

# Many-Valued Logics

## A Mathematical and Computational Introduction

Luis M. Augusto

© Individual author and College Publications 2017  
All rights reserved.

ISBN 978-1-84890-250-3

College Publications  
Scientific Director: Dov Gabbay  
Managing Director: Jane Spurr

<http://www.collegepublications.co.uk>

Printed by Lightning Source, Milton Keynes, UK

---

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system or transmitted in any form, or by any means, electronic, mechanical, photocopying, recording or otherwise without prior permission, in writing, from the publisher.

# Contents

Preface	xi
<b>1. Introduction</b>	<b>1</b>
1.1. Logics, classical and non-classical, among which the many-valued . . . . .	1
1.2. Logic and mathematics . . . . .	3
1.3. Logic and computation . . . . .	5
<b>I. THINGS LOGICAL</b>	<b>9</b>
<b>2. Logical languages</b>	<b>11</b>
2.1. Formal languages and logical languages . . . . .	11
2.2. Propositional and first-order languages . . . . .	13
2.3. The language of classical logic . . . . .	17
2.4. Clausal and normal forms . . . . .	18
2.4.1. Literals and clauses . . . . .	18
2.4.2. Negation normal form . . . . .	19
2.4.3. Prenex normal form . . . . .	19
2.4.4. Skolem normal form . . . . .	21
2.4.5. Conjunctive and disjunctive normal forms . . . . .	22
2.5. Signed logic and signed clause logic . . . . .	26
2.5.1. Signed logic . . . . .	26
2.5.2. Signed clause logic . . . . .	28
2.6. Substitutions and unification for FOL . . . . .	29
Exercises . . . . .	34
<b>3. Logical systems</b>	<b>39</b>
3.1. Logical consequence and inference . . . . .	39
3.2. Semantics and model theory . . . . .	42
3.2.1. Truth-functionality and truth-functional completeness . . . . .	43
3.2.2. Semantics and deduction . . . . .	46
3.2.3. Matrix semantics . . . . .	50
3.3. Syntax and proof theory . . . . .	53
3.3.1. Inference rules and proof systems . . . . .	54

## *Contents*

3.3.2. Syntax and deduction . . . . .	56
3.4. Adequateness of a deductive system . . . . .	58
3.5. The system of classical logic . . . . .	62
Exercises . . . . .	64
<b>4. Logical decisions</b>	<b>67</b>
4.1. Meeting the decision problem and the SAT . . . . .	67
4.1.1. The Boolean satisfiability problem, or SAT . . . . .	68
4.1.2. Refutation proof procedures . . . . .	69
4.2. Some historical notes on automated theorem proving . . . . .	70
4.3. Herbrand semantics . . . . .	72
4.4. Proving validity and satisfiability . . . . .	78
4.4.1. Truth tables . . . . .	79
4.4.2. Axiom systems . . . . .	79
4.4.3. Natural deduction . . . . .	80
4.4.4. The sequent calculus $\mathcal{LK}$ . . . . .	83
4.4.5. The DPLL procedure . . . . .	87
4.5. Refutation I: Analytic tableaux . . . . .	91
4.5.1. Analytic tableaux as a propositional calculus . . . . .	91
4.5.2. Analytic tableaux as a predicate calculus . . . . .	99
4.5.2.1. FOL tableaux without unification . . . . .	101
4.5.2.2. FOL tableaux with unification . . . . .	103
4.6. Refutation II: Resolution . . . . .	105
4.6.1. The resolution principle for propositional logic . . . . .	105
4.6.2. The resolution principle for FOL . . . . .	107
4.6.3. Completeness of the resolution principle . . . . .	108
4.6.4. Resolution refinements . . . . .	110
4.6.4.1. A-ordering . . . . .	111
4.6.4.2. Hyper-resolution and semantic resolution	115
4.6.5. Implementation of resolution in Prover9-Mace4 . . . . .	118
Exercises . . . . .	125
<b>II. MANY-VALUED LOGICS</b>	<b>131</b>
<b>5. Many-valued logics</b>	<b>133</b>
5.1. Some historical notes . . . . .	133
5.2. Many-valuedness and interpretation . . . . .	134
5.2.1. Suszko's Thesis . . . . .	134
5.2.2. Non-trivial many-valuedness . . . . .	136
5.2.3. Classical generalizations to the many-valued logics	137
5.3. Structural properties of many-valued logics . . . . .	141

## Contents

5.4.	The Łukasiewicz propositional logics . . . . .	142
5.4.1.	Łukasiewicz's 3-valued propositional logic $L_3$ . . . . .	142
5.4.2.	Tautologousness, contradictoriness, and entailment in $L_3$ . . . . .	148
5.4.3.	$n$ -valued generalizations of $L_3$ . . . . .	149
5.5.	Finitely many-valued propositional logics . . . . .	151
5.5.1.	Bochvar's 3-valued system . . . . .	151
5.5.2.	Kleene's 3-valued logics . . . . .	155
5.5.3.	Finn's 3-valued logic . . . . .	157
5.5.4.	Logics of nonsense: the 3-valued logics of Halldén, Åqvist, Segerberg, and Piróg-Rzepecka . . . . .	157
5.5.5.	Heyting's 3-valued logic . . . . .	162
5.5.6.	Reichenbach's 3-valued logic . . . . .	163
5.5.7.	Belnap's 4-valued logic . . . . .	164
5.5.8.	The finitely $n$ -valued logics of Post and Gödel . . . . .	166
5.5.8.1.	Post logics . . . . .	166
5.5.8.2.	Gödel logics . . . . .	169
5.6.	Fuzzy logics . . . . .	170
5.7.	Quantification in many-valued logics . . . . .	174
5.7.1.	Quantification in finitely many-valued logics . . . . .	174
5.7.2.	Quantification in fuzzy logics . . . . .	182
	Exercises . . . . .	187

### **III. REFUTATION CALCULI FOR MANY-VALUED LOGICS** **195**

6.	<b>The signed SAT for many-valued logics</b>	<b>197</b>
6.1.	From the MV-SAT to the signed SAT . . . . .	197
6.2.	From many-valued formulae to signed formulae . . . . .	200
6.2.1.	General notions and definitions . . . . .	200
6.2.2.	Transformation rules for many-valued connectives	205
6.2.3.	Transformation rules for many-valued quantifiers	208
6.2.4.	Transformation rules and preservation of structure	212
6.2.5.	Translation to clausal form . . . . .	213
	Exercises . . . . .	217
7.	<b>Signed tableaux for the MV-SAT</b>	<b>219</b>
7.1.	Introductory remarks . . . . .	219
7.2.	Signed analytic tableaux for classical formulae . . . . .	221
7.3.	Surma's algorithm . . . . .	224

*Contents*

7.4.	Signed tableaux for finitely many-valued logics . . . . .	229
7.4.1.	Propositional signed tableaux . . . . .	231
7.4.2.	FO signed tableaux . . . . .	240
7.5.	Signed tableaux for infinitely many-valued logics . . . . .	248
	Exercises . . . . .	256
<b>8.</b>	<b>Signed resolution for the MV-SAT</b>	<b>259</b>
8.1.	Introductory remarks . . . . .	259
8.2.	Signed resolution for finitely many-valued logics . . . . .	261
8.2.1.	Signed resolution proof procedures . . . . .	261
8.2.1.1.	Main rules . . . . .	261
8.2.1.2.	Refinements of signed resolution . . . . .	264
8.2.2.	The main theorem of signed resolution . . . . .	265
8.3.	Signed resolution for infinitely many-valued logics . . . . .	271
	Exercises . . . . .	283
<b>IV.</b>	<b>APPENDIX</b>	<b>287</b>
<b>9.</b>	<b>Mathematical notions</b>	<b>289</b>
9.1.	Sets . . . . .	289
9.2.	Functions, operations, and relations . . . . .	290
9.3.	Algebras and algebraic structures . . . . .	294
9.4.	Lattices . . . . .	297
9.5.	Graphs and trees . . . . .	303
	<b>Bibliography</b>	<b>305</b>
	<b>Index</b>	<b>319</b>