

Logical Consequences
Theory and Applications:
An Introduction

Luis M. Augusto

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Preface

Wishing to anchor my research more and more in logic, pure or applied (where *or* is to be read in the logical, inclusive, sense), I aimed to increase substantially my knowledge of (classes of) logics. For both the pure and applied aspects I had in mind, tackling the central notion of logical consequence appeared as the best way to achieve my objective.

My work in this topic benefited from a short stay in early 2015 as a visiting researcher at the University of Barcelona, where, at the libraries of the Faculties of Mathematics and of Philosophy, I was able to collect much of the material that allowed me to take the turnstiles by the horns.

Melvin Fitting, Peter Schotch, and Yde Venema read parts of the manuscript and their observations helped me to improve it considerably. My sincere thanks to them.

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