Burgess and the Bucket: The Emergence of Spacetime in Classical Theories of Gravitation*

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Abstract

The paper studies in detail a precise formal construction of spacetime from matter suggested by the logician John Burgess. We presuppose a continuous and perdurantistic matter ontology. The result is a systematic method to translate claims about the geometry of a flat relativistic, or classical, spacetime into claims about geometrical relations between matter points. The approach is extended to electric and magnetic fields by treating them as multifields defined on matter, rather than as fields in the vacuum. A few tentative suggestions are made to adapt the method to general relativity and to quantum theories.

1 Introduction

In his article "Synthetic Mechanics Revisited", John Burgess [1991] considers the question of whether our best physical theories can dispense with spacetime. He suggests a way to reduce facts about space to facts about matter, if certain existence assumptions about matter are made. The assumption he uses is that there exists what we may call a *plenum* of matter. A plenum of matter is a mereological aggregate of matter points that is apt to fill an open region of space, or that contains a part that is apt to fill an open region of space.

Let T_S be a substantival theory of space, such as the axiomatization of Euclidean geometry in [Tarski 1959]. The theory adopts as primitives two predicates of congruence and betweenness. Let T_R be a relational theory of matter with similar primitives as its substantivalist counterpart, except that they hold between matter points. For example, a matter point p_1 can be between matter points p_2 and p_3. The postulates of the theory will be discussed later. Burgess gives a very rough sketch of the following claim. There is a Morita extension T_R^+

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of T_R in which points of space, or unoccupied locations, are logically constructed from matter by abstraction. The idea is to code facts about unoccupied points within the plenum. An unoccupied point will be simulated by triples of matter points. In other words, points will be identified to equivalence classes of triples under a certain relationistically definable equivalence relation.

This claim can be made more precise as follows. Take a model $\mathfrak{M}$ of classical continuum mechanics in absolute Newtonian space, such that the matter fills up an open ball. Let $\mathfrak{M}'$ be its relationist reduct: a set of material points with the relations of betweenness and proportionality inherited from their spatial locations. We can extend $\mathfrak{M}'$ to form an intended model of T_R^+, with classes of triples and defined relations of congruence and betweenness on such classes, that contains an isomorphic copy of substantival space. Thus there is a clear sense in which space can be recovered from matter.

1.1 The philosophical significance of the construction

What is the philosophical significance of the construction? We should begin by distinguishing two positions we might have with respect to entities of some sort, eliminativism and reductionism. Eliminativism is the view that the entity in question does not exist at all. For example, scientists are eliminativist about caloric, phlogiston, and the aether. Reductionism is the view that the entity in question exists but is nonfundamental or ontologically derivative. Reductionism seems plausible in the case of life, tables, chairs and possibly consciousness.

Correspondingly, there are two sorts of relationism. Eliminativist relationism claims that there is no such thing as space or spacetime at all. Reductionist relationism claims that space or spacetime is ontologically derivative of matter. In his classic paper "*Can we dispense with spacetime* ?", Hartry Field [1984] begins by defining relationism as eliminativist relationism:

> It is tempting to put this doctrine by saying that there are no spacetime regions, but only aggregates of matter. [...]

but then pivots to a form including reductive relationism.

> [...] This formulation might be faulted, for a relationist might want to 'logically construct' regions out of aggregates of matter, and given such a 'logical construction' the relationist will assert that regions do exist. [Field 1984, 123] (reprinted in [Field 1989, 171])

The relationism to be explored in this paper is a relationism of the second variety. To telescope the two formulations, relationism is the view that there does not exist a spacetime over and above aggregates of matter and their interrelations. The reductive relationism we are interested in says that there is a spacetime, although it does not exist *over and above matter* (see [Field 1989, 171]). Both eliminativism and reductionism are present in the literature, without being sharply distinguished. As Caspar Jacobs [2024] points out in a recent reply to Teitel [2019], "Teitel defines substantivalism as the thesis that spacetime exists, rather than (as is more common in the contemporary literature) the claim that spacetime exists independently from matter" (see [Jacobs 2024, 1]). As the partisans of the dynamical interpretation put it, one debate is not about the geometrical facts, but about ontological priority or the direction of explanation (see [Read 2020, 3]). Minkowski spacetime need not be a glorious nonentity. Minkowski spacetime could be a glorious construction. As Brown [2005] puts it, spacetime may be a 'codification' of facts about particles or matter fields (see [Brown 2005, 25] [Brown and Read, 2021]).

1.2 The Method

The method of reduction, or of construction, we employ is the method of paraphrasing. The method of systematic paraphrase was originally introduced as a technique for avoiding having to admit the existence of some suspect entity or entities [Quine 1953, 65] [White 1948, 34] [Jackson 1980]. However, as William Alston [1958] and others have pointed out, the move makes little conceptual sense: if the sentence 'goblins exist' is synonymous with some harmless sentence, a speaker has to accept or deny them both.

A better interpretation of paraphrase is that it sweetens the pill, showing that the existence of the initially suspect entities is in fact unproblematic. We also understand it as a means to prove metaphysical priority. With the exception of eliminativists (see Churchland [1984]), materialists do not claim that a sentence like 'Joe is hungry' is *false*. They will claim that it is made true by facts about Joe's brain state, or its functional organization. Similarly, the reductive relationist does not deny that sentences like 'spacetime is eleven dimensional', 'spacetime is continuous', 'there is a point p midway between the Earth and the Moon' are *false*. The contention is that they are true or false in virtue of the arrangement of material bodies.

Gordon Belot [2011, 33] defines the task of the relationist as the quest to state truth conditions for geometrical statements. For a general conception of ground in terms of translation, see the discussion in [Sider forth., 5-7]. It is misleading to append adjectives like 'relationist' and 'substantivalist' to theories, and say that general relativity is a substantivalist theory. The theory posits the existence of regions, but a reductive relationist can literally believe it. Relationists need not be alienated from our best scientific theories. Reductionists do not *want* a rival theory, such as shape dynamics and the like.

1.3 Enriched Relationism

The approach of Burgess [1991] is a version of an approach known in the literature as 'enriched relationism' (see [Earman 1989], [Maudlin 1993], [Pooley 2013]). Traditional or *empoverished* relationists admit only metric relations between matter points at an instant of time. The *enriched* relationist maintains the light ontology but adopts a heavier ideology. A theory is relationist, in this enriched sense, if it adopts the same primitive relations as spacetime geometry, except that these hold between matter points or material bodies.

This paper answers the question of how much more ideology an enriched relationist has to adopt to obtain a physically adequate theory. The answer is that, if we assume a continuous matter, we can do with two predicates: a ternary predicate of betweenness and an octonary predicate of proportionality. Betweenness applies to three points on a line, including inertial trajectories. Proportionality replaces the more usual quaternary predicate of congruence. These notions allow the enriched relationist to define the difference between inertial and accelerated motions and respond to the famous bucket argument that Newton [1729] gives against relationism, and to which we shall return (see also [Earman 1989, chapt. 4] for a discussion of the bucket argument). However, Earman [1989] and Maudlin [1993] proceed by postulating that facts about material bodies determine an embedding of the geometry of matter into a spacetime structure. The spacetime in question can be considered purely mathematical [Huggett 2006], or otherwise 'fictional' [Maudlin 1993].

The physical laws are then formulated by referring to this fictional spacetime structure. This approach seems to us to be unsatisfactory, for various reasons. The theory one obtains is definitionally equivalent with special relativity, as usually presented, but the paraphrase turns geometrical statements into mixed mathematical statements referring to matter and 'platoplasm'. This raises the suspicion that we are trading spacetime substantivalism for pythagoreanism, or heavy duty platonism (see [Field 1989, 186-188]. The method of Burgess [1991] has the advantages of honest toil over theft. It provides us with an embedding by construction, rather than by *fiat*. That is, section (8) explains how to formulate a theory of matter, without quantifying over spacetime regions or abstract mathematical objects, and from which it is possible to recover spacetime geometry and classical gravitation theory. Moreover, Maudlin [1993] and Earman [1989] give up when it comes to general relativity. The methods of Burgess [1991] may allow us to go further. Our new proposal, to be presented in this paper, builds on Burgess [1991]. It proposes to construe fields as a distribution of properties to triples of matter points. This opens the way to an integration of gravity into the picture. However, this requires us to abandon the equivalence principle and view gravity as a spin two-field.

Burgess [1991] claims that his approach works just as well for classical and flat relativistic spacetime as it does for space. When it comes to the curved spacetimes of general relativity, he does not feign any hypothesis. His proof sketch, even for the case he discusses, is short and cryptic. His whole discussion is a paragraph long and we will see that it hides several technical difficulties. The first aim of this paper is to give a correct and intelligible reconstruction of his approach and fill in the details of a sophisticated enriched relationism. The second aim is to add a treatment of force fields.

1.4 Modal Implications

One concern of relationists has been *shifts*, such as the genuine but unrealized possibility that the world had been displaced of three meters in some direction in absolute space. It should be noted that these possibilities disappear.

Fundamentally, there is no space or spacetime in which to displace matter points. Another way to see it is that, given our definition of 'point', 'collinearity' and 'congruence', it is analytic that a point is at a certain distance and direction from the matter points out of which it is constructed. If we try to 'move' or 'rotate' the three matter points around an axis, then we end up 'rotating' the spacetime point as well, so that we end up on exactly the same possibility.

2 The primitives of T_{CM}

In this paper, we will describe the method by constructing a relationist version of classical mechanics. We will start from a theory T_{CM} of continuum mechanics in Galilean spacetime. We assume that its ontology consists of spacetime regions and aggregates of instantaneous matter points. Assume that the theory is formalized as a two sorted first order theory. It has two sorts of variables, for example ξ_i and p_j, for parts of spacetime and matter respectively.

We will construct a relationist theory T_R and recover $T_{CM} = T_R^+$ as a Morita extension of T_R. We will later explain how to do the same for a relativistic spacetime, and how to construct force fields defined in spacetime.

First of all, the theory T_{CM} contains:

- A binary predicate of parthood that applies to regions, Part $\xi_1\xi_2$, that holds between two regions of spacetime when one is a part of the other.

A point of spacetime can be defined in terms of parthood as a region that is mereologically simple. Galilean spacetime consists of a series of Euclidean spaces stacked on top of each other. Simultaneous points stand in relations of spatial congruence. Points at different times stand in relations of temporal congruence. There is also a global affine or inertial structure. The geometrical predicates of T_{CM} are the following three:

- A binary predicate of simultaneity $Sim\ \xi_1\xi_2$ which applies to two spacetime points ξ_1 and ξ_2 when they both lie in the same surface of simultaneity.

- A ternary predicate of betweenness $Between\ \xi_1\xi_2\xi_3$ that applies to three spacetime points, ξ_1, ξ_2 and ξ_3 when ξ_2 is between ξ_1 and ξ_3. This primitive captures the affine structure of Galilean spacetime.

- A quaternary predicate $Congruent_S\ \xi_1\xi_2\xi_3\xi_4$ of spatial distance that applies to four points ξ_1, ξ_2, ξ_3 and ξ_4 when ξ_1 and ξ_2 are simultaneous, ξ_3 and ξ_4 are simultaneous and the spatial distance between ξ_1 and ξ_2 is the same as the spatial distance between ξ_3 and ξ_4.

From the relation of spatial congruence and the relation of simultaneity, we can define the notion of temporal congruence (see [Field 2016, 53 n. 32]).

The main ontological assumption of the theory is that matter is continuous, and that it perdures through time. Familiar material objects like chairs and neutrinos are spacetime worms. They are the fusion or mereological sum of instantaneous and inextended matter points. The physical theory T_{CM} will also contain the following predicates:

- A binary predicate of location, Location $p_1\xi_1$, that holds between a matter point and the point of spacetime at which it is located.
- A binary predicate of parthood that applies to matter, Part p_1p_2, that holds between two aggregates of matter (or between a matter point and an aggregate of matter) when the former is a part of the latter.
- A binary predicate of genidentity that applies to matter points, Genidentical p_1p_2, that holds of two matter points when they are both temporal parts of the same perduring particle.

The predicate of location can also be assumed to hold between a mereological sum or fusion of matter points and a region of spacetime. We will informally use the words 'is occupied by' as the converse of 'is located at'. Therefore, a point of spacetime ξ is occupied by a matter point p if and only if the matter point p is located at point ξ. The point ξ will also be referred to as the location of p. We can then *define* a location predicate for aggregates of matter points. An aggregate p is located at region ξ if all its pointlike parts are located at a point of ξ and all points of ξ are occupied by mass points in p.

There will also be purely physical predicates for fields, or for forces, that we will leave unspecified. They can be assumed to be along the lines of the comparative predicates for fields axiomatized in Burgess [1984]. His examples concern only scalar fields. However, it is obvious how to extend them to vector and even tensor fields. Just break down these fields into components.

Galilean spacetime has recently been axiomatized in [Ketland, forthcoming].

To give axioms for classical continuum mechanics would take too much space. We assume that it can be done. The theory T_{CM}, or better, a theory closely related to it, should admit of representation theorems in the style of [Tarski 1959] and [Field 1980]. Since the theories are first-order theories, they admit many nonstandard models of many cardinalities. But the 'intended' or 'standard' models of the theory, which can be defined as the models that satisfy, as an *additional* constraint, a second order continuity axiom C, ought to be coordinatizable by models of classical continuum mechanics in the ordinary mathematical sense. The axioms C states, roughly, that any bounded set of points in a line has a least upper bound (see [Tarski and Givant 1999, 185]).

Moreover, the class of reference frames ought to be related by, and closed under, galilean transformations. The theory T_{CM} of continuum mechanics we shall describe is thus a first order theory, but, if we add to it the apparatus of second order variables and an axiom of continuity C, the models of the theory $T_{CM} + C$ must be *representable* by the mathematical structures in classical continuum mechanics (à la Tarski [1953]). This second order extension is used *only* as a way to single out the *intended* models of the theory.[1]

The ontology of the substantivalist theory T_{CM} is dualistic: it contains both spacetime and matter. The ontology of T_R will contain only matter. Every object will be assumed to be a fusion of instantaneous mass points.

Using the predicate of location, we can extend geometrical predicates from points to matter points. We will use the adverb 'material' or a subscript 'M' to indicate that the predicate holds between mass points when their locations satisfy the original. For example, a triple of points stands in the relation of material betweenness if and only if their locations are between each other:

Definition 1. *Material Between* $p_1 p_2 p_3 \leftrightarrow_{df} \exists \xi_1 \exists \xi_2 \exists \xi_3$ *(Location* $p_1 \xi_1 \wedge$ *Location* $p_2 \xi_2 \wedge$ *Location* $p_3 \xi_3 \wedge$ *Between* $\xi_1 \xi_2 \xi_3$*)*

We will refer to the predicate 'Between$_M$' as the material counterpart of the predicate 'Between'. In general, if $P(\xi_1, ..., \xi_n)$ is a predicate of L_{CM}, then we introduce a material counterpart as follows. If the extension of $P(\xi_1, ...\xi_n)$ in a model $\mathfrak{M}$ is the relation R, then the intended extension of the material counterpart $P_M(p_1, ...p_n)$ is the following relation:

$$R_M = \{\langle p_1, ..., p_n \rangle \mid \text{there are } \xi_1 ... \xi_n \in \mathcal{M}, \text{ such that } \mathfrak{M} \\ \models \text{Location } p_1 \xi_1, ..., \mathfrak{M} \models \text{ Location } p_n \xi_n, \text{ and} \langle \xi_1, ..., \xi_n \rangle \in R\}$$

The relationist language L_R takes predicates like these as primitives.

We define a relationist definable reduct of a model of mechanics as follows:

Definition 2. *A relationist definable reduct* $\mathfrak{N}$ *of* $\mathfrak{M}$ *is a structure* $\langle P, R_M^1, ..., R_M^n \rangle$ *with* $P = \{p \mid \mathfrak{M} \models \exists \xi_1$ *Located* $p \xi_1 \wedge \forall \xi_2 ($*Part* $\xi_2 \xi_1 \rightarrow \xi_1 = \xi_2)\}$ *is the set of matter points in* $\mathfrak{M}$ *and the* R_M^i *are definable relations in the structure* $\mathfrak{M}$*.*

The primitive ontology of T_R is monistic: it initially only quantifies over fusions of matter points. We will now see how to logically construct geometrical points, and in particular all the unoccupied points of spacetime.

For the construction to succeed, we need to take as primitive a relation of proportionality rather than a relation of congruence. Burgess [1991] assumes that a relation of proportionality is available. But it is easy to see that known definitions of proportionality in terms of congruence break down when unoccupied points are rejected, and that in fact no such definition can be given.

This is a manifestation of a curse that Field [1984] calls '*the problem of quantities*'. In Euclidean geometry, the notion of proportionality can be defined in terms of similar triangles (see also [Burgess and Rosen 1997, 109-110]). In school, we learned that corresponding sides in two similar triangles are proportional: the ratios between the lengths of the corresponding sides are the same. Consider a simple case of similar triangles, as illustrated in fig.1. Take a triangle $\triangle xyz$, and extend both $\overline{xy}$ and $\overline{xz}$ to form a new triangle $\triangle xy_1z_1$.

The side xy stands in the same proportion to xy_1 as xz stands to xz_1. The longer triangle may be uniformly doubled, or tripled, or stretched by any fixed

[1] One may also single out the intended models of T_{CM} *without second order logic*, as those in which every set of individuals in the universe of discourse has a fusion in the universe.

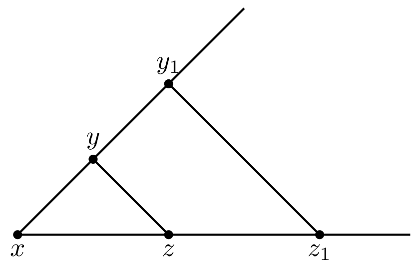

Figure 1: Triangle similarity

factor k. Four arbitrary segments l_1, l_2, l_3 and l_4 in substantival space are going to be proportional if and only if *there exists* a triangle of the form exhibited in fig. 1, and the segments l_1 and l_2 are congruent to respectively xy and xy_1, while l_3 and l_3 are congruent respectively to xz and xz_1. This gives a simple definition of proportionality in a metric substantivalist setting.

In a relationist theory like T_R we cannot presuppose that there is a triangle whose sides match any possible length. Only the substantivalist can assume that any segment of space can be infinitely extended. We cannot, of course, suppose that a tree, rigid rock or any material segment extends indefinitely. As a matter of fact, we can prove something stronger. We cannot define the notion of material proportionality from material betweenness and congruence.

Theorem 1. *There exists a model $\mathfrak{M} = \langle \mathcal{M}, Betweenness, Sim, Congruence \rangle$ of T_{CM} and a relationist definable reduct of $\mathfrak{M}$, $\mathfrak{M}' = \langle \mathcal{M}'$, Material Betweenness, Material Sim, Material Congruence $\rangle$, such that the relation of material proportionality is not a definable relation of the relationist structure $\mathfrak{M}'$.*

Proof. We follow Padoa's method (see [Hodges 1993, 65-66]). Consider a model $\mathfrak{M}$ in which there is exactly one plenum r and exactly one isolated matter point x which is not a part of r. Place the matter point at a distance d from the plenum that is greater than the distance between any two matter points in r. For example, place it at a distance d that is two times the diameter of the plenum. Consider a model $\mathfrak{N}$ just like $\mathfrak{M}$, except that x is at a distance d' greater than d, say three times the diameter of the plenum.

The relations of material congruence and material betweenness are the same in $\mathfrak{M}$ as in $\mathfrak{N}$. The two relationist reducts $\langle \mathcal{M}'$, Material Betweenness, Material Sim, Material Congruence$\rangle$ and $\langle \mathcal{N}'$, Material Betweenness, Material Sim, Material Congruence$\rangle$ are therefore two isomorphic structures. However, the two relationist reducts $\langle \mathcal{M}'$, Material Proportionality$\rangle$ and $\langle \mathcal{N}'$, Material Proportionality$\rangle$ are clearly not isomorphic. Therefore, material proportionality cannot be defined in terms of material congruence, simultaneity and betweenness holding between matter points. □

We will assume material spatial proportionality as a primitive. Another notion that cannot be defined in terms of material simultaneity and betweenness holding between matter points is the temporal order'$<_T$'.

The binary predicate '$<_T$' holds between two matter points when the first exists at an earlier time than the time at which the second exists. If you consider a model with only two nonsimultaneous matter points that are not on the same inertial trajectory, it should be obvious that betweenness and simultaneity by themselves do not fix which one precedes the other. If we call $\mathfrak{N}$ a relationist reduct without further specifications, we are referring to the structure $\langle P, Betw_M, <_T, Proportionality_M, Part_M, Genidentity\rangle$.

2.1 The language and ontology of T_R

The ontology of T_R is monistic: a perdurantistic ontology of aggregates of instantaneous matter points. The primitive notions of the language of L_R will be the material analogs for material bodies, that is, of matter points or aggregates thereof, of certain definable predicates of L_{CM}.[2] They are the predicates corresponding to the relations that feature in the relational reduct. Let us briefly review them in more detail. Intuitively speaking, matter points inherit a geometrical structure from the geometrical structure of their locations. For example, we may say that three matter points are collinear *if* their relations are materially collinear in spacetime, or, in symbols:

$$\exists\xi_1\exists\xi_2\exists\xi_3\ (Located\ p_1\xi_1 \wedge Located\ p_2\xi_2 \wedge Located\ p_3\xi_3 \wedge Collinear\ \xi_1\xi_2\xi_3)$$

The primitive predicates of L_R will be analogs of all the primitive predicates of T_{CM}, except that we do away with the location predicate and replace spatial and temporal congruence with proportionality. For example, the theory T_R speaks of absolute material temporal precedence between matter points:

1. The binary relation of temporal precedence '$p_1 <_T p_2$' holds of two matter points p_1 and p_2 if and only if p_1 temporally precedes p_2.

To capture the affine structure of configurations of matter, we assume a ternary predicate of material betweenness.

2. The ternary predicate 'Between $p_1p_2p_3$' applies to three matter points, p_1, p_2 and p_3 when p_2 is between p_1 and p_3 on a straight line in spacetime.

However, we do not take as primitive a predicate of spatial and temporal congruence between points. We need to assume as primitive notions the predicates spatial and temporal proportionality as applied to matter points:

3. The octonary predicate 'Proportional$_S$ $p_1p_2p_3p_4p_5p_6p_7p_8$' applies to eight matter points, when p_1 to p_4 are simultaneous, p_5 to p_8 are simultaneous,

[2]We should note that a material analog is not the same as the restriction of a predicate to matter points. The geometrical primitives of T_{CM} apply to spacetime and do not have matter points in their extension. But in a supersubstantival presentation of classical mechanics, in which matter points are *identified* with occupied matter points, this subtlety evaporates.

and, in mathematical terms, the following proportion holds:

$$\frac{d(p_1p_2)}{d(p_3p_4)} = \frac{d(p_5p_6)}{d(p_7p_8)}$$

where d is the Euclidean distance.

4. The octonary predicate 'Proportional$_T$ $p_1p_2p_3p_4p_5p_6p_7p_8$' applies to eight matter points, when the ratio of the time elapsed between p_1 and p_2 to that elapsed between p_3 and p_4 is the same as that of the time elapsed between p_5 and p_6 to that elapsed between p_7 and p_8:

$$\frac{d_t(p_1p_2)}{d_t(p_3p_4)} = \frac{d_t(p_5p_6)}{d_t(p_7p_8)}$$

(where d_t is the temporal distance).

We can define a predicate of proportionality to state the disjunction of spatial and temporal proportionality:

D0. Proportional $p_1p_2p_3p_4p_5p_6p_7p_8 \leftrightarrow_{df}$ Proportional$_S$ $p_1p_2p_3p_4p_5p_6p_7p_8 \vee$ Proportional$_T$ $p_1p_2p_3p_4p_5p_6p_7p_8$

It is important to note that the preceding notions apply only to matter points (and not to bodies, or aggregates of matter points). We also need a binary predicate of parthood that relates matter points to bodies:

5. The predicate 'Part p_1p_2', applies to a matter point, or an aggregate of matter points, when it is part of an aggregate of matter points.

Finally we take as primitive the binary predicate of genidentity.

6. The predicate 'Genidentical p_1p_2', applies to two matter points, when they are temporal parts of the same perduring particle.

In favorable circumstances, it may be possible to define genidentity from other physical primitives. If matter is discrete, two points are genidentical if there exists a material timelike curve connecting them. If matter is continuous, but charge or mass density are nowhere constant, the particle emanating from a point p could be defined as the only curve of constant mass density or charge.

If these quantities are constant in a neighborhood of p, or are not conserved, we may still employ features on the quantum state, at least in a quantum mechanical theory. But if all of this fails, the bucket argument transforms into the rotating disk argument against perdurantism (see [Armstrong 1980]). To solve the rotating bucket argument in a perdurantist theory, we will have to take the notion of genidentity as a primitive of our physical theory.

We can define the notion of a matter point in terms of proper parthood:

D1. Matter point $p_1 \leftrightarrow_{df} \forall p_2$ (Part $p_2 p_1 \rightarrow p_2 = p_1$)

The notion of temporal precedence allows us to define a notion of absolute simultaneity between matter points '$\text{Sim}_M\ p_1 p_2$' , which applies to matter points p_1 and p_2 if and only if they lie in the same hyperplane of simultaneity.

D2. $Sim\ p_1 p_2 \leftrightarrow_{df} \neg p_1 <_T p_2 \wedge \neg p_2 <_T p_1$

Our primitive of spatial material proportionality can be used to define the notion of spatial and temporal congruence holding of matter points. The distance between two matter points p_1 and p_2 is the same as the distance between two other matter points p_3 and p_4 if and only if it is the unit ratio: the two segments are proportional to the ratio of a segment to itself. Similarly, we assume as primitive the notion of temporal material proportionality. We define the notion of temporal material congruence in exactly the same fashion.

D3. $\text{Congruence}_S\ p_1 p_2 p_3 p_4 \leftrightarrow_{df} \text{Proportional}_S\ p_1 p_2 p_3 p_4 p_3 p_4 p_3 p_4$

D4. $\text{Congruence}_T\ p_1 p_2 p_3 p_4 \leftrightarrow_{df} \text{Proportional}_T\ p_1 p_2 p_3 p_4 p_3 p_4 p_3 p_4$

Using these primitives, the enriched relationist can distinguish between inertial and accelerated motion, and therefore avoid the famous 'bucket argument' (see [Maudlin 1993, 186-7] for a discussion). Consider a pointlike particle in the bucket. The theory construes such objects as spacetime worms.

The spacetime worm to which an instantaneous matter point belongs is the aggregate of all the points genidentical to it. The particle is moving inertially if and only if, if we pick three temporal parts, they stand in the relation of betweenness. But as soon as the bucket starts to rotate, and we pick three mass points, they cease to stand in the relation of betweenness.

3 The notion of a plenum

The existence of a plenum is a condition for the success of the construction in [Burgess 1991], and also a key ingredient in the construction itself. We clarify the notion and give an informal definition of it in the metalanguage.[3]

Definition 3. *An aggregate a of matter points in a model $\mathfrak{M}$ of T_{CM} is a ball of matter if and only if there is an open ball b in space, not confined to any three dimensional hyperplane embedded in our four dimensional Galilean spacetime*[4]*, such that every point of b is exactly occupied by a matter point in a and every matter point in a exactly occupies a point in b.*

Definition 4. *An aggregate a of matter points in a model $\mathfrak{M}$ of T_{CM} is plenum if and only if there in open connected region U of spacetime such that every point of U is exactly occupied by a matter point in a and every matter point in a exactly occupies a point in U.*

[3] The formal definition of the first order predicate 'Plenum x' can be found in [Babic, 2023].

[4] In the analogous construction of Euclidean space, the condition should read that the ball of matter is not confined to a plane.

The definitions above are stated in an ordinary mathematical language, and not in a formal language. The definition of the predicate 'Plenum x' in the formal language L_R is also straightforward, but extremely long and tedious.

4 The plenum-dependent construction

We are finally ready to explain the construction in the classical spacetime setting. The construction will be relative to a plenum r. This means that there will exist as many emergent spacetimes as there are plena r. We will discuss later the problem of amalgamating these disparate plenum-dependent spaces into a unique spacetime that does not depend on the choice of a plenum.

The key to the entire construction is the definition of indicating a matter point. This notion already foreshadows the coding of spacetime points by triples of matter points. We will say that a matter point p_4 is *indicated* by a triple $\langle p_1, p_2, p_3 \rangle$. The matter point p_2 must be materially between p_1 and p_3. The oriented segment $\overrightarrow{p_1 p_2}$ will point in a specific direction. The direction in spacetime is the same as the oriented segment $\overrightarrow{p_1 p_3}$. This is the same notion of indication as if one of us were to point a finger at an object in space.

We are supposed to look at an object in the continuation of the segment occupied by the finger. But finger pointing does not determine exactly the position of the objection on this line. How far are we supposed to look?

We are often tempted to stop at the first solid object that intersects the path, but such may not be the intention of the indicator. The matter is formally resolved by specifying the distance as a function of the triple $\langle p_1, p_2, p_3 \rangle$. We will stipulate that, if a matter point is indicated at all, it is the point that satisfies a certain equation. Three matter points p_1, p_2 and p_3 inside a plenum r indicate a fourth matter point p_4, which may or may not lie inside the plenum, just in case the following proportionality hold:

$$\frac{d_S(p_1, p_2)}{d_S(p_1, p_3)} = \frac{d_S(p_1, p_3)}{d_S(p_1, p_4)} \quad \text{(See fig. 2)}$$

Formally, this gives the following definition of spatial indication:

D5. Indicate$_S$ $p_1 p_2 p_3 p_4 \leftrightarrow_{df}$ Between $p_1 p_2 p_3 \wedge$ Between $p_2 p_3 p_4 \wedge$ Proportional$_S$ $p_1 p_2 p_1 p_3 p_1 p_3 p_1 p_4$

In spacetime, we can also point to a matter point in a temporal way. For example, we may launch a rocket straight from the Earth at some initial constant speed. We may click or light up a bulb to fix three matter points $\langle p_1, p_2, p_3 \rangle$ on this trajectory at the three different instants of time. A fourth point on the trajectory will be indicated once we decide how long the rocket has travel. The point p_4 will be stipulated to be the unique matter point on the line from p_1 to p_2 that satisfies the following temporal equation:

$$\frac{d_t(p_1, p_2)}{d_t(p_1, p_3)} = \frac{d_t(p_1, p_3)}{d_t(p_1, p_4)}$$

D6. Indicate$_T$ $p_1p_2p_3p_4 \leftrightarrow_{df}$ Between $p_1p_2p_3 \wedge$ Between $p_2p_3p_4 \wedge$ Proportional$_T$ $p_1p_2p_1p_3p_1p_3p_1p_4$

We define indication as the disjunction of temporal and spatial indication.

D7. Indicate $p_1p_2p_3p_4 \leftrightarrow_{df}$ Indicate$_T$ $p_1p_2p_3p_4 \vee$ Indicate$_S$ $p_1p_2p_3p_4$

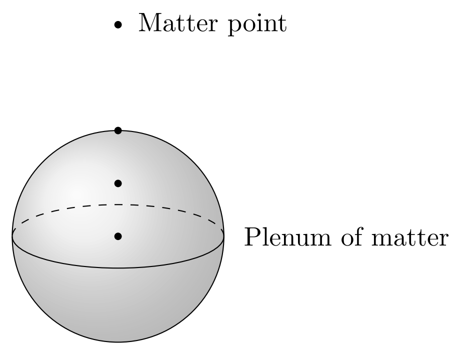

Figure 2: The relation of indication.

The relation of indication is functional. Given three matter points, if there is a fourth indicated by the first three, it is unique. Most of the time, there is no such matter point. Most triples of matter points will point to nothing.

This is the basis for the construction of spacetime from triples. We will use the triple as a substitute for the spacetime location that it indicates. We will see that triples can realize all the functional roles of points. However, the same point can be indicated by more than one triple. In the same way, a fictitious unoccupied point can be realized by more than one triple. We will identify points to a certain equivalence class of triples. The task of the next subsection is to define the appropriate equivalence relation in the language of T_R.

4.1 Equivalent Triples

Since we have at our disposal the predicate of indication, we can define an unoccupied point of spacetime ξ relative to a plenum r as an equivalence class, or a quotient sort (using the terminology of Morita equivalence) of triples of matter points belonging to a plenum r. The move is the familiar method of *abstraction*. It is analogous to the common method of constructing directions as equivalence classes of parallel lines, the natural numbers as equivalence classes of equinumerous classes and meanings as equivalence classes of synonymous expressions. We will write $\vec{p}$ in place of the three variables p_1, p_2 and p_3. For example, we define a quaternary predicate of parthood:

D8. Part $p_1p_2p_3r \leftrightarrow_{df}$ $Part\ p_1r \wedge Part\ p_2r \wedge Part\ p_3r$

In order to introduce the sort ξ, we need to define the relation of indicating the same unoccupied point without referring to points. Of course, if the two triples happen to indicate a real matter point, then they ought to stand in the equivalence relation if and only if they indicate the same matter point p_4. The idea of the definition is precisely to piggyback on the case in which there is a genuine matter point that is indicated. Two triples of matter points inside the same plenum r indicate the same point if it is possible to construct a tiny model in scale of the two triples that indicate the same point (see [Burgess 1991, 129]).

To construct a model in scale of $\langle p_1, p_2, p_3 \rangle$ and $\langle q_1, q_2, q_3 \rangle$, we need to find two other triples $\langle p'_1, p'_2, p'_3 \rangle$ and $\langle q'_1, q'_2, q'_3 \rangle$ that point to a single matter point p_4, and such that the two angles at the base of the triangle $\triangle p'_1 q'_1 p_4$ are equal to the angles $\widehat{p_1 q_1 q_2}$ and $\widehat{q_1 p_1 p_2}$. The condition is illustrated in the next picture.

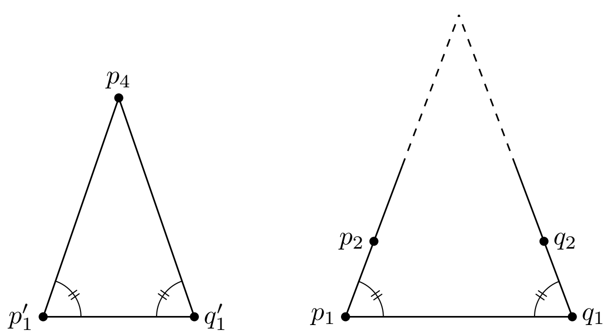

Figure 3: The base angles of $\triangle p'_1 q'_1 p_4$ are equal to $\widehat{p_1 q_1 q_2}$ and $\widehat{q_1 p_1 p_2}$

We can do so by picking two matter points p'_1 and q'_1 on the segment $\overline{p_1 q_1}$ and 'sliding' the two triples with a square. If p'_1 and q'_1 are between p_1 and q_1, and if the half lines pointed at by the triples $\langle p'_1, p'_2, p'_3 \rangle$ and $\langle q'_1, q'_2, q'_3 \rangle$ is parallel to those indicated by $\langle p_1, p_2, p_3 \rangle$ and $\langle q_1, q_2, q_3 \rangle$, then the condition will be satisfied (see fig.4 below for an illustration of this construction).

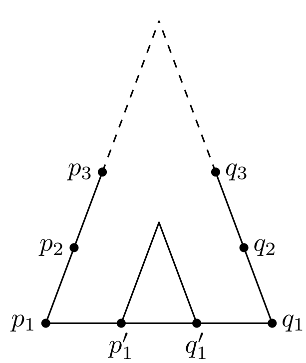

Figure 4: The points p'_1 and q'_1 are between p_1 and q_1.

The notion of material parallelism that is employed here poses no problem since it can be defined in terms of congruence. Two segments $\overline{xy}$ and $\overline{zw}$ are parallel if and only if there are two congruent segments $\overline{x'y'}$ and $\overline{z'w'}$ on the lines passing through $\overline{xy}$ and $\overline{zw}$ (that is, such that x' and y' are collinear to x and y, z' and w' are collinear to z and w) such that x', y', z' and w' form a rectangle. The last condition, rectangularity, means that the segments $\overline{x'z'}$ and $\overline{y'w'}$ and the segments $\overline{x'w'}$ and $\overline{y'z'}$ are spatially congruent.

D9. Parallel $p_1p_2p_3p_4 \leftrightarrow_{df} \exists p_5 \exists p_6 \exists p_7 \exists p_8$(Between $p_1p_2p_5 \wedge$ Between $p_2p_5p_6 \wedge$ Between $p_3p_4p_7 \wedge$ Between $p_4p_7p_8 \wedge$ Congruent$_S$ $p_5p_6p_7p_8$ $\wedge$ Congruent$_S$ $p_5p_7p_6p_8 \wedge$ Congruent$_S$ $p_5p_8p_6p_7$)

The second condition that a model in scale must satisfy is that all the distances must be shrunk by a fixed factor k. Let k be the ratio of the distance from p_1 to q_1 to the distance from the slid matter point p'_1 to matter point q'_1:

$$\frac{d_S(p_1,q_1)}{d_S(p'_1,q'_1)} = k \tag{1}$$

We want all other proportions between matter points in the new triples to be equal to k. This comes down to four claims of proportionality:

$$\frac{d_S(p_1,p_2)}{d_S(p'_1,p'_2)} = k \tag{2}$$

$$\frac{d_S(p_1,p_3)}{d_S(p'_1,p'_3)} = k \tag{3}$$

$$\frac{d_S(q_1,q_2)}{d_S(q'_1,q'_2)} = k \tag{4}$$

$$\frac{d_S(q_1,q_3)}{d_S(q'_1,q'_3)} = k \tag{5}$$

The equivalence relation of indicating the same unoccupied point of space-time will be called 'equi-indication' and noted as $\sim_r$. It can be formally as follows. Note that the equi-indication predicate has seven arguments: six variables for the matter points and a variable r for a particular choice of a plenum.

D10. $p_1p_2p_3 \quad \sim_r \ q_1q_2q_3 \ \leftrightarrow_{df}$ Plenum of matter $r \wedge$ *Part* $\vec{p}r \wedge$ *Part* $\vec{q}r \wedge$

$\exists\vec{p'}\exists\vec{q'}\exists p_4 \Big($*Part* $\vec{p'}r \wedge$ *Part* $\vec{q'}r \wedge$ *Part* $p_4r \wedge$

Parallel $p_1p_2p'_1p'_2 \wedge$ *Parallel* $p_2p_3p'_2p'_3 \wedge$ *Parallel* $q_1q_2q'_1q'_2 \wedge$

Parallel $q_2q_3q'_2q'_3 \wedge$ *Between* $p_1p'_1q_1 \wedge$ *Between* $p'_1q'_1q_1 \wedge$

Proportional $p_1p_2p'_1p'_2p_1q_1p'_1q'_1 \wedge$ *Proportional* $p_2p_3p'_2p'_3p_1q_1p'_1q'_1 \wedge$

$Proportional\ q_1q_2q_1'q_2'p_1q_1p_1'q_1' \ \wedge \ Proportional\ q_2q_3q_2'q_3'p_1q_1p_1'q_1' \ \wedge$

$Indicate\ p_1'p_2'p_3'p_4 \ \wedge \ Indicate\ q_1'q_2'q_3'p_4 \Big)$

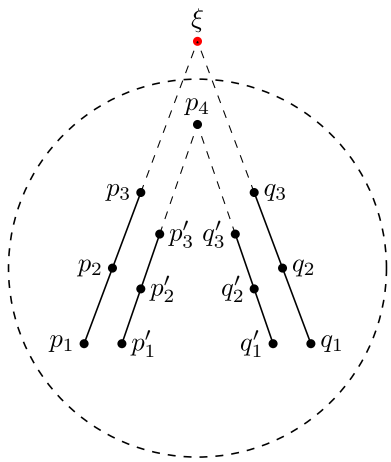

Figure 5: Two triples indicate the same unoccupied point ξ.

We can prove that the definition is adequate in the following way. We can first define an analogous notion of indication in the substantivalist theory. If spacetime exists, then every triple of spacetime points indicates a point. Every point is indicated by some triple of points. Moreover, a triple of matter points indicates a fourth matter point p if and only if their locations indicate the location of p. This holds in every model $\mathfrak{M}$ of classical mechanics T_{CM}.

Consider the matter points that in a model $\mathfrak{M}$ and the relationist reduct $\mathfrak{N}$ consisting of these matter points and the relations of material betweenness and congruence. We can show that they are equivalent in the relationist reduct if and only their spacetime time locations indicate the same point of spacetime in the original spacetime structure. In other words, we can say that two triples are equivalent in a relationally definable sense if and only if, in a possible world in which spacetime existed, they would in fact indicate the same point.

Theorem 2. *Let $\mathfrak{M}$ be a model of T_{CM} and $\mathfrak{N}$ a relationist definable reduct of $\mathfrak{M}$. Let $p_1, ...p_3, q_1, ..., q_3$ be six matter points in the domain of $\mathfrak{N}$ and $x_1, ..., x_3, y_1, ...y_3$ their locations in $\mathfrak{M}$. Then, we have that $\mathfrak{N} \models p_1p_2p_3 \sim_r q_1q_2q_3$ if and only if $\mathfrak{M} \models \exists z\ Indicate\ x_1x_2x_3z \wedge\ Indicate\ \ y_1y_2y_3z$.*

Proof. An exercise in similar triangles. □

4.2 The emergence of spacetime

Once we have the equivalence relation, we can then extend T_R to a theory T_R^+ where a quotient sort or abstraction is introduced. We follow the treatment in

[Halvorson 2019, ch.5]. The variables $\xi_r, \xi'_r, ...$ will range over the equivalence classes, or the spacetime points relative to a plenum r. We will call these 'r-points'. The language of the extension T_R^+ includes a function symbol

$$Ind(p_1, p_2, p_3, r)$$

which takes as arguments triples of matter points in a plenum r together and returns the r-point coded by the triple in the plenum r. We can read $Ind(\vec{p}, r)$ as 'the point of spacetime coded by $\vec{p}$ in r'. T_R^+ then contains all the axioms of T_R plus an abstraction principle, which says that two points ξ and ξ' are identical if and only if their corresponding triples satisfy the equivalence relation above, and an axiom which says that for any point, there is a triple which codes it:

$$\xi_r = \xi'_r \leftrightarrow p_1p_2p_3 \ \sim_r \ p_4p_5p_6 \tag{6}$$

$$\forall \xi_r \, \exists \, \vec{p} \; Ind(\vec{p}, r) = \xi_r \tag{7}$$

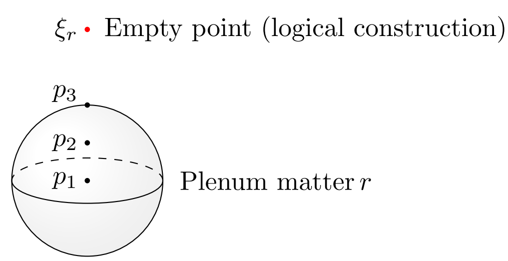

Figure 6: An r-point ξ coded by a triple of matter points $\vec{p}$ inside a plenum r

We can expand canonically a relationist reduct $\mathfrak{N}$ to include $r-$points.

Definition 5. *Let $\mathfrak{M}$ be a model of T_{CM} and $\mathfrak{N} = \langle \mathcal{N}, Sim_M, Between_M, Proportionality_M \rangle$ a relationist definable reduct of $\mathfrak{M}$. Then the canonical extension $\mathfrak{N}^+$ of $\mathfrak{N}$ is the structure*

$$\langle \mathcal{U}, \mathcal{N}, Sim_M, Between_M, Proportionality_M, Ind, \pi_1, \pi_2, \pi_3 \rangle$$

where $\mathcal{U} = \mathcal{N}^3 / Equi{-}indication$ is the partition of triples under equi-indication, Ind is the mapping of triples onto their equivalence class, and the π_i are projection functions from triple onto their ith components.

We can show that, if there is a plenum r, then there is a one-to-one correspondence between $r-$points and points of substantival spacetime:

Theorem 3. *Let $\mathfrak{M}$ be a model of T_{CM} and $\mathfrak{N}$ be a relationist definable reduct of $\mathfrak{M}$. Let $P \subset \mathcal{M}$ be $\{x \mid \mathfrak{M} \models Point\ x\}$. Let $\mathcal{U} = \mathcal{N}^3 / Equi - indication$ be again the partition of triples under equi-indication. If there is an r in $\mathfrak{M}$ such that $\mathfrak{M} \models Plenum\ of\ matter\ r$, there is a one-to-one correspondence between P and U.*

Proof. We define a function $f : U \mapsto P$ as follows. Let ξ be an $r-point$. Pick a triple $\langle p_1, p_2, p_3 \rangle$ in ξ and let their locations in $\mathfrak{M}$ be $\langle x_1, x_2, x_3 \rangle$. Consider the spatiotemporal predicate of indication between spacetime points

$$\textit{Indicate } xyzw$$

From standard geometrical reasoning, it follows that there is a unique point $x_4 \in \mathcal{N}$ such that $\mathfrak{M} \models$ Indicate $x_1x_2x_3x_4$. We set $f(\xi) = x_4$. Injectivity and the fact that f is a function follow both from *Theorem* 2. Surjectivity requires the condition that r be a plenum. Let $x_4 \in \mathcal{N}$ be an arbitrary point. Let p be a point in the interior of r and x its location. From the fact that r is a plenum, it follows that the segment $\overline{xx_4}$ intersects the plenum. There is then a subsegment $\overline{xy}$ of $\overline{xx_4}$ that is entirely occupied by matter. Let us suppose that x, y and x_4 are on an inertial trajectory. If they are simultaneous, the proof is similar, except that we replace the temporal distance with spatial distance. Let z be a point collinear to x and y that is at the following distance from x:

$$d_t(x,z) = \frac{d_t(x,y)^2}{d_t(x,x_4)}$$

The point z is between x and y, since $d_t(x,y) < d_t(x,x_4)$, implies $d_t(x,z) < d_t(x,y)$. Let $\langle x, y, z \rangle$ be the matter points located at $\langle x, y, z \rangle$ and let $\langle p_1, p_2, p_3 \rangle \in \xi$ be their equivalence class. It follows that $f(\xi) = x_4$. □

Definition 6. *We call the function $f : \mathcal{U} \rightarrow P$ defined in the proof of Theorem 3 the canonical function from $r-$points to points.*

5 Geometrical Relations

At this point, we have shown how to define the identity between $r-$points, and we have given an argument for the adequacy of the definition. We will now define geometrical notions of betweenness, congruence and simultaneity holding between $r-$points. We will also prove metatheorems to show their adequacy.

We will prefix the geometrical notions to be defined with an r in order to differentiate them from the materials notions, which apply to matter points, and the substantivalist notions, that apply to points rather than $r-$points. The key to the definitions is the same strategy of constructing a scale model. We will begin with the definition of simultaneity holding between two $r-$points.

We will first need a lemma that says that we can point to any $r-$point from any material point. If ξ is an $r-$point and p is a material point, then there exist two material points p_1 and p_2 such that the triple $\langle p, p_1, p_2 \rangle$ indicates ξ. This implies that, if ξ_1 and ξ_2 are two $r-$points and p is a material point, then there exist two matter points p_1 and p_2, and two matter points q_1 and q_2 in r, such that the triple $\langle p, p_1, p_2 \rangle$ indicates ξ_1 and the triple $\langle p, q_1, q_2 \rangle$ indicates ξ_2.

Lemma 1. *Let $\mathfrak{M}$ be a model of T_{CM} that contains a plenum r and $\mathfrak{N}$ a relationist definable reduct of $\mathfrak{M}$. Then $\mathfrak{N}^+ \models \forall\xi_1 \forall p_1 \forall r\ \exists p_2 \exists p_3\ Ind(p_1, p_2, p_3, r) = \xi_1$*

Proof. As in the proof of Theorem 3. □

Corollary 1. *Let $\mathfrak{M}$ be a model of T_{CM} that contains a plenum r and $\mathfrak{N}$ a relationist definable reduct of $\mathfrak{M}$. Then $\mathfrak{N}^+ \models \forall\xi_1\forall\xi_2\forall p\forall r\ \exists p_1\exists p_2\exists q_1\exists q_2$ $(Ind(p, p_1, p_2, r) = \xi_1 \wedge Ind(xp, q_1, q_2, r) = \xi_2)$*

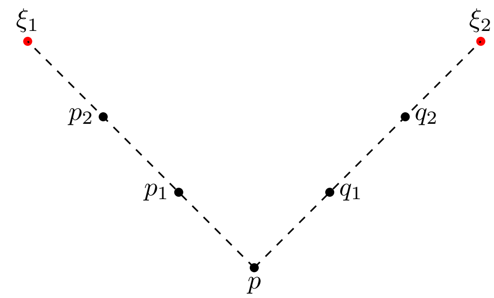

Figure 7: Corollary 1

Consider two $r-$points ξ_1 and ξ_2 indicated by two triples $\langle p, p_1, p_2\rangle$ and $\langle p, q_1, q_2\rangle$. We are using the lemma to reduce the general case to a case in which the two triples have a single origin p. We want to find out whether ξ_1 and ξ_2 should be counted as simultaneous or not. The first scenario is when one of ξ_1 and ξ_2 is located in the simultaneity slice as the matter points that indicate it. If p is simultaneous to p_2, then the point ξ_1 is located in the same simultaneity slice as p and p_1. Therefore, ξ_2 is simultaneous to ξ_1 iff p is simultaneous also to q_2. A symmetric condition holds if p is simultaneous to q_2.

The second scenario is when both $\langle p, p_1, p_2\rangle$ and $\langle p, q_1, q_2\rangle$ are arranged on inertial trajectories. In such a case, the strategy is once again to construct a tiny scale model inside the plenum. Consider the triangle $\triangle p\xi_1\xi_2$. We want to rescale it by a factor k, so as to find a similar triangle within the region of spacetime occupied by matter. We want four points p_1', p_2', q_1', and q_2' on the same sides, that indicate material points, and such that the time elapsed is shrinked by some uniform factor. For example, let

$$k = \frac{d_t(p, p_1')}{d_t(p, p_1)} \tag{8}$$

We want all the other proportion to be k as well. In other words, we want a selection of matter points that satisfies the following equations:

$$\frac{d_t(p, q_1')}{d_t(p, q_1)} = k \tag{9}$$

$$\frac{d_t(p, p_2')}{d_t(p, p_2)} = k \tag{10}$$

$$\frac{d_t(p, q_2')}{d_t(p, q_2)} = k \tag{11}$$

D11. r-Sim $\xi_1\xi_2 r \leftrightarrow_{df} \exists p \exists p_1 \exists q_1 \exists p_2 \exists q_2 \Big(\text{Ind}(p, p_1, p_2, r) = \xi_1 \wedge$

$\text{Ind}(p, q_1, q_2, r) = \xi_2 \wedge \text{Sim}_M\ pp_2 \wedge \text{Sim}_M\ pq_2 \Big) \bigvee$

$\Big(\neg\text{Sim}_M\ pp_2 \wedge \neg\text{Sim}_M\ pq_2 \wedge \exists p_1' \exists p_2' \exists q_1' \exists q_2' \exists p_3 \exists q_3 \ (\text{Between}_M\ p_1 p_2' p$

$\wedge\ \text{Between}_M\ p_2' p_1' p \wedge \text{Between}_M\ q_1 q_2' q \wedge \text{Between}_M\ q_2' q_1' q \wedge \text{Proportional}$

$pp_1' pp_1 pp_2' pp_2 \wedge \text{Proportional}\ pp_1' pp_1 pq_1' pq_1 \wedge \text{Proportional}\ pp_1' pp_1 pq_2' pq_1' \wedge$

$\text{Indicate}\ pp_1' p_2' p_3 \wedge \text{Indicate}\ pq_1' q_2' q_3 \wedge \text{Part}\ p_3 r \wedge \text{Part}\ q_3 r \wedge \text{Sim}_M p_3 q_3) \Big)$

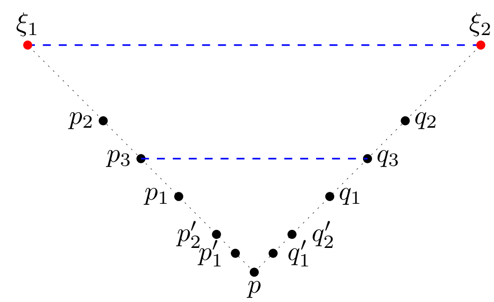

Figure 8: The matter point p_3 is simultaneous to q_3 iff ξ_1 is simultaneous to ξ_2.

Theorem 4. *Let $\mathfrak{M}$ be a model of T_{CM} containing a plenum and $\mathfrak{N}$ be a relationist definable reduct of $\mathfrak{M}$. Let $\mathfrak{N}^+$ be the canonical extension of $\mathfrak{N}$ and f the canonical function relative to certain plenum of matter r. Then for all ξ_1, ξ_2*

$$\mathfrak{N}^+ \models r\text{-}Sim\ \xi_1\xi_2 r \textit{ iff } \mathfrak{M} \models Sim\ f(\xi_1)f(\xi_2)$$

Proof. Another exercise in similar triangles. □

Let us turn to congruence. Consider four r-points ξ_1, ξ_2, ξ_3 and ξ_4 that determine two segments. We want to lay down the conditions that need to be satisfied in order for them to be of equal length. The spatial and temporal cases are analogous. We can suppose that we are in the case of the lemma. There are two points p and q that play the role of the two vertices. The triple $\langle p, p_1, p_2 \rangle$ indicates ξ_1. The triple $\langle p, q_1, q_2 \rangle$ indicates ξ_2. The triple $\langle q, p_3, p_4 \rangle$ indicates ξ_3. The triple $\langle q, q_3, q_4 \rangle$ indicates ξ_4. They form two triangles (fig. 9). We want to know when the opposing sides, in red, should count as equal.

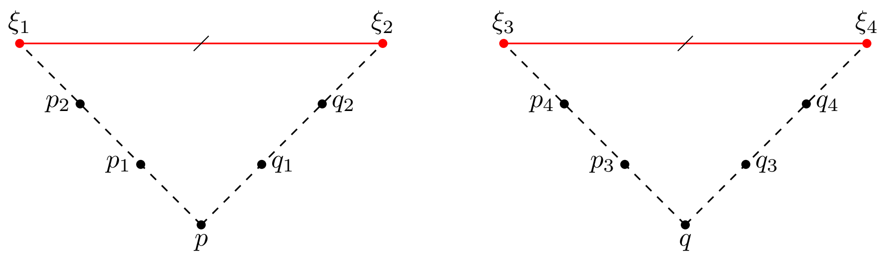

Figure 9: The triples indicating ξ_1, ξ_2, ξ_3 and ξ_4.

The strategy is once again to rescale the two triangles by a factor k, so as to find a similar triangle within the region of spacetime occupied by matter. We want to find eight points $p'_1, p'_2, q'_1, q'_2, p'_3, p'_4, q'_3$, and q'_4 on the same sides, that indicate material points, and such that the distances are shrinked by some uniform factor. Let us suppose that they are on inertial trajectories and let

$$k = \frac{d_t(p, p'_1)}{d_t(p, p_1)} \tag{12}$$

We want all the other proportions to be k as well. In other words, we want a choice of matter points that satisfy the following equations:

$$\frac{d_t(p, q'_1)}{d_t(p, q_1)} = k \tag{13}$$

$$\frac{d_t(p, p'_2)}{d_t(p, p_2)} = k \tag{14}$$

$$\frac{d_t(p, q'_2)}{d_t(p, q_2)} = k \tag{15}$$

$$\frac{d_t(q,p_3')}{d_t(q,p_3)} = k \tag{16}$$

$$\frac{d_t(q,p_4')}{d_t(q,p_4)} = k \tag{17}$$

$$\frac{d_t(q,q_3')}{d_t(q,q_3)} = k \tag{18}$$

$$\frac{d_t(q,q_4')}{d_t(q,q_4)} = k \tag{19}$$

We can find shrinked triples that indicate four matter points p_5, p_5' and q_5, q_5'. The two opposing sides of the two small material triangles are going to be shrinked by the same factor of k. The segments $\overline{p_5p_5'}$ and $\overline{q_5q_5'}$ are therefore going to be congruent if and only if the two $r-$segments $\overline{\xi_1\xi_2}$ and $\overline{\xi_3\xi_4}$ are congruent. Note that just like the identity, this notion of congruence is relative to the choice of a plenum r. The definition can be formalized as follows:

D12. r-Congruent$_S$ $\xi_1\xi_2\xi_3\xi_4 r \leftrightarrow_{df} \forall p\forall p_1\forall p_2\forall q_1\forall q_2\forall q\forall p_3\forall p_4\forall q_3\forall q_4$

$\Big($ Ind$(p,p_1,p_2,r) = \xi_1 \wedge$ Ind$(p,q_1,q_2,r) = \xi_2 \wedge$ Ind$(q,p_3,p_4,r) = \xi_3 \wedge$

Ind$(q,q_3,q_4,r) = \xi_4 \wedge$ r-Sim $\xi_1\xi_2 \wedge$ r-Sim $\xi_3\xi_4 \longrightarrow$

$\exists p_1'\exists p_2'\exists q_1'\exists q_2'\exists p_3'\exists p_4'\exists q_3'\exists q_4'\exists p_5\exists p_5'\exists q_5\exists q_5'$ $\Big($ Between$_M$ $p_1p_2'p \wedge$ Between$_M$

$p_2'p_1'p \wedge$ Between$_M$ $q_1q_2'p \wedge$ Between$_M$ $q_2'q_1'p \wedge$ Between$_M$ $p_3p_4'q \wedge$ Between$_M$

$p_4'p_3'q \wedge$ Between$_M$ $q_3q_4'q \wedge$ Between$_M$ $q_4'q_3'q \wedge$ Proportional $pp_1'pp_1pq_1'pq_1$

$\wedge$ Proportional $pp_1'pp_1pp_2'pp_2 \wedge$ Proportional $pp_1'pp_1pq_2'pq_2' \wedge$ Proportional

$pp_1'pp_1qp_3'qp_3 \wedge$ Proportional $pp_1'pp_1qp_4'qp_4 \wedge$ Proportional $pp_1'pp_1qq_3'qq_3$

$\wedge$

Proportional $pp_1'pp_1qq_4'qq_4 \wedge$ Indicate $pp_1'p_2'p_5 \wedge$ Indicate $pq_1'q_2'p_5' \wedge$ Indicate $qp_3'p_4'q_5 \wedge$ Indicate $qq_3'q_4'q_5' \wedge$ Part $p_5r \wedge$ Part $p_5'r \wedge$ Part $q_5r \wedge$ Part

$q_5'r \wedge$ Congruent$_{MS}$ $p_5p_5'q_5q_5'$ $\Big)\Big)$

In order to show the adequacy of the definition of spatial congruence of $r-$points, we prove another metatheorem.

Theorem 5. *Let $\mathfrak{M}$ be a model of T_{CM} that contains a plenum and $\mathfrak{N}$ be a relationist definable reduct of $\mathfrak{M}$. Let $\mathfrak{N}^+$ be the canonical extension of $\mathfrak{N}$ and f the canonical function relative to certain plenum of matter r. Then for all ξ_1,*

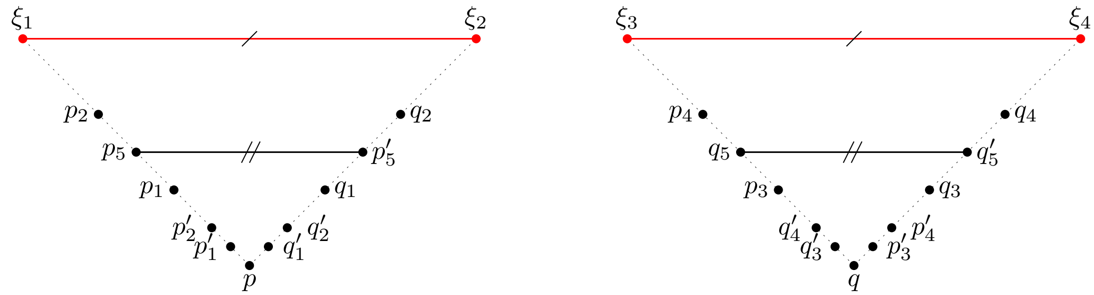

Figure 10: $\overline{\xi_1\xi_2}$ is congruent to $\overline{\xi_3\xi_4}$ if and only if $\overline{p_5p_5'}$ is congruent to $\overline{q_5q_5'}$.

ξ_2, ξ_3, ξ_4

$$\mathfrak{N}^+ \models \textit{r-Congruent}_S\ \xi_1\xi_2\xi_3\xi_4 r \ \textit{ iff } \mathfrak{M} \models \textit{Congruent}_S\ f(\xi_1)f(\xi_2)f(\xi_3)f(\xi_4)$$

Proof. Another exercise in similar triangles. □

A similar definition can be given of temporal congruence. Suppose that ξ_1 and ξ_2 are not simultaneous and that neither are $r-$points ξ_3 and ξ_4. We want to find out whether the time elapsed between the two pairs of material events is the same. We can find in both cases a point of the plenum, respectively p and q, that is simultaneous to neither. This leaves us once again with two triangles. The sides of such a triangle are all inertial trajectories. The strategy is to give truth conditions for the claim of temporal congruence in terms of the existence of a small similar triangle within the plenum, such that the opposing sides are temporally congruent. Details are omitted.

Alfred Tarski [1999, 202-3] has shown that betweenness can be defined from spatial congruence and logical notions, in Euclidean geometry, if the dimension of space is greater than or equal to two and we assume the Circle Axiom (see also [Tarski 1999, fn. 4] for a discussion of the Circle Axiom). We can give an analogous definition of spatiotemporal betweenness in terms of spatial and temporal congruence. We first need to define the spatial and temporal shorter-or-equal relation. The relation '$\xi_1\xi_2 \leq_S \xi_3\xi_4$' holds of two pairs of simultaneous points, each of which determines a segment, if the first segment is shorter or congruent to the second.

D13. $\xi_1\xi_2\ r- \leq_S\ \xi_3\xi_4 r \leftrightarrow_{df}$ r-Sim $\xi_1\xi_2 \wedge$ r-Sim $\xi_3\xi_4 \wedge$ $\forall\xi_5$(r-Congruent$_S$ $\xi_3\xi_5\xi_4\xi_5 r \rightarrow \exists\xi_6$(r-Congruent$_S$ $\xi_6\xi_1\xi_6\xi_2 r \wedge$ r-Congruent$_S$ $\xi_6\xi_2\xi_3\xi_5 r$)

D14. $\xi_1\xi_2\ r- \leq_T\ \xi_3\xi_4 r \leftrightarrow_{df} \forall\xi_5$(r-Congruent$_T$ $\xi_3\xi_5\xi_4\xi_5 r \rightarrow \exists\xi_6$(r-Congruent$_T$ $\xi_6\xi_1\xi_6\xi_2 r \wedge$ r-Congruent$_T$ $\xi_6\xi_2\xi_3\xi_5 r$)

D15. $\xi_1\xi_2\ r- \leq\ \xi_3\xi_4 r \leftrightarrow_{df} \xi_1\xi_2\ r- \leq_S\ \xi_3\xi_4 r \vee \xi_1\xi_2\ r- \leq_T\ \xi_3\xi_4 r$

Then we define the notion of betweenness. An r-point ξ_2 is between ξ_1 and ξ_3 when for all ξ_4 if $\xi_1\xi_2 \leq \xi_4\xi_1$ and $\xi_4\xi_3 \leq \xi_2\xi_3$, then ξ_4 is identical to ξ_2:

D16. r-Between $\xi_1\xi_2\xi_3 r \leftrightarrow \forall\xi_4\ (\xi_4\xi_1\ r{-} \leq \xi_1\xi_2 \wedge \xi_4\xi_3\ r{-} \leq \xi_2\xi_3 \rightarrow \xi_4 = \xi_2)$

The last predicate we need to construct spacetime is the predicate of location between matter points and r-points. A matter point is located at an r-point ξ if and only if it is indicated by the same triples of matter points that indicate ξ:

D17. r-Located $p\xi r \leftrightarrow \exists p_1 \exists p_2 \exists p_3$ (Indicate $p_1p_2p_3p \wedge \mathrm{Ind}(p_1, p_2, p_3, r) = \xi$)

Theorem 6. *Let $\mathfrak{M}$ be a model of T_{CM} containing a plenum and $\mathfrak{N}$ be a relationist definable reduct of $\mathfrak{M}$. Let $\mathfrak{N}^+$ be the canonical extension of $\mathfrak{N}$ and f the canonical function relative to a certain plenum of matter r. Then for all ξ_1 and for all matter points p:*

$$\mathfrak{N}^+ \models \textit{r-Located}\, p\xi r \textit{ iff } \mathfrak{M} \models \textit{Located}\, p f(\xi)$$

Proof. By definition. □

We can put together the three metatheorems of this section into a single metatheorem that buttresses the claim that we have logically constructed spacetime.

Theorem 7. *Let $\mathfrak{M}$ be a model of T_{CM} containing a plenum and $\mathfrak{N}$ be a relationist definable reduct of $\mathfrak{M}$. Let $\mathfrak{N}^+$ be the canonical extension of $\mathfrak{N}$ and $r \in M$ such that*

$$\mathfrak{M} \models \textit{ Plenum of matter } r.$$

Then, the canonical function f_r is an isomorphism between $\mathfrak{M}$ and $\mathfrak{N}^+$.

Proof. An immediate consequence of Theorems 2-6. □

6 The plenum-independent construction

In the previous section, we have constructed a spacetime relative to the choice of a plenum r. There are, therefore, as many distinct spacetimes as there are plena. There is nothing that deserves to be called 'spacetime' without qualification. The problem arises of giving a construction that does not require an r.

We do not want to suppose that there is a privileged plenum, or a '*marvellous ball*'. Therefore, the strategy is to aggregate together all these spacetimes. We will identify spacetime points to equivalence classes of $r-$points. For the purposes of this section, an $r-$point could be treated as an equivalence class of four tuples $\langle p_1, p_2, p_3, r\rangle$. Actually, we will manage to define an equivalence relation $\sim$ between triples. As long as $\langle p_1, p_2, p_3\rangle$ is part of some plenum, then it does not matter which one, to determine whether $\langle p_1, p_2, p_3\rangle \sim \langle q_1, q_2, q_3\rangle$. We want to define, in the relationistic language, an equivalence relation of six arguments that can hold also between triples that are part of different plena.

The strategy we adopt is inspired by the construction of the intersubjective world in *Der Logische Aufbau der Welt* (see [Carnap 1968, §147]). To check whether $p_1p_2p_3 \sim q_1q_2q_3$, we want first to construct the r-locations of q_1, q_2 and q_3 relative to a plenum r that contains p_1, p_2 and p_3. These are the locations of q_1, q_2 and q_3 from the point of view of the matter points p_1, p_2 and p_3. These r-locations are equivalence classes of triples from r. We can fix three triples of matter points in the plenum r that materially indicate q_1, q_2 and q_3 (see fig. 11).

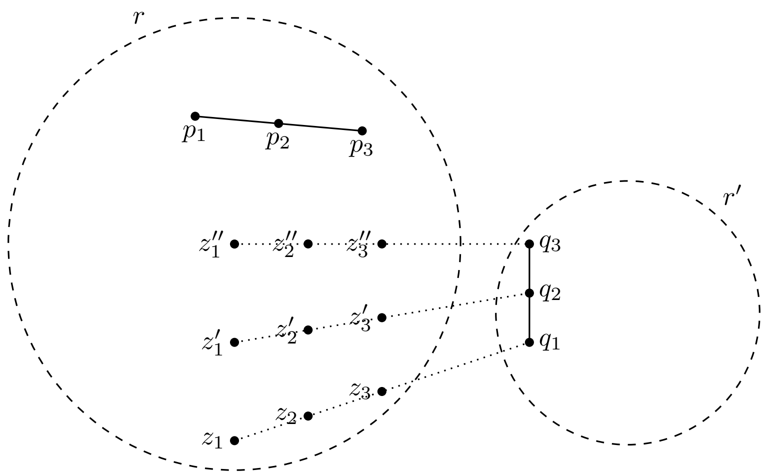

Figure 11: The triples $\vec{z}$, $\vec{z'}$, $\vec{z''}$ indicate respectively q_1, q_2, and q_3.

The second step in the program is to define a notion of indication between $r-$points relative to the plenum r. The definition merely swaps the notions of material betweenness and congruence with their $r-$relative counterparts:

D18. r-Indicate$_S$ $\xi_1\xi_2\xi_3\xi_4 r \leftrightarrow_{df}$ r-Between $\xi_1\xi_2\xi_3 r \wedge$ r-Between $\xi_2\xi_3\xi_4 r \wedge$ r-Proportional$_S$ $\xi_1\xi_2\xi_1\xi_3\xi_1\xi_3\xi_1\xi_4 r$

D19. r-Indicate$_T$ $\xi_1\xi_2\xi_3\xi_4 r \leftrightarrow_{df}$ r-Between $\xi_1\xi_2\xi_3 r \wedge$ r-Between $\xi_2\xi_3\xi_4 r \wedge$ r-Proportional$_T$ $\xi_1\xi_2\xi_1\xi_3\xi_1\xi_3\xi_1\xi_4 r$

D20. r-Indicate $\xi_1\xi_2\xi_3\xi_4 r \leftrightarrow_{df}$ r-Indicate$_T$ $\xi_1\xi_2\xi_3\xi_4 r \vee$ r-Indicate$_S$ $\xi_1\xi_2\xi_3\xi_4 r$

The triples that materially indicate the matter points q_1, q_2 and q_3 belong to three $r-$points ξ_1, ξ_2 and ξ_3. These are the $r-$locations of q_1, q_2 and q_3. The three matter points p_1, p_2 and p_3 will be counted as equivalent to q_1, q_2 and q_3 if and only if the $r-$point ξ such that $\langle p_1, p_2, p_3\rangle \in \xi$ is $r-$indicated by ξ_1, ξ_2 and ξ_3.

D21. $p_1p_2p_3 \sim p_4p_5p_6 \leftrightarrow_{df} \exists r \exists r' \exists \xi \exists \xi_1 \exists \xi_2 \exists \xi_3 \exists \vec{q} \exists \vec{q'} \exists \vec{q''}$ (Plenum of Matter $r \wedge$ Part $p_4 r \wedge$ Part $p_5 r \wedge$ Part $p_6 r \wedge Ind(\vec{p}, r) = \xi \ \wedge \ Indicate_M\ \vec{q} p_4 \ \wedge \ Indicate_M\ \vec{q'} p_5 \ \wedge Indicate_M\ \vec{q''} p_6 \ \wedge Ind(\vec{q}, r) = \xi_1 \ \wedge \ Ind(\vec{q'}, r) = \xi_2 \ \wedge \ Ind(\vec{q''}, r) = \xi_3 \ \wedge \ r - Indicate\ \xi_1\xi_2\xi_3\xi$)

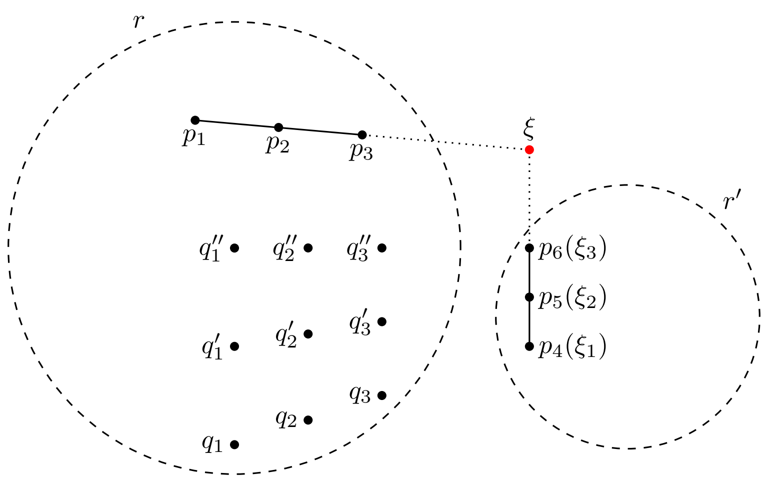

Figure 12: The equivalence relation $\sim$

We can prove that $\sim$ is an equivalence relation on any relationist definable reduct of a model of classical mechanics. The result follows from:

Theorem 8. *Let $\mathfrak{M}$ be a model of T_{CM} that contains a plenum and $\mathfrak{N}$ be a relationist definable reduct of $\mathfrak{M}$. Let $p_1, p_2, p_3, q_1, q_2, q_3$ be six matter points and $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6$ be their locations in $\mathfrak{M}$. Then, we have that $\mathfrak{N} \models p_1p_2p_3 \sim q_1q_2q_3$ if and only if*
$\mathfrak{M} \models \exists\xi_7$ *(Indicate $\xi_1\xi_2\xi_3\xi_7 \wedge$ Indicate $\xi_4\xi_5\xi_6\xi_7$).*

Proof. An easy consequence of Theorem 7. □

The equivalence relation $\sim$ can be used to introduce by abstraction an ulterior sort of entities $\alpha, \alpha', \ldots$. These are the constructed entities that will play the official role of spacetime points. The definitions of the emergent predicates of congruence, simultaneity, and betweenness that apply to the entities of sort $\alpha, \alpha', \ldots$, can be easily recovered from the $r-$notions that apply to r-points. The construction depends on the following lemma:

Lemma 2. *Let $\mathfrak{M}$ be a model of T_{CM} containing a plenum and $\mathfrak{N}$ be a relationist definable reduct of $\mathfrak{M}$. Let r be a plenum such that $\mathfrak{M} \models$ Plenum of Matter r. Let $p_1, p_2, p_3, p_4, p_5, p_5, p_6$ be six matter points that are part of the plenum r. Then, we have that*
$\mathfrak{N} \models p_1p_2p_3 \sim_r q_1q_2q_3$ *if and only if* $\mathfrak{N} \models p_1p_2p_3 \sim q_1q_2q_3$.

Proof. Immediate from Theorem 2 and Theorem 8. □

This lemma implies that every $r-$point ξ is included in a point. For every ξ, there is an α such that $\xi \subset \alpha$. The lemma also implies that the α is unique. For suppose that $\xi_1 \subset \alpha$ and $\xi_2 \subset \alpha$. Let $\langle p_1, p_2, p_3\rangle \in \xi_1$ and $\langle q_1, q_2, q_3\rangle \in \xi_2$. It follows that $\langle p_1, p_2, p_3\rangle \in \alpha_1$ and $\langle q_1, q_2, q_3\rangle \in \alpha$ by inclusion. Therefore,

$q_1q_2q_3 \sim p_1p_2p_3$. It follows that $q_1q_2q_3 \sim_r p_1p_2p_3$ and, therefore, $\xi_1 = \xi_2$. Finally, we can show that every α contains at least one $r-$point. This follows from the isomorphism theorem between $r-$spacetime and substantival spacetime. It is easy to see how to define geometric relations between the α. A tuple of points will stand in a geometric relation if and only if the $r-$points that they contain stand in the appropriate $r-$relation. For example:

D22. Between $\alpha_1\alpha_2\alpha_3 \leftrightarrow_{df} \exists r \exists \xi_1 \exists \xi_2 \exists \xi_3$ $(\xi_1 \subset \alpha_1 \wedge \xi_2 \subset \alpha_2$ $\xi_3 \subset \alpha_3 \wedge$ r-Between $\xi_1\xi_2\xi_3 r)$

Let $\mathfrak{M}$ be a model of classical continuum mechanics containing a plenum. Let $A = P^3/\sim$ be the partition of the set of triples of matter points in $\mathfrak{M}$ under the plenum-independent equivalence relation of this section. We will call $\mathfrak{N}^{++}$ the structure

$$\langle A, P, Sim, Between, Congruent, Location \rangle.$$

The remarks should have made obvious the following theorem:

Theorem 9. *The structures $\mathfrak{M}$ and $\mathfrak{N}^{++}$ are isomorphic.*

7 Minkowski spacetime

Minkowski spacetime can be constructed from continuous matter in exactly the same way as a classical spacetime. We only need to verify that every pair of points can be connected to a point within a plenum by either spacelike or timelike segments. We also need to verify that the theorems about similar triangles that we have used extend to triangles in a flat relativistic spacetime, as long as the sides are not lightlike. If two triangles $\triangle xyz$ and $\triangle pqw$ have equal the angles $\hat{x}$ and $\hat{p}$, and the two sides $\overline{xy}$ and $\overline{xz}$ are proportional to $\overline{pq}$ and $\overline{pw}$ with a factor of k, then also the opposite sides $\overline{yz}$ and $\overline{qw}$ are multiples of a factor k. Proportionality claims refer to the relativistic interval.

The simplest case is when $x = p$ and one triangle is the continuation of each other. In this scenario, we can set up a coordinate system with origin $o = x = p$. If the sides of the first triangle are the four-vectors $\vec{v}_1$ and $\vec{v}_2$, then the sides of the second triangle are the vectors $\lambda \cdot \vec{v}_1$ and $\lambda \cdot \vec{v}_2$. Since the relativistic interval is linear, the length of the opposite side is:

$$||\lambda \cdot \vec{v}_1 - \lambda \cdot \vec{v}_2|| = \lambda \cdot ||\vec{v}_1 - \vec{v}_2||$$

The general case about the norm derived from a multilinear form in an affine space is left to the reader. The proof is only slightly less obvious.

The special case above is the one that is needed to define geometrical predicates of $r-$points and then points. In the spacetime of special relativity, there is only one predicate of points to be defined: the binary predicate of causal connectibility. All others, including congruence and betweenness, can be defined from causal connectibility (see [Goldblatt 1987] and [Pambuccian 2007]).

The same strategy is used as in the definition of congruence in a classical spacetime. Consider two $r-$points, or equivalence classes of triples, ξ_1 and ξ_2 and suppose that we want to figure out whether they are causally connectible: whether the interval between them is lightlike or timelike. The answer is positive if and only if it is possible to construct a similar triangle within the plenum such that the opposing side of this material triangle is lightlike or timelike.

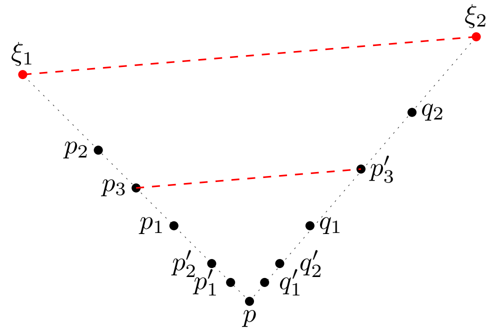

Figure 13: The r-point ξ_1 is causally connectible to ξ_2

The relativistic construction is in all other respects analogous to the classical spacetime construction. Further details will be omitted.

8 Postulates

Until now all of the results have been semantical. We have not specified an alternative relational theory to replace the substantival formulation of classical continuum mechanics T_{CM}. However, specifying such a theory is not difficult. Such a theory will also be first order, just like the first order substantivalist theory T_{CM} (see sec.2). We can show its existence as follows.

The constructions that we have employed are in [Halvorson and Barrett 2016][Halvorson 2019]. Morita equivalent theories are theories that can be extend into an identical theory using these sorts of procedures. Morita equivalence can also be equivalently formulated in terms of translations, or generalized reconstruals (see [Halvorson 2019, §5.4 and §7.5]). This means that there is a translation f from the language of T_{CM} to the relationist language L_R. We can therefore, take the postulates of T_R to be the translation of those of T_{CM}:

$$T_R = \{f(\phi) \mid \phi \in T_{CM}\} \tag{20}$$

8.1 Morals

Let us consider a last time the philosophical morals of the construction. Should we say, for instance, that spacetime points have been shown to be 'surplus structure' or 'descriptive fluff'? It seems to us that the answer is negative.

These terms designate parts of the mathematics that are gauge, that is, that do not represent features of the world. However, the construction in this paper is thoroughly nominalistic. Spacetime regions are physical objects, and their arrangement is *definable* from that of matter; they are thus, in our book, ontologically derivative, but not unreal, or gauge, or surplus in that sense.

Moreover, since the theories T_{CM} and T_R are equivalent, does it not follow that the debate between the enriched relationist and the substantivalist has been shown to be a verbal dispute? Our answer is that the debate about *existence* has been dissolved[5], but not that about *priority* of matter over spacetime. There is *one* theory and not two, since T_R and T_{CM} are theoretically equivalent. But the relationist presentation T_R is a perspicuous presentation of the *fundamental* ontology of T_{CM}, in the sense that it singles out a minimal base, from which the unoccupied points in T_{CM} can be recovered.

9 Further physics

In this final section, let us briefly discuss how the scheme may be extended further, to deal with force fields and more contemporary physics.

9.1 Fields

Burgess [1991] deals with one of the main problems for relationism: the problem of grounding the distinction between inertial and accelerated motion. This problem seems to us to have been solved, at least for classical field theories. However, Burgess [1991] entirely ignores another problem for relationism: that of incorporating fields (see [Field 1984, 40-42]). For example, substantival spacetime seems to be needed as the repository of electromagnetic radiation, propagating in the vacuum, where there is no charge and mass.

There have been two standard solutions to the problem of fields. The first is to abandon field theories in favour of action-at-a-distance theories. Field [1984, 40] presents this as the only viable option for the relationist. Theories positing direct interparticle forces have some defenders (see [Mundy 1989; Lazarovici 2018]). The second common approach is to reify fields as some sort of gelatinous substance in their own right (see for example [Pooley 2013, 37]).

The difficulty with the first is that constructing such action-at-a-distance models to replace all of our best classical and quantum field theories is an ambitious project. It is a project that may fail, even in the most promising case, that of electrodynamics (see [Wald 2022, 9/f] for a critique of action at a distance). The main difficulty with the reification strategy is that the difference between positing fields and positing spacetime, and therefore the motivation for relationism, become unclear (see [Field 1984, 41-42] and [Maudlin 1993, 200]). The parsimony of relationism seems lost, as soon as queer entities are admitted.

[5] Dissolved, that is, in classical mechanics and special relativity, under the assumption of a continuous matter distribution. A brief discussion of how the matter stands with contemporary physics, and some ideas for further research, are in the next section.

Such controversies cannot be decided in the present paper. But we think that it is prudent for the relationist to have an alternative approach to fields.

Such controversies cannot be decided in the present paper. But we think that it is prudent for the relationist to have an alternative approach to fields.

We would like to propose a third option based on the constructions in this paper. We believe that it suffers from none of these drawbacks. The idea is to treat electromagnetic fields as multifields defined on matter.

9.2 The multi-field approach

Multifields have been discussed in the recent literature on the ontology of the wave function (see [Romano & Hubert 2018], [Chen 2019] and [Romano 2020]). Consider the wave function ψ of a system of spinless particles in a classical spacetime. In the position basis, the wave function can be modelled as an assignment of complex numbers to points of configuration space.

The idea of the multifield is to treat it instead as an assignment of complex numbers to fusions of points [Chen 2019], or as a relation between points of spacetime and complex numbers [Romano & Hubert 2018]. We will treat similarly the electromagnetic field as a multifield defined on continuous matter. The field will be represented by an assignment $F_{\alpha\beta}$ of a tensor to each triple of matter points. The derivative field on the $r-$spaces, and then on spacetime can be constructed in the obvious fashion. For example, if ξ is an $r-$point:

$$F_{\alpha\beta}(\xi) = F(p_1, p_2, p_3) \text{ where } Ind(p_1, p_2, p_3) = \xi$$

This is a mathematical, or 'platonistic' function, but it is not too difficult to convert it into a comparative predicate (*à la* Burgess [1984]).

9.3 Gravitation

The techniques that have been used to deal with flat spacetimes won't generalize to the curved spacetimes of general relativity. The facts about a bounded portion of spacetime do not determine the geometrical facts about points far away. For example, it is impossible to detect, in a flat region of spacetime occupied by matter, the presence or absence of a crease far away.

Even if we fix a specific spacetime background *ab initio*, its geometry may be inhospitable to the sort of constructions we want to do. For example, there is no concept of similar triangles in spherical geometry. The only way we see at present to integrate general relativity into the approach is to abandon the equivalence principle, and therefore distinguish inertial and gravitational structure. The gravitational field ought to be treated as a force field evolving in a fixed flat background. Minkowski spacetime is the natural arena for such a theory. This is the approach of Weinberg [1972].

The suggestion on behalf of the relationist is to integrate the gravitational field into the picture by treating it as a tensorial multifield defined on some charged matter dust. However, this may be repugnant to many relativists.

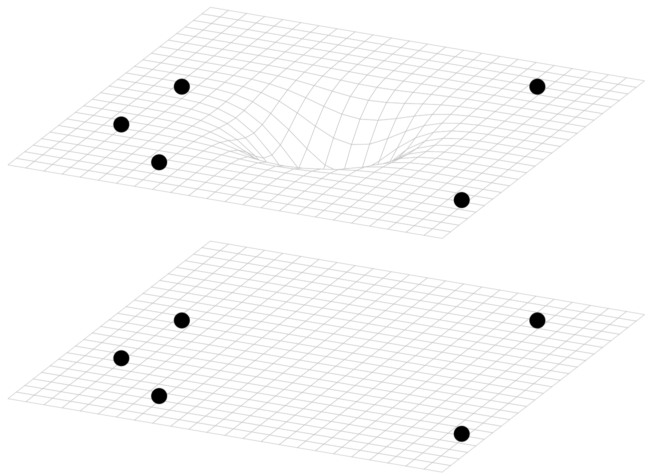

Figure 14: The balls of matter are on flat regions

A way to sweeten the pill may be to distinguish some sort of fundamental flat spacetime from an emergent curved spacetime. This distinction seems to be suggested by the framework of string theory (see [Huggett 2017] and [Huggett & Vistarini 2015]). String theorists distinguish fundamental space from 'target space'. Although the equivalence principle is rejected at the fundamental level, it is recovered in some limit for the emergent spacetime. In this sense, [Huggett & Vistarini 2015, 7] claim that string theory is 'background independent'.

The reference to string theory raises the question of how the approach can be extended to quantum theories. A detailed discussion of this problem is beyond the scope of the paper. Clearly, supersubstantivalist formulations dealing only with operators defined on regions of spacetime (see [Wallace and Timpson 2010]) and theories with discrete pointlike particles are inhospitable to the present framework. This excludes the simplest form of the pilot wave theory and objective collapse models with flashes. The most promising bet is an objective collapse model with a mass density, or a descendant of the pilot wave theory that posits a continuous beable. In any theory that posits a continuous field - maybe a charge density or a fermion-number density - that is continuous, but not defined on the totality of spacetime, we can treat the points at which it is defined as material pointlike particles. The wavefunction will then be a multifield on the beables. Just triplicate the arity of the usual multifield.

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