition of states in a glutty world.

Definition 5 $real(p \vee_Q q, w)=true$ iff pPs or qPs or $(p \vee_Q q)Ps$ for some $s \in w_Q$, and $real(p \vee_Q q, w)=false$ otherwise.

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6 Compare the following citation and interpret "independent" as "both cannot be simultaneously observed": "*Orthogonality* between rays is an important concept for quantum mechanics: orthogonal rays refer to states that are independent of one another. The different possible position states of a particle are all orthogonal to one another, as are all possible momentum states. But position states are not orthogonal to momentum states. ... The general rule *R* for a measurement (or observation) requires that the different aspects of a quantum system that can be simultaneously magnified to the classical level--and between which the system must then choose--must always be orthogonal" (Penrose, 1999: 336).

7 The orthodox or realistic interpretation can be found in this quotation if we take the expression "not anywhere" to mean "at no particular position but nevertheless somewhere": "Suppose I do measure the position of the particle, and I find it to be at the point *c*. Question: Where was the particle just before I made the measurement? ... [...] 2. The orthodox position: The particle wasn't really anywhere. It was the act of measurement that forced the particle to *take a stand* (though how and why it decided on the point *c* we dare not ask). [...] This view (the so-called Copenhagen interpretation) is associated with Bohr and his followers. Among physicists it has always been the most widely accepted position" (Griffiths, 2012:89).

8 An example from the literature: "In quantum mechanics ...the state of a system is represented by an element in a complex Hilbert space ... (which is a vector space, equipped with a scalar product, that is complete in the norm induced by the scalar product). In particular, this means that for any two states (e.g., for a spin-1/2 system, the states of spin-up and spin-down in direction *x*), an arbitrary linear combination (or *superposition*) is also a possible state: $|\varphi\rangle = \alpha|+x\rangle + \beta|-x\rangle$ " (Bacciagaluppi, 2009:52).

Is the best possible knowledge logically complete?

In 1935. Schrödinger's formulated the mereological principle of best possible knowledge:

> Best possible knowledge of a whole does not include best possible knowledge of its parts ... (Schrödinger, [1935] 1980: 338)

The best possible knowledge is not complete since, inter alia, it is possible that maximal knowledge of a disjunction does not imply knowledge of any of its disjuncts provided that the disjunction in question is a quantum one. Knowledge can be the best possible even if no disjunct of a known disjunction is known. So, the epistemic situation described by proposition (6) below can occur in the best possible knowledge in Schrödinger's sense, reading $K_Q\varphi$ as "in the best possible knowledge in Schrödinger's sense it is known that φ ".

$$K_Q(p \vee_Q q) \wedge \neg K_Q p \wedge \neg K_Q q \tag{6}$$

The rejection of Tractarian possibility of complete knowledge (Proposition 1) is a corollary of the theses on the possibility of superposition and its representation (see Definition 5).

Suppose that for some glutty world w_Q contains the descriptum $des(p \vee_Q p')$ for propositions that represent orthogonal states $des(p)$ and $des(p')$. The best possible knowledge of world w_Q implies the absence of knowledge whether p or p' hold. So, the best possible empirical knowledge is not a complete knowledge: if a disjunction with propositions describing orthogonal states is true in a glutty world, none of its disjuncts can be known.[9]

Is logic prior to any experience?

In Tractatus propositions are understood as truth-functions while logic is conceived as a condition of possibility of any experience of reality.

> 5. A proposition is a truth-function of elementary propositions. (An elementary proposition is a truth-function of itself.)
> 5.552 [...] Logic is prior to every experience--that something is so.
> 5.557 [...] ...logic and its application must not overlap.
> (Wittgenstein, [1921] 1974)

Non-truth functional character of quantum disjunction

The semantic definitions for connectives of the quantum logic can be obtained by taking " φ is true" to mean "the vector that represents the system lies within the subspace that represents φ " and by associating each quantum *n*-ary connective with a function from *n* subspaces to a subspace of the high-dimensional complex vector space (Maudlin, 2005). Subspaces can be partially ordered by inclusion since it is a reflexive, transitive and antisymmetric relation. Since any non-empty finite set of subspaces has the lowest upper bound (supremum) and the greatest lower bound (infimum), the partial ordering forms a lattice. In their seminal paper on quantum logic Birkhoff and von Neumann (1936) identify this ordering relation with the consequence relation.[10]

9 Putnam draws the same conclusion but from different premises: "A system has no complete description in quantum mechanics; such thing is a logical impossibility, [...]" (Putnam, 1975:185).

10 Implication is a partial ordering: "... the experimental propositions concerning a system S correspond to a family of subsets of its phase-space Σ , in such a way that "*x* implies *y*" (*x* and *y* being any two experimental propositions) means that the subset of Σ corresponding to *x* is contained set-theoretically in the subset corresponding to *y*" (Birkhoff and von Nemann, 1936: 827). Implication structure forms a lattice: "In any calculus of propositions, it is natural to imagine that there is a weakest proposition implying, and a strongest proposition

Quantum conjunction is an easy case. It is interpreted as the intersection of subspaces and consequently, the logical behavior of the quantum conjunction is analogous to the classical conjunction. The hard case is quantum disjunction, which is true iff the vector that represents the system lies within the subspace that represents the disjunctive proposition $\varphi \vee_Q \psi$. In other words, the quantum disjunction is true iff the vector that represents the system lies within the subspace that is *spanned* by the subspace that represents the proposition φ and the subspace that represents the proposition ψ .[11] The logical behaviour of the quantum disjunction *is not* analogous to the classical one. Quantum disjunction is not just the union of the subspaces associated with the disjuncts but their span. Therefore, it is possible for a disjunction to be true while none of disjuncts is true. This fact makes quantum disjunction similar to classical proof-theoretical notion of disjunction: provability of a disjunction does not imply provability of at least one disjunct. On the other hand, the quantum disjunction is not similar to the intuitionistic proof-theoretical notion of disjunction or to the dialogue-logic disjunction. In intuitionistic logic provability of a disjunction demands provability of at least one disjunct, while in dialogue logic the disjunction stands for the right of the opponent of the disjunction to seek from the proponent the justification for one disjunct chosen by the proponent.

Suppose that propositions φ and ψ are contradictory so that one is observed just in case the other is not observed. Adopt the realistic interpretation under which it makes sense to claim that the quantum disjunction $\varphi \vee_Q \psi$ has a truth-value prior to observation and that it is true. According to the semantic definition, the vector V that represents the system must lie in the subspace $||\varphi \vee_Q \psi||$. If $V \notin ||\varphi|| \cup ||\psi||$, then $\varphi \vee_Q \psi$ is true although neither φ nor ψ is true. Since, under realistic interpretation, the vector V represents the system, i.e. the reality, then there must be some state of affairs s that is represented by the vector., but s makes neither φ nor ψ true since $V \notin ||\varphi|| \cup ||\psi||$. However, by the initial, supposition if φ is not true, then ψ must be true and vice versa. So, s makes contradictories true, which is impossible and, therefore, it does not exist. However, s is represented by V and, therefore, must exist. There are different ways of avoiding paradox by revising Tractarian postulates. For example, one can revise ontology by allowing disjunctive composition of states of affairs, or revise epistemology and reject realism, or revise logic by admitting connectives which are not truth-functional. If we keep quantum logic and opt for realism regarding the possibility of representing (the possibility of truth) of the sentence describing a non-observable states of affairs, then we are forced to revise our ontology and logic.

For the anti-realist the quantum disjunction can be regarded as an "inference ticket" allowing the deduction from " $\varphi \vee_Q \psi$ holds" to "the measurement will show that either φ or ψ is the case". On the other hand, the realist must give an account of truth-makers of the quantum disjunction. A disjunctive state of affairs or disjunctive fact can play the role of a truth-maker for quantum disjunction. Not everybody accepts the disjunctive type of composition of states of affairs or superposition. The most famous opposition to the ontology of disjunctive states was expressed in Schrödinger's thought experiment about a cat that is both alive and dead. Notwithstanding the persuasiveness of Schrödinger's fable, the price for rejecting the ontology of disjunctive states ("blurred model") is anti-realism. Beyond the realist vs. anti-realist dispute, the admission of the quantum disjunction together with its semantics defined over high-dimensional complex vector space, forces the rejection of the Tractarian logical postulate, namely that complex sentence is a truth-function of its constituent parts.

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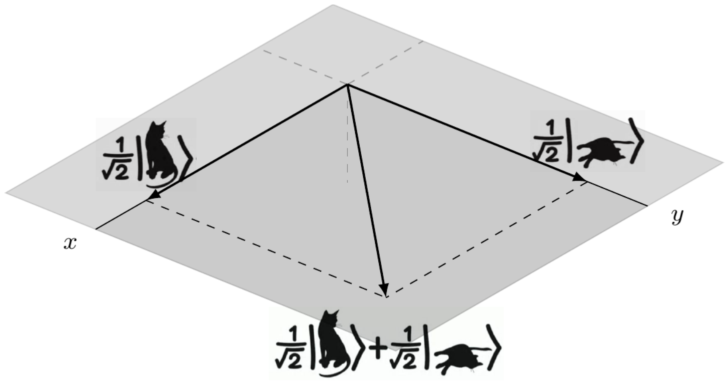
implied by a given pair of propositions" (Birkhoff and von Nemann, 1936: 827).

11 Denote the subspace that represents proposition φ by $||\varphi||$. The subspace $||\varphi||$ is a set of vectors. The span of two subspaces $span(||\varphi||, ||\psi||)$ is the set of all linear combinations of their vectors: $span(||\varphi||, ||\psi||) = \{x + y : x \in ||\varphi|| \wedge y \in ||\psi||\}$. Finally, $||\varphi \vee_Q \psi|| = span(||\varphi||, ||\psi||)$.



$$\|\text{Alive} \vee_Q \text{Dead}\| = \{x + y \mid x \in \|\text{Alive}\| \wedge y \in \|\text{Dead}\|\}$$

"One can even set up quite ridiculous cases. A cat is penned up in a steel chamber, along with the following device (which must be secured against direct interference by the cat): in a Geiger counter, there is a tiny bit of radioactive substance, so small, that perhaps in the course of the hour one of the atoms decays, but also, with equal probability, perhaps none; if it happens, the counter tube discharges and through a relay releases a hammer that shatters a small flask of hydrocyanic acid. If one has left this entire system to itself for an hour, one would say that the cat still lives if meanwhile no atom has decayed. The Ψ -function of the entire system would express this by having in it the living and dead cat (pardon the expression) mixed or smeared out in equal parts. It is typical of these cases that an indeterminacy originally restricted to the atomic domain becomes transformed into macroscopic indeterminacy, which can then be resolved by direct observation. That prevents us from so naively accepting as valid a "blurred model" for representing reality. In itself, it would not embody anything unclear or contradictory. There is a difference between a shaky or out-of-focus photograph and a snapshot of clouds and fog banks." (Schrödinger, [1935] 1980)

Figure 2: The famous Schrödinger's thought experiment coupled with an illustration for geometrical semantics of quantum logic.

Logical pluralism

The maintenance of external consistency between logic and quantum mechanics requires of logic to take into account empirical considerations. An early account of the new, non-Tractarian conception of logic was given by Jan Łukasiewicz who famously put the thesis on empirical character of logic in 1936:

> All a priori systems, as soon as they are applied to reality, become natural-science hypotheses which have to be verified by facts in a similar way as is done with physical hypotheses. (Łukasiewicz, [1936] 1970:233)

The thesis on empirical character of logic seems to be self-evident from the pluralistic standpoint in philosophy of logic: if there is a multitude of conflicting logical systems, then some external criterion must be used to decide which system to use or, rather, to develop.[12] Applying and modifying Łukasiewicz's thesis one can arrive at the following reformulation: laws of classical logic, such as distributivity of conjunction over disjunction, as soon as they are applied to the model of quantum reality, become natural-science hypotheses and share their destiny.

Two roles of logic in the development of scientific theory can be differentiated. First, a formal model of some part of reality is built relying on some founding logical theory, e.g., classical logic, and, second, the semantics of sentences about that part of reality is defined over its model. Thus there are two logics, the one *before the model*, i.e., the logic used in building it, and the one *after the model*, i.e., the logic of sentences whose meaning is defined over the model. Quantum logic as a logic coming after the model becomes an empirical theory if its semantics is defined over the

12 Putnam has noticed that redefinition of the position of geometry within the unified science provides a clear historical precedent, analogous in many ways to repositioning of logic: "It makes as much sense to speak of *physical logic* as of *physical geometry*. We live in a world with a non-classical logic" (Putnam 1975:184).

model of reality. Thus, the logic after the model is a theory whose claims are refutable on external grounds. Some of the logical laws, such as distributivity of conjunction over disjunction, may fail because they do not apply to the model. Putnam's (1975) claim that quantum logic is *the* logic is challenged by the fact of logical pluralism, the logic used in building the model need not be the same as the logic read-off the model.

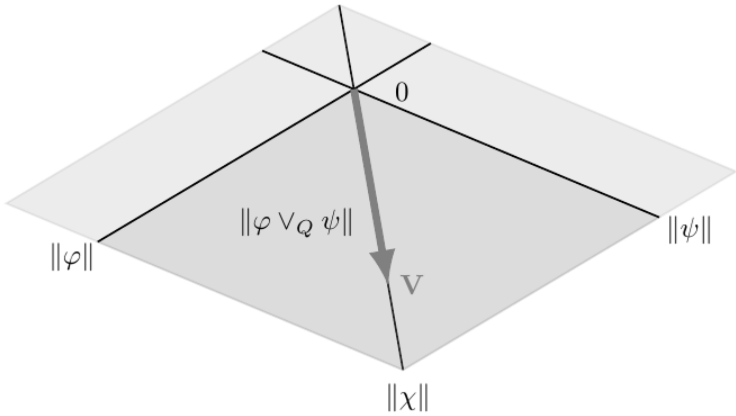


Figure 3: A counterexample for distributivity in two-dimensional "geometrical semantics": the three black lines represent propositions φ, ψ, χ ; the gray arrow **V** represent the system; the plane represents proposition $\varphi \vee_Q \psi$. The origin point 0 represents necessarily false proposition since zero-dimensional point cannot contain any vector. Proposition $\chi \wedge_Q (\varphi \vee_Q \psi)$ is true since $V \in \|\chi \wedge_Q (\varphi \vee_Q \psi)\|$. Proposition $(\chi \wedge_Q \varphi) \vee_Q (\chi \wedge_Q \psi)$ is false since $\|\chi \wedge_Q \varphi\| = \|\chi \wedge_Q \psi\| = 0$ and $\|(\chi \wedge_Q \varphi) \vee_Q (\chi \wedge_Q \psi)\| = 0$.

Focusing on logic as a theory after the model, the question arises as to whether quantum logic is the logic of the language of quantum theory. If so, the quantum disjunction would be the disjunction of quantum theory. If not, then we must allow more than one disjunction-like connective in the logic of the language of quantum theory. How many disjunctions do we need for the language of quantum mechanics? We are inclined to go along with Maudlin's remark that the language of a theory may use more than one logical term from the same category.

> Using the usual technique for deriving predictions from quantum states, one would say that if "The particle is at r_1 $\cup$ the particle is at r_2 " is true, then an experiment designed to locate the particle will be certain to "find" it either at r_1 or at r_2 , where the "or" is the classical disjunction. That is, the truth of the join of the propositions implies the truth of a classical disjunction regarding the result of a "measurement". This is an implication from a proposition using a quantum connective to a proposition formulated with a classical connective − it is an inference that cannot even be formulated if the classical disjunction is unavailable. A fortiori it cannot be an inference which could in any way suggest that the classical connective is expendable. (Maudlin, 2005:172)

From the pluralistic standpoint it is acceptable to permit the two disjunction-type operators for the language of quantum mechanics, both quantum disjunction and classical disjunction. This suggests a richer notion of logical pluralism: not only (i) logical terms can differ in meaning with respect to theories or discourses, but also (ii) similar logical terms can have different meaning in the language of one and the same theory. The language of quantum dynamic logic developed by Baltag and Smets (2012) can accommodate the varieties of disjunction like operators within the same formal system. In their approach non-classical features of the quantum behavior are explained by "non-classical nature of quantum-information-extracting actions (quantum tests)" (Baltag and Smets, 2012:771). [13]

13 This work is based on the talks given in 2013 and 2014 at "Physics and Philosophy" conferences at the University of Split. We wish to thank the conferences participants for questions, comments and criticism, in particular to Gabriela Bašić, Ljudevit Hanžek, Luca Malatesti, Tim Maudlin, Višnja Maudlin, Franjo Sokolić, Marko Uršič and

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