Many-Valued Logics

A Mathematical and Computational Introduction

Second Edition

Luis M. Augusto

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Preface to the 1st edition

Although the title of this book indicates two main components, to wit, mathematical and computational, we wish to emphasize that this is first and foremost a book on many-valued logics. As such, the reader who might just be interested in the various many-valued logics will find abundant material in that which is the central part of this book, to wit, Part II. In this, the several main many-valued logics are presented from the viewpoints of their semantical properties, and their semantics are characterized mostly by means of truth tables and logical matrices. From the proof-theoretical viewpoint, for many of these logics partial or full axiomatizations are provided.

This Part II is, mainly for the objective of self-containment, preceded by an extensive introduction to "things" logical: logical languages, systems, and decisions. In this Part I, important notions such as logical consequence, adequateness of a logical system, and matrix semantics, among others, are rigorously defined. Further notions regarding decision procedures in classical logic are introduced, namely the satisfiability and validity problems (abbreviated as SAT and VAL, respectively), and they are put into relation with the several proof systems available. In Part III, we chose to approach the many-valued satisfiability problem (MV-SAT) via refutation, i.e. by proving unsatisfiability of a set of formulae (a theory), because of the many advantages, computational and other, that this proof procedure presents for the many-valued logics. This explains our extensive discussion of the analytic tableaux and resolution calculi for classical logic in this Part I. However, we wanted the reader to acquire a good grasp also of validity- and satisfiability-testing methods, and we thus provide sufficient material on this topic.

Part III deals with well-studied computational, or potentially computerizable, approaches to the many-valued logics, namely as far as automated deduction is concerned. Here, signed (clause) logic has a central place (Chapter 6). In effect, both signed tableaux and signed resolution are automatable proof calculi that can very naturally be applied to many-valued logics, reason why we chose to elaborate on them at length. This elaboration constitutes Chapters 7 and 8, for signed tableaux and

signed resolution, respectively.

We did not wish to emphasize the mathematical component, as this could scare away many a potential reader. Nevertheless, we believe that logic is a branch of mathematics, and as such it is both mathematically motivated and justified. (Incidentally, we do not think that logic is a foundation for mathematics–the reverse is true, in our opinion.) Given these two somehow conflicting views, we decided that the best thing to do would be to provide the mathematically literate, or just simply interested, readers with an Appendix in which the mathematical bases of our approach are expounded to the required extent. This is Part IV.

Logic is a subject that requires (many) years before one grasps satisfactorily what it is actually all about. This stage will arguably not be reachable without hands-on practice, much in the same way, perhaps, that many other fields require extensive, often arduous, practice. In this belief, we provide the readers with a large selection of exercises.[1]

In effect, this book is conceived for the reader who wishes to *do* something with many-valued logics, especially–but by no means only–in a computational context. With this we associate fields as diverse as cognitive modeling and switching theory. The literature on many-valued logics is abundant, but the "standard" monographs (Bolc & Borowik, 1992; Malinowski, 1993; Rescher, 1969; Rosser & Turquette, 1952), while not being obsolete, are now of mainly theoretical and historical interest. Some more recent work claiming to have practical applications in mind (e.g., Bolc & Borowik, 2003) follows to a great extent in the footsteps of these earlier monographs. On the other hand, recent work both in mathematical logic (e.g., Gottwald, 2001; Hähnle, 2001) and in philosophical logic / the philosophy of logic (e.g., Bergmann, 2008) provides little or no material for the practical applications mentioned.

Surprisingly–or just plainly intriguingly–, the last comprehensive discussion of modern applications of many-valued logics dates from 1977 (Dunn & Epstein). The fact is that many-valued logics are today in more demand than ever before, due to the realization that inconsistency in information (and knowledge bases are frequently inconsistent) "is respectable and is most welcome" (Gabbay, 2014). We now know and accept that it is often the case that theories have truth-value gaps (some propositions appear to be neither true nor false), truth-value gluts (some propositions appear to be both true and false), vague concepts (e.g., cold, young, tall, sufficient), indefinite functions, lacunae, etc. Moreover, theories are in constant updating processes, and human and arti-

[1] For reasons of timing, solutions to the exercises are not provided in this edition. In due time, there will be an Appendix with these solutions, which may actually be (also) posted online.

ficial reasoners require formal means to keep pace with these often fast and/or unpredictable processes, namely in order to review beliefs: what yesterday was a true/false proposition may today be neither true nor false, or both; what yesterday was clear-cut is now vague or indeterminate. The many-valued logics provide us with a powerful formalism to tackle these cases in terms of reasoning. In fact, this scenario calls for not only a practical-based approach to the many-valued logics per se, but also for hybridizations of these with other formal systems, as well as with more quantitative(-like) cognitive approaches (e.g., D'Avila Garcez, Lamb, & Gabbay, 2009). Importantly, the practical applications of many-valued logics have now gone beyond the mere industrial applications (e.g., fuzzy logics in washing machines), and promise to be of import for "higher," cognitive modeling. This *new logic* entails, in particular, a reappraisal of both "psychologism" and "formalism"–if not also of "logicism"–, especially because it proposes a reappraisal of what a cognitive, or reasoning, agent is (see, e.g. Gabbay & Woods, 2001). In order for this emerging work (Gabbay & Woods, 2003; 2005) to be further carried out and to incorporate the many-valued logics, we need a solid mathematical-computational grasp of these logics.

We hope the present book will provide the motivation for a new comprehensive treatment of the modern applications of many-valued logics that will include the *new-logic* factor as a central aspect of modern logic and its applications. Thus, a book like this on many-valued logics is now very timely, for *the times, they are a-changin'*.

I wish to thank Dov M. Gabbay for including this book in the excellent Studies in Logic series of College Publications. My thanks go also to Jane Spurr for a smooth publication process.

Madrid, Summer 2017

Luis M. S. Augusto

Preface to the 2nd edition

Generally, the present 2nd edition corrects detected addenda and errata, uniformizes the notation, has improved figures, and introduces and/or defines some concepts that were either missing or not adequately defined in the 1st edition (e.g., *quantifier closure*, *parameter*, *free for*, *fuzzy set*).

The most significant change was in the notation. Because in this book there is a large variety of logical systems with corresponding object languages a meta-metalanguage was thought useful for utmost generality. This meta-metalanguage is easily made evident by the use of Greek letters for formulae and sets of formulae; the use of the same Roman letters for all the object languages gives homogeneity to the metalanguage-level presentation of the diverse object languages. Still with respect to this aspect, a new Proposition (3.4.7) was introduced, as it helps to clarify what appears to be, perhaps mostly for historical reasons, an inevitable notational ambiguity in modern first-order logic.

The Index was completely revised with the aim of being as comprehensive as possible. This said, only for names of authors are pages given exhaustively; for the concepts, only the page in which they are defined is given, unless a concept has more than one definition or occurs in distinct contexts.

Although the original structure of the book is unaltered (i.e. no chapters, sections, etc. were deleted or newly added), throughout the main text changes–often substantial–were made with enhanced clarity in view, so that paragraphs or sentences in the first edition might have been significantly changed or even entirely deleted, while new paragraphs were added. In particular, many new propositions and examples were added in this edition. This, together with some typographical improvements, contributed to the increase in the number of pages.

Another innovation is the isolation of algorithms in shaded boxes. In this edition, I dispense entirely with pseudo-code for the formulation of algorithms, preferring a simple listing or enumeration of steps or rules to be applied. The readers can easily turn this formulation into their own (pseudo-)code.

Finally, for the same reason invoked in the 1st edition, no solutions are given for any of the exercises.

I acknowledge the readiness of College Publications to publish the present 2nd edition.

Madrid, June 2020

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Index

This index is a hybrid of an *Index nominum* and an *Index rerum.* With respect to the latter, only the most general, positive, concept is given; for example, for *(un)satisfiability* and *(un)satisfiable*, look under *satisfiability.* (An exception is *indeterminacy.*) For complex expressions (e.g., *many-valued satisfiability problem*), look under the final noun (e.g., *Problem, Many-valued satisfiability*). Exceptions are complex expressions beginning with one or more proper nouns, which should be looked under the (first) name (e.g., *Belnap's 4-valued system*; *Hintikka set*; *Church-Turing theorem*) and expressions beginning with central concepts (e.g., *Truth-value gap*). For most concepts, only the page where they are defined or first occur significantly is given. Abbreviations are provided in brackets, and recurrent abbreviations are also listed as entries (e.g., CNF). Logical systems are listed both by the first name/concept and by the first letter denoting them (e.g, both *Post's n-valued system* and "P_n" are distinct entries in the Index). As for the *Index nominum*, only proper nouns associated with fundamental aspects (e.g., theorems, problems, structures) and/or logical systems are listed; for these, all pages of occurrence are given. The letters Å and Ł are alphabetically processed as A and a variant of L, respectively, and the Greek letter Π is so as P.

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