Studies in Logic 67

Many-Valued Logics

A Mathematical and Computational Introduction

Luis M. Augusto

Studies in Logic

Volume 67

Many-Valued Logics

A Mathematical and Computational Introduction

Volume 56
Dualities for Structures of Applied Logics
Ewa Orłowska, Anna Maria Radzikowska and Ingrid Rewitzky

Volume 57
Proof-theoretic Semantics
Nissim Francez

Volume 58
Handbook of Mathematical Fuzzy Logic, Volume 3
Petr Cintula, Petr Hajek and Carles Noguera, eds.

Volume 59
The Psychology of Argument. Cognitive Approaches to Argumentation and Persuasion
Fabio Paglieri, Laura Bonelli and Silvia Felletti, eds

Volume 60
Absract Algebraic Logic. An Introductory Textbook
Josep Maria Font

Volume 61
Philosophical Applications of Modal Logic
Lloyd Humberstone

Volume 62
Argumentation and Reasoned Action. Proceedings of the 1st European Conference on Argumentation, Lisbon 2015. Volume I
Dima Mohammed and Marcin Lewiński, eds

Volume 63
Argumentation and Reasoned Action. Proceedings of the 1st European Conference on Argumentation, Lisbon 2015. Volume II
Dima Mohammed and Marcin Lewiński, eds

Volume 64
Logic of Questions in the Wild. Inferential Erotetic Logic in Information Seeking Dialogue Modelling
Paweł Łupkowski

Volume 65
Elementary Logic with Applications. A Procedural Perspective for Computer Scientists
D. M. Gabbay and O. T. Rodrigues

Volume 66
Logical Consequences. Theory and Applications: An Introduction.
Luis M. Augusto

Volume 67
Many-Valued Logics: A Mathematical and Computational Introduction
Luis M. Augusto

Studies in Logic Series Editor
Dov Gabbay dov.gabbay@kcl.ac.uk

Many-Valued Logics

A Mathematical and Computational Introduction

Luis M. Augusto

ISBN 978-1-84890-250-3

College Publications
Scientific Director: Dov Gabbay
Managing Director: Jane Spurr

http://www.collegepublications.co.uk

Printed by Lightning Source, Milton Keynes, UK

Contents

Preface **xi**

1. Introduction **1**
1.1. Logics, classical and non-classical, among which the many-valued . . . 1
1.2. Logic and mathematics . . . 3
1.3. Logic and computation . . . 5

I. THINGS LOGICAL **9**

2. Logical languages **11**
2.1. Formal languages and logical languages . . . 11
2.2. Propositional and first-order languages . . . 13
2.3. The language of classical logic . . . 17
2.4. Clausal and normal forms . . . 18
2.4.1. Literals and clauses . . . 18
2.4.2. Negation normal form . . . 19
2.4.3. Prenex normal form . . . 19
2.4.4. Skolem normal form . . . 21
2.4.5. Conjunctive and disjunctive normal forms . . . 22
2.5. Signed logic and signed clause logic . . . 26
2.5.1. Signed logic . . . 26
2.5.2. Signed clause logic . . . 28
2.6. Substitutions and unification for FOL . . . 29
Exercises . . . 34

3. Logical systems **39**
3.1. Logical consequence and inference . . . 39
3.2. Semantics and model theory . . . 42
3.2.1. Truth-functionality and truth-functional completeness . . . 43
3.2.2. Semantics and deduction . . . 46
3.2.3. Matrix semantics . . . 50
3.3. Syntax and proof theory . . . 53
3.3.1. Inference rules and proof systems . . . 54

3.3.2. Syntax and deduction 56
3.4. Adequateness of a deductive system 58
3.5. The system of classical logic 62
Exercises 64

4. Logical decisions **67**
4.1. Meeting the decision problem and the SAT 67
4.1.1. The Boolean satisfiability problem, or SAT 68
4.1.2. Refutation proof procedures 69
4.2. Some historical notes on automated theorem proving . . . 70
4.3. Herbrand semantics 72
4.4. Proving validity and satisfiability 78
4.4.1. Truth tables 79
4.4.2. Axiom systems 79
4.4.3. Natural deduction 80
4.4.4. The sequent calculus $\mathcal{LK}$ 83
4.4.5. The DPLL procedure 87
4.5. Refutation I: Analytic tableaux 91
4.5.1. Analytic tableaux as a propositional calculus . . . 91
4.5.2. Analytic tableaux as a predicate calculus 99
4.5.2.1. FOL tableaux without unification 101
4.5.2.2. FOL tableaux with unification 103
4.6. Refutation II: Resolution 105
4.6.1. The resolution principle for propositional logic . . 105
4.6.2. The resolution principle for FOL 107
4.6.3. Completeness of the resolution principle 108
4.6.4. Resolution refinements 110
4.6.4.1. A-ordering 111
4.6.4.2. Hyper-resolution and semantic resolution 115
4.6.5. Implementation of resolution in Prover9-Mace4 . . 118
Exercises 125

II. MANY-VALUED LOGICS **131**

5. Many-valued logics **133**
5.1. Some historical notes 133
5.2. Many-valuedness and interpretation 134
5.2.1. Suszko's Thesis 134
5.2.2. Non-trivial many-valuedness 136
5.2.3. Classical generalizations to the many-valued logics 137
5.3. Structural properties of many-valued logics 141

5.4. The Łukasiewicz propositional logics 142
5.4.1. Łukasiewicz's 3-valued propositional logic Ł$_3$ . . . 142
5.4.2. Tautologousness, contradictoriness, and entailment in Ł$_3$ 148
5.4.3. n-valued generalizations of Ł$_3$ 149
5.5. Finitely many-valued propositional logics 151
5.5.1. Bochvar's 3-valued system 151
5.5.2. Kleene's 3-valued logics 155
5.5.3. Finn's 3-valued logic 157
5.5.4. Logics of nonsense: the 3-valued logics of Halldén, Åqvist, Segerberg, and Piróg-Rzepecka 157
5.5.5. Heyting's 3-valued logic 162
5.5.6. Reichenbach's 3-valued logic 163
5.5.7. Belnap's 4-valued logic 164
5.5.8. The finitely n-valued logics of Post and Gödel . . . 166
5.5.8.1. Post logics 166
5.5.8.2. Gödel logics 169
5.6. Fuzzy logics 170
5.7. Quantification in many-valued logics 174
5.7.1. Quantification in finitely many-valued logics 174
5.7.2. Quantification in fuzzy logics 182
Exercises 187

III. REFUTATION CALCULI FOR MANY-VALUED LOGICS 195

6. The signed SAT for many-valued logics 197
6.1. From the MV-SAT to the signed SAT 197
6.2. From many-valued formulae to signed formulae 200
6.2.1. General notions and definitions 200
6.2.2. Transformation rules for many-valued connectives 205
6.2.3. Transformation rules for many-valued quantifiers . 208
6.2.4. Transformation rules and preservation of structure 212
6.2.5. Translation to clausal form 213
Exercises 217

7. Signed tableaux for the MV-SAT 219
7.1. Introductory remarks 219
7.2. Signed analytic tableaux for classical formulae 221
7.3. Surma's algorithm 224

7.4. Signed tableaux for finitely many-valued logics 229
7.4.1. Propositional signed tableaux 231
7.4.2. FO signed tableaux 240
7.5. Signed tableaux for infinitely many-valued logics 248
Exercises 256

8. Signed resolution for the MV-SAT **259**
8.1. Introductory remarks 259
8.2. Signed resolution for finitely many-valued logics 261
8.2.1. Signed resolution proof procedures 261
8.2.1.1. Main rules 261
8.2.1.2. Refinements of signed resolution 264
8.2.2. The main theorem of signed resolution 265
8.3. Signed resolution for infinitely many-valued logics 271
Exercises 283

IV. APPENDIX **287**

9. Mathematical notions **289**
9.1. Sets 289
9.2. Functions, operations, and relations 290
9.3. Algebras and algebraic structures 294
9.4. Lattices 297
9.5. Graphs and trees 303

Bibliography **305**

Index **319**

List of Figures

1.1.1.Some main non-classical logics. 4

2.4.1.Tseitin transformations for the connectives of L. 25
2.5.1.A partially ordered set. 27
2.6.1.Unifying the pair $\langle P(a, x, h(g(z))), P(z, h(y), h(y))\rangle$ . . 33

3.4.1.Adequateness of a deductive system $\mathrm{L} = (\mathsf{L}, \Vdash)$. 61

4.3.1.Closed semantic tree of $C = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5\}$ in Example 4.3.3. 76
4.3.2.A closed semantic tree. 77
4.4.1.Proof of $\vdash \phi \to \phi$ in $\mathscr{L}$ 79
4.4.2.$\phi, \forall x(\phi) \to \chi \vdash \forall x(\chi)$: Proof in the axiom system $\mathcal{L}^*$. . . 80
4.4.3.$\mathcal{NK}$ proof of $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$. . . 83
4.4.4.Proof in $\mathcal{LK}$ of a FO validity. 87
4.4.5.Proof in $\mathcal{LK}$ of axiom $\mathscr{L}2$ of the axiom system $\mathscr{L}$. . . . 88
4.4.6.A DPLL proof procedure. 90
4.5.1.Analytic tableaux expansion rules: $\alpha\beta$-classification. . . . 94
4.5.2.A propositional tableau proof. 96
4.5.3.Analytic tableaux expansion rules: $\gamma\delta$-classification. . . . 100
4.5.4.A FO tableau proof without unification. 102
4.5.5.A FO tableau with unification. 105
4.6.1.A refutation tree. 107
4.6.2.A refutation failure tree. 109
4.6.3.Hyper-resolution of $\Xi = (\mathcal{C}_3; \mathcal{C}_1, \mathcal{C}_2)$. 116
4.6.4.Input in Prover9-Mace4. 120
4.6.5.Output by Prover9-Mace4. 120
4.6.6.Output by Prover9-Mace4. 121
4.6.7.Schubert's steamroller in natural language. 122
4.6.8.Schubert's steamroller in FOL. 123
4.6.9.Proof of Schubert's steamroller by Prover9-Mace4. 124

5.2.1.The homomorphic interpretation $h \in Hom(\mathfrak{L}, \mathfrak{A})$ and the valuation $val_h : F \longrightarrow W_2$. 135
5.4.1.Proof of $\vdash_{\mathrm{L}3} \neg P \to (P \to Q)$ in $\mathscr{L}_3$. 146
5.4.2.Proof of $\vdash_{\mathrm{L}\aleph} P \to ((P \to Q) \to Q)$ in $\mathscr{L}_\aleph$. 151

5.5.1.The lattices $\mathcal{A}4$ (1.) and $\mathcal{L}4$ (2.). 164
5.7.1.Proof of $\vdash_{\mathrm{L}_3^*} (\forall x)\, P\,(x) \rightarrow (\exists x)\, P\,(x)$ in $\mathscr{L}_3^*$. 178
5.7.2.A fuzzy binary relation. 185

6.2.1.*cnfs* for the connective $\rightarrow_{\mathrm{L}3}$. 209
6.2.2.Transformation rules for signed formulae of L_n. 214
6.2.3.Clause transformation process of a signed FO many-valued formula. 216

7.1.1.General tableau rule schema for finite tableaux. 219
7.2.1.Signed tableaux extension rules: $\alpha\beta$-classification. 222
7.2.2.Signed tableaux extension rules: $\gamma\delta$-classification. 222
7.2.3.A signed tableau proof of a CPL formula. 223
7.4.1.Signed tableau rules for some connectives of L_3. 236
7.4.2.A signed tableau proof in L_3 of the classical tautology *modus ponens.* 238
7.4.3.Signed tableau rules for the logical system MK. 241
7.4.4.FO tableau rules for filters and ideals. 247
7.4.5.Signed tableau rules for quantified formulae in MK. 248
7.5.1.Signed fuzzy tableaux. 254

8.2.1.A signed resolution procedure. 267

9.2.1.A partially ordered set. 292
9.4.1.Join table of 2^A. 299
9.4.2.Meet table of 2^A. 300
9.4.3.The lattice $(\mathcal{S}, \cup, \cap)$. 300
9.4.4.The non-distributive lattices $\mathcal{L}_1$ and $\mathcal{L}_2$. 301

Preface

Although the title of this book indicates two main components, to wit, mathematical and computational, we wish to emphasize that this is first and foremost a book on many-valued logics. As such, the reader who might just be interested in the various many-valued logics will find abundant material in that which is the central part of this book, to wit, Part II. In this, the several main many-valued logics are presented from the viewpoints of their semantical properties, and their semantics are characterized mostly by means of truth tables and logical matrices. From the proof-theoretical viewpoint, for many of these logics partial or full axiomatizations are provided.

This Part II is, mainly for the objective of self-containment, preceded by an extensive introduction to "things" logical, i.e. logical languages, systems, and decisions. In this Part I, important notions such as logical consequence, adequateness of a logical system, and matrix semantics, among others, are rigorously defined. Further notions regarding decision procedures in classical logic are introduced, namely the satisfiability and validity problems (abbreviated as SAT and VAL, respectively), and they are put into relation with the several proof systems available. In Part III, we chose to approach the many-valued satisfiability problem (MV-SAT) via refutation, i.e. by proving unsatisfiability of a set of formulae (a theory), because of the many advantages, computational and other, that this proof procedure presents for the many-valued logics. This explains our extensive discussion of the analytic tableaux and resolution calculi for classical logic in this Part I. However, we wanted the reader to acquire a good grasp also of validity- and satisfiability-testing methods, and we thus provide sufficient material on this topic.

Part III deals with well-studied computational, or potentially computerizable, approaches to the many-valued logics, namely as far as automated deduction is concerned. Here, signed (clause) logic has a central place (Chapter 6). In effect, both signed tableaux and signed resolution are automatizable proof calculi that can very naturally be applied to many-valued logics, reason why we chose to elaborate on them at length. This elaboration constitutes Chapters 7 and 8, for signed tableaux and signed resolution, respectively.

We did not wish to emphasize the mathematical component, as this

could scare away many a potential reader. Nevertheless, we believe that logic is a branch of mathematics, and as such it is both mathematically motivated and justified. (Incidentally, we do not think that logic is a foundation for mathematics–the reverse is true, in our opinion.) Given these two somehow conflicting views, we decided that the best thing to do would be to provide the mathematically literate, or just simply interested, readers with an Appendix in which the mathematical bases of our approach are expounded to the required extent. This is Part IV.

Logic is a subject that requires (many) years before one grasps satisfactorily what it is actually all about. This stage will arguably not be reachable without hands-on practice, much in the same way, perhaps, that many other fields require extensive, often arduous, practice. In this belief, we provide the readers with a large selection of exercises.[1]

In effect, this book is conceived for the reader who wishes to *do* something with many-valued logics, especially–but by no means only–in a computational context. With this we associate fields as diverse as cognitive modeling and switching theory. The literature on many-valued logics is abundant, but the "standard" monographs (Bolc & Borowik, 1992; Malinowski, 1993; Rescher, 1969; Rosser & Torquette, 1952), while not being obsolete, are now of mainly theoretical and historical interest. Some more recent work claiming to have practical applications in mind (e.g., Bolc & Borowik, 2003) follows to a great extent in the footsteps of these earlier monographs. On the other hand, recent work both in mathematical logic (e.g., Gottwald, 2001; Hähnle, 2001) and in philosophical logic / the philosophy of logic (e.g., Bergman, 2008) provides little or no material for the practical applications mentioned.

Surprisingly–or just plainly intriguingly–, the last comprehensive discussion of modern applications of many-valued logics dates from 1977 (Dunn & Epstein). The fact is that many-valued logics are today in more demand than ever before, due to the realization that inconsistency in information (and knowledge bases are frequently inconsistent) "is respectable and is most welcome" (Gabbay, 2014). We now know and accept that it is often the case that theories have truth-value gaps (some propositions appear to be neither true nor false), truth-value gluts (some propositions appear to be both true and false), vague concepts (e.g., cold, young, tall, sufficient), indefinite functions, lacunae, etc. Moreover, theories are in constant updating processes, and human and artificial reasoners require formal means to keep pace with these often fast and/or unpredictable processes, namely in order to review beliefs: what

[1] For reasons of timing, solutions to the exercises are not provided in this edition. In due time, there will be an Appendix with these solutions, which may actually be (also) posted online.

yesterday was a true/false proposition may today be neither true nor false, or both; what yesterday was clear-cut is now vague or indeterminate. The many-valued logics provide us with a powerful formalism to tackle these cases in terms of reasoning. In fact, this scenario calls for not only a practical-based approach to the many-valued logics per se, but also for hybridizations of these with other formal systems, as well as with more quantitative(-like) cognitive approaches (e.g., D'Avila Garcez, Lamb, & Gabbay, 2009). Importantly, the practical applications of many-valued logics have now gone beyond the mere industrial applications (e.g., fuzzy logics in washing machines), and promise to be of import for "higher," cognitive modeling. This *new logic* entails, in particular, a reappraisal of both "psychologism" and "formalism"–if not also of "logicism"–, especially because it proposes a reappraisal of what a cognitive, or reasoning, agent is (see, e.g. Gabbay & Woods, 2001). In order for this emerging work (Gabbay & Woods, 2003; 2005) to be further carried out and to incorporate the many-valued logics, we need a solid mathematical-computational grasp of these logics.

We hope the present book will provide the motivation for a new comprehensive treatment of the modern applications of many-valued logics that will include the *new-logic* factor as a central aspect of modern logic and its applications. Thus, a book like this on many-valued logics is now very timely, for *the times, they are a'changing.*

I wish to thank Dov M. Gabbay for including this book in the excellent Studies in Logic series of College Publications. My thanks go also to Jane Spurr for a smooth publication process.

Madrid, Summer 2017

Luis M. Augusto