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# Foundations of Mathematics

**by**

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**Abstract**

Analytical philosophy defines mathematics as an extension of logic. This research will restructure the progress in mathematical philosophy made by analytical thinkers like Wittgenstein, Russell, and Frege. We are setting up a new theory of mathematics and arithmetic's familiar to Wittgenstein's philosophy of language. The analytical theory proposed here proves that mathematics can be defined with non-logical terms, like numbers, theorems, and operators. We'll explain the role of the arithmetical operators and geometrical theorems to be foundational in mathematics. Our position states that mathematical operators and theorems are tautologies. Here numbers are arbitrary signs of magnitudes whose meaning is arbitrated by the mathematical operator. We will prove that set theory in mathematics doesn't form a branch of logic and operators arbitrarily form sets into units. Herein are the foundations of numbers.

## Introduction

The main hypothesis of this research states that the principles of mathematics are its operators and theorems. Both operators and theorems are of the tautological order. Tautologies in logic and mathematics don't have a meaning of their own. The function of mathematical tautologies is to create a reflexive representation of all signs in the symbolic order and to operationalize all expressions into magnitudes. The meaning of numbers is derived from their combination with tautological operators. According to this research, the apodictic and *a priori* nature of mathematics is not found in knowledge or logic. Instead, the founding of the 'universal' nature of all mathematical entities hides in the tautology of the operators. The 'universality' of the geometrical theorems also originates from their tautological nature.

We can summarize all of the arguments on the *a priori* nature of knowledge with the simplest of mathematical formulas: 1+1=2, and try to see where it goes wrong. We'll prove that there are inherent inconsistencies in the argumentative basis of philosophy when dealing with the apodictic nature of knowledge. All apparent apodictic conclusiveness in *a priori* knowledge comes from misinterpreting the validity that originally belongs to the mathematical domain. There's a hidden shift of meanings that migrate between the two disciplines of philosophy and mathematics.

All mathematical formulas that yield a certain result are based on a certain kind of operation, like addition or multiplication of numbers. However, when philosophy asserts that 1+1=2 is an apodictic proposition of the *a priori* domain, universally valid, then this is no longer a mathematical expression, but an empirical and philosophical one. Philosophy here shows a tendency to extend the certainty of mathematical operations into the certainty of cognitive operations which are different. Let us debrief on this for a moment. In Bertrand Russell, we find the following analysis:

> Returning to the problem of *a priori* knowledge, which we left unsolved when we began the consideration of universals, we find ourselves in a position to deal with it in a much more satisfactory manner than was possible before. Let us revert to the statement that 'two and two are four'. It is obvious, given what has been said, that this proposition states the relation between the universal 'two' and the universal 'four'. This suggests a proposition that we shall now endeavor to establish. Namely, all *a priori knowledge deals exclusively with the relations of universals*.[i]

The true question here would be if we could truly conclude on any philosophical residue in mathematical propositions. Namely, does mathematics resonate with anything other than certain operations that yield certain results? Mathematics doesn't omit anything of the 'universal' or 'eternal' order in numbers or figures. A simple analysis of Russell's arguments can conclude this. Namely, what philosophers are trying to do in these instances is to expand the certainty of mathematics to the certainty of knowledge. This is true of philosophy since the time of Plato. The propositions of math are certain. What's uncertain here is their universal being, for no mathematical operation speaks of it. Let's try and asses this on the simplest syllogism forms, to see if any mathematical certainty would grant our knowledge a certainty of its own:

**The major statement would be a broad one:**

<u>All triangles have three sides and three interior angles.</u>

**The minor premise scales down the major premise to something local, exact, or familiar.**

<u>This figure is a three-sided polygon.</u>

**The conclusion connects the universal truth of the major premise to the immediate example of the minor premise.**

<u>Then this three-sided polygon is a triangle.</u>

We see that logical propositions, when given in the form of syllogism, may express mathematical truths. However, this apodictic certainty doesn't apply to our knowledge. Math is silent when it comes to the validity of knowledge, and it remains correct even when knowledge is wrong. If we look at the syllogism and analyze the arguments, we'll find that there is no mention of any universals in the major, minor, or conclusive statements. The syllogism isn't true because of universality's sake. It's true because it logically deduces certain statements. The universality of the three-sided triangles is based on the Pythagorean Theorem, namely of not having more than three sides rather than 'always' having three sides. Both terms "always" and "universally" are absent from the arguments.

The goal of this research is to set aside all cognitive and logical implications in theorizing about the nature of mathematics.

## 1. THE APODICTIC IN THE *A PRIORI*

Whenever arithmetic's operates with either natural, negative, or rational numbers, etc., is something explicitly agreed upon by the use of its operators and numerical set formations. Each stage of math is only apodictic to a point where a community has agreed about the symbolic use of its signs and their structures. The meaning of any currently used expressions in arithmetic's and geometry is dictated by a set of rules that are universally valid only in the area that has an arbitrary layout whose signs are founded by the practicing community of mathematicians. Mathematics is apodictic because each set of its signification is strictly regulated by a set of tautological operators and theorems. Its meaning is self-encapsulated by its formulas, operators, and theorems that are without the usual representative function of meaning found in cognition and language. The meaning of mathematics is founded on calculations and measurements.

The system of mathematics isn't universally valid and apodictic because it has universality imprinted in its nature, or because certainty was somehow innate to us or *a priori* given in our knowledge.

Here we theorize that arithmetic's forms into determined numerical sequences only as part of arithmetical operations of additions and fractions, like n+1 or $\frac{1}{10}$. Numbers are never arbitrarily given on the progressive scale without being grouped into either decimal or duodecimal sets, or even transfinite sets. Arithmetical sequences are never given without a decimal structure, and they are always grouped into tens or dozens. This is due to the strict use of arithmetical operators. An arithmetical set is not a logical concept but an arithmetical fractionalized formula, expressed as a $\frac{1}{10}$.

Consequently, we state that arithmetical fractionalized sets, halves, quarters, or decimals, must have preceded the formation of the number 1, derived by the primary use of arithmetical operation, like division or fraction. Thus, numbers are impossible without arithmetical operators and geometrical shapes are impossible without geometrical theorems. Therefore, both mathematical theorems and operators must precede and condition the formation of shapes and numbers.

Mathematics is only rigorous and apodictic because of its strict arbitrary layout, where it's every layer of operationalization and signification is something strictly agreed upon by the community of mathematicians and the use of its operators and formulas. Thus, mathematics is a form of syntax like logic, with a base of tautological rules that express its validity. Mathematical operators, like division (÷), are tautologies, similar to the law of contradiction in logic. Operators in math are as senseless to the result a certain calculation produces as much as according to Wittgenstein, logical tautologies are to the 'state of affairs' in propositional expressions.[ii] Both operators of logic and arithmetic's are tautologies. There is no more meaning in stating the principle of contradiction, formally expressed as a tautology "~ (p, ~p)", than there is in defining the addition sign as the addition of numbers, or expressing the multiplication sign as multiplication of numbers. There isn't a meaning in stating the same thing twice, that's why arithmetical operators are also tautologies.

This, however, doesn't conclude that mathematical rules expressed as operators or theorems don't have meaning at all. Like the rules of logic, they find their meaning in supplying the symbolical scaffolding in expressing arithmetical formulas and geometrical shapes, instead of making descriptions like in language. Thus, the more appropriate word for describing logical and mathematical tautologies is senseless instead of meaningless. Meaning is obtained every time there is a shift between the representation of one symbolic order with another symbolic order, like when an arithmetical operator formulates an arithmetical formula or when a geometrical theorem forms a shape. A sense, however, is something that's carried by an expressed combination of symbols, pointing to another set of symbols using an operator. Since rules of logic, mathematical theorems, and operators can be expressed independently from any order or combination alike, they are tautological forms, referred to as senseless but we'll use this term interchangeably with meaningless.

Wittgenstein stated this as far as the rules of language and logic were concerned but it can be extended in mathematics as well.

Namely, what does Wittgenstein imply when he says a 'language game'?[iii] He implies a community of speakers that asserts its use of meanings according to a logical syntax to a certain *state of affairs*.[iv] Since languages form on the expressive function of propositions and words, every community that has a language is supposed to have logic as well. In Wittgenstein, this is because even though different languages have different meanings and sense, the fact that each language expresses thoughts and facts into named objects and a system of propositions, speaks of a universally applied logic underlying each language as something *a priori*.[v] According to Wittgenstein, however, this *a priori* isn't the underlying metaphysical order found in nature or mind, but simply means that logic is not part of the descriptive propositions but rather part of the structural syntax forming the objective languages. *The symbolism of logic can be expressed across different languages because it's meaningless and tautological*. There isn't a translation of the laws of logic because they don't carry a meaning. In our research, mathematics is also universal because it's non-translatable like logic, and based on tautological operators that yield identical results.

According to Wittgenstein, the case of the universality of logic is clear, and we'll apply this in mathematics as well, without inferring that: "Mathematics is a method of logic."[vi]. The following elaboration is given by Wittgenstein:

> My fundamental thought is that the logical constants do not represent. That the *logic* of the facts cannot be represented.[vii] The same argument goes for mathematics for it also cannot be represented by another generalized form.[viii]

Thus, according to Wittgenstein, the similarity between logic and mathematics is the fact that they aren't representative of another codex of expressions or a different set of rules. Universality then is something that can be achieved plainly, when a certain form of symbolism has no representative function to its statements and is not in any way descriptive of the state of affairs to which it neither corresponds or doesn't. All that logical rules state is the possibility or impossibility of our meanings about the world. This universality isn't abstract or ideal.

This universality found in logic and mathematics is peculiar since it underlies the nature of mathematics as being simplistic rather than general. Universality here is in no way associated with generality and it only can be found in what is elemental and doesn't require as many specifications as descriptive languages do. A community of speakers of a language arbitrarily sets all its layers of signification, like the meaning of names. If there was a universal language in the truest sense of the word, it would either be a manual 'sign language' or a language that has nothing objective to say. We either must sacrifice the sense of a language expressed in its plurality of words, or abolish any objective representation a language has in order to achieve universality as found in logic and mathematics.

However, Wittgenstein left any such insight about mathematics unfinished and went on to ground mathematics in logic. Finding the meaning of mathematics in the peculiar argumentative set in his TLP can be simple. Namely, what he showed here is that mathematics doesn't have to be further reduced to logic, but it can operate on an equal level as logical propositions. The problem is that he didn't formulate an accurate theory of numbers because, similarly to Frege, he didn't consider them to be thoughts. If numbers were thoughts, then each sign number would have a different meaning, concluded Frege, and since meanings must be found on reference, he further argued that there must be a Platonic realm where numbers have their reference.

Similarly to Frege, Wittgenstein pursued the meaning of mathematics externally, but not in Platonism but rather in logicism.

> According to Wittgenstein, we ascertain the truth of both mathematical and logical propositions by the symbol alone (i.e., by purely formal operations), without making any ('external', non-symbolic) observations of the state of affairs or facts in the world.[ix]

However, to what extent can we truly argue in favor of a mathematical game *a priori* without grounding it in Platonic objectivism or logic, as Frege did? Wittgenstein is against objectivity in mathematics.

There is nothing mysterious about the assumed hidden entity of numbers other than the fact that their meaning is in strict arbitration of sequenced numerical quantities, arranged by the use of the arithmetical operators. Numbers are very much like letters except for the fact that rather than expressing an articulated sound of the alphabet, they express the point order of a growing or descending scaled magnitude. This scale is not infinite but derived by a fractional unit of the decimal kind. Number 1 has no meaning when added to infinity but only when being a part of a decimal set, 1/10.

When enumeration by addition becomes too complex, we may express any sort of magnitudes by the use of the exponents. As long as we can state what a point is on a certain scale, we know what the meaning of a number is as a representation of a certain point in an arithmetically operationalized order. Arithmetical operators form sets of natural, rational, irrational, decimal, or duodecimal numbers and arbitrate any numerical order or magnitude. Later we'll establish the difference between decimals and duodecimals sets, which will be proven to be a crucial argument in this research.

It's the same with geometrical shapes like lines and squares, which aren't perfect shapes and objects of the platonic realm but rather something set into action by strictly arbitrated tautological statements or theorems regarding the regularity of certain shapes. If numbers express the magnitude of an order, shapes express the regularity of an order. This order is expressed as sets of numbers in arithmetic's, either decimal or duodecimal, or as shapes of space in either Euclidean, hyperbolic, or spherical geometry.

The layout of mathematics is arbitrated through the layout of its axioms and principles.

The Pythagorean theorem $A^2+B^2=C^2$ is expressed as a tautology similarly to logical identity and its function is to shape geometrical space, similarly to how logical rules arrange logical space and propositions.

Kant originally suggested that no mathematical proposition is to be considered analytically, or that the number 4 doesn't contain 2+2 or vice versa.[x] Thus, Kant concluded that mathematics is synthetical rather than analytical. This is true and Kant probably influenced Wittgenstein in his philosophy. There is a certain type of correspondence between arithmetical operators with numbers and between geometrical theorems with shapes.

Our position, however, is not about mere tautological forms but about the extension of these analytical principles formed as tautologies in a similar manner as applied to language by Wittgenstein: "What the picture must have in common with reality in order to be able to represent it after its manner – rightly or falsely – is its form of representation.".[xi]

Thus, we can conclude that numbers are magnitudes of order formally represented by the arithmetical operators and that shapes are spatial magnitudes formally represented by geometrical theorems. Therefore, both mathematical operators and theorems have a certain form of representation. However, it's not reality that mathematical principles represent but the symbolic order of magnitude.

The working of mathematics is like language except that it's purely symbolic, unlike language.

In mathematics, theorems and operators yield the rules according to which certain estimate of an operationalized order of magnitude shows the result. The object of mathematical propositions is the result, represented as a number or a shape, and never reality per se.

This, however, doesn't imply that the making of the mathematical object is immaterial or Platonic, but rather that the rules are tautological. The same goes for logic where the fact that logical principles are expressed with letters, like identity being A=A, doesn't imply anything about the psychological nature of these principles.

Thus, we say that mathematical operations yield their result apodictically because the domain of their determinacy is arbitrated only to the meaning of symbolical statements. If the extent of this meaning is full, the truth value of the symbolic order is apodictic. This can be represented with the formulas:

If x=3 and y=2+1, then x=y

When the symbolic meaning of both x and y is full, both terms are equal since the arithmetical meaning is a function of transferring one set of terms (numbers) that belong to one variable term to another variable term. When the constant terms defining the variable terms are equal, we say that the variable terms are identical as well.

Because the meaning of the rules of mathematics or its operators is tautological, the expressed propositions of the symbolic order of the magnitude of numbers or the regularity of shapes are apodictic. Namely, there is no margin of error in yielding a proof for mathematical propositions like for empirical ones, since the 'picture and the pictured' – the assertion and its result – in arithmetic's are of the same symbolic order that the tautological operators and theorems of expressing mathematical propositions make possible. The proof of a valid theorem is apodictic only because of the tautological rules that are applied to the proposition, arbitrating its meaning into an expressed result. Neither numbers nor shapes are apodictic by themselves. What's apodictic are the expressions formed by either the Pythagorean Theorem or the arithmetical operators.

Numbers and triangles don't have any hidden meaning of their own. Every possibility of deviation in the meaning of a mathematical expression to that of the result is due to either the felicity of the theorem or its miscalculation. Therefore, only mathematics yields the most appropriate form of correspondence between formulas and their results. Every other theory of truth from the time of Aristotle strives to achieve this form of correspondence between statement and fact/product.

## 2. MATHEMATICAL MEANINGS AND PLATONIC TRUTHS

The assumed ideal existence of mathematical entities is based on the way we comprehend certain states of meaning found in arithmetical expressions and their products. A number's meaning is founded separately from any assumed pre-existing arithmetical sequence. These sequences form into expressions only with the use of operators and not plainly by acts of enumeration or logical relations. All arithmetical expressions are made from operators and number signs, which are their objects and results. The apodictic or universal sequence of the result or the referent from the use of an operator or a theorem is what creates the belief in the ideal type of existence of mathematical entities. If there was anything of the universal *a priori* and eternal order to be found in math, then this represents the sequence of the signs or shapes based on the applied operators and theorems that are tautological. Every form of mathematical sign or a geometrical shape is an arbitrated set and it's certain only to the point of expressing the tautological theorems and operators in their arbitrated field of application.

In regards to logicism, mathematical propositions, when logically analyzed, should not include the operation of additions, etc., and the result as part of a single proposition, and should consider the result as a formula that expresses the value of the formulas of the arithmetical operations. Formulas like 2+2=4 don't form single expressions of the logical kind, but rather the adding formula should be regarded separately from its result. Thus, we can compute a formal correspondence of an analytical kind between one and the other. If we consider a mathematical proposition to be of a unified type, such as 2+2=4, this will infer a type of a logical proposition like $p \supset q$, where *q* would be true if *p* is true, something that's false since mathematical formulas aren't propositional. The only tautologies in mathematics are its operators and theorems.

A conditional statement is true when the truth of the antecedent is a condition for the truth of the consequent, and the truth of the consequent is a necessary condition for the truth of the antecedent. However, in mathematics, we can't assume that the operation 2+2 is represented by *p* since it's neither true nor false without the second operation given in the result that's represented by *q*. There are no conditional propositions of the logical type in mathematics, for neither 2+2 (if *p*) is true/false, if =4 (if *q*) is true/false, or vice-versa. Their validity comes from correlating the expressions arbitrated by the tautological signs in use, or in this instances the addition operator and the equality sign. One sign doesn't necessitate the other sign, and both have to be expressed for the complex formula to be valid and defined. Thus, we can't say that arithmetical propositions of the type a+b=c can be logically expressed as $p \supset q$, since elementary mathematical expressions include one operator at a time. An addition formula doesn't express an antecedence conditioning a consequence, the result, or the consequence, the result, doesn't determine the truth value of the antecedents. Therefore, arithmetical formulas are very different from logical propositions because one arithmetical operation doesn't necessitate any other symbolically speaking. Each mathematical expression that performs according to a different operation is separate. Therefore, the result is never part of the preceding operation, as according to Kant, or 4 is not part of 2+2.[xii]

It's different for geometry, where the Pythagorean theorem, $A^2+B^2=C^2$, can be regarded as a function of identity since it doesn't express different quantities or magnitudes but identical measurements of the sum of the areas of the two squares on the legs that equal the area of the square on the hypotenuse.

Mathematical propositions of geometry or arithmetic's have their meaning given in the result. Arithmetical operators and geometrical theorems are the scaffoldings of numbers and shapes into certain results that can be derived in an apodictic symbolic order. The Pythagorean Theorem would be un-referential without a right triangle, and the triangle would be geometrically senseless without the theorem. The Bedeutung of the triangle is hidden in the Pythagorean Theorem. The statement that the sum of the areas of the two squares on the legs (a and b) equals the area of the square on the hypotenuse (c) has an analytical meaning formally represented by the triangle. Otherwise, the theorem is void of meaning similarly to arithmetical operations, i.e. multiplication (x) or addition (+), when isolated and without numbers to calculate or infer a result. Mathematical meanings dwell in the result.

Namely, the reference of the result or product is the asserting mathematical proposition, either the geometrical theorem or the arithmetical operation formula, and the meaning of this proposition is in the result. Thus, the meaning of mathematics is self-encapsulated like the meaning of language, as in Wittgenstein, but only in a much stricter sense. The difference between language propositions and mathematical ones is the one of order in the representing symbols since mathematics is symbolic and not factual or descriptive. Thus, the first order of representation is that instead of propositions and symbols representing (picturing) facts, one group of symbols represents another, like when the Pythagorean Theorem represents triangles or the addition of numbers represents the number shown in the result from the numerical sequence. The picturing of a right triangle isn't for propaedeutic purposes only but for the symbolic verification of the Pythagorean Theorem. However, triangles in turn, although they represent the referent of the theorem, have their meaning held by the proposition expressing the Pythagorean Theorem. We sense here a particular kind of resemblance characteristic for analytical truths because the meaning of the proposition and its results is self-encapsulated by the interchange of meanings via the operation performed. Thus although geometrical theorems are tautological, they have meaning by extension having a referent in the drawn figures.

The order of reference in mathematical propositions is that of the product (referent) where, unlike in empirical propositions of correspondents, the referent product is not of the objective '*state of affairs'*, as according to Wittgenstein's philosophy of language, but something of the symbolic order. The mathematical proposition pictures the referent and the referent reverses the referral to the asserting proposition, yielding it a meaning, uncharacteristic for tautologies. It's rather like a backward-going order characteristic for analytical truths, where the result refers to the arithmetical or geometrical statements and where tautologies gain meaning through symbolical extensions.

Consequently, we find that arithmetical expressions represent a state of symmetrical transformations between the expressed proposition and its result.

We may express this similarly to a physical formula:

$\Omega\varphi(x, y) = \varphi(x', y')$

where Ω denotes a mathematical operator of unity. If x and y are to be considered as propositional elements of a certain arithmetical operation or geometrical theorem, then the stripes in x' and y' are the product forms of the non-invariant transformation relation marked as $\varphi$. The transformation relation marked as $\varphi$, forms mathematical expressions that are expressed in the result. Therefore, 2+2 doesn't have to yield 4 and can be 3+1 or 5-1, etc.

The Pythagorean Theorem is regarded as a proposition that transforms into a certain figure of a right triangle where lengths and angles can be measured. A theorem is not a figure per se but it's a symbolical scaffolding.

A number like 4 has its reference in any possible operation of arithmetic's, and the right triangle has its reference in the Pythagorean Theorem but not in hyperbolic and spherical geometries. Mathematical results have their reference either in the arithmetical operation or in geometrical propositions. This is uncharacteristic for propositions of language that have their referent in the object or fact.

The confusion, so to speak, in stating the existence of mathematical entities, is caused by comparing mathematical analysis to analyzing propositions of language or logic. Namely, if statements of speech refer to objects, the assumption is that mathematical propositions also refer to something objective, possibly of the Platonic order. However, we concluded against this reasoning since we found the meaning of the mathematical result in the operator or theorem, and the referent of the tautological operator or theorem can be found in their result. In mathematics, only the result is the referent as a form represented by the proposition. Thus, we can say that arithmetical operators or geometrical theorems are tautologies because they don't contain the combination that produces the resulting referent.

Since complex mathematical propositions that contain the operation and the sign, unlike operators or theorems alone, aren't tautological truths, we can't say that a complex mathematical proposition is of the logical form of $x=y$ or $x \neq y$. Thus, $x$ can only be represented by the formulaic calculation or theorem and $y$ by the result or figure. As we argued before, propositional logic isn't applicable here, and numbers alone aren't propositions.

Thus instead of holding the logical analysis that mathematical propositions are of the kind 2+2=4, the propositional part is expressed by the number addition, and the objective part is the result. Then the result 4 forms the sense of the additional expression, taken referentially since it confirms the validity of the proposition, for each mathematical statement must be either true or false depending on the referent. This would've been simpler to grasp if mathematics ever yielded senseless propositions, like 2{]&=4, where {] is an undefined operator and & is an unknowable variable. The fact that senseless propositions aren't part of mathematics speaks of the simpler symbolical order in math compared to logic and language.

The important thing, however, is to take the mathematical propositions of the old philosophical review, where the result was taken to be part of the proposition, and break them apart to make the result a referent of the proposition now expressed only as a form of the first order of stating the operation. Thus, the truth validity of the mathematical proposition would be carried by the referent, which is the result.

By making this philosophical turn we confirm an argument in favor of the analytical character of mathematical propositions since the meanings of both the expressed operation and the following result are interchangeable but not identical ($x \neq y$), since their sense differs. Namely, although the result has its meaning implied by the proposition, it varies in a sense, like when the triangle is drawn, or when number 4 is calculated based on either a geometrical or an arithmetical operation. In mathematics, we can conclude that there must be a shift of meaning in the sense of the propositions and the sense of the referent, for their meaning is interchangeably analytical. The result must point back to the asserting proposition and the asserting proposition must assert the result.

Truth-values of math are directional like language because the referent isn't true/false apart from the mathematical proposition, that's why we concluded that its meaning is derived from the preceding proposition.

This can be exemplified using the Pythagorean Theorem in hyperbolic and elliptic geometry, where the theorem would be wrong if it doesn't reformulate the fifth postulate of the straight line with either a negatively or positively curved line. Here the condition that the triangle be right is replaced with the condition that two of the curved angles sum to the third, say instead of $A^2+B^2=C^2$ will have $\mathit{A}^2+\mathit{B}^2=\mathit{C}^2$, where the italic letters would signify the curved nature of space.

### 3. MATHEMATICAL OPERATORS DETERMINE THE MEANING OF NUMBERS

The laws of arithmetic's are tautological but their objects may be either representations or physical objects. Mathematical rules like addition, multiplication, etc. equally apply to physical objects as well as to the so-called mental objects. Two and two apples will be four apples always as two and two are four. As to Frege's objections that numbers can't represent positions in a series, the same argument holds. For the same number to appear at the same position in a series, we don't have to imagine the entire numerical sequence because of a single number. We can avoid imagining the entire numerical sequence preceding any large number by applying operators and exponentials, but the very idea of a number is impossible to learn without the grasping of a countable series, contrary to Frege.[xiii] Whether they are applied on a physical ruler, abacus, or an imaginary scale of numbers, it's irrelevant to the rules of arithmetic's since they're tautological. The same holds for logic where the rules of deduction, identity, and contradiction, can be apodictically expressed with the intervention of physiology, for when we state that A=A or that Socrates is mortal, we inarguably exclaim a physical sound or a memory of a man to express a certain thought that's tautological and *a priori*.

However, although the rules of logic and mathematics are tautologically *a priori*, their expressions are not and persist as *a priori* only as far as there is a physical audience present to psychologically comprehend the meaning of the terms. Comprehension or understanding is a psychological act that yields certain meanings through sign manipulation. Tautologies, although meaningless, can shift meanings between two or more physical or mental things that can designate a scale or magnitude. Any consistent transfer of signs by the use of the tautological operators designates the magnitude of the sign transfer of numbers from 1, 2, …10, to ad infinitum where number sets are replaced with imaginary transfinite sets. The vision of 10 doesn't give us the right to say that $10^{10}$ yields the same type of vision, however, the exponential operator doesn't represent a vision of a kind and it's a strict arithmetical operation.

The process of shifting different signs into formulas, shapes, or sets, is done through comprehension, which is a psychological act that arbitrates meanings using non-psychological and tautological operators and theorems.

All signs are either meaningless and tautological, or meaningful and transferable.

Mathematical and logical operators have no meaning of their own and have to mediate a meaning between signs, quantities, and propositions. A combination of the two is possible.

A meaning is the ability of a sign to represent or replace another sign. The shift and translation between signs is done by comprehension. No sign represents another sign without anyone understanding it. The directional use of signs described by Wittgenstein is only possible through comprehension.

In regard to Frege's argument against Euclid that numbers can't be "units" or predicates like words are,[xiv] our theory states that the meaning of numbers is never independently fixed, either to an object, to a series, to logic, or a subjective image. Numbers involve learning of the rules and operations of arithmetic's and grasping the magnitude of their scaling. Thus, numbers have transitional meanings of scaled quantities scaled by arithmetical operators.

Therefore, when we say that a number is a scale or a position in a series, Frege's argument,[xv] that this cannot be so because we can't remember the place in a series or the entire numerical series every time we conceive a number, is invalid. This is because a number is a point in a series of magnitudes according to arithmetical rules and operators. Just because we memorize a point this doesn't mean we have to evoke from memory the entire sequence to which it points. When we think of a million, we don't have to evoke the images of a million points in our imagination but we still mark it as one million, distinguishing it clearly with a point mark from all numbers that precede it or follow it.

We can't think of a million fingers every time we think of a million, but we may think of a million or any other large or even infinite number as a decimal unit or a set times itself, even an infinite one, as long as we have the right set of operators and signs to do so.

This can be proven by using the example of the duodecimal system of the base 12 number system where the use of decimal arithmetic's would be proven inaccurate. The duodecimal arithmetical sequence will confirm that all numbers like one, ten, thousand, and million would inaccurately express the decimal concepts of numbers and quantities if considered as belonging to the duodecimal order. The following argumentation will prove that numbers are arithmetically operationalized sets or systems of magnitudes and that no two numbers from the decimal and duodecimal sequence of sets, even when signed the same, express the same quantity.

If numbers were non-arbitrary and objective, then a number would always have the same meaning or calculation. Let's take 10 with the power of an exponent 10, $10^{10}$, and express it in the duodecimal system. A $10^{10}$ of the duodecimal system would not mean ten billion but 8,916,100,448,256, or eight point nine and something trillion of our decimal order.

Whether it's per hundred or per hundred-twenty, makes all the difference in arithmetic's and it's clear that there is an arbitrary decision on what magnitudinal numbers represent and that only our subsequent calculations founded on arbitrary sets fractionalized in a certain way, like into 1/4 or 1/8, will calculate these results with accuracy.

However, the most important consequence of the duodecimal arithmetical sequence is that even identically designated numbers have different meanings. Let's designate a scale of the duodecimal or dozenal primes down to their percentages and compare the results between the decimal and the duodecimal systems to find out if numbers are truly universal. The duodecimal order can be of the type:

1, 2, 3, 4, 5, 6, 7, 8, 9, *X*, *E*, 10

Where X is the decimal 10, E is 11, and 10 is 12

Then

11, 12, 13, 14, 15, 16, 17, 18, 19, 2*X*, 2*E*, 20,

Here 11 is our 13… and 2X is 22, 2E is 23, and 20 would be 24.

In such circumstances, the arithmetical formulas of the decimal system of numbers would not work in the duodecimal system since the former system's addition 20+20 would yield 40 but it won't match the duodecimal 40, since decimally it states 48. We may designate numbers however, we'd like, but when it comes to expressing the arithmetical sequence, a strict arbitration must proceed as to whether it's a system of tens, twelves, or any other order we can imagine. This proves that numbers are fractions of sets rather than bundles or classes that make up a set, meaning that numbers are derived by the mathematical operations that precede or condition them into forming strict sequences.

Then we have the following sequences:

… … …

81, 82, 83, 84, 85, 86, 87, 88, 89, 8E, 8X, 90

91, 92, 93, 94, 95, 96, 97, 98, 99, 9X, 9E, 100

Here 84 is the decimal 100, 90 represents our 108, and 99 is the decimal 117. 100 would then be 120. Roughly speaking all our numbers would have to convert to represent the right products, proving the argument that numbers are arbitrary symbols within a numerical sequence that's arbitrarily given.

The same result goes for their percentages right to the conclusion that the decimal 1 doesn't equal the duodecimal 1 (1≠1) except by sign, although they are equal under different terms of the agreement that doesn't take the duodecimal system coherently but only if applied to our very decimal world, like when 1-foot measures 12 inches, but there are 5280 feet in a mile. 1-foot here is a perfect match to the decimal 1 (1=1) of the decimal count, but it shouldn't be. Thus, we see that the inches-foot ratio belongs to an extinct measurement of the duodecimal order that was probably lost in translation since there isn't a following duodecimal set to where a foot belongs to, like when a meter belongs to the decimal sequence of a kilometer, 1/1000. The standard measurement usually gives 3 feet per yard, concealing the true magnitude of the duodecimal order.

The percentage rate will change as well since percent literary means per hundred. The duodecimal 100 is either 120 or it may take another expression depending on our arbitration. This proves that numbers are magnitudes of quantities expressed in decimal or dozenal sets. Thus, the duodecimal 50% would be greater than ½ of a decimal number, and the duodecimal 1 must mark a decimal 1.2 meaning that 100% of the decimal order means a sum smaller than the duodecimal one. The duodecimal 100 translates to the decimal percentage of 120% or 120/120.

The fact that the duodecimal fractions and percentages express bigger quantities than decimal ones, speaks of the magnitudinal type of sequencing of all types of numbers, natural, irrational, etc. The reason why there is a duodecimal ½ or 50% that expresses an identical sense in the decimal order suggests the permanence of the fractional operator, which is tautological.

This would be clearer if we properly match the initial numbers of the two series and see whether *X* and *E* are the same as the decimal ($_{d1}$) 10 and 11, or whether the duodecimal ($_{d2}$) 10 is decimal ($_{d1}$) 10. We can try this in the following example:

$1_{d1}, 2_{d1}, 3_{d1}, 4_{d1}, 5_{d1}, 6_{d1}, 7_{d1}, 8_{d1}, 9_{d1}, 10_{d1}, 11_{d1}, 12_{d1}, \ldots$

$1_{d2}$, $2_{d2}$, $3_{d2}$, $4_{d2}$, $5_{d2}$, $6_{d2}$, $7_{d2}$, $8_{d2}$, $9_{d2}$, $X_{d2}$, $E_{d2}$, $10_{d2}$, …

If we now match each number of the two series forming a column, we can see that no two numbers can be identical other than in sign, thus proving the hypothesis that numbers represent magnitudes in a sequence because the two numerical sequences are derived through fractions.

Numbers that express the same fractions and percentages in the two different series only prove the arbitrated nature of the number signs and that their signification is mute unless mediated by operators and fractions.

Nevertheless, why would someone want a single number to depend on an order? Perhaps the problem is that the inventors of the duodecimal system, or any other, never thought of expressing increasing quantities found with enumeration. Think of it this way, if you were an ancient Babylonian going to the store to buy fruit, only to find out that the salesperson didn't know how to count but only to fractionalize? This would be unusual since we infer that numbers begin with the act of counting, or according to Russell: "The act of counting consists in establishing a one-one correlation between the set of objects counted and the natural numbers (excluding 0) that are used up in the process.".[xvi] However, is there a possibility that a number was initially part of an order like when pizza slices come in 1/4 or 1/8? Nothing goes against this argument and counting may have begun by splitting things apart and setting a one-one correlation between the counted set of divided or sliced objects and the natural numbers that are used in the process as quarters, eights, decimals, or dozens, $\frac{1}{10}$. This way we can see the link between an operator and its derived number.

All signs, mathematical or lingual, are numb and of an arbitrary nature set according to the circumstances and context they have in a system or a scale. If there was an exam day where a student had to use the duodecimal system instead of the decimal one, and he didn't remember it well or confused it with the decimal one, his calculations would all be wrong.

## 4. ENUMERATION AND NUMBERS

From here, we can deduce the meaning of a number through analysis of units and scales associated with objects. Counting isn't a simple psychological process because it relies on the use of operators, and operators from what we concluded are tautological and therefore senseless unlike signs arbitrated through psychological processes and meaning. However, Frege gives the following argument against representing units as numbers:

> If we use 1 to stand for each of the objects to be numbered, we make the mistake of assigning the same symbol to different things. However, if we provide the 1 with different strokes, it becomes unusable for arithmetic's.[xvii]

Frege's argumentation here is forwarded with the numerical definitions of the type: 4=1+3. Namely, he argues that if numbers are units then they must be either different or identical, and therefore can't arithmetically express a single result.[xviii] Further, he argues, that if 1=1 is a unity of identity, like Socrates and Socrates, then 1+1=1, since both numbers represent the same thing.[xix] Therefore, number 1 can't represent an identical object, as unity does in apples, for one apple is the same one apple. If, however, 1≠1 in the unity theory, then we must put a stroke on top of 1, every time we express it arithmetically to gain the desired result, like 1’+1’’+1’’’=3.

The problem here from what we concluded before, is the inconsideration between the arbitrated nature of the numerical sequence and the tautology of its operators. Namely, definitions of numbers are not tautologies like definitions of operators. The meaning of an operator is the same in every instance. The meaning of a number hides in the arithmetical formula, therefore when the same numbers add together, we express them as separate units, and only then can we get a different result: 1+1=2.

It’s clear that any misunderstanding of the role of arithmetical operators as tautological, and the analytical definition of numbers as having certain meanings, can lead to confusion that the meaning of a number is the same as that of an operator, i.e. tautological, falsely concluding thereof, that numbers have a nontransferable meaning like tautologies do. The meaning of a number is always transferable to any other number using an arithmetical operator, even when we use two identical numbers. The meaning of an operator, however, isn’t transferable and is always the same no matter what number signs we use it with, or how many times we repeat it in certain formulas, its meaning will not change under different circumstances. Hence 1+1+1+1=4, where the meaning of the same numbers accumulates because the meaning of the operator remains. Neither Frege nor the empirical theorists of arithmetic’s that he mentions in his *Foundations*, ever argued anything of the sort that the use of an identical operator should be stroked to be differentiated from the other identical operators in formulas, like 1+’1+’’1+’’’1=4, since there is no such meaning in operators because they are tautologies like logical principles.

Frege's argumentation that numbers can't be ideas of any sort is based on the undefined use of arithmetical operations in asserting the true nature of numbers.[xx] The fact that we can't form an idea of the number 0 by stating something like there is 0 number of mammal crocodiles, or form an idea of a huge number by imagining the length of the Milky Way by taking a piece of rod and extending it in our syntheticall apperception, isn't an argument against the scaled nature of numbers as simple arbitrary signs of sets. It's similar to how words and sentences form their meaning from whole language systems, even when a language is boundless. The only difference between the use of words in forming sentences, and the use of numbers in forming sequences, formulas or bigger numbers, is the nature of the operators in use. Where the rules of syntax state the order and use of different types of word signs, like the use of a subject and a verb, the operators of arithmetic's state the scale and succession of the number signs. Both the operators of language and arithmetic's make up the scaffolding in either descriptive meaning, found in language, or meaning of magnitude, found in mathematics.

The difference between arithmetic's and language is the accuracy with which it asserts its results. In arithmetic's the results are large numbers given by the combination of signs and operators, that can't be simply visualized. The results in arithmetic's never surpass the realm of the stated signs, and it merely demonstrates them in a different order, even when the order is infinite. When we say one million dollars, we may think of it simply as being rich or we may imagine a single dollar and have it multiplied by the exponential, 1$ x $10^6$, where the exponent is an operation. Even when some of us can't grasp the meaning of this operation, we still may think of a million as 1000x1000 or simply think that a million is like a 10 made up of 10 hundred thousand, 10x100x1000, but we must think of a sign like 1 that we originally imagined it as representing an apple or a bead on the abacus but based on an operator.

Thinking in any form is an act of psychology. The fact that we can't think of billion beads on the abacus, doesn't mean that we erased the abacus from our memory, it's just that we improved our arithmetical abilities by relying more on operators in place of signs, like when $1 \times 10^9$ comes to my mind, I've perfected arithmetic's and not that of the sign numbers but of its operations and have forgotten all about the abacus and the beads where simple additions and subtractions occur. Arithmetical skills are like any other and reside in the physiology of the human brain.

Besides 1 the other important number is 10, since both 1 and 10 will have a different scale in a duodecimal ($_{d1}$) system than in a decimal ($_{d2}$) one. Even if 1+1=2 in both numerical systems, this doesn't prove that $_{d1}1={}_{d2}1$, but only that the addition formula is identical, and not the number. However, this will be inconsistent with the operationalized nature of 1 as a symbol representing a fraction of either a decimal set or a duodecimal one. If $_{d1}1={}_{d2}1$ in both systems, then the duodecimal $\frac{1}{10_{d1}} = \frac{1}{10_{d2}}$, which is false. This proves the arbitrary nature of the number 1, and it shows the magnitudinal type of its fractionalized order in the separate numerical systems, and therefore confirming the definition of a number as an arithmetically operationalized scale.

## 5. ARITHMETICAL SIGNS

We continue our analysis of the meaning of numbers by reconsidering the validity of Frege's logical definition of the number 0, as a "concept not identical to itself".[xxi] If we prove that Frege's definition of 0 applies to any other number of either the decimal or duodecimal sets, then it will be fair to say that any logical grounding of arithmetic's lacks universality and is incomplete in its foundational work. The proof that Ferege's definition of number 0 is applicable to any other numbers than 0 of both the decimal and duodecimal order, would mean that the two numerical sequences must form separate domains of calculations, two parallel worlds so to speak.

This would mean that the validity of logicism in mathematics can be revoked. Frege argues that if there is a concept *F* whose object *a* permit the insertion of anything whatsoever, then the meaning of *a* is not identical to *a*.[xxii]

We can attempt to use this same logical definition of 0 on the numbers of the decimal and duodecimal order to see where the project of the logical foundation of math goes wrong.

When we concluded that $_{d1}1$ is unequal to $_{d2}1$, we also made a logical statement of the Fregean type that both number 1s are part of the same concept that's not identical to itself, like when defining 0 as being *a* and not being *a*. However, this won't suffice since both number 1s from the duodecimal and decimal sequence are forbidden under the concept that's not identical to itself. Frege writes that:

> Now it must be possible to prove, by means of what has already been laid down, that every concept under which no concept falls is equal to every other concept under which no other concept falls, and to them alone; from which it follows that 0 is the number that belongs to any such concept, and that no object falls under any concept if the number which belongs to that concept is 0.[xxiii]

Thus we conclude that the logical definition of 0 given by Frege, isn't valid because the universe of the duodecimal order and that of the decimal order would have their respective numbers, like $_{d1}1$ and $_{d2}1$, included in its definition and equal to each other. However, all non-zero numbers are forbidden under the concept that's not identical to itself, or 0, unless they represent Frege's definition of the number 0.

The other proof that all numbers from the two arithmetical sequences that are marked the same aren't the same comes from the argument that if $_{d1}1=_{d2}1$, then so would any other number, like $_{d1}10=_{d2}10$, in the two sequences, which is not the case of the ten and the dozen. No number other than 0 can fall under the "concept that's not identical to itself" which proves that $_{d1}1\neq_{d2}1$ and that 1 is a fractionalized unit of 10, either as a dozen or a ten.

It's not that $_{d1}1$ and $_{d2}1$ are not identical concepts, because they have identical sense. What's different then is their meaning since it's not found in logical sets but in arithmetical sets where numbers are fractions of decimal-like sets.

Arithmetical sequences are formed based on the types of operations performed in either decimal or duodecimal sets. Thus, the objects or meaning of any two identical arithmetical concepts are defined according to their scale in the sequences of sets in d1*n* or d2*n*, and not in logics.

This, however, underlies the entire basis of Frege's logical theory of numbers by refuting his definition of logical equality. Namely, Frege defines logical equality on which his entire theory of numbers is based, with the following arguments:

> This reduces one-one correlation to a purely logical relationship, and enables us to give the following definition:
>
> the expression "the concept *F* is equal to concept *G*" is to mean the same as the expression
>
> "there exists a relation *φ* which correlates one to one the objects falling under the concept *F* with the objects falling under the concept *G*".[xxiv]

That this correlation isn't just logical but also arbitrary, can be proved with the use of the duodecimal sequence, where the referred to "one-one correlation" of identity through "the relation *φ*", was proven to be something strictly agreed upon on the type of the sets used, where numbers are exponents of operations of either decimal or duodecimal sets.

Number 1 is a mute sign like any language sign except that it designates a quantity on a scale and not a description. This means that the identity of the concepts *F* and *G* is arbitrated through the agreed-upon propositional structure of the relation *φ* between the number objects *a* and *a*. In other words, this is the same as using language expressions where different meanings are expressed based on stating one identical proposition in different circumstances. Some propositions must use different objects to express identical truth values or facts. A language game may allow under certain conditions the proposition of the type 'the weather is fine' as accurate, like when talking to your mom over the phone, but if given on Google or TV weather forecast, a proposition of this type would not be accurate, for part of the audience may prefer warm weather over a cold one, and all would certainly want to know whether it would rain or not, therefore a more elaborate proposition would do.

It is the same with numbers whose meaning is not innate to their signs. The difference is that unlike sentences and word signs that change meaning under different circumstances, numbers change meanings depending on an exact set of agreed-upon rules, like the use of arithmetical operators and also on the certain type of numerical sequences founded on operationalized sets, either a duodecimal or decimal or any other kind our imagination may invent.

But why is language so different? The thing about language is that it doesn't rely on symbolism as much as arithmetic's does, and requires factual or temporal confirmation of its expressions, unlike arithmetic's.

This we'll further analyze on Frege's concept of the number series.

Another problematic aspect of Frege's theory is his definition of the relation of the adjacent members of the series of natural numbers that he logically derives as standing next to each other.[xxv] The formulas that Frege gives here have been standardized in the arithmetic's of numerical series, of the type: $n$=$m$+1; where:

> The proposition: "there exists a concept *F*, and an object falling under it *x*, such that the number which falls under the concept *F* is *n* and the number which belongs to the concept 'falling under *F* but not identical with *x*' is *m*" is to mean the same as "*n* follows in the series of natural numbers directly after *m*".[xxvi]

More contemporary formulas explain this with the use of the arithmetical operators, as Frege in part does. The current model is: $a_n=a_{n-1}+(n-1)d$, which explains the number series generated by adding multiples of a common difference, $d$, to previous terms. The only terms that are strictly defined here are the addition, subtraction, and the equality sign, and only after the definition of the numbers follows. It's simply impossible to define a number without the use of arithmetical operators.

If, instead, we use just the common difference, $d_1$, to explain the numerical set, then the duodecimal set will yield 10 as

$a_{10}=a_e+1d$,

instead of the decimal formula of

$a_{10}=a_9+1d$

Thus stating an adjacent series of numbers will simply not do here when it's only grounded on logic, because the only apodictic terms here are the tautologies of the operators and not the logical identity of numbers. The sets of dozens and tens are agreed upon arithmetical terms of sets that are further fractionalized in 1s, 2s, etc. The arithmetical operators are the only things of *a priori* and apodictic nature, in Wittgenstein's sense, in any numerical theory.

## 6. FREGE'S PLATONISM

The use of the identity principle in Frege is what founded his logicism in the first place, and this was the cause of the unwanted migration of Platonism in his philosophy. Namely, through the solution of the problem proposed by our mathematical theory, we'll completely render any manifestations of Platonism as unnecessary and locate its root cause.

In this regard, Frege himself properly noticed the problem and explicitly refuted all notions of numbers as a series of spatial magnitudes of a sort. The problem is that under the disguise of logic, Frege assumed that to avoid the trap of spatial description of numbers, logic is given a role to build a theory dealing with concepts and their relations, where any notion of space is forbidden. However, once you enter a number theory using logic, this position no longer holds, since numbers remain objects in geometrical space.

Thus, his statement that "the relation of the adjacent members of the series of natural numbers that stand next to each other"[xxvii], is not a logical one or even an arithmetical but rather a geometrical one based on the concept that no sequence of numbers can naturally be given as adjacent in any form even when formulated with logical relations. The very idea of an adjacent series is a product of geometry, not logic. The same goes for geometry, where two identical angles or triangles don't become logical terms because of the relation of identity they have, a concept that's defined by logic.

A proper relation of any number sequence lies on the agreed numerical system in use. This could also be expressed on any number series, as 2, 4, etc., and it's based on the arithmetical operators. A closer look at Frege's proposition about the natural sequence of numbers can be clearly shown now and see where it goes wrong by defining arithmetic's as logical in the quoted passage.[xxviii]

The grounding of the theory of numbers on the logical concepts of identity and contradiction comes out of the need to opt-out from any psychological interpretation of numbers as 'sense ideas' but it fails in resolving the remaining issue regarding Platonism. The formulation given by Frege that there is a series of natural numbers where *n* follows *m*, does not define the series itself or the fact why certain numbers follow certain order. Once we follow this line of thinking without defining first the series and the order, we immediately fall into the Platonic trap where numbers somehow preexist us as objects do, and also the unnecessary assertion of the 'existence' of *F*, as stated in Frege's quoted passage.[xxix] Existence as such isn't a logical concept but a descriptive one, and should be avoided in logical definitions unless they are non-logical and based on geometry and description, as we saw earlier in Frege's reinstating of geometry where existence makes sense since he deals with descriptions.

If instead we reformulate the terms of the definition and state the propositions differently, where *F* is to be defined as the addition formula, then we can eliminate everything that goes into the unwanted sphere of existence or Platonism. Thus, we reformulate the proposition for the formula *F* as follows:

The formula *F* has a concept *x* and an object falling under it such that the number that stands before the addition sign, and is part of the concept *x*, is *n* and the object that belongs to the formula *F* but is not identical with *x*' is *m*, means the same as "*n* makes the series of natural numbers by directly following after *m* through *F*". This would be valid with both even and uneven numbers where not only arithmetical sequences can be expressed, but also any other numerical series. If we keep Wittgenstein's definition that numbers are exponents of operations, this can be defined in the following way:

x is in the left hand product side of the formula *F*, and all other signs are exponents of *F*.

$Fa_x = a_m + nd$.

This is a clean arithmetical definition of the sequence of numbers, where *F* marks the addition formula, *x* is the product of the adding signs marked as *m*, that's placed before the addition sign in *F*, and *n*, placed as the second sign that comes after or in the right hand corner of the addition sign. Since place here designates the position of the number signs on either the left- or right-hand corner of the formula, there is no assumption of any kind about the nature or certainty of existence of numbers, for it can either be descriptive or physical, where the only conceptual form of existence that's independent of sensual or physical objects, is the addition sign of *F* that's tautological.

This formula also sets numbers as point places of an arithmetical expression in a certain operation that's not spatially coordinated, saving it from the intrusion of empirical or geometrical spatiality. The fact that numbers are successors in several series of editing *n*'s, in the type of additional succession as *n*+1, makes the point place sequence of numbers non-spatial. The fact that smaller numbers count on the left-hand side of the numerical succession, has nothing to do with any spatial direction of numbers, like on a ruler, but is purely the place of numbers in an arithmetical expression, *n*+1, where *n* marks any preceding number in the operational sequence.

Therefore, we can infer that all forms of Platonism are rooted in the misconception of the existence of numbers conceived as apodictic and *a priori*. Any certainty in the use of numbers comes from the use of the arithmetical operators.

## 7. NUMBERS ARE NUMB

This, however, shouldn't be confused with the meaning of numbers being magnitudes along. Saying that a number is a magnitude is like inferring that arithmetic's is a mere enumeration. This, however, is incorrect and all arithmetical sequences are founded on sets, which can be decimal or any other, derived by various operators and exponents. Scientific notations are an example of how operationalized high numbers work but the same process goes undisrupted in all of arithmetic's where even smaller numbers like 1 are exponents of the fraction, $\frac{1}{10}$.

Rational and negative numbers are clear instances of how we obtain numbers with the use of arithmetical operators. If negative numbers were originally derived via the process of subtraction, and division originally derived rational numbers, then why not assume that operators ground natural numbers as well?

This way, any definition of numbers that assumes their independence from material or psychological sets or units, is of no consequence here, for a mortgagor may be in debt, having a negative amount of cash, whether the amount he owes is expressed in negative numbers or negative dollars. The representation of the concept of number, whatever it may be, is of no consequence here since net subtraction and division are foundational in estimating someone's debt. This will overcome the intrusions of both Platonism and empiricism in defining mathematics.

A number then is a fraction of magnitude or an exponent of an operation as Wittgenstein states.[xxx] From what we concluded in the previous chapters on the function of sign numbers and formulas, we're setting a definition of numbers based on arithmetical operations. We categorically state that numbers of form sets but deny that these sets are based on logical concepts of any kind or that logic forms the basis of mathematics. We can show these clearly by analyzing Russell's work *Introduction to Mathematical Philosophy*. Russell's definitions rely on the usage of terms that define the vogue concept of 'succession' as a logical type of relation. These terms are order, posterity, heritage, mathematical induction, etc.[xxxi]

Wittgenstein was the first to observe the limitations of the logical definition of numbers, stating that numbers are exponents of operations, but he never fully developed a complete mathematical analysis. Wittgenstein further concludes that the theory of classes is altogether superfluous in math and refutes the definition of numbers as general concepts.[xxxii]

The main problem with mathematical induction is based not on Russell's concept of numbers as bundles of identical terms, but in overlooking where mathematical induction first sneaks in. What Russell didn't realize by not defining a number as a member of either a dozenal or a decimal set, is that his definition of numbers includes mathematical induction at the start of stating that numbers are bundles of a certain type or represent classes of terms.[xxxiii] Mathematical induction doesn't happen after he defines what a number is and the concept of even a single number like 0, is based on mathematical induction.

Namely, a number isn't something apart from a set and its order, where succession truly steps in. We saw this in the example where decimal and duodecimal numbers aren't equal, or $1 \neq 1$. This is because 1 is a fractional part of a set, either a decimal or a duodecimal, like a decimal or a fraction, and not just a plain figure that adds itself ad infinitum. By making a number a part of an arithmetical set, we categorically state that the base of a number is the operator, in a similar way as we said that the base of a right triangle is the Pythagorean Theorem, or that the meaning of a word is founded by its syntax and context.

The case of transfinite ordinals only strengthens our claim, since these are sets of an infinite type, unlike decimal or duodecimal sets that are of a finite type, made up of 1's in $\frac{1}{10}$ fractional sets.

The question of what comes first, whether decimal or dozenal sets, operationalized into one's, or number 1 that was added into a decimal type of sets, can be arguably decided in favor of the decimal-like sets since division is a simpler, natural occurring act that proceeds addition. Although conclusive historical proof of whether division proceeded addition lacks, the fact that more advanced numerical sequences were either decimal or duodecimal, speaks in favor of the fractional type of arithmetic's used that's based on the division of decimal-like sets into one's, rather than of addition of ones into the decimal type of sets.

The fact that number 0 was introduced into the arithmetical sequence centuries after number 1, a number that precedes 1 in the sequence, only confirms our case that numbers aren't derived by following the common sequential scaling of $n$+1 but represent a case of the primacy of $\frac{1}{n}$. Here the numerator is defined as the number of parts of $n$ and can be any number. This would certainly be a historically more appropriate definition of numbers since people didn't learn to count or invented the arithmetical sequence before using numbers as portions of something that developed into operators.

Only a theory where numbers are considered as exponents of operators, in Wittgenstein's sense, can fairly claim to give a pure arithmetical definition of numbers irrespective of their assumed transcendental or empirical nature. Arithmetic's isn't about origins, and it fairly existed independently of the philosophical controversies regarding the nature of numbers because it's not grounded on any such uncertainties. The certainty of mathematics is of an operational kind and not one of origination or application.

In defining numbers, a common mistake is to think that we have to start in the common sequential order of beginning with number 1 or zero ad infinitum, 0, 1, 2, and so on. Because we learn arithmetic's through counting, this doesn't constitute its true meaning. When we begin defining numbers by defining 0, we accidentally set up a series of definitions that are somehow based on the definition of 0. This would be an inappropriate case of a number theory because 0 was not part of arithmetic's thousands of years before it was discovered fairly late in history.

The logical theory of numbers is based on the idea that there's a sequence or posterity, as Russell calls it, in the order of numbers. If we take numbers first, order comes second, and operators third or later. The fact that operators may precede numbers is something basic, since the use of operators isn't based on numbers, and people can calculate without having an idea of numbers, either using the abacus or with fingers. However, numbers and relations are impossible without the use of operators, like in $n$+1.

Although it's not false to say that numbers are of a sequential order, starting from 0 and so on, the more appropriate approach would be to say that numbers are orders or magnitudes of certain sets of dozens or decimals, like when 1 is $\frac{1}{10}$ or 1/100. Then divisions and fractions would derive the meaning of numbers, instead of the late-developed logical form of posterity. The late invention and use of 0 becomes clear and related to the use of more complicated mathematical operations and magnitudes of numbers. Its Indian inventor, Brahmagupta, invented 0 by also inventing the negative numbers.[xxxiv] The two series of negative and positive numbers would be indistinguishable without 0 in the middle. Negative numbers were probably originally derived by reasoning on the result of subtracting natural numbers. Thus, we obtain a positive and a negative set, proving that 0 comes between the two sets. All numbers are impossible terms without the prior use of a sequential numerical set. In the invention of 0, these were the negative and the positive sets.

But why would there be a set before a part? I don’t think that an example from history or early childhood would substitute for a more proper explanation.

Let’s now prove why numbers are derived from sets:

Let *F* be the type of the formula expressed with a fractional operator, $F\ \frac{m}{n}$, where *m*, the numerator, whatever number it may be, is the part of the fraction. Then *n* or the denominator is the set, either a decimal or a duodecimal, $\frac{1}{10}$. Thus, an arithmetical sequence expresses the applied arithmetical operation of a fraction in a desired series and order. The base of the number 1 is then purely arithmetical and designates it as a part of an arithmetical set of the decimal 10, meaning that “*n* makes up for the series or sets of natural numbers by directly including *m* as a fractional unit of the denominator represented in *n*. *F* then is the formula of a fractional relation between natural numbers and sets.

Let’s now closely examine Russell’s key method of mathematical induction and posterity to see why it goes wrong.

> The process of mathematical induction, by means of which we defined the natural numbers, is capable of generalisation. We defined the natural numbers as the "posterity" of 0 with respect to the relation of a number to its immediate successor. If we call this relation N, any number *m* will have this relation to *m*+1. A property is "hereditary with respect to N," or simply "N-hereditary," if, whenever the property belongs to a number *m*, it also belongs to *m* + 1, i.e. to the number to which *m* has the relation N.[xxxv]

The obvious problem in defining arithmetic's with seemingly logical terms like "posterity" and "heredity" is that they firstly denote the given arithmetical sequence with non-arithmetical relations, and secondly, explain it in a manner of generalized terms that have nothing to do with arithmetic's. Both heredity and posterity designate a biological process of succession of certain traits to future generations. If numbers were generations, and arithmetical numeration was a generalization derived from mathematical induction, then the foundational relation in arithmetic's would be that of progression. However, this would only work in a sequential order of natural numbers expressed with the formula, n+1. But a sequence of the progressive kind that involves posterity or heredity can't explain fractions, decimals, and negative numbers. If Russell's philosophy of mathematics is to be conclusive, then the appropriate terms for describing fractions, decimals, negative numbers, etc. would be ancestral and predecessorial, which would be wrong since neither negative numbers nor rational numbers, although preceding the order of natural numbers, are their forbearers of any kind.

Any number that stands in the series or sequence of numbers before the number that follows it, whether a rational, natural, a decimal, or negative number, is a sequentially smaller by a magnitude of the applied operator, and not by precession or heredity.

A magnitude is properly expressed when formulated through a mathematical operation other than enumeration. A certain number takes a definite order only by the use of the operators and not through any sort of hereditary relations.

In the relation expressed as n+1, the key term is not posterity of 1 to any future number, but addition as such, which is not a general term or a form of heredity, but a tautology that forms the syntax of any addition formula until "and so on".

In Russell, any definition of a number is based on operations such as mathematical induction and heredity. When Russell defines numbers before a succession, he clearly states that a number is a bundle or a class of classes with the same number of terms.[xxxvi] The only obvious problem in this definition is that it already assumes what a number is before defining it as a bundle with the same number of terms. Namely, there can't be a class of classes with identical number of terms based on logical identity before defining what a number is and what a class is. Both a class and a subclass then presuppose the meaning of numbers, making the argument circular. If we reformulate this into an effective argument, we'd have to reverse the order of the class members as that which comes prior to any class of classes or their member classes. In this instance, all classes are generalized terms from their members by the process of empirical induction.

If the main class precedes the minor classes, then it has to precede it either in a succession or in posterity. In the reverse case, the member terms would have to precede the main class inductively. Both arguments would render the use of the concept of class in math useless.

In Wittgenstein, we find the same logical argument about the nature of numbers. The problem of logicism begins when mathematics is derived from logic, something that can't be proven.

Wittgenstein assumes that there are certain things called numbers and a certain thing called the concept of a number, where the more general term is used as a definition of the simpler terms. Wittgenstein states that:

> The concept number is nothing else than that which is common to all numbers, the general form of number. The concept number is the variable number. And the concept of equality of numbers is the general form of all special equalities of numbers.[xxxvii]

Here we see how inconsistent a logical definition of numbers is, and with the introduction of broad logical concepts, regarded independently from numbers, we turn the entire mathematical domain into a logical one. Thus in Wittgenstein, we see there is: a) a mutual concept of a number common to all numbers; b) a concept number that's a variable number; c) the concept of equality of numbers representing all equalities of numbers. This is how mathematics is turned into logic.

In our definition, a number is something that belongs or follows a certain order of numbers that's formed by a set, through the use of the arithmetical operators, which are tautological and only thus necessitate a certain well-defined order. Numbers are not logical in any sense. Thus, the concept number common to all numbers is replaced by arithmetical formulas of the mathematical order expressed as a decimal or a duodecimal set, $\frac{1}{10}$ or an exponent of operators. Unless arithmetic's can be entirely derived from logic, there is no use in inventing general concepts common to numbers, independently from the arithmetical operators that can't be reduced to general logical concepts.

## 8. NUMBER ONE

We can also state our number theory on the case of number 1, which we'll define using the old formula of a sequence, but with our formulation of the concept *F* that represents the addition formula, thus avoiding not only psychologism but also Platonism. In defining number 1, Frege keeps his arguments clear from psychology by stating concepts of the logical form stated about 0, which are identical or un-identical to each other. In Frege, if the number 0 is defined as a concept that's not identical to itself, then the concept that's identical to the concept of 0, is number 1.[xxxviii] I don't want to go through all the details of Frege's theory but just point out that his theory of numbers is based on the concept which is identical to the concept of 0 (number 1) and the concept which is not identical to itself (number 0). Although these two different concepts make sense in grounding arithmetic's in logic, they are not mathematically apodictic or *a priori* concepts of the arithmetical sequence, since two different concepts where numbers can be found, aren't necessarily apodictic concepts of the arithmetical sequence. Namely, nothing in the concept that's identical to the concept 0 states that it must be the concept of the number that directly follows the concept of the number that's not identical to itself, or that 1 follows 0 in the sequence. Other than the addition operator, which can't be logically defined because it's arithmetical, any number may be assigned in the concept identical to 0. Therefore, this is not a proper definition of 1 and can be another number.

To avoid both Platonism and psychologism, a more suiting definition would be of the form given above on the number sequence, where the product 1 will be written as follows,

$1=Fa_x=a_m+\mathrm{n}d$,

Here *F* is the type of the formula expressed with the addition operator, and 1 or *x* is the desired product. The object, *a,* which comes before the addition operator is 0 and the number, *d*, that follows the operator is 1. Thus, any arithmetical sequence expresses an applied arithmetical operation of the desired series and order. The base of the concept 1 is then to be regarded as the adding symbol to 0, meaning the same as "*n* makes the series of natural numbers by directly following after *m* in the formula of the type *F*". Without the use of the arithmetical operators, no two numbers can concur in a sequence.

This proves our point that operators precede or condition the meaning of a number or its place in a sequence.

The scale and quantity of number 1 will thus be arbitrary and only defined with the use of fractionalized sets in a chosen numerical sequence. Any number whatsoever can follow 0, and the fact that 1 is the number that follows it, is something strictly agreed upon by the use of numerical signs and operators in the system of sequencing in place.

## 9. TRANSFINITE NUMBERS

A common flaw in any non-logical philosophy of mathematics is the explanation of transfinite numbers, which are theoretically based on logical set theory where mathematics is proven as its branch. We'll try to overcome the obstacle of the transfinite set theory by showing that numerical sets are the base of all arithmetical sequences and that the concept of a set is something introduced in arithmetic's from its beginning, through the use of fractionalized systems. A set is not a logical construct but a mathematical one, formed by the use of the arithmetical operators, primarily division but only in finite sets.

Russell gives the following definitions of numbers: "In the first place, numbers themselves form an infinite collection, and cannot therefore be defined by enumeration.".[xxxix] Here Russell assumes Frege's hypothesis that the meaning of numbers of dwells outside of physical or psychological representation. However, what's important for the operationalization of mathematical expressions, are its operators and not mere numbers, thus the act of enumerating falls short to the sets produced by the arithmetical operators and their products. Enumeration is the product of arithmetical operations. We said that small numbers like 1 are the product forms of fractionalized sets, 1/10, and large numbers like a million, which are best expressed as either $10^6$ or 1000×1000, are exponents of operators. Large numbers can't be formed without the proper use of operators and exponentiation.

However, how can these be a case for grounding transfinite numbers when even a large number like $10^{12}$, isn't an infinite number, or a number preceding it, and still even this large number isn't countable, from first to last?

The transfinite theory of numbers presents a confirmation of our arithmetical philosophy, where numbers are considered as mere signs that are operationalized into expressions. Thus, a transfinite number is an operation of an uncountable and infinite number. There is no other meaning to transfinite numbers other than representing operations of abstract numbers that transcend any form of enumeration.

Namely, there is no meaning of a number beyond a certain arithmetical series, like 1 doesn't represent an apple or isn't applied to an apple independently of the arithmetical sequences made by division or addition. Perhaps it's not as plain to demonstrate that 1 is a function of the fractional operation of 1/10 as it is with larger numbers. The role of scientific notations or exponentials is more practical because it's used in orders of numbers of very high magnitude that are practically non-comprehensive.

Transfinite numbers are such instances of numbers where the un-expressible orders are summarized according to infinite sets of order or magnitude, where the *n* number set is replaced with ω, the first infinite set. Since an infinite number set doesn't have a finite set preceding it, transfinite numbers are examples of infinitary arithmetic, where only addition, multiplication, and exponentiation are applied, and subtraction/division are forbidden. Thus, what transfinite numbers represent is a case of infinitely extended numerical sets by either a finite or an infinite order, where instead of having the common order of 1/10 or n+1, we have ω+1, ω+n, ω×n, and $\omega^{\omega}$.[xl]

This means that transfinite cardinals can be conceived as any group of cardinal sets with the exception that they can only be expressed through infinitary arithmetic and are therefore impossible to fractionalize. The number ω doesn't necessarily have an inherent infinite meaning and can be conceived as a number set without meaning outside of infinitary arithmetic. The same goes for finite arithmetic where we said that even finite numbers don't have an inherent meaning outside of certain operations. Arithmetical sequences are formulas of the type n+1.

Therefore, the foundational work in forming transfinite numbers is carried by the operator and not the number itself, which can be conceived as a form of fictitious set that can't be fractionalized. If we use a fictitious number instead of an infinite number where only infinitary arithmetic applies, we'll have a similar result as with transfinite ordinals. Whether numbers are finite, or infinite is irrelevant here just as long as we don't apply fractionalization, division, and subtraction. Therefore, infinity plays a marginal role in a number theory where infinitary arithmetic comes first.

This confirms our argument about operationalized arithmetic's where operators condition the formation of numbers, and numbers have an arbitrary role. It's the same with transfinite numbers, which don't have numerical meaning but can be used in mathematics because numbers come second to the use of infinitary arithmetic.

Because set theory is the basis of ordinal numbers extended to infinity, it's commonly believed that set theory is part of mathematical logical theory, founded on grouping numbers, like any objects, into sets. Our claim here is that set theory is not formally conditioning number sequences of any order and it's based on the formation of numbers alone. All that infinite numbers represent is a hypothetical number theory of a fictious kind, where mathematical expressions are operationalized into sets, and founded on infinitary arithmetic. Here only addition, multiplication, and exponentiation can group the ω sets.

The main condition of forming the ω sets is based on calculating sets alone without calculating their content.

Russell explains the problem with transfinite numbers as follows:

> But when we come to consider infinite classes, we find that enumeration is not even theoretically possible for beings that only live for a finite time. We cannot enumerate all the natural numbers: they are 0, 1, 2, 3, *and so on*. At some point we must content ourselves with "*and so on*".[xli]

Mathematicians have found a way to formulate this 'so on' by beginning a new series of numbers that are transfinite ordinals. The solution that we'll propose here will be simple and we won't go through the entire theory, including the transfinite cardinals.

We can begin an infinite set by applying some of the formulas as earlier in defining the sequence of natural numbers. Let Russell's "*and so on*" mark the end of all finite numbers. Since there's no end we'll mark it with ω. Because numbers are ordinals, expressing a certain magnitude by the use of the operators, the next step is to begin a new order representing a different set of numbers, but ordered in the same way. Since it's a transfinite number, we express it as a perpetual serial growth of $f(a) = \omega(a+1)$, right up to the last representable point in the transfinite order of $\omega^{\omega}$.[xlii]

## Conclusion

Any order of numbers, both finite and transfinite, originates from sets. All finite numbers come in arbitrary sets of tens, hundreds, and thousands, etc. However, as we stated before, there are also dozenal sets, depending on the numerical agreement in use, where percentages, fractions, and sequence formulas are greater than in the decimal sets. This holds even when expressed with the same symbols. Transfinite ordinals are then the same type of expressions based on the familiar numerical order but founded by different mathematical operators. Here instead of a decimal or a duodecimal set, which varies in order, we have sets that have an infinite order. The difficulty then in grasping transfinite ordinals arises from our definition of numbers as something with self-encapsulating meaning irrelevant to the arithmetical operations in use. Instead of conceiving a number as a count or as an object of a concept, we can simply state it to represent a symbolic function in an arithmetical operation of a certain type, expressing orders in sets. Then all numbers, finite and transfinite, are product symbols of certain sets, made by multiplication, division, exponentiation, etc.

The strictness of arithmetic's arises from the consistent application of formulas that combine symbols with finite or infinite meaning, expressed as definite quantities. All mathematical symbols are operationally definite.

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