

# Approximating Propositional Calculi by Finite-valued Logics

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## Abstract

*Bernays introduced a method for proving underivability results in propositional calculi  $\mathbf{C}$  by truth tables. It is investigated how far this method can be carried using (1) one or (2) an infinite sequence of many-valued logics. It is shown that the best candidate matrices for (1) can be computed from the calculus, and how sequences for (2) can be found for certain classes of logics. Furthermore, it is shown that the set of formulas which cannot be proved underivable by Bernays' method is non-trivial and decidable for propositional linear logic.*

## 1 Introduction

The question of what to do when face to face with a new logical calculus is an age-old problem of mathematical logic. One usually has, at least at first, no semantics. For example, intuitionistic propositional logic was constructed by Heyting only as a calculus; semantics for it were proposed much later. Currently we face a similar situation with Girard's linear logic. The lack of semantical methods makes it difficult to answer questions such as: Are statements of a certain form (un)derivable? Are the axioms independent? Is the calculus consistent? For logics closed under substitution many-valued methods have often proved valuable since they were first used for proving underivabilities by Bernays [2] in 1926 (and later by others, e.g., McKinsey and Wajsberg; see also [16, § 25]). For the above-mentioned underivability question it is necessary to find many-valued matrices for which the given calculus is sound. If a formula is not a tautology under such a matrix, it cannot be derivable in the calculus. It is also necessary, of course, that the matrix has as few tautologies as possible in order to be useful.

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Such “optimal” approximations of a given calculus may also have applications in computer science. In the field of artificial intelligence many new (propositional) logics have been introduced. They are usually better suited to model the problems dealt with in AI than traditional (classical, intuitionistic, or modal) logics, but many have two significant drawbacks: First, they are either given solely semantically or solely by a calculus. For practical purposes, a proof theory is necessary; otherwise computer representation of and automated search for proofs/truths in these logics is not feasible. Second, most of them are intractable, and hopelessly so, provided the polynomial hierarchy does not collapse. For instance, many nonmonotonic formalisms have been shown to be hard for classes above NP [6]. Although satisfiability in many-valued propositional logics is (as in classical logic) NP-complete [15], this is still (probably) much better.

On the other hand, it is evident from the work of Carnielli [3] and Hähnle [10] on tableaux, and Rousseau, Takahashi, and Baaz et al. [1] on sequents, that finite-valued logics are, from the perspective of proof *and* model theory, very close to classical logic. Therefore, many-valued logic is a very suitable candidate if one looks for approximations, in some sense, of given complex logics.

What is needed are methods for obtaining finite-valued approximations of the propositional logics at hand. It turns out, however, that a shift of emphasis is in order here. While it is the *logic* we are actually interested in, we always are given only a *representation* of the logic. Hence, we have to concentrate on approximations of the representation, and not of the logic per se.

What is a representation of a logic? The first type of representation that comes to mind is a calculus. Hilbert-type calculi are the simplest conceptually and the oldest historically. We will investigate the relationship between such calculi on the one hand and many-valued logics or recursive sequences of many-valued logics on the other hand. The latter notion has received considerable attention in the literature in the form of the following two problems: Given a calculus  $\mathbf{C}$ ,

- (1) find a minimal (finite) *normal* matrix for  $\mathbf{C}$  (relevant for non-derivability and independence proofs), and
- (2) find a sequence of finite-valued logics whose intersection equals the theorems of  $\mathbf{C}$ , and its converse, given a sequence of finite-valued logics, find a calculus for its intersection (exemplified by Jaśkowski’s sequence for intuitionistic propositional calculus, and by Dummett’s extension axiomatizing the intersection of the sequence of Gödel logics, respectively).

For (1), of course, the best case would be a finite-valued logic  $\mathbf{M}$  whose tautologies *coincide* with the theorems of  $\mathbf{C}$ .  $\mathbf{C}$  then provides an axiomatization of  $\mathbf{M}$ . This of course is not always possible, at least for *finite*-valued logics. Lindenbaum [14, Satz 3] has shown that any logic (in our sense, given by a set of rules and closed under substitution) can be characterized by an *infinite*-valued logic. For a discussion of related questions see also Rescher [16, § 24].

In the following we will consider these questions in a general setting. Consider a propositional Hilbert-type calculus  $\mathbf{C}$ . First of all, an optimal (i.e., minimal under set inclusion of the tautologies)  $m$ -valued logic for which  $\mathbf{C}$  satisfies reasonable soundness

properties can be computed. We call such a logic *normal* for  $\mathbf{C}$ . The next question is, can we find an approximating sequence of  $m$ -valued logics in the sense of (2)? It is shown that this is impossible for undecidable calculi  $\mathbf{C}$ , and possible for all decidable logics closed under substitution. This leads us to the investigation of the *many-valued closure*  $MC(\mathbf{C})$  of  $\mathbf{C}$ , i.e., the set of formulas which are true in all covers of  $\mathbf{C}$ . In other words, if some formula can be shown to be underivable in  $\mathbf{C}$  by a Bernays-style many-valued argument, it is not in the many-valued closure. Using this concept we can classify calculi according to their many-valued behavior, or according to how good they can be dealt with by many-valued methods. In the best case  $MC(\mathbf{C})$  equals the theorems of  $\mathbf{C}$  (This can be the case only if  $\mathbf{C}$  is decidable). Otherwise  $MC(\mathbf{C})$  is a proper superset of the theorems of  $\mathbf{C}$ .

We show that  $MC(\mathbf{C})$  is decidable if  $\mathbf{C}$  is *analytic* (This does not imply that  $\mathbf{C}$  itself is decidable; e.g., cut-free propositional linear logic is known to be undecidable). Two axiomatizations  $\mathbf{C}$  and  $\mathbf{C}'$  of the same logic may have different many-valued closures  $MC(\mathbf{C})$  and  $MC(\mathbf{C}')$  while being model-theoretically indistinguishable. Hence, the many-valued closure can be used to distinguish between  $\mathbf{C}$  and  $\mathbf{C}'$  with regard to their proof-theoretic properties.

Finally, we investigate some of these questions for other representations of logics, namely for decision procedures and finite Kripke models. In these cases approximating sequences of many-valued logics whose intersection equals the given logics can always be given.

## 2 Propositional Logics

**2.1. DEFINITION** A *propositional language*  $\mathcal{L}$  consists of the following:

- (1) propositional variables:  $X_0, X_1, X_2, \dots, X_j, \dots$  ( $j \in \omega$ )
- (2) propositional connectives of arity  $n_j$ :  $\Box_0^{n_0}, \Box_1^{n_1}, \dots, \Box_r^{n_r}$ . If  $n_j = 0$ , then  $\Box_j$  is called a *propositional constant*.
- (3) Auxiliary symbols:  $(, )$ , and  $,$  (comma).

Formulas and subformulas are defined as usual. We denote the set of formulas over a language  $\mathcal{L}$  by  $\text{Frm}(\mathcal{L})$ . By  $\text{Var}(A)$  we mean the set of propositional variables occurring in  $A$ .

**2.2. DEFINITION** A *propositional Hilbert-type calculus*  $\mathbf{C}$  in the language  $\mathcal{L}$  is given by

- (1) A finite set  $A(\mathbf{C}) \subseteq \text{Frm}(\mathcal{L})$  of axioms.
- (2) A finite set  $R(\mathbf{C})$  of rules of the form

$$\frac{A_1 \quad \dots \quad A_n}{A} r$$

where  $A, A_1, \dots, A_n \in \text{Frm}(\mathcal{L})$

A formula  $F$  is a *theorem* of  $\mathbf{L}$  if there is a derivation of  $F$  in  $\mathbf{C}$ , i.e., a finite sequence

$$F_1, F_2, \dots, F_s = F$$

of formulas s.t. for each  $F_i$  either

- (1)  $F_i$  is a substitution instance of an axiom in  $A(\mathbf{C})$ , or
- (2) there are  $F_{k_1}, \dots, F_{k_n}$  with  $k_j < i$  and a rule  $r \in R(\mathbf{C})$ , s.t.  $F_{k_j}$  is a substitution instance of the  $j$ -th premise of  $r$ , and  $F_i$  is a substitution instance of the conclusion.

If  $F$  is a theorem of  $\mathbf{C}$  we write  $\mathbf{C} \vdash F$ . The set of theorems of  $\mathbf{C}$  is denoted by  $\text{Thm}(\mathbf{C})$ .

**2.3. Remark** The above notion of a propositional rule is the one usually used in axiomatizations of propositional logic. It is, however, by no means the only possible notion. For instance, Schütte's rules

$$\frac{A(\top) \quad A(\perp)}{A(X)} \quad \frac{C \leftrightarrow D}{A(C) \leftrightarrow A(D)}$$

where  $X$  is a propositional variable, and  $A$ ,  $C$ , and  $D$  are formulas, does not fit under the above definition.

**2.4. DEFINITION** A propositional calculus is called *analytic* iff for every rule

$$\frac{A_1 \dots A_n}{A} r$$

it holds that  $\text{Var}(A_1) \subseteq \text{Var}(A), \dots, \text{Var}(A_n) \subseteq \text{Var}(A)$ .

**2.5. Remark** Note that analytic calculi here need *not* have a strict subformula property, in contrast to the notion in sequent calculus. Cut-free sequent calculi can easily be encoded in analytic Hilbert-type calculi. Henceforth, whenever we refer to a sequent calculus we always mean its encoding according to the following construction.

- (1) Sequences of formulas can be coded using a binary operator  $\cdot$ . The sequent arrow can simply be coded as a binary operator  $\rightarrow$ . We have the following rules, to assure associativity of  $\cdot$ :

$$\frac{X \cdot \left( (U \cdot (V \cdot W)) \cdot Y \right) \rightarrow Z}{X \cdot \left( ((U \cdot V) \cdot W) \cdot Y \right) \rightarrow Z} \quad \frac{\left( X \cdot (U \cdot (V \cdot W)) \right) \cdot Y \rightarrow Z}{\left( X \cdot ((U \cdot V) \cdot W) \right) \cdot Y \rightarrow Z}$$

as well as the respective rules without  $X$ , without  $Y$ , without both  $X$  and  $Y$ , with the rules upside-down, and also for the right side of the sequent (20 rules total).

- (2) To avoid logical rules acting on sequences instead of formulas, a formula marker  $^F$  is introduced. Logical axioms then take the form  $X^F \rightarrow X^F$ .

- (3) The usual sequent rules can be coded using the above constructions.
- (4) Some sequent rules require restrictions on the form of the side formulas in a rule, e.g., the  $R!$  rule in classical linear logic:

$$\frac{!\Pi \rightarrow A, ?\Gamma}{!\Pi \rightarrow !A, ?\Gamma} R!$$

We introduce operators  $!$  and  $?$  s.t.

- (a)  $A!$  and  $B?$  can be introduced only on  $A \equiv !C$  and  $B \equiv ?D$ , respectively;
- (b)  $!$  and  $?$  distribute over  $\cdot$ ; and
- (c)  $!$  and  $?$  can always be canceled.

$R!$  would then take the form

$$\frac{X! \rightarrow A \cdot Y?}{X! \rightarrow !A \cdot Y?}$$

It is easily seen that the resulting Hilbert calculus is analytic in the sense of Definition 2.4 if the original sequent calculus was. This also shows that this notion of analyticity does not entail decidability, since for instance cut-free propositional linear logic  $\mathbf{LL}$  can be coded in an analytic Hilbert calculus.  $\mathbf{LL}$ , however, is undecidable [13].

**2.6. EXAMPLE** Intuitionistic propositional logic is axiomatized by the following calculus  $\mathbf{IPC}$ :

(1) Axioms:

- $a_1$   $A \supset A \wedge A$   
 $a_2$   $A \wedge B \supset B \wedge A$   
 $a_3$   $A \supset B \supset (A \wedge C \supset B \wedge C)$   
 $a_4$   $(A \supset B) \wedge (B \supset C) \supset (A \supset C)$   
 $a_5$   $B \supset (A \supset B)$   
 $a_6$   $A \wedge (A \supset B) \supset B$   
 $a_7$   $A \supset A \vee B$   
 $a_8$   $A \vee B \supset B \vee A$   
 $a_9$   $(A \supset C) \wedge (B \supset C) \supset (A \vee B \supset C)$   
 $a_{10}$   $\neg A \supset A \supset B$   
 $a_{11}$   $(A \supset B) \wedge (A \supset \neg B) \supset \neg A$   
 $a_{12}$   $A \supset (B \supset A \wedge B)$

(2) Rules (in usual notation):

$$\frac{A \quad A \supset B}{B} \text{MP}$$

Gentzen's sequent calculus  $\mathbf{LJ}$  without cut gives an analytical axiomatization.

**2.7. DEFINITION** A *propositional logic*  $\mathbf{L}$  in the language  $\mathcal{L}$  is a subset of  $\text{Frm}(\mathcal{L})$  closed under substitution.

Every propositional calculus  $\mathbf{C}$  defines a propositional logic, namely  $\text{Thm}(\mathbf{C}) \subseteq \text{Frm}(\mathcal{L})$ , since  $\text{Thm}(\mathbf{C})$  is closed under substitution. Not every propositional logic, however, is axiomatizable, let alone finitely axiomatizable by a Hilbert calculus. For instance, the logic

$$\{\Box^k(\top) \mid k \text{ is the Gödel number of a true sentence of arithmetic}\}$$

is not axiomatizable, whereas the logic

$$\{\Box^k(\top) \mid k \text{ is prime}\}$$

is certainly axiomatizable (it is even decidable), but not by a Hilbert calculus using only  $\Box$  and  $\top$ . (It is easily seen that any Hilbert calculus for  $\Box$  and  $\top$  has either only a finite number of theorems or yields arithmetic progressions of  $\Box$ 's.)

**2.8. DEFINITION** A *propositional finite-valued logic*  $\mathbf{M}$  is given by a set of truth values  $V(\mathbf{M}) = \{1, 2, \dots, m\}$ , the set of *designated truth values*  $V^+(\mathbf{M}) \subseteq V(\mathbf{M})$ , and a set of truth functions  $\tilde{\Box}_j: V(\mathbf{M})^{n_j} \rightarrow V(\mathbf{M})$  for all connectives  $\Box_j \in \mathcal{L}$  with arity  $n_j$ .

The corresponding subset of  $\text{Frm}(\mathcal{L})$  of true formulas is the set of tautologies of  $\mathbf{M}$ , defined as follows.

**2.9. DEFINITION** A *valuation*  $\mathbf{I}$  is a mapping from the set of propositional variables into  $V(\mathbf{M})$ . A valuation  $\mathbf{I}$  can be extended in the standard way to a function from formulas to truth values.  $\mathbf{I}$  *satisfies* a formula  $F$ , in symbols:  $\mathbf{I} \models_{\mathbf{M}} F$ , if  $\mathbf{I}(F) \in V^+(\mathbf{M})$ . In that case,  $\mathbf{I}$  is called a *model* of  $F$ , otherwise a *countermodel*. A formula  $F$  is a *tautology* of  $\mathbf{M}$  iff it is satisfied by every valuation. Then we write  $\mathbf{M} \models F$ . We denote the set of tautologies of  $\mathbf{M}$  by  $\text{Taut}(\mathbf{M})$ .

**2.10. EXAMPLE** The sequence of  $m$ -valued Gödel logics  $\mathbf{G}_m$  is given by  $V(\mathbf{G}_m) = \{0, 1, \dots, m-1\}$ , the designated values  $V^+(\mathbf{G}_m) = \{0\}$ , and the following truth functions:

$$\begin{aligned} \tilde{\neg}_{\mathbf{G}_m}(v) &= \begin{cases} 0 & \text{for } v = m-1 \\ m-1 & \text{for } v \neq m-1 \end{cases} \\ \tilde{\vee}_{\mathbf{G}_m}(v, w) &= \min(a, b) \\ \tilde{\wedge}_{\mathbf{G}_m}(v, w) &= \max(a, b) \\ \tilde{\supset}_{\mathbf{G}_m}(v, w) &= \begin{cases} 0 & \text{for } v \geq w \\ w & \text{for } v < w \end{cases} \end{aligned}$$

This sequence of logics was used in [8] to show that intuitionistic logic cannot be characterized by a finite matrix.

In the remaining sections, we will concentrate on the relations between calculi  $\mathbf{C}$ , logics  $\mathbf{L}$ , and many-valued logics  $\mathbf{M}$ . The objective is to find many-valued logics  $\mathbf{M}$  (or sequences thereof) that, in a sense, approximate the calculus  $\mathbf{C}$  and/or the logic  $\mathbf{L}$ .

The following well-known product construction is useful for characterizing the “intersection” of many-valued logics.

**2.11. DEFINITION** Let  $\mathbf{M}$  and  $\mathbf{M}'$  be  $m$  and  $m'$ -valued logics, respectively. Then  $\mathbf{M} \times \mathbf{M}'$  is the  $mm'$ -valued logic where  $V(\mathbf{M} \times \mathbf{M}') = V(\mathbf{M}) \times V(\mathbf{M}')$ ,  $V^+(\mathbf{M} \times \mathbf{M}') = V^+(\mathbf{M}) \times V^+(\mathbf{M}')$ , and truth functions are defined component-wise. I.e., if  $\square$  is an  $n$ -ary connective, then

$$\tilde{\square}_{\mathbf{M} \times \mathbf{M}'}(w_1, \dots, w_n) = \langle \tilde{\square}_{\mathbf{M}}, \tilde{\square}_{\mathbf{M}'} \rangle.$$

For convenience, we define the following: Let  $\mathbf{I}$  and  $\mathbf{I}'$  be valuations of  $\mathbf{M}$  and  $\mathbf{M}'$ , respectively.  $\mathbf{I} \times \mathbf{I}'$  is the valuation of  $\mathbf{M} \times \mathbf{M}'$  defined by:  $(\mathbf{I} \times \mathbf{I}')(X) = \langle \mathbf{I}(X), \mathbf{I}'(X) \rangle$ . If  $\mathbf{I}^\times$  is a valuation of  $\mathbf{M} \times \mathbf{M}'$ , then the valuations  $\pi_1 \mathbf{I}^\times$  and  $\pi_2 \mathbf{I}^\times$  of  $\mathbf{M}$  and  $\mathbf{M}'$ , respectively, are defined by  $\pi_1 \mathbf{I}^\times(X) = v$  and  $\pi_2 \mathbf{I}^\times(X) = v'$  iff  $\mathbf{I}^\times(X) = \langle v, v' \rangle$ .

**2.12. LEMMA**  $\text{Taut}(\mathbf{M} \times \mathbf{M}') = \text{Taut}(\mathbf{M}) \cap \text{Taut}(\mathbf{M}')$

*Proof.* Let  $A$  be a tautology of  $\mathbf{M} \times \mathbf{M}'$  and  $\mathbf{I}$  and  $\mathbf{I}'$  be valuations of  $\mathbf{M}$  and  $\mathbf{M}'$ , respectively. Since  $\mathbf{I} \times \mathbf{I}' \models_{\mathbf{M} \times \mathbf{M}'} A$ , we have  $\mathbf{I} \models_{\mathbf{M}} A$  and  $\mathbf{I}' \models_{\mathbf{M}'} A$  by the definition of  $\times$ . Conversely, let  $A$  be a tautology of both  $\mathbf{M}$  and  $\mathbf{M}'$ , and let  $\mathbf{I}^\times$  be a valuation of  $\mathbf{M} \times \mathbf{M}'$ . Since  $\pi_1 \mathbf{I}^\times \models_{\mathbf{M}} A$  and  $\pi_2 \mathbf{I}^\times \models_{\mathbf{M}'} A$ , it follows that  $\mathbf{I}^\times \models_{\mathbf{M} \times \mathbf{M}'} A$ . ■

The definition and lemma are easily generalized to the case of finite products  $\prod_i \mathbf{M}_i$  by induction.

When looking for a logic with as small a number of truth values as possible which falsifies a given formula we can use the following construction.

**2.13. PROPOSITION** Let  $\mathbf{M}$  be any many-valued logic, and  $A_1, \dots, A_n$  be formulas not valid in  $\mathbf{M}$ . Then there is a finite-valued logic  $\mathbf{M}' = \Phi(\mathbf{M}, A_1, \dots, A_n)$  s.t.

- (1)  $A_1, \dots, A_n$  are not valid in  $\mathbf{M}'$ ,
- (2)  $\text{Taut}(\mathbf{M}) \subseteq \text{Taut}(\mathbf{M}')$ , and
- (3)  $|V(\mathbf{M}')| \leq \xi(A_1, \dots, A_n)$ , where  $\xi(A_1, \dots, A_n) = \prod_{i=1}^n \xi(A_i)$  and  $\xi(A_i)$  is the number of subformulas of  $A_i + 1$ .

This holds also if  $\mathbf{M}$  has infinitely many truth values, provided  $V(\mathbf{M})$ ,  $V^+(\mathbf{M})$  and the truth functions are recursive.

*Proof.* We first prove the proposition for  $n = 1$ . Let  $\mathbf{I}$  be the interpretation in  $\mathbf{M}$  making  $A_1$  false, and let  $B_1, \dots, B_r$  ( $\xi(A_1) = r + 1$ ) be all subformulas of  $A_1$ . Every  $B_i$  has a truth value  $t_i$  in  $\mathbf{I}$ . Let  $\mathbf{M}'$  be as follows:  $V(\mathbf{M}') = \{t_1, \dots, t_r, \top\}$ ,  $V^+(\mathbf{M}') = V^+(\mathbf{M}) \cap V(\mathbf{M}') \cup \{\top\}$ . If  $\square \in \mathcal{L}$ , define  $\tilde{\square}$  by

$$\tilde{\square}(v_1, \dots, v_n) = \begin{cases} t_i & \text{if } B_i \equiv \square(B_{j_1}, \dots, B_{j_n}) \\ & \text{and } v_1 = t_{j_1}, \dots, v_n = t_{j_n} \\ \top & \text{otherwise} \end{cases}$$

(1) Since  $t_r$  was undesigned in  $\mathbf{M}$ , it is also undesigned in  $\mathbf{M}'$ . But  $\mathbf{I}$  is also a truth value assignment in  $\mathbf{M}'$ , hence  $\mathbf{M}' \not\models A_1$ .

(2) Let  $C$  be a tautology of  $\mathbf{M}$ , and let  $\mathbf{J}$  be an interpretation in  $\mathbf{M}'$ . If no subformula of  $C$  evaluates to  $\top$  under  $\mathbf{J}$ , then  $\mathbf{J}$  is also an interpretation in  $\mathbf{M}$ , and  $C$  takes the

same truth value in  $\mathbf{M}'$  as in  $\mathbf{M}$  w.r.t.  $\mathbf{J}$ , which is designated also in  $\mathbf{M}'$ . Otherwise,  $C$  evaluates to  $\top$ , which is designated in  $\mathbf{M}'$ . So  $C$  is a tautology in  $\mathbf{M}'$ .

(3) Obvious.

For  $n > 1$ , the proposition follows by taking  $\Phi(\mathbf{M}, A_1, \dots, A_n) = \prod_{i=1}^n \Phi(\mathbf{M}, A_i)$  ■

Algebraic constructions can be used for simplifications of many-valued logics. For example, a many-valued logic  $\mathbf{M}$  has the same tautologies as a homomorphic image  $\mathbf{M}'$ , if the induced congruence  $C$  on  $V(\mathbf{M})$  satisfies the following condition:

$$\text{if } U \in C \text{ then } V^+(\mathbf{M}) \cap U = \emptyset \text{ or } V^+(\mathbf{M}) \cap (V(\mathbf{M}) \setminus U) = \emptyset.$$

### 3 Many-valued Covers for Calculi

We are looking for many-valued logics  $\mathbf{M}$  s.t.  $\text{Thm}(\mathbf{C}) \subseteq \text{Taut}(\mathbf{M})$ .  $\mathbf{M}$  must, however, behave “normally” with respect to  $\mathbf{C}$ , i.e.,  $\mathbf{C}$  must remain sound whenever we add new operators and their truth tables to  $\mathbf{M}$  or add tautologies as axioms to  $\mathbf{C}$ .

**3.1. DEFINITION** An  $m$ -valued logic  $\mathbf{M}$  is *normal* for a calculus  $\mathbf{C}$  (and  $\mathbf{C}$  *strongly sound* for  $\mathbf{M}$ ) if

- (\*) All axioms  $A \in A(\mathbf{C})$  are tautologies of  $\mathbf{M}$ , and for every rule  $r \in R(\mathbf{C})$ : if a valuation satisfies the premises of  $r$ , it also satisfies the conclusion.

$\mathbf{M}$  is then called a *cover* for  $\mathbf{C}$ .

We would like to stress the distinction between strong soundness, a.k.a. normality, and soundness. The latter is the familiar property of a calculus to produce only valid formulas as theorems. This “plain” soundness is what we actually would like to investigate in terms of approximations. More precisely, when looking for a finite-valued logic that approximates a given calculus, we are content if we find a logic for which  $\mathbf{C}$  is sound. It is, however, not possible in general to test if a calculus is sound for a given finite-valued logic. It *is* possible to test if it is strongly sound. For this pragmatic reason we consider only normal matrices for the given calculi. The next proposition characterizes the normal matrices in terms of strong soundness conditions. These are reasonable conditions which one expects to hold of a “normal” matrix.

**3.2. PROPOSITION**  $\mathbf{C}$  is *strongly sound* for a many-valued logic  $\mathbf{M}$  if  $\text{Thm}(\mathbf{C}') \subseteq \text{Taut}(\mathbf{M}')$  for all  $\mathbf{M}'$  and  $\mathbf{C}'$ , where

- (1)  $\mathbf{M}'$  is obtained from  $\mathbf{M}$  by adding truth tables for new operations, and
- (2)  $\mathbf{C}'$  is obtained from  $\mathbf{C}$  by adding tautologies of  $\mathbf{M}'$  to as axioms.

*Proof.* If: First of all,  $\mathbf{C}$  is sound for  $\mathbf{M}$ : Let  $\mathbf{C} \vdash F$ . We show that  $\mathbf{M} \models F$  by induction on the length  $l$  of the derivation in  $\mathbf{C}$ :

$l = 1$ : This means  $F$  is a substitution instance of an axiom  $A$ .

$l > 1$ .  $F$  is the conclusion of a rule  $r \in R(\mathbf{C})$ . If  $r$  is

$$\frac{A_1 \quad \dots \quad A_k}{A} \quad r$$



and  $X_1, X_2, \dots, X_n$  are all the variables in  $A, A_1, \dots, A_k$ , then the inference has the form

$$\frac{A_1[B_1/X_1, \dots, B_n/X_n] \quad \dots \quad A_k[B_1/X_1, \dots, B_n/X_n]}{F = A[B_1/X_1, \dots, B_n/X_n]}$$

Let  $\mathbf{I}$  be a valuation of the variables in  $F$ , and let  $v_i = \mathbf{I}(B_i)$  ( $1 \leq i \leq n$ ). By induction hypothesis, the premises of  $r$  are valid. This implies that, for  $1 \leq i \leq k$ , we have  $\{X_1 \mapsto v_1, \dots, X_n \mapsto v_n\} \models A_i$ . By hypothesis then,  $\{X_1 \mapsto v_1, \dots, X_n \mapsto v_n\} \models A$ . But this means that  $\mathbf{I} \models F$ . Hence,  $\mathbf{M} \models F$ .

Moreover,  $\mathbf{C}$  satisfies conditions (1) and (2) above.

Only if: Every axiom is derivable in  $\mathbf{C}$ . By soundness, it is a tautology of  $\mathbf{M}$ , which is just what (\*) says. Now let  $r \in R(\mathbf{C})$  be a rule, let  $\mathbf{I}$  be an interpretation which makes the premises  $A_1, \dots, A_k$  of  $r$  true, and let  $A$  be the conclusion of  $r$ .  $\mathbf{I}$  assigns truth values  $v_1, \dots, v_l$  to the variables  $X_1, \dots, X_l$  in  $r$ . Let  $\mathbf{M}'$  be the  $m$ -valued logic resulting from  $\mathbf{M}$  by extending the language by the constants  $V_1, \dots, V_l$  with values  $v_1, \dots, v_l$ , respectively. Let  $\sigma$  be the substitution mapping  $X_i$  to  $V_i$ . The formulas  $A_1\sigma, \dots, A_l\sigma$  and (by  $r$  also)  $A\sigma$  are derivable in the extension  $\mathbf{C}'$  of  $\mathbf{C}$  by the axioms  $A_1\sigma, \dots, A_l\sigma$ . By (1) and (2),  $\mathbf{C}'$  is sound, so  $A\sigma$  is a tautology in  $\mathbf{M}'$ . Consequently,  $\mathbf{I} \models A$  in  $\mathbf{M}$ . ■

**3.3. COROLLARY** *If  $\mathbf{C}$  is strongly sound for  $\mathbf{M}$  and  $r$  is a directly dependent rule of  $\mathbf{C}$  (i.e.,  $r$  can be simulated by the rules of  $\mathbf{C}$ ) then  $\mathbf{C} + r$  is also strongly sound for  $\mathbf{M}$ .*

**3.4. PROPOSITION** *It is decidable if a given propositional calculus is strongly sound for a given  $m$ -valued logic.*

Note also that for usual calculi, Property (\*) is relatively easy to check. For instance, modus ponens is strongly sound iff, whenever  $A$  is true,  $A \supset B$  is true iff  $B$  is true; necessitation is strongly sound if  $\Box X$  is true whenever  $X$  is true.

**3.5. EXAMPLE** The **IPC** is strongly sound for the  $m$ -valued Gödel logics  $\mathbf{G}_m$ . For instance, take axiom  $a_3$ :  $B \supset A \supset B$ . This is a tautology in  $\mathbf{G}_m$ , for assume we assign some truth values  $a$  and  $b$  to  $A$  and  $B$ , respectively. We have two cases: If  $a \leq b$ , then  $(A \supset B)$  takes the value  $m - 1$ . Whatever  $b$  is, it certainly is  $\leq m - 1$ , hence  $B \supset A \supset B$  takes the designated value  $m - 1$ . Otherwise,  $A \supset B$  takes the value  $b$ , and again (since  $b \leq b$ ),  $B \supset A \supset B$  takes the value  $m - 1$ .

Modus ponens passes the test: Assume  $A$  and  $A \supset B$  both take the value  $m - 1$ . This means that  $a \leq b$ . But  $a = m - 1$ , hence  $b = m - 1$ .

Now consider the following extension  $\mathbf{G}_m^\top$  of  $\mathbf{G}_m$ :  $V(\mathbf{G}_m^\top) = V(\mathbf{G}_m) \cup \{\top\}$ ,  $V^+(\mathbf{G}_m^\top) = \{m - 1, \top\}$ , and the truth functions are given by:

$$\tilde{\Box}_{\mathbf{G}_m^\top}(\bar{v}) = \begin{cases} \top & \text{if } \top \in \bar{v} \\ \tilde{\Box}_{\mathbf{G}_m}(\bar{v}) & \text{otherwise} \end{cases}$$

for  $\Box \in \{\neg, \supset, \wedge, \vee\}$ . Neither **IPC** nor **LJ** are strongly sound for  $\mathbf{G}_m^\top$ , but **LJ** without cut is.

**3.6. EXAMPLE** Consider the following calculus  $\mathbf{K}$ :

$$X \leftrightarrow_{\circ} X \quad \frac{X \leftrightarrow Y}{X \leftrightarrow_{\circ} Y} r_1 \quad \frac{X \leftrightarrow X}{Y} r_2$$

It is easy to see that the corresponding logic consists of all instances of  $X \leftrightarrow_{\circ}^k X$  where  $k \geq 1$ . This calculus is only strongly sound for the  $m$ -valued logic having all formulas as its tautologies. But if we leave out  $r_2$ , we can give a sequence of many-valued logics  $\mathbf{M}_n$ , for each of which  $\mathbf{K}$  is strongly sound: Take for  $V(\mathbf{M}_n) = \{0, \dots, n-1\}$ ,  $V^+(\mathbf{M}_n) = \{0\}$ , with the following truth functions:

$$\begin{aligned} \widetilde{\circ}v &= \begin{cases} v+1 & \text{if } v < n-1 \\ n-1 & \text{otherwise} \end{cases} \\ v \widetilde{\leftrightarrow} w &= \begin{cases} 0 & \text{if } v < w \text{ or } v = n-1 \\ 1 & \text{otherwise} \end{cases} \end{aligned}$$

Obviously,  $\mathbf{M}_n$  is a cover for  $\mathbf{K}$ . On the other hand,  $\text{Taut}(\mathbf{M}_n) \neq \text{Frm}(\mathcal{L})$ , e.g., any formula of the form  $\circ(A)$  takes a (non-designated) value  $> 0$  (for  $n > 1$ ). In fact, every formula of the form  $\circ^k X \leftrightarrow X$  is falsified in some  $\mathbf{M}_n$ .

## 4 Optimal Covers

By Proposition 3.4 it is decidable if a given  $m$ -valued logic  $\mathbf{M}$  is a cover of  $\mathbf{C}$ . Since we can enumerate all  $m$ -valued logics, we can also find all covers of  $\mathbf{C}$ . Moreover, comparing two many-valued logics as to their sets of tautologies is decidable, as the next theorem will show. Using this result, we see that we can always generate optimal covers for  $\mathbf{L}$ .

**4.1. DEFINITION** For two many-valued logics  $\mathbf{M}_1$  and  $\mathbf{M}_2$ , we write  $\mathbf{M}_1 \leq \mathbf{M}_2$  iff  $\text{Taut}(\mathbf{M}_1) \subseteq \text{Taut}(\mathbf{M}_2)$ .

$\mathbf{M}_1$  is *better* than  $\mathbf{M}_2$ ,  $\mathbf{M}_1 \triangleleft \mathbf{M}_2$ , iff  $\mathbf{M}_1 \leq \mathbf{M}_2$  and  $\text{Taut}(\mathbf{M}_1) \neq \text{Taut}(\mathbf{M}_2)$ .

**4.2. THEOREM** Let two logics  $\mathbf{M}_1$  and  $\mathbf{M}_2$ ,  $m_1$ -valued and  $m_2$ -valued respectively, be given. It is decidable whether  $\mathbf{M}_1 \triangleleft \mathbf{M}_2$ .

*Proof.* It suffices to show the decidability of the following property: There is a formula  $A$ , s.t. (\*)  $\mathbf{M}_2 \models A$  but  $\mathbf{M}_1 \not\models A$ . If this is the case, write  $\mathbf{M}_1 \triangleleft^* \mathbf{M}_2$ .  $\mathbf{M}_1 \triangleleft \mathbf{M}_2$  iff  $\mathbf{M}_1 \triangleleft^* \mathbf{M}_2$  and not  $\mathbf{M}_2 \triangleleft^* \mathbf{M}_1$ .

We show this by giving an upper bound on the depth of a minimal formula  $A$  satisfying the above property. Since the set of formulas of  $\mathcal{L}$  is enumerable, bounded search will produce such a formula iff it exists. Note that the property (\*) is decidable by enumerating all assignments. In the following, let  $m = \max(m_1, m_2)$ .

Let  $A$  be a formula that satisfies (\*), i.e., there is a valuation  $\mathbf{I}$  s.t.  $\mathbf{I} \not\models_{\mathbf{M}_1} A$ . W.l.o.g. we can assume that  $A$  contains at most  $m$  different variables: if it contained more, some of them must be evaluated to the same truth value in the counterexample  $\mathbf{I}$  for  $\mathbf{M}_1 \not\models A$ . Unifying these variables leaves (\*) intact.

Let  $B = \{B_1, B_2, \dots\}$  be the set of all subformulas of  $A$ . Every formula  $B_j$  defines an  $m$ -valued truth function  $f(B_j)$  of  $m$  variables where the values of the variables which actually occur in  $B_j$  determine the value of  $f(B_j)$  via the matrix of  $\mathbf{M}_2$ . On the other hand, every  $B_j$  evaluates to a single truth value  $t(B_j)$  in the countermodel  $\mathbf{I}$ .

Consider the formula  $A'$  constructed from  $A$  as follows: Let  $B_i$  be a subformula of  $A$  and  $B_j$  be a proper subformula of  $B_i$  (and hence, a proper subformula of  $A$ ). If  $f(B_i) = f(B_j)$  and  $t(B_i) = t(B_j)$ , replace  $B_i$  in  $A$  with  $B_j$ .  $A'$  is shorter than  $A$ , and it still satisfies (\*). By iterating this construction until no two subformulas have the desired property we obtain a formula  $A^*$ . This procedure terminates, since  $A'$  is shorter than  $A$ ; it preserves (\*), since  $A'$  remains a tautology under  $\mathbf{M}_2$  (we replace subformulas behaving in exactly the same way under all valuations) and the countermodel  $\mathbf{I}$  is also a countermodel for  $A'$ .

The depth of  $A^*$  is bounded above by  $m^{m^m+1} - 1$ . This is seen as follows: If the depth of  $A^*$  is  $d$ , then there is a sequence  $A^* = B'_0, B'_1, \dots, B'_d$  of subformulas of  $A^*$  where  $B'_k$  is an immediate subformula of  $B'_{k-1}$ . Every such  $B'_k$  defines a truth function  $f(B'_k)$  of  $m$  variables in  $\mathbf{M}_2$  and a truth value  $t(B'_k)$  in  $\mathbf{M}_1$  via  $\mathbf{I}$ . There are  $m^{m^m}$   $m$ -ary truth functions of  $m$  truth values. The number of distinct truth function-truth value pairs then is  $m^{m^m+1}$ . If  $d \geq m^{m^m+1}$ , then two of the  $B'_k$ , say  $B'_i$  and  $B'_j$  where  $B'_j$  is a subformula of  $B'_i$  define the same truth function and the same truth value. But then  $B'_i$  could be replaced by  $B'_j$ , contradicting the way  $A^*$  is defined. ■

**4.3. COROLLARY** *It is decidable if two many-valued logics define the same set of tautologies. The relation  $\trianglelefteq$  is decidable.*

*Proof.*  $\text{Taut}(\mathbf{M}_1) = \text{Taut}(\mathbf{M}_2)$  iff neither  $\mathbf{M}_1 \triangleleft^* \mathbf{M}_2$  nor  $\mathbf{M}_2 \triangleleft^* \mathbf{M}_1$ . ■

Let  $\simeq$  be the equivalence relation on  $m$ -valued logics defined by:  $\mathbf{M}_1 \simeq \mathbf{M}_2$  iff  $\text{Taut}(\mathbf{M}_1) = \text{Taut}(\mathbf{M}_2)$ , and let  $\text{MVL}_m$  be the set of all  $m$ -valued logics over  $L$ . By  $\mathcal{M}_m$  we denote the set of all sets  $\text{Taut}(\mathbf{M})$  of tautologies of  $m$ -valued logics  $\mathbf{M}$ . The partial order  $\langle \mathcal{M}_m, \subseteq \rangle$  is isomorphic to  $\langle \text{MVL}_m / \simeq, \trianglelefteq / \simeq \rangle$ .

**4.4. PROPOSITION**  *$\langle \mathcal{M}_m, \subseteq \rangle$  is a finite complete partial order.*

*Proof.* The set of  $m$ -valued logics  $\text{MVL}_m$  is obviously finite, since there are at most  $m^{n_1} m^{n_2} \dots m^{n_c}$  different  $m$ -valued matrices for  $C$ .  $\triangleleft$  is a partial order on  $\text{MVL}_m / \simeq$  with the smallest element  $\perp := \text{Frm}(L)$  and the largest element  $\top := \emptyset$ . ■

The “best” logic is the one without theorems, generated by a matrix where no connective takes a designated truth value *anywhere*. The “worst” logic is the one where every formula of  $L$  is a tautology, it is generated by a matrix where every connective takes a designated truth value *everywhere*.

In every complete partial order over a finite set, there exist lub and glb for every two elements of the set. Hence,  $\langle \mathbf{M}, \triangle, \nabla, \perp, \top \rangle$  is a finite complete lattice, where  $\triangle$  is the lub in  $\trianglelefteq$ , and  $\nabla$  is the glb in  $\trianglelefteq$ . Since  $\trianglelefteq$  is decidable and  $\mathbf{M}$  can be automatically generated the functions  $\triangle$  and  $\nabla$  are computable.

**4.5. PROPOSITION** *The optimal (i.e., minimal under  $\triangleleft$ )  $m$ -valued covers of  $C$  are computable.*

*Proof.* Consider the set  $C(\mathbf{C})$  of  $m$ -valued covers of  $\mathbf{C}$ . Since  $C(\mathbf{C})$  is finite and partially ordered by  $\leq$ ,  $C(\mathbf{C})$  contains minimal elements. The relation  $\leq$  is decidable, hence the minimal covers can be computed. ■

**4.6. EXAMPLE** By Example 3.5, **IPC** is strongly sound for  $\mathbf{G}_3$ . The best 3-valued approximation of **IPC** is the 3-valued Gödel logic. In fact, it is the only 3-valued approximation of *any* sound calculus  $\mathbf{C}$  (containing modus ponens) for **IPL** which has less tautologies than **CL**. This can be seen as follows: Consider the fragment containing  $\perp$  and  $\supset$  ( $\neg B$  is usually defined as  $B \supset \perp$ ). Let  $\mathbf{M}$  be some 3-valued strongly sound approximation of  $\mathbf{C}$ . By Gödel's double-negation translation,  $B$  is a classical tautology iff  $\neg\neg B$  is true intuitionistically. Hence, whenever  $\mathbf{M} \models \neg\neg X \supset X$ , then  $\text{Taut}(\mathbf{M}) \supseteq \mathbf{CL}$ . Let 0 denote the value of  $\perp$  in  $\mathbf{M}$ , and let  $1 \in V^+(\mathbf{M})$ . We distinguish cases:

- (1)  $0 \in V^+(\mathbf{M})$ : Then  $\text{Taut}(\mathbf{M}) = \text{Frm}(\mathcal{L})$ , since  $\perp \supset X$  is true intuitionistically, and by modus ponens:  $\perp, \perp \supset X/X$ .
- (2)  $0 \notin V^+(\mathbf{M})$ : Let  $u$  be the third truth value.
  - (a)  $u \in V^+(\mathbf{M})$ : Consider  $A \equiv ((X \supset \perp) \supset \perp) \supset X$ . If  $\mathbf{I}(X)$  is  $u$  or  $1$ , then, since everything implies something true,  $A$  is true (Note that we have  $Y, Y \supset (X \supset Y)/X \supset Y$ ). If  $\mathbf{I}(X) = 0$ , then (since  $0 \supset 0$  is true, but  $u \supset 0$  and  $1 \supset 0$  are both false),  $A$  is true as well. So  $\text{Taut}(\mathbf{M}) \supseteq \mathbf{CL}$ .
  - (b)  $u \notin V^+(\mathbf{M})$ , i.e.,  $V^+(\mathbf{M}) = \{1\}$ : Consider the truth table for implication. Since  $B \supset B$ ,  $\perp \supset B$ , and something true is implied by everything, the upper right triangle is 1. We have the following table:

$\supset$	0	$u$	1
0	1	1	1
$u$	$v_1$	1	1
1	$v_0$	$v_2$	1

Clearly,  $v_0$  cannot be 1. If  $v_0 = u$ , we have, by  $((X \supset X) \supset \perp) \supset Y$ , that  $v_1 = 1$ . In this case,  $\mathbf{M} \models A$  and hence  $\text{Taut}(\mathbf{M}) \supseteq \mathbf{CL}$ . So assume  $v_0 = 0$ .

- (i)  $v_1 = 1$ :  $\mathbf{M} \models A$  (Note that only the case of  $((u \supset 0) \supset 0) \supset u$  has to be checked).
- (ii)  $v_1 = u$ :  $\mathbf{M} \models A$ .
- (iii)  $v_1 = 0$ : With  $v_2 = 0$ ,  $\mathbf{M}$  would be incorrect ( $u \supset (1 \supset u)$  is false). If  $v_2 = 1$ , again  $\mathbf{M} \models A$ . The case of  $v_2 = u$  is the Gödel logic, where  $A$  is not a tautology.

Note that it is in general impossible to algorithmically construct a  $\leq$ -minimal  $m$ -valued logic  $\mathbf{M}$  (i.e., given independently of a calculus) with  $\mathbf{L} \subseteq \text{Taut}(\mathbf{M})$ , because, e.g., it is undecidable whether  $\mathbf{M}$  is empty or not: e.g., take

$$\mathbf{L} = \begin{cases} \{\Box^k(\top)\} & \text{if } k \text{ is the least solution of } D(x) = 0 \\ \emptyset & \text{otherwise} \end{cases}$$

where  $D(x) = 0$  is the diophantine representation of some undecidable set.

## 5 Sequential Approximations of Calculi

In the previous section we have shown that it is always possible to obtain the best  $m$ -valued covers of a given calculus, but there is no way to tell *how good* these covers are. In this section, we investigate the relation between sequences of many-valued logics and the set of theorems of a calculus  $\mathbf{C}$ . Such sequences are called *sequential approximations* of  $\mathbf{C}$  if they verify all theorems and refute all non-theorems of  $\mathbf{C}$ . Put another way, this is a question about the limitations of Bernays' method. On the negative side an immediate result says that calculi for undecidable logics do not have sequential approximations. If, however, a propositional logic is decidable, it also has a sequential approximation (independent of a calculus). However, they all have a uniquely defined *many-valued closure*, whether they are decidable or not. This is the set of all sentences which cannot be proved underivable using a Bernays-style many-valued argument. If a calculus has a sequential approximation, then the set of its theorems equals its many-valued closure. If it does not, then its closure is a proper superset. Different calculi for one and the same logic may have different many-valued closures according to their degree of analyticity.

**5.1. DEFINITION** Let  $\mathbf{C}$  be a calculus and let  $\mathbf{A} = \langle \mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \dots, \mathbf{M}_j, \dots \rangle$  ( $j \in \omega$ ) be a sequence of many-valued logics s.t.

- (1)  $\mathbf{A}$  is given by a recursive procedure,
- (2)  $\mathbf{M}_i \leq \mathbf{M}_j$  iff  $i \geq j$ , and
- (3)  $\mathbf{M}_i$  is a cover for  $\mathbf{C}$ .

$\mathbf{A}$  is called a *sequential approximation* of  $\mathbf{C}$  iff  $\text{Thm}(\mathbf{C}) = \bigcap_{j \in \omega} \text{Taut}(\mathbf{M}_j)$ . We say  $\mathbf{C}$  is *approximable*, if there is such a sequential approximation for  $\mathbf{C}$ .

Condition (2) above is technically not necessary. Approximating sequences of logics in the literature (see next example), however, satisfy this condition. Furthermore, with the emphasis on "approximation," it seems more natural that the sequence gets successively "better."

**5.2. EXAMPLE** Consider the sequence  $\mathbf{G} = \langle \mathbf{G}_i \rangle_{i \geq 2}$  of Gödel logics and intuitionistic propositional logic  $\mathbf{IPC}$ .  $\text{Taut}(\mathbf{G}_i) \supset \text{Thm}(\mathbf{IPC})$ , since  $\mathbf{G}_i$  is a cover for  $\mathbf{IPC}$ . Furthermore,  $\mathbf{G}_{i+1} \triangleleft \mathbf{G}_i$ . This has been pointed out by [8], for a detailed proof see [9, Satz 3.4.1]. It is, however, not a sequential approximation of  $\mathbf{IPC}$ : The formula  $(A \supset B) \vee (B \supset A)$ , while not a theorem of  $\mathbf{IPL}$ , is a tautology of all  $\mathbf{G}_i$ . In fact,  $\bigcap_{j \geq 2} \text{Taut}(\mathbf{G}_j)$  is the set of tautologies of the infinite-valued Gödel logic  $\mathbf{G}_\aleph$ , which is axiomatized by the rules of  $\mathbf{IPC}$  plus the above formula. This has been shown in [5] (see also [9, § 3.4]). Hence,  $\mathbf{G}$  is a sequential approximation of  $\mathbf{G}_\aleph = \mathbf{IPC} + (A \supset B) \vee (B \supset A)$ .

Jaśkowski [12] gave a sequential approximation of  $\mathbf{IPC}$ . That  $\mathbf{IPC}$  is approximable is also a consequence of Theorem 6.7, with the proof adapted to Kripke semantics for intuitionistic propositional logic, since  $\mathbf{IPL}$  has the finite model property [7, Ch. 4, Theorem 4(a)].

The natural question to ask is: Which calculi are approximable? First we give the unsurprising negative answer for undecidable calculi.

**5.3. PROPOSITION** *If  $\mathbf{C}$  is undecidable, then it is not approximable.*

*Proof.* If  $\mathbf{C}$  were approximable, there were a sequence  $\mathbf{A} = \langle \mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \dots \rangle$  s.t.  $\bigcap_{j \geq 2} \text{Taut}(\mathbf{M}_j) = \text{Thm}(\mathbf{C})$ . If  $N$  is a non-theorem of  $\mathbf{C}$ , then there would be an index  $i$  s.t.  $N$  is false in  $\mathbf{M}_i$ . But this would yield a semi-decision procedure for non-theorems of  $\mathbf{C}$ : Try for each  $j$  whether  $N$  is false in  $\mathbf{M}_j$ . If  $N$  is a non-theorem, this will be established at  $j = i$ , if not, we may go on forever. This contradicts the assumption that the non-theorems of  $\mathbf{C}$  are not r.e. ( $\mathbf{C}$  is undecidable and the theorems are r.e.). ■

**5.4. EXAMPLE** This shows that a result similar to that for **IPC** cannot be obtained for full propositional linear logic.

If  $\mathbf{C}$  is not approximable (e.g., if it is undecidable), then the intersection of all covers for  $\mathbf{C}$  is a proper superset of  $\text{Thm}(\mathbf{C})$ . This intersection has interesting properties.

**5.5. DEFINITION** The *many-valued closure*  $MC(\mathbf{C})$  of a calculus  $\mathbf{C}$  is the set of formulas which are true in every many-valued cover for  $\mathbf{C}$ .

$MC(\mathbf{C})$  is unique, since it obviously equals  $\bigcap_{\mathbf{M} \in S} \text{Taut}(\mathbf{M})$  where  $S$  is the set of all covers for  $\mathbf{C}$ . It is also approximable, an approximating sequence is given by

$$\begin{aligned} \mathbf{M}_1 &= \mathbf{M}'_1 \\ \mathbf{M}_i &= \mathbf{M}_{i-1} \times \mathbf{M}'_i \end{aligned}$$

where  $\mathbf{M}'_i$  is an enumeration of  $S$ .

The many-valued closure, however, need not be trivial (i.e., equal to  $\text{Frm}(\mathcal{L})$ )—even for undecidable  $\mathbf{C}$ .

**5.6. PROPOSITION** *If  $\mathbf{C}$  is analytical then  $MC(\mathbf{C})$  is decidable.*

*Proof.* Assume  $\mathbf{C}$  is analytical. A decision procedure for  $A \in MC(\mathbf{C})$  is given by the following: Enumerate all many-valued logics  $\mathbf{M}_i$  in order of increasing number of truth values. Check if  $\mathbf{C}$  is strongly sound for  $\mathbf{M}_i$  (decidable by Proposition 3.4). If it is strongly sound, then check whether  $\mathbf{M}_i \models A$ . If not, terminate with  $A \notin MC(\mathbf{C})$ . By Proposition 2.13, we only have to search until all many-valued logics with number of truth values  $\leq \xi(A)$  have been checked, provided  $\mathbf{C}$  is strongly sound for  $\mathbf{M}' = \Phi(\mathbf{M}, A)$ . Since  $A$  must be a non-tautology of some cover  $\mathbf{M}$  of  $\mathbf{C}$  for  $A \notin MC(\mathbf{C})$  to hold, we can assume that  $\mathbf{M}$  is a cover of  $\mathbf{C}$ . Since  $\text{Taut}(\mathbf{M}) \subseteq \text{Taut}(\mathbf{M}')$ , all axioms of  $\mathbf{C}$  are tautologies in  $\mathbf{M}'$ . Let

$$\frac{A_1 \dots A_n}{A} r$$

be a rule in  $\mathbf{C}$ , and let  $\mathbf{J}$  be an interpretation in  $\mathbf{M}'$  making each  $A_j$  true. If  $\mathbf{J}$  maps no variable to  $\top$ ,  $\mathbf{J}$  is also an interpretation in  $\mathbf{M}$ . Then, since  $\mathbf{C}$  is sound for  $\mathbf{M}$ ,  $A$  is true under  $\mathbf{J}$  (in both  $\mathbf{M}$  and  $\mathbf{M}'$ ). Otherwise, if  $\mathbf{J}$  assigns  $\top$  to some variable  $X$ ,  $A$  is true under  $\mathbf{J}$  since  $X$  occurs in  $A$  (recall that  $\mathbf{C}$  is analytical). So  $\mathbf{C}$  is strongly sound for  $\mathbf{M}'$ . ■

**5.7. COROLLARY** *The many-valued closure of cut-free propositional linear logic  $\mathbf{LL}$  is decidable.*

**5.8. COROLLARY** *If  $\mathbf{C}$  is analytic and decidable, then  $MC(\mathbf{C}) = \text{Thm}(\mathbf{C})$ .*

*Proof.* Certainly  $\text{Thm}(\mathbf{C}) \subseteq MC(\mathbf{C})$ . Let  $A \notin \text{Thm}(\mathbf{C})$ . Then the (infinite-valued) Lindenbaum logic  $\mathbf{L}(\mathbf{C})$  [14, Satz 3] for  $\mathbf{C}$  falsifies  $A$ . Since  $\mathbf{C}$  is decidable,  $\mathbf{L}(\mathbf{C})$  is effectively given.  $\mathbf{L}(\mathbf{C})$  satisfies (\*). It is easy to see that  $\Phi(\mathbf{L}(\mathbf{C}), A)$  also satisfies (\*). By Proposition 2.13 and the argument of the above proof, there is a finite-valued cover for  $\mathbf{C}$  falsifying  $A$ . Hence,  $A \notin MC(\mathbf{C})$ . ■

The last corollary can be used to uniformly obtain semantics for decidable analytic Hilbert calculi.

## 6 Sequential Approximations of Other Representations

Propositional logic can also be given by effective representations other than calculi. A decidable logic, for instance, may be represented by a decision procedure. Logics with Kripke semantics which have the finite model property can be given by the r.e. sequence of their finite models. In this section, we investigate the question of sequential approximation for these representations.

**6.1. PROPOSITION** *For every decidable propositional logic  $\mathbf{L}$  there is a sequence  $\mathbf{A}$  of many-valued logics  $\mathbf{M}_i$  satisfying*

- (1)  $\mathbf{A}$  is given by a recursive procedure,
- (2)  $\mathbf{M}_i \trianglelefteq \mathbf{M}_j$  iff  $i \geq j$ , and
- (3)  $\mathbf{L} \subseteq \text{Taut}(\mathbf{M}_i)$ ,

s.t.  $\mathbf{L} = \bigcap_{i \geq 2} \text{Taut}(\mathbf{M}_i)$ .

*Proof.* The proof uses an argument similar to that of Lindenbaum [14, Satz 3]. Let  $\text{Frm}_i(\mathcal{L}) \subset \text{Frm}(\mathcal{L})$  be the set of formulas of depth  $\leq i$  (which is finite up to renaming of variables). To every formula  $F \in \text{Frm}(\mathcal{L})$  we assign a code  $[F]$ , yielding the sets  $[\text{Frm}_i(\mathcal{L})]$  for all  $i \in \omega$ . We construct a sequential approximation of  $\mathbf{L}$  as follows:  $V(\mathbf{M}_i) = [\text{Frm}_i(\mathcal{L})] \cup \{\top\}$ , with the designated values  $V^+(\mathbf{M}_i) = [\text{Frm}_i(\mathcal{L})] \cap [\mathbf{L}] \cup \{\top\}$ . The truth tables for  $\mathbf{M}_i$  are given by:

$$\begin{aligned} \tilde{\square}_{\mathbf{M}_i}(v_1, \dots, v_n) &= \\ &= \begin{cases} [\square(F_1, \dots, F_n)] & \text{if } v_j = [F_j] \text{ for } 1 \leq j \leq n \\ & \text{and } \square(F_1, \dots, F_n) \in \text{Frm}_i(\mathcal{L}) \\ \top & \text{otherwise} \end{cases} \end{aligned}$$

$\mathbf{M}_i$  is constructed in such a way as to agree with  $\mathbf{L}$  on all formulas of depth  $\leq i$ , and to make all formulas of depth  $> i$  true. Hence,  $\text{Taut}(\mathbf{M}_i) \supseteq \mathbf{L}$ , and  $\mathbf{M}_i \trianglelefteq \mathbf{M}_{i+1}$ . Every formula  $F$  false in  $\mathbf{L}$  is also false in some  $\mathbf{M}_i$  (namely in all  $\mathbf{M}_i$  with  $i \geq$  the depth of  $F$ ). ■

Note that it is in general impossible to algorithmically construct a  $\triangleleft$ -minimal  $m$ -valued logic  $\mathbf{M}$  with  $\mathbf{L} \subseteq \text{Taut}(\mathbf{M})$ , because, e.g., it is undecidable whether  $\mathbf{M}$  is empty or not: e.g., take

$$\mathbf{L} = \begin{cases} \{\Box^k(\top)\} & \text{if } k \text{ is the least solution of } D(x) = 0 \\ \emptyset & \text{otherwise} \end{cases}$$

where  $D(x) = 0$  is the diophantine representation of some undecidable set.

The following definitions are taken from [4].

**6.2. DEFINITION** A modal logic  $\mathbf{L}$  has as its language  $\mathcal{L}$  the usual propositional connectives plus two unary modal operators:  $\Box$  (necessary) and  $\Diamond$  (possible). A Kripke model for  $\mathcal{L}$  is a triple  $\langle W, R, P \rangle$ , where

- (1)  $W$  is any set: the set of worlds,
- (2)  $R \subseteq W^2$  is a binary relation on  $W$ : the accessibility relation,
- (3)  $P$  is a mapping from the propositional variables to subsets of  $W$ .

A modal logic  $\mathbf{L}$  is characterized by a class of Kripke models for  $\mathbf{L}$ .

This is called the *standard semantics* for modal logics (see [4, Ch. 3]). The semantics of formulas in standard models is defined as follows:

**6.3. DEFINITION** Let  $\mathbf{L}$  be a modal logic,  $\mathcal{K}_{\mathbf{L}}$  be its characterizing class of Kripke models. Let  $K = \langle W, R, P \rangle \in \mathcal{K}_{\mathbf{L}}$  be a Kripke model and  $A$  be a modal formula.

If  $\alpha \in W$  is a possible world, then we say  $A$  is *true in  $\alpha$* ,  $\alpha \models_{\mathbf{L}} A$ , iff the following holds:

- (1)  $A$  is a variable:  $\alpha \in P(X)$
- (2)  $A \equiv \neg B$ : not  $\alpha \models_{\mathbf{L}} B$
- (3)  $A \equiv B \wedge C$ :  $\alpha \models_{\mathbf{L}} B$  and  $\alpha \models_{\mathbf{L}} C$
- (4)  $A \equiv B \vee C$ :  $\alpha \models_{\mathbf{L}} B$  or  $\alpha \models_{\mathbf{L}} C$
- (5)  $A \equiv \Box B$ : for all  $\beta \in W$  s.t.  $\alpha R \beta$  it holds that  $\beta \models_{\mathbf{L}} B$
- (6)  $A \equiv \Diamond B$ : there is a  $\beta \in W$  s.t.  $\alpha R \beta$  and  $\beta \models_{\mathbf{L}} B$

We say  $A$  is *true in  $K$* ,  $K \models_{\mathbf{L}} A$ , iff for all  $\alpha \in W$  we have  $\alpha \models_{\mathbf{L}} A$ .  $A$  is *valid in  $\mathbf{L}$* ,  $\mathbf{L} \models A$ , iff  $A$  is true in every Kripke model  $K \in \mathcal{K}_{\mathbf{L}}$ . By  $\text{Taut}(\mathbf{L})$  we denote the set of all formulas valid in  $\mathbf{L}$ .

Many of the modal logics in the literature have the *finite model property (fmp)*: for every  $A$  s.t.  $\mathbf{L} \not\models A$ , there is a finite Kripke model  $K = \langle W, R, P \rangle \in \mathcal{K}$  (i.e.,  $W$  is finite), s.t.  $K \not\models_{\mathbf{L}} A$  (where  $\mathbf{L}$  is characterized by  $\mathcal{K}$ ). We would like to exploit the fmp to construct sequential approximations. This can be done as follows:



**6.4. DEFINITION** Let  $K = \langle W, R, P \rangle$  be an effectively given finite Kripke model. We define the many-valued logic  $\mathbf{M}_K$  as follows:

- (1)  $V(\mathbf{M}_K) = \{0, 1\}^W$ , the set of 0-1-sequences with indices from  $W$ .
- (2)  $V^+(\mathbf{M}_K) = \{1\}^W$ , the singleton of the sequence constantly equal to 1.
- (3)  $\sim_{\mathbf{M}_K}, \tilde{\vee}_{\mathbf{M}_K}, \tilde{\wedge}_{\mathbf{M}_K}, \tilde{\supset}_{\mathbf{M}_K}$  are defined componentwise from the classical truth functions
- (4)  $\tilde{\square}_{\mathbf{M}_K}$  is defined as follows:

$$\tilde{\square}_{\mathbf{M}_K}(\langle w_\alpha \rangle_{\alpha \in W})_\beta = \begin{cases} 1 & \text{if for all } \gamma \text{ s.t.} \\ & \beta R \gamma, w_\gamma = 1 \\ 0 & \text{otherwise} \end{cases}$$

- (5)  $\tilde{\diamond}_{\mathbf{M}_K}$  is defined as follows:

$$\tilde{\diamond}_{\mathbf{M}_K}(\langle w_\alpha \rangle_{\alpha \in W})_\beta = \begin{cases} 1 & \text{if there is a } \gamma \text{ s.t.} \\ & \beta R \gamma \text{ and } w_\gamma = 1 \\ 0 & \text{otherwise} \end{cases}$$

Furthermore,  $\mathbf{I}_K$  is the valuation defined by  $\mathbf{I}_K(X)_\alpha = 1$  iff  $\alpha \in P(X)$  and  $= 0$  otherwise.

**6.5. LEMMA** Let  $\mathbf{L}$  and  $K$  be as in Definition 6.4. Then the following hold:

- (1) Every valid formula of  $\mathbf{L}$  is a tautology of  $\mathbf{M}_K$ .
- (2) If  $K \not\models_{\mathbf{L}} A$  then  $\mathbf{I}_K \not\models_{\mathbf{M}_K} A$ .

*Proof.* Let  $B$  be a modal formula, and  $K' = \langle W, R, P' \rangle$ . We prove by induction that  $\text{val}_{\mathbf{I}_{K'}}(B)_\alpha = 1$  iff  $\mathcal{X}' \models_{\mathbf{L}} B$ :

$B$  is a variable:  $P'(B) = W$  iff  $\mathbf{I}_K(B)_\alpha = 1$  for all  $\alpha \in W$  by definition of  $\mathbf{I}_K$ .

$B \equiv \neg C$ : By the definition of  $\sim_{\mathbf{M}_K}$ ,  $\text{val}_{\mathbf{I}_K}(B)_\alpha = 1$  iff  $\text{val}_{\mathbf{I}_K}(C)_\alpha = 0$ . By induction hypothesis, this is the case iff  $\alpha \not\models_{\mathbf{L}} C$ . This in turn is equivalent to  $\alpha \models_{\mathcal{X}'} B$ . Similarly if  $B$  is of the form  $C \wedge D$ ,  $C \vee D$ , and  $C \supset D$ .

$B \equiv \square C$ :  $\text{val}_{\mathbf{I}_K}(B)_\alpha = 1$  iff for all  $\beta$  with  $\alpha R \beta$  we have  $\text{val}_{\mathbf{I}_K}(C)_\beta = 1$ . By induction hypothesis this is equivalent to  $\beta \models_{\mathbf{L}} C$ . But by the definition of  $\square$  this obtains iff  $\alpha \models_{\mathbf{L}} B$ . Similarly for  $\diamond$ .

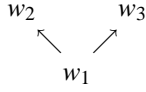
(1) Every valuation  $\mathbf{I}$  of  $\mathbf{M}_K$  defines a function  $P_{\mathbf{I}}$  via  $P_{\mathbf{I}}(X) = \{\alpha \mid \mathbf{I}(X)_\alpha = 1\}$ . Obviously,  $\mathbf{I} = \mathbf{I}_{P_{\mathbf{I}}}$ . If  $\mathbf{L} \models B$ , then  $\langle W, R, P_{\mathbf{I}} \rangle \models_{\mathbf{L}} B$ . By the preceding argument then  $\text{val}_{\mathbf{I}}(B)_\alpha = 1$  for all  $\alpha \in W$ . Hence,  $B$  takes the designated value under every valuation.

(2)  $A$  is not true in  $K$ . This is the case only if there is a world  $\alpha$  at which it is not true. Consequently,  $\text{val}_{\mathbf{I}_K}(A)_\alpha = 0$  and  $A$  takes a non-designated truth value under  $\mathbf{I}_K$ . ■

The above method can be used quite in general to construct many-valued logics from Kripke structures for not only modal logics, but also for intuitionistic logic. Kripke semantics for  $\mathbf{IPL}$  are defined quite similar, with the exception that  $\alpha \models A \supset B$

iff  $\beta \models A \supset B$  for all  $\beta \in W$  s.t.  $\alpha R \beta$ . **IPL** is then characterized by the class of all finite trees [7, Ch. 4, Thm. 4(a)]. Note, however, that for intuitionistic Kripke semantics the form of the *assignments*  $P$  is restricted: If  $w_1 \in P(X)$  and  $w_1 R w_2$  then also  $w_2 \in P(X)$  [7, Ch. 4, Def. 8]. Hence, the set of truth values has to be restricted in a similar way. Usually, satisfaction for intuitionistic Kripke semantics is defined by satisfaction in the *initial* world. This means that every sequence where the first entry equals 1 should be designated. By the above restriction, the only such sequence is the constant 1-sequence.

**6.6. EXAMPLE** The Kripke tree with three worlds



yields a five-valued logic  $\mathbf{T}_3$ , with  $V(\mathbf{T}_3) = \{000, 001, 010, 011, 111\}$ ,  $V^+(\mathbf{T}_3) = \{111\}$ , the truth table for implication

$\supset$	000	001	010	011	111
000	111	111	111	111	111
001	010	111	010	111	111
010	001	001	111	111	111
011	000	001	010	111	111
111	000	001	010	011	111

$\perp$  is the constant 000,  $\neg A$  is defined by  $A \supset \perp$ , and  $\vee$  and  $\wedge$  are given by the componentwise classical operations.

The Kripke chain with four worlds corresponds directly to the five-valued Gödel logic  $\mathbf{G}_5$ . It is well known that  $(X \supset Y) \vee (Y \supset X)$  is a tautology in all  $\mathbf{G}_m$ . Since  $\mathbf{T}_3$  falsifies this formula (take 001 for  $X$  and 010 for  $Y$ ), we know that  $\mathbf{G}_5$  is not the best five-valued approximation of **IPL**.

Furthermore, let

$$\begin{aligned} O_5 &= \bigwedge_{1 \leq i < j \leq 5} (X_i \supset X_j) \vee (X_j \supset X_i) \text{ and} \\ F_5 &= \bigvee_{1 \leq i < j \leq 5} (X_i \supset X_j). \end{aligned}$$

$O_5$  assures that the truth values assumed by  $X_1, \dots, X_5$  are linearly ordered by implication. Since neither  $010 \supset 001$  nor  $001 \supset 010$  is true, we see that there are only four truth values which can be assigned to  $X_1, \dots, X_5$  making  $O_5$  true. Consequently,  $O_5 \supset F_5$  is valid in  $\mathbf{T}_3$ . On the other hand,  $F_5$  is false in  $\mathbf{G}_5$ .

**6.7. THEOREM** Let  $\mathbf{L}$  be a modal logic characterized by a r.e. set of finite Kripke models, and  $\langle A_1, A_2, \dots \rangle$  an enumeration of its non-theorems. A sequential approximation of  $\mathbf{L}$  is given by  $\langle \mathbf{M}_1, \mathbf{M}_2, \dots \rangle$  where  $\mathbf{M}_1 = \mathbf{M}_{K_1}$ , and  $\mathbf{M}_{i+1} = \mathbf{M}_i \times \mathbf{M}_{K_{i+1}}$  where  $K_i$  is the smallest finite model s.t.  $K_i \not\models_{\mathbf{L}} A_i$

*Proof.* (1)  $\text{Taut}(\mathbf{M}_i) \supseteq \text{Taut}(\mathbf{L})$ : By induction on  $i$ : For  $i = 1$  this is Lemma 6.5 (1). For  $i > 1$  the statement follows from Lemma 2.12, since  $\text{Taut}(\mathbf{M}_{i-1}) \supseteq \text{Taut}(\mathbf{L})$  by induction hypothesis, and  $\text{Taut}(\mathbf{M}_{K_i}) \supseteq \text{Taut}(\mathbf{L})$  again by Lemma 6.5 (1).

(2)  $\mathbf{M}_i \leq \mathbf{M}_{i+1}$  from  $A \cap B \subseteq A$  and Lemma 2.12.

(3)  $\text{Taut}(\mathbf{L}) = \bigcap_{i \geq 1} \text{Taut}(\mathbf{M}_i)$ . The  $\subseteq$ -direction follows immediately from (1). Furthermore, by Lemma 6.5 (2), no non-tautology of  $\mathbf{L}$  can be a member of all  $\text{Taut}(\mathbf{M}_i)$ , whence  $\supseteq$  holds. ■

**6.8. Remark** Note that Theorem 6.7 does not hold in general if  $\mathbf{L}$  is not finitely axiomatizable. This follows from Proposition 5.3 and the existence of an undecidable recursively axiomatizable modal logic which has the fmp (see [17]). Note also the condition in Theorem 6.7 that there is an enumeration of the non-theorems of  $\mathbf{L}$ . Since finitely axiomatizable logics with the fmp are decidable ([11]), there always is such an enumeration for the logics we consider.

This theorem can also be used to show that the many-valued closure of a calculus for a modal logic with the fmp equals the logic itself, provided that the calculus contains modus ponens and necessitation as the only rules. (All standard axiomatizations are of this form.)

## 7 Conclusion

The main open problem, especially in view of possible applications in computer science, is the complexity of the computation of optimal covers. One would expect that it is tractable at least for some reasonable classes of calculi which are syntactically characterizable, e.g., analytic calculi.

A second problem is in how far approximations can be found for first-order logics and calculi. One obstacle, for instance, is that it is difficult to check whether a matrix is normal for a given calculus, in particular if the rules of the calculus are not “monadic” in the sense that they manipulate more than one variable at a time. In any case, a systematic treatment only seems feasible for many-valued logics with, at most, distribution quantifiers [3].

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