A Theory of Necessities

Anonymized

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Abstract

We develop a theory of necessity operators within a version of higher-order logic that is neutral about how fine-grained reality is. The theory is axiomatized in terms of the primitive of *being a necessity*, and we show how the central notions in the philosophy of modality can be recovered from it. Various questions are formulated and settled within the framework, including questions about the ordering of necessities under strength, the existence of broadest necessities satisfying various logical conditions, and questions about their logical behaviour. We also wield the framework to probe the conditions under which completely reductive theories of necessities are possible.

Keywords: Broadest necessity; higher-order metaphysics; higher-order logic; modal metaphysics.

The philosophy of modality often finds itself preoccupied with the notion of metaphysical necessity. But there are many other necessities that are worthy of study. Some of these are restrictions of metaphysical necessities, such as physical necessity or various practical necessities concerning what we can do. However there are, arguably, other necessities which are not restrictions of metaphysical necessities. According to some philosophers, epistemic necessities, certain tense operators, determinacy operators, or counterfactual necessity are *not* restrictions of metaphysical necessity.[1] According to these views, the philosophy of modality is not simply the study of restrictions of metaphysical necessity. As such, many questions about the structure of necessities remain open:

> Is there a necessity which is a restriction of every necessity?
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> For any two necessities, is there a further necessity which they are both restrictions of? Or a necessity which is a restriction of both?
>
> Is there a *broadest* necessity: a necessity which every necessity is a restriction of?
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> If there is a broadest necessity, what is its logic?
>
> Can necessities be reductively defined in purely logical or in non-modal terms?

In this paper we will introduce a general framework for theorizing about necessities in higher-order logic. Within this system one can say what it means for one necessity to be broader than another, and prove that there are (possibly several) necessities that are as broad as any other necessity, and that these necessities obey the principles of S4.

[1]See, respectively: Chalmers [6], Fine [11] and Kaplan [19], Bacon [2], Nolan [24].

A similar project is undertaken in Bacon [1], which attempts to uphold a form of *modal logicism*: definitions of *necessity*, *broader than* and *the broadest necessity* are offered in purely logical terms. Roberts [29] shows, in this framework, how other necessities can be understood as restrictions of the broadest necessity. However, the adequacy of the definitions, and the results about the broadest necessity, depend essentially on the background theory of propositional granularity assumed in those papers: a system that included identities, like $A = (A \wedge B) \vee A$ and $\exists x(A \vee B) = (\exists x A \vee \exists x B)$, that correspond to provable biconditionals in the underlying logic. Such identities are contentious, and rejected by philosophers interested in more fine-grained pictures.[2]

In this paper we aim to provide a *grain-neutral* theory of necessities. But to remain grain-neutral, we have found it necessary to take at least one modal term as primitive. In this paper we proceed by taking the notion of *being a necessity operator* as primitive, and axiomatize it directly yielding a general theory of necessities. The theory does not imply any of the aforementioned propositional identities; indeed we will show that it is conservative over a minimal theory of higher-order logic that does not encode any particular vision of granularity.

The theory also brings to salience a distinction between two sorts of extensionalism that are often conflated. One is a theory of granularity, which we call Fregeanism, that maintains that propositions, properties and relations are individuated by their extensions. The other is a fundamentally modal principle we are calling Quineanism, which maintains that every necessity is a truth-functional operator, and which is completely neutral about how propositions, properties and relations are individuated.[3] While Fregeanism entails Quineanism as we will see, the converse is not true. Rather Fregeanism is the result of adding Intensionalism — the view that necessarily equivalent entities are the same — to Quineanism. Indeed, given any extension of our theory of necessities, there is an intensional view corresponding to the result of adding Intensionalism to that theory. As a limiting case, when you add Intensionalism to the theory of necessities itself you get a very natural theory of granularity, Classicism (appearing in, e.g. [2], [4]), which we believe deserves special attention.

In section 1 we outline the background framework of higher-order logic, and present a theory, H_0, that we believe is sufficiently grain-neutral. In section 2 we introduce our theory of necessities, and explain its axioms and their motivation. In section 3 we establish some facts about the ordering of necessities, including the fact that there is a minimal element — the *broadest necessity* — and we establish some facts about its logic. We also explore the notion of a relative necessity and prove the aforementioned conservativeness result. Section 4 explores some strengthenings of the theory including the forms of extensionalism mentioned above. In section 5 we compare our theory with that of Williamson [36], Roberts [30], and Dorr, Hawthorne and Yli-Vakkuri [9]. In section 6 we explore some connections between our grain-neutral theory and the aforementioned reductive one, and then outline some general conditions under which a reductive theory of necessities is possible.

[2] e.g. Dorr [8], Fine [12] Goodman [15], Soames [32], Zeng [37].

[3] Quineanism is consistent with many very fine-grained pictures of reality. But surprisingly, it is even consistent with a conception of propositions in which they are sets of possible worlds, even though such views are often assumed to admit lots of necessities that are not truth-functional.

1 Higher-order logic

In modal logic, a modality is typically regimented with a sentential operator expression representing an English phrase like *it is necessary that* or *it is possibly that*: an expression that can combine grammatically with a sentence to form another sentence. A language with particular sentential operator expressions may be sufficient for articulating the theory of a particular necessity, but in order to formulate a theory of necessities *in general* we will need to quantify into the position that operator expressions occupy and to employ expressions with more complicated types, such as expressions which combine with operator expressions to form sentences. We therefore believe that the appropriate framework for this investigation is higher-order logic. What follows is a brief introduction to higher-order logic.

In higher-order logic, expressions fall into different grammatical categories, called *types*. There are basic types e and t, corresponding to the category of names and sentences respectively. And whenever σ and τ are types, there is a functional type $(\sigma \to \tau)$ of expressions that combines with expressions of type σ to form an expression of type τ. In what follows we shall adopt the convention of omitting brackets from types that are associated to the right: i.e. $\sigma_1 \to \sigma_2 \to \cdots \to \sigma_n$ is short for $(\sigma_1 \to (\sigma_2 \to (\cdots \to \sigma_n \ldots)))$. Thus operator expressions have type $t \to t$, expressions that combine with operator expressions to form sentences — operator predicates — have type $(t \to t) \to t$, and so on. For each type σ, we have a set of specified *constants* $c_1, c_2, \ldots$, which may or may not be empty, and a set of infinitely many *variables* $x_1, x_2, \ldots$. *Terms* of a higher-order language will be built from those constants and variables recursively (we use $M, N, O, \ldots$ as meta-linguistic variables and '$M : \sigma$', for example, means M is a term of type σ):

- If M is a constant or a variable of type σ, then $M : \sigma$;
- If $M : \sigma \to \tau$ and $N : \sigma$, then $(MN) : \tau$;
- If $M : \tau$ and x is a variable of type σ, then $(\lambda x.M) : \sigma \to \tau$.

With terms we follow the convention of omitting brackets associated to the left, i.e. $M_1M_2 \ldots M_n$ is short for $((\ldots(M_1M_2)\ldots)M_n)$. And we often write $\lambda x_1x_2 \ldots x_n.M$ for $\lambda x_1.(\lambda x_2.(\ldots(\lambda x_n.M)\ldots))$. We will omit brackets as we see fit, provided no ambiguities arise.

Given a λ-term $\lambda x.N$, N is the *scope* of λx. An occurrence of a variable x in a term is *free* if it is not in the scope of λx. A variable x is said to be *free* in a term M if it has some free occurrences in M.[4] A term is *closed* if no variable is free in it and *open* otherwise. We use $M[N_1/x_1, \ldots, N_n/x_n]$ for the result of *substituting* $N_1, \ldots, N_n$ for each free occurrence of $x_1, \ldots, x_n$ in M simultaneously (note that N_i and x_i must belong to the same type).[5]

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[4]Let FV be the function mapping each term to the set of all variables free in it. Then we have: $\mathrm{FV}(c) = \varnothing$, $\mathrm{FV}(x) = \{x\}$, $\mathrm{FV}(MN) = \mathrm{FV}(M) \cup \mathrm{FV}(N)$, and $\mathrm{FV}(\lambda x.M) = \mathrm{FV}(M) \setminus \{x\}$.

[5]The notion of substitution can be defined as follows (let $\bar{N} = N_1, \ldots, N_n$ and $\bar{x} = x_1, \ldots, x_n$):

- $x_i[\bar{N}/\bar{x}] = N_i$;
- $M[\bar{N}/\bar{x}] = M$ when M is a c or a $y \notin \{x_1, \ldots, x_n\}$;
- $MN[\bar{N}/\bar{x}] = M[\bar{N}/\bar{x}]N[\bar{N}/\bar{x}]$;
- $(\lambda x_i.M)[\bar{N}/\bar{x}] = \lambda x_i.M[N_1/x_1, \ldots, N_{i-1}/x_{i-1}, N_{i+1}/x_{i+1}, \ldots, N_n/x_n]$;
- $(\lambda y.M)[\bar{N}/\bar{x}] = \lambda y.M[\bar{N}/\bar{x}]$ when $x_i \in \mathrm{FV}(M)$ and $y \in \mathrm{FV}(N_i)$ for no i;
- $(\lambda y.M)[\bar{N}/\bar{x}] = (\lambda z.M[z/y])[\bar{N}/\bar{x}]$ when $x_i \in \mathrm{FV}(M)$ and $y \in \mathrm{FV}(N_i)$ for some i, where $z \notin \mathrm{FV}(M) \cup \mathrm{FV}(N_1) \cup \cdots \cup \mathrm{FV}(N_n)$.

Note that this is not the typical way to define substitution. We do so just because we want to choose the system H_0 as our background theory. If we defined substitution in the usual way, we would need, for

Two terms are said to be *immediately* β*-equivalent* if one of them is $(\lambda x.M)N$ and the other is $M[N/x]$ for some M and N. Two terms are said to be *immediately* η*-equivalent* if one of them is $\lambda x.Mx$ and the orther is M for some M, where x is not free in M. Two terms are $\beta\eta$*-equivalent* if one can be gotten from the other by replacing immediately β or η-equivalent terms for n times ($n \geq 0$).[6] It is not hard to see that $\beta\eta$-equivalent terms share the same type.

From now on, let's focus on languages containing a logical constant $\forall_\sigma$ of type $(\sigma \to t) \to t$ for each σ and the logical constant $\to$ of type $t \to t \to t$. We use $A, B, C, \ldots$ in particular as meta-linguistic variables for terms of type t. Following the conventions, we write $A \to B$ for $\to AB$, write $A_1 \to A_2 \to \cdots \to A_n$ for $(A_1 \to (A_2 \to (\cdots \to A_n \ldots)))$, and $\forall_\sigma x A$ for $\forall_\sigma(\lambda x.A)$. Other logical terms can be defined accordingly:

$$\begin{array}{lll} \bot := \forall_{t\to t}\forall_t & \vee := \lambda pq.(\neg p \to q) & \exists_\sigma := \lambda X.\neg\forall_\sigma x \neg Xx \\ \top := \bot \to \bot & \wedge := \lambda pq.\neg(p \to \neg q) & =_\sigma := \lambda xy.\forall_{\sigma\to t}X(Xx \to Xy) \\ \neg := \lambda p.(p \to \bot) & \leftrightarrow := \lambda pq.(p \to q) \wedge (q \to p) & \end{array}$$

We shall drop the superscript from $\forall_\sigma$, $\exists_\sigma$ or $=_\sigma$ when it is clear from context; and we shall write, for example, $\forall x_1 \ldots x_n A$ for $\forall x_1 \ldots \forall x_n A$.

Sometimes we will provide English glosses on expressions in higher-order languages. For example, we may gloss $\forall X(WX \to Xp)$ as 'every operator X having the property W applies to the proposition p'. This talk should not be understood as providing any translation from a higher-order language to English; rather, it should only be understood as a way of indicating a particular sentence of higher-order logic.[7] Another thing we should clarify here is that in the interest of readability, we will not distinguish carefully between use and mention. For instance, when the context is clear enough, we may use X of type $t \to t$ for an operator expression which is a term but in other contexts we may use X for the corresponding operator which is a wordly matter.

Theories will be treated as sets of formulae — i.e. terms of type t. An *axiomatic system* of higher-order logic is a collection of axioms and rules, and it determines a theory as the smallest set containing those axioms and closed under those rules. Given, for example, a theory T, a (schematic) formula A and an inferential rule R, we'll use $T \oplus A \oplus R$ for the result of adding A to T and closing under R plus the original rule(s) of T.

The weakest axiomatic system of higher-order logic studied in this paper, $\mathsf{H_0}$, has the following axioms and rules:

PC All theorems of propositional calculus;

UI $\forall_\sigma F \to Fa$;

β_E $(\lambda x_1 \ldots x_n.A)N_1 \ldots N_n \leftrightarrow A[N_1/x_1, \ldots, N_n/x_n]$;

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example, an extension of $\mathsf{H_0}$ containing α, a principle about grain, which says that α-equivalence suffices for identity (see below).

[6] Two terms are *immediately* α*-equivalent* if one of them is $\lambda x.M$ and the orther is $\lambda y.M[y/x]$ for some M, where y is not free in M. Two terms are α*-equivalent* if one can be gotten from the other by replacing immediately α-equivalent terms for zero or more times. It can be proved that two terms are α-equivalent only if they are $\beta\eta$-equivalent. (Hint: Since it is required that y is not free in M, $\lambda x.M$ is immediately η-equivalent to $\lambda y.(\lambda x.M)y$ and $(\lambda x.M)y$ is immediately β-equivalent to $M[y/x]$.)

[7] The indication relation may not preserve meaning, or even truth: the sentence 'Alice possesses some property' indicates the sentence $\exists X.Xa$, but we understand the latter sentence in such a way that it would be true if there were no properties. For more discussions, see Prior [27] and Williamson [35], ch. 5.9.

mp If $\vdash A \to B$ and $\vdash A$, then $\vdash B$;

Gen If $\vdash A \to Fx$, then $\vdash A \to \forall_\sigma F$, provided x is not free in A.

Note that by our definition of $=_\sigma$, the reflexivity of identity and Leibniz's Law are theorems of $\mathsf{H_0}$:

Ref $M =_\sigma M$;

LL $M =_\sigma N \to A[M/x] \to A[N/x]$.

The system $\mathsf{H_0}$ can be given a sound and complete semantics using the model theory of Muskens [23]. $\mathsf{H_0}$ is equivalent to Muskens sequent calculus ITL, which has a sound and complete semantics, in the sense that one can derive the sequent $\Gamma \Rightarrow \Sigma$ in ITL iff one can derive a contradiction in $\mathsf{H_0}$ from $\Gamma, \neg\Sigma$, where $\neg\Sigma = \{\neg A \mid A \in \Sigma\}$.[8]

$\mathsf{H_0}$ is fairly neutral about how fine-grained reality is; for instance the only identities it implies are trivial self-identities.[9] It can be strengthened by adding axioms or rules reflecting certain assumptions of grain. Consider the following one:

$\beta\eta$ $A \leftrightarrow B$ whenever A and B are $\beta\eta$-equivalent.

Let H be the result of replacing β_{E} in $\mathsf{H_0}$ with $\beta\eta$. H is an extension of $\mathsf{H_0}$ because β_{E} can be derived from $\beta\eta$ in H.[10] Also note that within H, $\beta\eta$ is equivalent to such a seemingly stronger principle:[11]

$\beta\eta^*$ $M = N$ whenever M and N are $\beta\eta$-equivalent.

So the extended system H says something about grain: $\beta\eta$-equivalence implies identity. For instance, the proposition that *Mary loves Mary*, formalized Lmm, is therefore identical to $(\lambda x.Lxm)m$, $(\lambda x.Lmx)m$, $(\lambda x.Lxx)m$ and $(\lambda x.Lmm)m$. Someone who adopted a very fine-grained account of propositions might reject these identities on the grounds that they each ascribe different properties to Mary: *loving Mary*, *being loved by Mary*, *loving oneself* and *being such that Mary loves Mary* respectively.

Still, $\beta\eta$ is a relatively modest grain constraint. There are rules reflecting some more contentious ideas:

E If $\vdash A \leftrightarrow B$, then $\vdash A =_t B$;

ζ If $\vdash Mx =_\tau Nx$, then $\vdash M =_{\sigma\to\tau} N$.[12]

[8] Roughly each sequent rule in ITL, from $\Gamma \Rightarrow \Sigma$ to $\Gamma' \Rightarrow \Sigma'$, is admissible in the sense that if $\Gamma, \neg\Sigma$ is inconsistent in $\mathsf{H_0}$ then so is $\Gamma', \neg\Sigma'$. Conversely, for each axiom A of $\mathsf{H_0}$, the sequent $\Gamma \Rightarrow A$ is derivable in ITL, and the rules of mp and Gen correspond to admissible sequent inferences, e.g. if $\Gamma \Rightarrow A$ and $\Gamma \Rightarrow A \to B$ are derivable in ITL then so is $\Gamma \Rightarrow B$.

[9] But note that the notorious Russell-Myhill argument can be run within $\mathsf{H_0}$, which means that certain structural views about grain (for example, those asserting the claim $\forall XYxy(Xx = Yy \to X = Y \land x = y)$) are ruled out by $\mathsf{H_0}$. See e.g. Uzquiano [33], Dorr [8], §6 or Goodman [14]. But the Russell-Myhill argument can be run in many different logics provided certain plausible assumptions. So we tend to think that the structural views ruled out are themselves very unattractive.

[10] To give the derivation precisely requires one get into the fine mechanics of the definition of α-equivalence (see note 6); we omit the argument for brevity.

[11] By Leibniz's Law, $A = B$ only if $A \leftrightarrow B$. Conversely, when M and N are $\beta\eta$-equivalent, so are $M = M$ and $M = N$.

[12] The name comes from the ζ rule for the equational λ-calculus (see Hindley and Seldin [17]).

Let $\mathsf{HE} = \mathsf{H} \oplus \mathsf{E}$ and $\mathsf{HE}\zeta = \mathsf{HE} \oplus \zeta$. HE straightforwardly articulates the idea that logical equivalence suffices for identity between propositions, and $\mathsf{HE}\zeta$ does the same for arbitrary relations.[13] In Bacon and Dorr [4] it is shown that it can be equivalently axiomatized by a set of closed equations, comprised of some equations imitating the theory of Boolean algebras governing the truth functional connectives, and some equations capturing an adjunctive relation between the quantifiers and the k combinator $\lambda xy.x$. (If we are in a restricted setting where all non-basic types end in t, it can be even shown that closing $\mathsf{H_0}$ under E and ζ yields $\mathsf{HE}\zeta$ as well: the rough idea is that with E and ζ, β_{E} allows one to prove the identities that were previously only provable with $\beta\eta$; see Proposition 6.2.) We will henceforth also refer to $\mathsf{HE}\zeta$ as *Classicism* (following [3], [4]).

By the arguments in Bacon [1], we can see that in HE (and thus $\mathsf{HE}\zeta$) the operator $\Box_\top := \lambda p.(p = \top)$ has the behaviour of a broadest necessity satisfying a logic of at least $\mathsf{S4}$. But the systems HE and $\mathsf{HE}\zeta$ are not grain neutral: the rule E, for instance, ensures identities like $A \wedge B = B \wedge A$, $A = \neg\neg A$, $(A \wedge B) \vee A = A$ and so on. Moreover, these theories contain many intensionalist theses to the effect that propositions and properties are individuated by necessary equivalence:

Propositional Intensionalism $\Box_\top(A \leftrightarrow B) \rightarrow A = B$;

Property Intensionalism $\Box_\top \forall x(Fx \leftrightarrow Gx) \rightarrow F = G$.

For instance, since HE is closed under the rule E, we know it contains the identities (i) $((A \leftrightarrow B) \rightarrow A) = ((A \leftrightarrow B) \rightarrow B)$, (ii) $(\top \rightarrow A) = A$ and (iii) $(\top \rightarrow B) = B$ (since the corresponding biconditionals are tautologies, and thus belong to HE). If $(A \leftrightarrow B) = \top$ and given (i), we may use Leibniz's Law to infer that $(\top \rightarrow A) = (\top \rightarrow B)$ and thus that $A = B$ using (ii) and (iii), thus establishing Propositional Intensionalism. Property Intensionalism is established in a completely parallel fashion, using E and ζ to turn open propositional equivalences into property identities.[14]

2 Being a necessity

In this section we present, informally, some constraints for being a necessity operator, which will provide a basis for the formal axiomatization of our theory of necessities.

Our formal theory will be formulated in the language of higher-order logic with a further constant, Nec of type $(t \rightarrow t) \rightarrow t$, representing our primitive notion of *being a necessity operator*. In what follows we will refer to the language of pure higher-order logic by $\mathcal{L}$, and the augmented language with $\mathcal{L}^{\mathrm{Nec}}$.

[13] Here logical equivalence is taken to include not only all provable equivalences in the background theory H, but also logical equivalences one can derive using these two further rules. However, in Bacon and Dorr [4] it is shown that there isn't really any distance between these ideas: merely adding identities between things provably equivalent in H would yield the same theory as closing under our stronger rules.

[14] Using the rules E and ζ, we can show (i) $\lambda y.(\forall x(Fx \leftrightarrow Gx) \rightarrow Fy) = \lambda y.(\forall x(Fx \leftrightarrow Gx) \rightarrow Gy)$, where y is free in neither F nor G. This is because $(\lambda y.(\forall x(Fx \leftrightarrow Gx) \rightarrow Fy))y \leftrightarrow (\lambda y.(\forall x(Fx \leftrightarrow Gx) \rightarrow Gy))y$ is derivable in H, with the help of $\beta\eta$. Similarly, by using $\beta\eta$, E and ζ, we can get (ii) $\lambda y.(\top \rightarrow Fy) = \lambda y.Fy$ and (iii) $\lambda y.(\top \rightarrow Gy) = \lambda y.Gy$. So given the assuming that $\forall x(Fx \leftrightarrow Gx) = \top$ we can infer that $\lambda y.Fy = \lambda y.Gy$ from (i)-(iii), and thus that $F = G$ by $\beta\eta$. A more general version of Property Intensionalism, $\Box_\top \forall x(Rx_1 \ldots x_n \leftrightarrow Sx_1 \ldots x_n) \rightarrow R = S$, can be proved in a similar way.

2.1 Conditions for being a necessity

Let us begin with some necessary conditions for an operator to be a necessity. According to a widely accepted modal intuition, a necessity operator satisfies, at least, the normal modal logic K. Within a propositional modal language, this logic can be axiomatized by extending propositional calculus with one modal axiom plus one rule of proof:

K $\Box(A \to B) \to \Box A \to \Box B$;

N If $\vdash A$, then $\vdash \Box A$.

They suggest two plausible necessary conditions that an operator must satisfy if it is a necessity operator. We will, moreover, posit that together they are sufficient.

The K axiom suggests that we should demand that necessity operators are closed under modus ponens. This just means that if p and q are propositions, and X is a necessity operator that applies to $p \to q$ and p, then X must apply to q too. But this is not enough. An operator can be closed under modus ponens for all sorts of contingent reasons. For instance, the operator *Alice said that* might be closed under modus ponens because Alice has said nothing (so that what she has said is vacuously closed under modus ponens). We shouldn't count this operator as a necessity: even though it is in fact closed under modus ponens it is possible (physically possible, say) that Alice failed to say all the consequences of things she's said that can be inferred using modus ponens. More generally, if an operator possibly fails to be closed under modus ponens in any other sense of 'possibly', it will not count as a necessity either. Thus we require necessities to satisfy a more robust condition we will call being *Closed*, namely that the operator should be not only closed under modus ponens, but necessarily closed under modus ponens, for any candidate notion of 'necessity':

Closure Every necessity operator is Closed.

The principle plausibly is true for any of the candidate notions we mentioned in the introduction, and we assert that it is true more generally of all necessity operators.

Because higher-order logic affords us the ability to quantify into sentence position, we can formulate the property of being an operator X that is closed under modus ponens, or, in other words, being an operator obeying the modal axiom K, with a single universal generalization:

$$K := \lambda X.\forall pq(X(p \to q) \to Xp \to Xq).$$

And since we can also quantify into operator position, we spell out what it means for a proposition p to be necessary in every sense as $\forall X(\mathrm{Nec}\, X \to Xp)$. Indeed, this notion of *being necessary in all senses* is so important, we shall introduce a shorthand for it:

$$L := \lambda p.\forall X(\mathrm{Nec}\, X \to Xp).$$

Thus our definition of being Closed becomes:

$$\mathrm{Closed} := \lambda X.(KX \wedge LKX).$$

Closure can then be formalised by the principle $\mathrm{Nec}\, X \to \mathrm{Closed}\, X$, which ought to be a consequence of our theory of necessities. One might wonder why we appeal to both KX and LKX when formalising the condition Closed. Shouldn't being necessary in every sense imply being true? Yes, we believe so. But also note this means that at least some necessities are factive in the sense that whenever Xp it is the case that p. At the current stage, we haven't introduced enough information about necessities to guarantee this, so we will simply bake it into the definition for now. It will turn out in our complete theory that being necessary in every sense implies being true; so the putative difference between LA and $A \wedge LA$ disappears (see section 2.2).

The necessitation rule N ensures that $\Box A$ is a theorem of the logic K whenever A is a theorem of K. Those who accept the rule of necessitation often do so by way of a more general principle stating that whenever A is a logical truth, then so is $\Box A$ — the rule of necessitation then being justified by the fact that the axioms of K are logically true and other rules of inference preserve logical truth. The notion of logical truth is a feature of sentences not propositions, but we have a natural worldly analogue of logical truth, namely: *being necessary in every sense of necessity.*[15] So the worldly analogue of this principle about logical truth is that a necessity X must satisfy the principle that if p is necessary in every sense, then so is Xp. However, as with Closure, a necessity shouldn't contingently satisfy this principle, thus we say that an operator X is *Logical* just in case it is necessary in every sense that if p is necessary in every sense, so is Xp, and then endorse the requirement:

Logicality Every necessity operator is Logical.

We may similarly define what it is for an operator to be Logical in higher-order logic:

$$N := \lambda X.\forall p(Lp \to LXp);$$
$$\text{Logical} := \lambda X.(NX \wedge LNX).$$

We propose that these two conditions are in fact not only necessary conditions, but sufficient for being a necessity operator. In our previous notation, this can be formalised:

Necessity $\text{Nec}\, X \leftrightarrow \text{Logical}\, X \wedge \text{Closed}\, X$.

Indeed, this will be the central axiom of our theory of necessities.

At this juncture we must emphasize the difference between giving necessary and sufficient conditions for an operator to be a necessity, and giving a *definition* of what it is to be a necessity. Our principle Necessity does *not* provide us with a definition of Nec because it involves the term L and therefore the term Nec on the right-hand-side (contained in our definitions of Closed and Logical). If we could give a definition without invoking Nec on the right-hand-side, we would have succeeded in giving a definition of Nec[16]; a project we

[15]One may try to directly define an operator applying to all and only propositions expressed by some logical truth(s) without appealing to Nec. Under the assumption of Classicism (i.e. $\mathsf{HE}\zeta$), for example, all theorems of $\mathsf{HE}\zeta$ express the same proposition, $\top$, so $\Box_\top := \lambda p.(p = \top)$ is such an operator. But if propositions are structured this project will be harder. We can, for instance, characterize the operator *being expressed by some theorem of propositional calculus*, using a complex term of pure higher-order logic:

$$\begin{aligned}\text{PC} := \lambda p.\forall X((&\forall pq X(p \to q \to p) \wedge \\ &\forall pqr X((p \to q \to r) \to (p \to q) \to p \to r) \wedge \\ &\forall pq X((\neg q \to \neg p) \to p \to q) \wedge \\ &\forall pq (X(p \to q) \to Xp \to Xq)) \to Xp).\end{aligned}$$

However, a finite definition of the theorems of higher-order logic is not possible because there are infinitely many logical constants — $\forall_\sigma$ for each σ. This same limitation applies to wider conceptions of logical truth that extend the theorems of higher-order logic (such as the theorems of our theory of necessities).

[16]Or at least, a definition of a predicate whose extension is just the necessity operators, which is good enough for most purposes.

suspect is impossible in a completely grain-neutral setting.[17]

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Before moving on, let us make a few brief methodological remarks. In presenting this theory, we do not conceive of ourselves as doing conceptual analysis on the word 'necessity' as it is used in philosophy. For one thing, it is a technical term, and has slightly different uses in different parts of philosophy. For instance, in metaphysics 'necessity' seems to be reserved for operators that are at least factive, i.e. obey the T axiom ($\Box p \to p$) of modal logic, whereas in linguistics and philosophy of language the word 'necessity' is used more liberally to include non-factive deontic modalities, such as those expressed by 'ought' in some contexts. Our view is that this is an entirely terminological issue: we just see our target to be the notion of a normal operator — the worldly analogue of an operator expression governed by the modal logic K. Other starting points would be equally acceptable to us. For instance, Bacon [1] works with an even weaker notion that builds in Logicality, but does not require Closure.

Similarly, one might take as a starting point a stronger notion. For example, if one wants to work with the notion of *objective necessities* studied by Williamson [36] and Roberts [30], one might wish to add the requirement that every necessity is factive in every sense of necessity. (But someone may disagree and argue that there is an operator of *having an objective chance of 1*, which is an objective but non-factive necessity.) Another particularly salient option in this direction is to strengthen the Closure condition. This condition ensures that given finitely many propositions, if each of them is X-necessary, so are their logical consequences. It's worth noting that we do not impose the stronger condition that necessities are closed under infinitary consequence since no analogous principle follows from our two principles of the modal logic K. (And as with the case of factivity, one might wish to include operators like *having an objective chance of 1* among the necessities, which are not closed under infinitary consequence.[18]) If we wished instead to characterize the worldly analogue of the stronger notion of an infinitely closed normal operator, we could similarly add a stronger condition $\text{Closed}^{\infty} X$, capturing a stronger form of closure.[19] However, we see little reason to take those stronger notions of being a necessity as primitive, as we can simply define them in the present theory.

2.2 Necessitation

Let us explore some further elements that we think should be part of our theory of necessities. Like the rule N for the logic K, we might demand that anything derivable in the theory of

[17] Again, if we assume Classicism, the operator $\Box_\top$ would suffice to serve all functions of L. This is basically because all theorems of higher-order logic express the same proposition according to this theory. So replacing all occurrances of L in $\text{Logical}\, X \wedge \text{Closed}\, X$ with $\Box_\top$ would give us a definition of $\text{Nec}\, X$ (see more discussion in section 6.1). The same strategy doesn't work in a grain-neutral setting. As we have explained in note 15, since we can't define, in the pure language $\mathcal{L}$, an operator applying to all theorems of higher-order logic, we take L-truth as an analogue of logical truth. Consequently, L should apply to all theorems of higher-order logic, and we do guarantee this by the rule Necessitation introduced in the next section. But if we could characterize L in $\mathcal{L}$, we would de facto define an operator applying to all theorems of higher-order logic in $\mathcal{L}$.

[18] This operator is not closed under infinite conjunction introduction: the chance of a point-sized dart missing a given point on a unit disc is 1, but the chance of it missing every point (the conjunction of these propositions) is 0.

[19] The rough idea can be understood as follows. Say that a collection of propositions represented by a propositional property X of type $t \to t$ entails p if every proposition entailing every member of X entails p, and say that X is Closed^{∞} if X applies to any proposition entailed by X relative to every sense of necessity. We will have more discussion on infinite closure in sections 3.1 and 3.4.

necessities should itself be necessary in any given sense of necessity. We can ensure this by demanding that our theory of necessities be closed under a rule of necessitation:

Necessitation If $\vdash A$, then $\vdash \mathrm{Nec}\, X \to XA$.

As with the rule N, this rule may be given a similar justification. Given the rule Gen, and the axiom UI, $\vdash \mathrm{Nec}\, X \to XA$ is equivalent to $\vdash \forall X(\mathrm{Nec}\, X \to XA)$, or given our notational conventions, just $\vdash LA$. Restated this way, the rule takes on a more familiar form of necessitation.

The combination of Necessity and Necessitation is a very powerful theory, which has many substantial consequences about necessities. Let $\mathsf{TN_0}$ be the theory $\mathsf{H_0} \oplus$ Necessity $\oplus$ Necessitation. One theorem of $\mathsf{TN_0}$ is that the operator L is Closed.

Proposition 2.1. $\vdash_{\mathsf{TN_0}} \mathrm{Closed}\, L.$

Proof. By Necessity, we know that $\mathrm{Nec}\, X \to \forall pq(X(p \to q) \to Xp \to Xq)$ for each X. It is not hard to see this implies $\forall X(\mathrm{Nec}\, X \to X(p \to q)) \to \forall X(\mathrm{Nec}\, X \to Xp) \to \forall X(\mathrm{Nec}\, X \to Xq)$, which amounts to $L(p \to q) \to Lp \to Lq$, for all p and q. Once we get KL, Necessitation will then give us LKL. □

Another important theorem of $\mathsf{TN_0}$ is the principle below, which states that the operator *it is true that* is a necessity (we adopt the convention of writing I for the identity combinator $\lambda p.p$):

Identity $\mathrm{Nec}\, I$.

In $\mathsf{H_0}$, every A is provably equivalent to IA. I is therefore a trivial operator. However, although I is intuitively a necessity, this requires some justification:

Proposition 2.2. $\vdash_{\mathsf{TN_0}}$ *Identity.*

Proof. By applying Necessitation to the $\mathsf{H_0}$ theorem $p \to Ip$ we have $L(p \to Ip)$. We just showed that L is closed under modus ponens, thus we can get $\forall p(Lp \to LIp)$. Also note that I is closed under modus ponens. So by Necessitation again, we have $\mathrm{Logical}\, I$ and $\mathrm{Closed}\, I$. Thus, according to Necessity, I is a necessity. □

Recall that when we formalise, for example, the idea that one operator is Closed, we appeal to both LKX and KX. Seemingly this is redundant because it is tempting to think that L is factive. But we point out for L to be factive, there must be some factive necessities. We've seen above that Identity provides us the existence of a factive necessity and therefore the factiveness of L. So now we can derive the following principle in $\mathsf{TN_0}$ as well:[20]

Necessity′ $\mathrm{Nec}\, X \leftrightarrow LNX \wedge LKX$.

[20] One tricky thing is that if we replace Necessity with Necessity′ as an axiom, we cannot directly get Identity. However, in a great many contexts Identity turns out to be derivable even if we have only Necessity′. For instance, assume the principle $\beta\eta$ of section 1 is accepted. Then note that Lp and LIp are $\beta\eta$-equivalent. So $\beta\eta$ will give us $Lp \to LIp$ and therefore LNI. Even if you're the sort of person who rejects $\beta\eta$ because you believe propositions are structured somehow, we think you should accept the principle that necessarily p if and only if necessarily it is true that p: $\mathrm{Nec}\, X \to (Xp \leftrightarrow XIp)$. This also suffices to prove Identity: Because Lp implies $\mathrm{Nec}\, X \to Xp$, this principle helps us to get $\mathrm{Nec}\, X \to XIp$, from which LIp follows. So we can have LNI.

Let's see one more theorem of $\mathsf{TN_0}$, which will be cited later. It says that if X and Y are necessities, then their composition $\lambda p.XYp$ is also a necessity.

$$\circ := \lambda XY\lambda p.XYp.$$

Proposition 2.3. $\vdash_{\mathsf{TN_0}} \mathrm{Nec}\, X \to \mathrm{Nec}\, Y \to \mathrm{Nec}(X \circ Y)$.

Proof. From NX, we have $LYp \to LXYp$. By Necessitation and the closure of L, we have $LYp \to L(X \circ Y)p$. So given NY, we have $\forall p(Lp \to L(X \circ Y)p)$ and hence $N(X \circ Y)$. If we necessitate this reasoning and distribute L, we can get $LNX \to LNY \to LN(X \circ Y)$. Moreover, observe that the conjunction of KX and XKY implies $K(X \circ Y)$. So by Necessitation and the closure of L, we have $LKX \to LXKY \to LK(X \circ Y)$. Next, note that from NX, we have $LKY \to LXKY$. So we can get $NX \wedge LKX \to LKY \to LK(X \circ Y)$. Therefore, Necessity lets us conclude that if both X and Y are necessities, so is $X \circ Y$. $\square$

Finally, note that $\mathsf{TN_0}$ allows us to talk about possibilities. We may define a term Pos of type $(t \to t) \to t$, which means *being a possibility operator*, as follows:

$$\mathrm{Pos} := \lambda X.\exists Y(\mathrm{Nec}\, Y \wedge L\forall p(Y\neg p \leftrightarrow \neg Xp)).$$

This definition guarantees that the dual operator of a necessity (possibility) must be a possibility (necessity).[21] Whenever X is a necessity, we may use $X^{\diamond}$ for the possibility $\lambda p.\neg X\neg p$.

Although the theory $\mathsf{TN_0}$ is strong enough, it is *not* our final theory. One more axiom is needed. We motivate it in the following section.

2.3 Mix-and-Match

Our final axiom imposes a closure condition on necessities. As emphasized in the introduction, we are attempting to capture a very liberal conception of necessity in which any operator with the right sort of formal behaviours counts as a necessity. Thus, for instance, if X and Y are necessities, then the operator *being X-necessary if snow is white and Y necessary if snow isn't white* is also a necessity. In fact, this result is already a consequence of the theory $\mathsf{TN_0}$ introduced above.[22]

The final axiom generalizes this idea: whenever W is necessarily a property of necessities, the operator of *possessing all the W-necessities*, $\lambda p.\forall X(WX \to Xp)$, is a necessity too. Adopting the notation

$$L_W := \lambda p\forall X(WX \to Xp),$$

our principle may be formalised as follows:

Mix-and-Match $L\forall X(WX \to \mathrm{Nec}\, X) \to \mathrm{Nec}\, L_W$.

[21]If X is a necessity, then it directly follows from the definition that its dual $\lambda p.\neg X\neg p$ is a possibility. If X is a possibility, observe that by our definition, $\lambda p.\neg X\neg p$ is L-necessarily coextensive with some necessity Y. It is easy to check that by Necessity′, when two operators are necessarily coextensive in every sense, one is a necessity only if the other is also a necessity.

[22]We prove that the operator $O = \lambda p.((q \to Xp) \wedge (\neg q \to Yp))$ is a necessity whenever both X and Y are necessities: Given the tautology $Xp \to q \to Xp$, by Necessitation, we have $LXp \to L(q \to Xp)$ and therefore $(Lp \to LXp) \to Lp \to L(q \to Xp)$. The same reasoning applies to $Ip \to \neg q \to Ip$ and we can therefore get $(Lp \to LYp) \to Lp \to L(\neg q \to Yp)$. So we have $(Lp \to LXp) \wedge (Lp \to LYp) \to Lp \to L(q \to Xp) \wedge L(\neg q \to Yp)$. Observe that $L(q \to Xp) \wedge L(\neg q \to Yp)$ implies LOp. Then by Necessitation again, $\mathrm{Logical}\, X \wedge \mathrm{Logical}\, Y$ implies $\mathrm{Logical}\, O$. A similar strategy can be employed to show that X and Y are Closed only if O is Closed.

Although the principles preceding Mix-and-Match encode a liberal conception of necessity, Mix-and-Match does not follow from them. The reason is that, for all we have said so far, it is possible, in some sense of 'possible', for there to exist *new* necessities — necessities which do not actually exist, as well as their dual possibilities. And moreover, it might be possible for things to be possible in these new senses of possibility that are not in fact possible for any actually existing kind of possibility. Now if W were a property of necessities which possibly contains new kinds of necessity like this, then there would be things that are possible according to some W-possibility but not possible according to any actually existing notion of possibility. Roughly, Mix-and-Match ensures that if something is possible according to a merely possible sort of possibility, it is in fact possible in some sense.

We can leverage these observations to find other assumptions from which Mix-and-Match can be derived. A strong assumption like this is the assumption that there simply cannot be any new necessities. We may formulate this principle in terms of the Barcan formula restricted to necessities:

$\mathbf{BF}_{\text{Nec}}$ $\forall X(\text{Nec}\, X \to LA) \to L\forall X(\text{Nec}\, X \to A)$.

The informal reason that this principle entails Mix-and-Match should be clear from the above.[23]

However, we think this is an overly restrictive assumption: if there could have been 'alien' fundamental properties, there could be new laws and nomic necessities corresponding them (see our discussion in section 5). An alternative and less contentious route to Mix-and-Match is simply the idea articulated above — that if something is possible according to some merely possible notion of possibility it is possible according to some actual possibility. Reformulating this in its contrapositive form allows us to state this principle with our preferred primitive, Nec:

$$\forall Z(\text{Nec}\, Z \to Zp) \to \forall X(\text{Nec}\, X \to X\forall Y(\text{Nec}\, Y \to Yp)).$$

This of course just has the form of the $\mathsf{4}$ axiom ($\Box p \to \Box\Box p$) for L. Given the assumption that this principle is itself necessary in every sense of necessity — $L\forall p(Lp \to LLp)$ — we may prove Mix-and-Match. Suppose that W is necessarily a property of necessities. We must show that L_W is also a necessity. L_W is easily seen to be Closed, since W necessarily consists only of Closed operators. It is also Logical because: by the $\mathsf{4}$ axiom for L, if p is necessary in every sense, then it is necessary, in every sense, that p possesses all necessities. This means it is necessary, in every sense, that p possesses every W-operator, since, necessarily, W-operators are necessities.[24]

The connection to the $\mathsf{4}$ axiom does bring to salience a competing picture — suggested in Fritz [13], Clarke-Doane [7], Roberts [28] — in which the space of possibilities are indefinitely extensible in something analogous to the way that the set-theoretic hierarchy is sometimes alleged to be. Roberts [30], for instance, formulates of the idea as follows, where $X \leq Y$ stands for Roberts notion of a necessity X being as broad as Y (we introduce the notion in the present framework in section 3):

[23]A formal deduction from the L-necessitated version of BF_{Nec} to Mix-and-Match can be run in TN_0: Suppose Lp holds. By Necessity, it implies $\forall X(\text{Nec}\, X \to LXp)$. Then BF_{Nec} lets us derive $L\forall X(\text{Nec}\, X \to Xp)$, which amounts to LLp given the closure of L. So by Necessitation $L\text{BF}_{\text{Nec}}$ implies $L\forall p(Lp \to LLp)$. See note 24 for the proof that the latter implies Mix-and-Match.

[24]Here's the formal argument in TN_0: From $\forall X(WX \to \text{Nec}\, X)$, we have $Lp \to L_W p$. Therefore by Necessitation and the $\mathsf{4}$ axiom for L, $L\forall X(WX \to \text{Nec}\, X)$ implies $L\forall p(LLp \to LL_W p)$, which then implies $L\forall p(Lp \to LL_W p)$ given the necessitated version of the $\mathsf{4}$ axiom, and this amounts to Logical L_W.

Extensibility $L\forall X(\mathrm{Nec}\, X \to \neg L\neg\exists Y(\mathrm{Nec}\, Y \wedge X \leq Y \wedge Y \not\leq X))$.

So understood, Extensibility says that it's necessary in every sense that for any necessity, it's possible in some sense that there is a strictly broader notion of necessity. In such a picture, the 4 axiom for L is not valid, because it can be possible that there's a new sort of possibility in which p is true without there being any actual sense of possibility in which p is true.

Extensibility is not merely the view that there could have been new sorts of necessity — a view we find eminently plausible. It is much more radical: it entails, for instance, that there could have been new necessities strictly broader than any actually existing necessities. But we feel there is a direct argument against such a view. For consider the operator of it being possible, in some sense of 'possible', that p. We contend that being possible in some sense of possibility is itself a kind of possibility. However Extensibility entails that it's possible, in some sense of possibility, that there is a notion of possibility strictly wider than it. That is to say, it's possible, in some sense, that there's a proposition p, and a notion of possibility, X, such that (i) it's X-possible that p, and (ii) it's not possible in any sense that p. But this is clearly incoherent.

This argument rested, of course, on the assumption that being possible in some sense of possibility is itself a sense of possibility, or equivalently to the assumption that L is a necessity. However, we think this principle is compelling in its own right. This brings us to a final principle from which Mix-and-Match could be derived, namely the assumption that being necessary in every sense of necessity is itself a form of necessity.

L-Necessity $\mathrm{Nec}\, L$.

Clearly, given Necessity, $\mathrm{Nec}\, L$ implies Logical L, which is just to say that $L\forall p(Lp \to LLp)$, so our argument for Mix-and-Match from the 4 axiom goes through. Indeed, this implication goes in the other direction, so that $\mathrm{Nec}\, L$ could be substituted for Mix-and-Match, as an alternative axiomatization of our theory.

Let us end with one final thought on the view that modal notions are indefinitely extensible. In our motivating discussion we often appealed to the idea that a genuinely Logical (or Closed) operator shouldn't contingently have the property $\lambda X.\forall p(Lp \to LXp)$, namely N, and we secured this by requiring that it be necessary for every *actual* necessity that the operator in question has N. We have seen that necessities are closed under composition (Proposition 2.3), so that this condition also ensures that if a proposition is necessary in every sense, then the result of prefixing any finite string of necessities to that proposition is also true. But if your view is that not only could there have been necessities that don't in fact exist, but there could have been necessities broader than any actual necessity, conditions stated in terms of being necessary for every actually existing necessity (or even every finite string of actually existing necessities) seems insufficiently strong. If X is a necessity, it shouldn't be possible, in some sense, that it is contingent in some sense that it applies only to truths. For X to be *truly* Logical, on this picture, it should be the case, speaking crudely, that for any string of necessities $Z_1, Z_2, Z_3, \ldots$ which may not all actually exist, but are such that Z_1 exists, Z_2 Z_1-possibly exists, Z_3 $(Z_1 \circ Z_2)$-possibly exists, etc, that the p be $(Z_1 \circ \cdots \circ Z_n)$-necessary. One way to capture this is to say that p is not only necessary in every sense, but necessary in every sense that it's necessary in every sense, necessary in every sense that it's necessary in every sense that it's necessary in every sense, and so on ad infinitum. We can encode this using Church's numerals: a Church numeral is an operation n of type $(t \to t) \to (t \to t)$ that takes an operator X as it's argument, and returns the operator that applies X to a proposition n times, $\lambda p.\underbrace{X \ldots X}_{n} p$

$$0 := \lambda X.X;$$
$$\text{suc} := \lambda n.\lambda X.\lambda p.(nX)Xp;$$
$$\text{ChurchNum} := \lambda n.\forall W(W0 \wedge \forall m(Wm \to W(\text{suc}\, m)) \to Wn).$$

So we think the view under consideration should not be giving the operator L the theoretical role we have been assigning it here, but instead the operator of having all finite iterations of L:

$$L^* := \lambda p.\forall n(\text{ChurchNum}\, n \to (nL)p).$$

Indeed, if you simply replace L with L^* in $\mathsf{TN_0}$, and make a modest modal assumption about the Church numerals — roughly that there couldn't have been any 'non-standard' Church numerals (i.e. Church numerals that don't in fact exist) — you can prove that L^* satisfies the $\mathsf{4}$ axiom. Since L^* is easily seen to be Closed in the modified sense, and the $\mathsf{4}$ axiom guarantees its Logicality, we can directly show that L^* is a necessity: so in this reinterpreted theory there is no need to make this extra assumption.[25] So we think the availability of the operator L^*, and the fact that it behaves like a genuine modality, provides us with a powerful argument against the modal indefinite extensibilist.

3 The theory of necessities

Putting this together we are now in a position to state our theory of necessities. As noted, we adopt the following definitions:

- $L_W := \lambda p.\forall X(WX \to Xp)$;
- $L := \lambda p.\forall X(\text{Nec}\, X \to Xp)$;
- $K := \lambda X.\forall pq(X(p \to q) \to Xp \to Xq)$;
- $N := \lambda X.\forall p(Lp \to LXp)$;
- $\text{Closed} := \lambda X.(KX \wedge LKX)$;
- $\text{Logical} := \lambda X.(NX \wedge LNX)$.

[25]The modest assumption about the Church numerals is simply this: the property of being a Church numeral is modally rigid, which we can spell out in terms of the Barcan formula and its converse for quantifiers restricted to the Church numerals:

Numerical Rigidity $\forall n(\text{ChurchNum}\, n \to L^*X) \leftrightarrow L^*\forall n(\text{ChurchNum}\, n \to X)$.

The reason this principle is necessary is slightly surprising. It is easy to prove, by induction on the Church numerals, that if something is a Church numeral it is L^* necessarily so, and so this property cannot shrink across modal space. However using the model theory in [1], we were able to find models in which the Church-numerals expand: in the actual world they consist of the standard Church numerals, but there are non-actual worlds in which you can iterate an operator a 'non-standard' number of times.

The reader may wonder why we did not take this route over the one we have presently taken. The reason is that, although we think the assumption of Numerical Rigidity is extremely plausible, it is a substantive metaphysical principle, and by assuming it we would no longer be able to prove all of our conservativity results. For instance, we wouldn't be able to show that our theory is interpretable in Classicism (since that theory also does not prove the rigidity of the Church numerals).

Let TN be $\mathsf{H_0} \oplus$ Necessity $\oplus$ Mix-and-Match $\oplus$ Necessitation:

Necessity $\text{Nec}\, X \leftrightarrow \text{Logical}\, X \wedge \text{Closed}\, X$;

Mix-and-Match $L\forall X(WX \rightarrow \text{Nec}\, X) \rightarrow \text{Nec}\, L_W$;

Necessitation If $\vdash A$ then $\vdash LA$.

Before we start to explore our theory TN, let's define a useful notion. Say that a proposition p *entails* a proposition q if the former necessarily implies the latter relative to all senses of necessity, i.e. $L(p \rightarrow q)$. This notion of *entailment* can be naturally generalized so that it can apply to any item of a type that ends in t:

$$\leq_\sigma := \lambda XY.L\forall x_1 \ldots x_n(Xx_1 \ldots x_n \rightarrow Yx_1 \ldots x_n),$$

where $\sigma = \sigma_1 \rightarrow \cdots \rightarrow \sigma_n \rightarrow t$ and $x_1, \ldots, x_n$ are of types $\sigma_1, \ldots, \sigma_n$ respectively.

3.1 Basic results

We will begin by proving some basic results involving the notion Nec, which we introduced informally as *being a necessity*.

Firstly, observe that by definition:

$$L = \lambda p.\forall X(\text{Nec}\, X \rightarrow Xp) = L_{\text{Nec}}.$$

Since $L\forall X(\text{Nec}\, X \rightarrow \text{Nec}\, X)$ is trivial, Mix-and-Match allows us to conclude that L, the operator of *possessing all necessities*, is itself a necessity!

One may wonder what general principles L obeys. Here, we show that the modal logic governing L is at least as strong as $\mathsf{S4}$. In section 6.1, it will be shown that no non-theorem of $\mathsf{S4}$ can be derived in the modal fragment of TN (although it is consistent with TN that the theorems of stronger modal logics are in fact true). The rule of necessitation is provided by the rule Necessitation. Given our axiom Necessity, it is an immediate consequence of L's being a necessity that it obeys the modal axioms K and $\mathsf{4}$. The fact that L obeys T is just an immediate corollary of I's being a necessity, which has already been shown in section 2.2.

An extremely significant consequence of our theory TN is that L is not only a necessity, but the *broadest necessity*. One necessity can be broader than another. For instance, philosophers typically judge metaphysical necessity to be broader than physical necessity, and this in turn to be broader than various kinds of practical necessities. But what does it mean, in general, for one necessity operator to be broader than another? Let's turn to the notion of *being as broad as*, since the notion of *being broader than* can be easily understood in terms of it: X is broader than Y if X is as broad as Y but not vice versa.

Certainly if necessity X is as broad as necessity Y, then a proposition is X-necessary only if it is also Y-necessary. However, this relation between necessities could obtain just by coincidence. If X were genuinely broader than Y, it wouldn't be contingent that every X-necessary proposition is a Y-necessary proposition: the inclusion should be necessary.[26]

[26]Consider the operator $O := \lambda p.((A \rightarrow \Box_{meta}p) \wedge (\neg A \rightarrow Ip))$, where $\Box_{meta}$ is metaphysical necessity and A is the proposition that Biden is the President of the U.S. It is a necessity (since we have shown in section 2.3 that $\lambda p.((q \rightarrow Xp) \wedge (\neg q \rightarrow Yp))$ is a necessity whenever X and Y are necessities). Moreover, in the actual world, every proposition has O is metaphysically necessary. But O might, in many possible circumstances, apply to some propositions which are not metaphysically necessary (in those circumstances). We are reluctant to think O is as broad as $\Box_{meta}$.

So we say that X is *as broad as* Y if in every sense of necessity, a proposition is X-necessary only if it is Y-necessary: $L\forall p(Xp \to Yp)$; in other words, X entails Y: $X \leq_{t\to t} Y$.[27] A broadest necessity is a necessity that is necessarily as broad as all necessities in every sense of necessity:

$$\text{BroadestNec} := \lambda Z.(\text{Nec}\, Z \wedge L\forall X(\text{Nec}\, X \to Z \leq X)).$$

Theorem 3.1. $\vdash_{\mathsf{TN}}$ BroadestNec L.

Proof. We know L is a necessity. Then observe that $\forall p(Lp \to LXp) \to \forall p(Lp \to Xp)$ is a theorem of TN because L is factive. So by Necessitation and the closure of L, $L\forall p(Lp \to LXp) \to L \leq X$. Given Necessity, we have $\text{Nec}\, X \to L \leq X$, and by using Necessitation again, we have $L\forall X(\text{Nec}\, X \to L \leq X)$. □

It's worth noting that there might be many equally broadest necessities: they might contain different constituents, for instance. However it strikes us that there is something especially natural about the definition of L — namely that it is nearly built into the definition that it is as broad as any necessity — so that the title of 'the broadest necessity' seems particularly apt for this operator.

Let's continue to prove more results concerning L. Because L is closed under modus ponens, an implication is that L is closed under *finite* entailment. Given finitely many propositions $p_1, \ldots, p_n$, they jointly entail the proposition p just in case $p_1 \wedge \cdots \wedge p_n \leq_t p$. So if every p_i is L, we can get $L(p_1 \wedge \cdots \wedge p_n)$ and then derive Lp.[28] As we discussed in section 2.1 however, to deal with cases of *infinite* entailment, we need a more general characterization of entailment. Say a collection of propositions represented by a propositional property X entails p if every proposition entailing every member of X entails p, i.e. $\forall q(\forall r(Xr \to q \leq r) \to q \leq p)$.[29] Accordingly, there is a stronger notion of being closed: an operator X is closed in this sense just in case X necessarily applies to every proposition entailed by X in every sense of necessity:

$$\text{Closed}^{\infty} := \lambda X.L\forall p(\forall q(\forall r(Xr \to q \leq r) \to q \leq p) \to Xp).$$

Surprisingly, we can prove that L also satisfies Closed^{∞} (a fact that cannot be proven of an arbitrary necessity in TN alone).

Proposition 3.2. $\vdash_{\mathsf{TN}} \text{Closed}^{\infty} L$.

Proof. Suppose we have $\forall q(\forall r(Lr \to q \leq r) \to q \leq p)$. An instance of it just amounts to $\forall r(Lr \to L(\top \to r)) \to L(\top \to p)$. Since $r \to \top \to r$ is a tautology, by Necessitation and the closure of L, we have $\forall r(Lr \to L(\top \to r))$. Then we get $L(\top \to p)$. Lp will be derived from $L(\top \to p)$ plus $L\top$. The whole reasoning can be necessitated, which will give us $\text{Closed}^{\infty} L$. □

[27] Here we deviate slightly from Bacon [1], where the following definition of the as broad as relation is presented instead $\lambda XY.\forall Z(\text{Nec}\, Z \to \forall pZ(Xp \to Yp))$. They are equivalent, in that paper, given the Functionality principle (or the Barcan formula for L). But in the context of the weaker principle Modalized Functionality (discussed in the appendix of that paper), and in the context of this paper, they are not equivalent. Roughly, in these contexts there could have (in some sense of 'could have') been more propositions than there in fact are: our definition requires that according to every possibility, all existing X-propositions are Y, whereas the definition in [1] only requires the inclusion to hold for the actually existing propositions. But intuitively, an operator cannot be as broad as another if it's possible that a proposition falls under the first but not the second.

[28] Thus Necessity and Theorem 3.1 jointly imply that every necessity is closed under finite entailment.

[29] This definition performs well because it guarantees, by the transitivity of $\leq$, that p's being entailed by X is inconsistent with its entailing a proposition which is not entailed by X.

Another important property of L is that it satisfies the converse Barcan formula for each type σ:

CBF$_\sigma$ $L\forall_\sigma x\, A \to \forall_\sigma x\, LA$.

The type e instance of this principle is a well-known theorem of first-order modal logic. The derivation at other types is entirely parallel: since an instance of UI yields $\forall_\sigma x A \to A$, by Necessitation and the closure of L, $L\forall_\sigma x A \to LA$ and then by Gen, we have $L\forall_\sigma x A \to \forall_\sigma x LA$.[30]

The converse Barcan formula tells us that if something exists, it does so necessarily. This is one of the surprising consequences of combining quantificational logic with modal logic. It effectively boils down to the fact that we have chosen classical logic, rather than a free logic, as our basic quantificational theory. Some philosophers may wish to avoid this consequence by weakening the theory H_0 along the lines of a free logic, although we will not pursue that line of inquiry here.[31]

One particular consequence of the converse Barcan formula for type $(t \to t) \to t$ is that necessity operators necessarily exist. But you may wonder whether necessity operators are necessarily necessity operators, as the principle below states:

Persistence $\mathrm{Nec}\, X \to L\, \mathrm{Nec}\, X$.

The answer is "Yes".

Proposition 3.3. $\vdash_{\mathsf{TN}}$ *Persistence.*

Proof. We have Necessity$'$: $\mathrm{Nec} \leftrightarrow LNX \wedge LKX$. Given the closure of L, this amounts to $\mathrm{Nec} \leftrightarrow L(NX \wedge KX)$. Necessitating it and then distributing the L operator will give us $L\,\mathrm{Nec} \leftrightarrow LL(NX \wedge KX)$. By the $\mathsf{4}$ axiom for L, we also have $L(NX \wedge KX) \to LL(NX \wedge KX)$. □

As in the case of basic first-order modal logic, our theory does not prove the Barcan formula:

BF$_\sigma$ $\forall_\sigma x LA \to L\forall_\sigma x A$.

This means that, although once something exists it does so necessarily, new things can come into existence. Prior [25] noted that given the B axiom ($p \to \Box\neg\Box\neg p$) one can derive the Barcan formula from the converse Barcan formula. However the B axiom for L is not a theorem of TN either.[32] Another observation due to Prior is that the B axiom guarantees

[30] Indeed, this reasoning works for any necessity — one can show by analogous reasoning that $\mathrm{Nec}\, X \to X\forall_\sigma x\, A \to \forall_\sigma x\, XA$ is a theorem of TN.

[31] For more discussion of this in the context of first-order modal logic, see Linsky and Zalta [21], Williamson [34]. Bacon and Dorr [4] contains discussion of these issues in higher-order logic in the context of Classicism. There it is shown — given certain background assumptions, the most important of which is the assumption that *being true* entails *being entailed by a truth* — that even if the official quantifiers of the theory obey a free logic, one can still define 'unrestricted' quantifiers satisfying UI, and by extension the converse Barcan formula. So the necessity of existence is hard to avoid when one is explicitly talking about existence in the unrestricted sense.

[32] This can be established as follows. Theorem 6.1 provides us with a translation of $\mathcal{L}^{\mathrm{Nec}}$ to $\mathcal{L}$, that takes theorems of TN to theorems of Classicism, and that maps any modal principle involving L to something equivalent in Classicism to the corresponding modal principle involving $\Box_\top$. But by the model theoretic techniques in Bacon [1], any modal sentence that can be refuted in a Kripke frame can be refuted in a corresponding model of Classicism built over that frame. So the B axiom for $\Box_\top$ is not a theorem of Classicism, and thus not a theorem of TN.

the necessity of distinctness, but again, without it the necessity of distinctness is not a theorem.[33] So in our theory we cannot prove such a principle:

ND$_\sigma$ $x \neq_\sigma y \to L(x \neq_\sigma y)$.

We will consider strengthenings of the theory with principles such as the B axiom for L in section 4.

We may also derive forms of the converse Barcan formula for quantifiers restricted by certain properties, including both of the following:

CBF$_L$ $L\forall p(Lp \to A) \to \forall p(Lp \to LA)$;

CBF$_{\text{Nec}}$ $L\forall X(\text{Nec}\, X \to A) \to \forall X(\text{Nec}\, X \to LA)$.

Intuitively, CBF_L says that the extension of L cannot shrink and CBF_{Nec} says that the extension of Nec cannot shrink. They follow, given our previous observations, from the $\mathsf{4}$ axiom for L and the persistence of necessities.[34]

3.2 Necessities and modal logics

In this section we will introduce, for every finitely axiomatizable modal logic, a corresponding notion of necessity satisfying that logic. It will turn out that for some logics, but not all logics, there exists broadest necessities satisfying that logic. In particular, we will see that the operator of possessing all $\mathsf{S5}$-necessities is itself an $\mathsf{S5}$-necessity, and is thus a broadest such necessity among that class.

Let $\mathcal{L}^\Box$ be the higher-order language equipped with a necessity operator constant $\Box$ of type $t \to t$ and $\mathcal{L}^\Box_\text{P}$ the propositional modal fragment of $\mathcal{L}^\Box$[35]; so $\mathcal{L}^\Box_\text{P}$ amounts to a propositional modal language. For every $A \in \mathcal{L}^\Box_\text{P}$ where $p_1, \dots, p_n$ are all propositional variables that occur in it, let $A^\sharp$ be $L\forall p_1 \dots p_n A$. Given a normal modal logic $\mathsf{M} \subseteq \mathcal{L}^\Box_\text{P}$, an operator O is said to be an M-necessity if $A^\sharp[O/\Box]$ holds for all $A \in \mathsf{M}$, where $A^\sharp[O/\Box]$ is the result of substituting O for each occurrence of $\Box$ in $A^\sharp$.[36] This natural idea can be captured in our theory of necessities so long as the logic M is finitely axiomatizable. By a 'finitely axiomatizable' normal modal logic, we simply mean one that can be obtained by adding finitely many axioms, $A_1, \dots, A_n \in \mathcal{L}^\Box_\text{P}$, to K and closing under the rules of K. The property of *being an* M*-necessity*, M, can then be defined in this way:

$$\text{M} := \lambda X.(LNX \land LKX \land A_1^\sharp[X/\Box] \land \dots \land A_n^\sharp[X/\Box]).$$

For instance, the property of *being an* $\mathsf{S5}$*-necessity* is just $\lambda X.(LNX \land LKX \land \mathsf{5}^\sharp[X/\Box])$, where $\mathsf{5}^\sharp[X/\Box]$ is $L\forall p(\neg X\neg p \to X\neg X\neg p)$. The adequacy of our definition is secured by the following result, which says, roughly, that for any theorem of M, TN proves the corresponding theorem about any particular M-necessity.

Proposition 3.4. *Given a normal modal logic* $\mathsf{M} \subseteq \mathcal{L}^\Box_\text{P}$ *which is finitely axiomatizable, if* $\vdash_\mathsf{M} A$*, then* $\vdash_\mathsf{TN} \text{M}\, X \to A^\sharp[X/\Box]$.

[33] Prior's original observation in [25] is presented in the context of the system $\mathsf{S5}$. He later presents an argument, attributed to E. J. Lemmon that uses only the B axiom [26] p.146.

[34] In fact, we have $\vdash_{\mathsf{TN}_0} CBF_L \leftrightarrow \forall p(Lp \to LLp)$ and $\vdash_{\mathsf{TN}_0} CBF_{\text{Nec}} \leftrightarrow$ Persistence.

[35] More precisely, $\mathcal{L}^\Box_\text{P}$ may be defined as the smallest set containing $\bot$ ($:= \forall_{t\to t}\forall_t$), $\to$, $\Box$ plus infinitely many t-type variables $p, q, \dots$, and closed under the term-forming rule of application: if $M : \sigma \to \tau$ and $N : \sigma$, then $(MN) : \tau$

[36] The precise definition of this substitution is similar to the one in note 5.

Proof. By induction on the length of a derivation in M. In particular, when A is derived from some B through N, $A[X/\Box]^\sharp$ amounts to $L\forall p_1 \dots p_n XB[X/\Box]$. $\mathrm{M}\,X$ implies $LNX \wedge LKX$, so by Necessity′, it implies $\mathrm{Nec}\,X$. Then by Theorem 3.1, $\mathrm{M}\,X$ and $L\forall p_1 \dots p_n B[X/\Box]$ jointly imply $\forall p_1 \dots p_n XB[X/\Box]$. Given the $\mathsf{4}$ axiom for L, $\mathrm{M}\,X$ and $B^\sharp[X/\Box]$ imply $A^\sharp[X/\Box]$. So the induction hypothesis $\mathrm{M}\,X \to B^\sharp[X/\Box]$ will let us conclude that $\mathrm{M}\,X \to A^\sharp[X/\Box]$. □

So every necessity is a K-necessity. Consequently L_{K} and L are L-necessarily coextensive and L_{K} is itself a K-necessity.[37] In fact it can be shown that L_{M} is an M-necessity for any finitely axiomatizable M included in $\mathsf{S4}$:

Proposition 3.5. *Given a normal modal logic* $\mathsf{M} \subseteq \mathsf{S4}$ *which is finitely axiomatizable,* $\vdash_{\mathsf{TN}} \mathrm{M}\,L_{\mathrm{M}}$.

Proof. Note that L is an $\mathsf{S4}$-necessity and, since $\mathsf{M} \subseteq \mathsf{S4}$, an M-necessity. So L_{M} entails L by definition. Conversely, L_{M} is a necessity by Mix-and-Match, so L entails L_{M} too. Since L and L_{M} are necessarily coextensive in every sense of necessity and the former is an M-necessity, so is the latter. □

Could such a result hold for all finitely axiomatizable normal modal logics? The answer is negative. Consider for example $\mathsf{S4.2}$, axiomatized over $\mathsf{S4}$ by adding the G axiom ($\neg\Box\neg\Box p \to \Box\neg\Box\neg p$). There's no way to prove that $L_{\mathrm{S4.2}}$ satisfies G. Indeed, there are models in which there is no broadest $\mathsf{S4.2}$-necessity at all.[38]

The good news, however, is that this result indeed holds for both B and $\mathsf{S5}$. L_{B} and L_{S5} obey N and K because they are necessities; they obey T because I is an $\mathsf{S5}$-necessity. The big task to show that L_{B} obeys the B axiom and L_{S5} obeys the $\mathsf{5}$ axiom.

Proposition 3.6. $\vdash_{\mathsf{TN}} p \to L_{\mathrm{B}}\neg L_{\mathrm{B}}\neg p$.

Proof. Suppose p is true. For each B-necessity X, we can get $X\neg X\neg p$ from p. Due to the $\mathsf{4}$ axiom for L, X is necessarily a B-necessity in every sense of necessity. So we have $L\,\mathrm{B}\,X \wedge X\neg X\neg p$, from which, by Theorem 3.1, we can get $X\,\mathrm{B}\,X \wedge X\neg X\neg p$ and therefore $X\exists X(\mathrm{B}\,X \wedge \neg X\neg p)$. □

But to finish the whole proof for L_{S5}, we have to make a detour. Let's start with the following definition:

$$\mathrm{S5}^* := \lambda Z.\forall W(\forall X(\mathrm{S5}\,X \to WX) \wedge \forall YY'(WY \wedge WY' \to W(Y \circ Y')) \to WZ).$$

Intuitively $\mathrm{S5}^*$ mimics the smallest collection containing all $\mathsf{S5}$-necessities and closed under the composition of operators: so basically the finite strings of compositions of $\mathsf{S5}$-necessities.

Proposition 3.7. $\vdash_{\mathsf{TN}} \mathrm{S5}^*\,X \wedge \mathrm{S5}^*\,Y \to \mathrm{S5}^*(X \circ Y)$.

[37] Note that in a fine-grained setting L_{K} may not be identical to L because $\lambda X.(LNX \wedge LKX)$ may not be identical to Nec. But it's still easy to see that they are necessarily coextensive in every sense.

[38] To show this negative result, we may exploit reasoning about ordinary Kripke models as outlined in note 32. The rough idea is this: suppose that the structure of the broadest necessity, L, can be represented by a Kripke frame that consists of three worlds in a forking structure — $W = \{0, 1, 2\}$, $R = \{(0,0), (0,1), (0,2), (1,1), (2,2)\}$. G characterises convergent frames, and the only convergent sub-relations of R are the identity relation, $R \setminus \{(0,1)\}$ and $R \setminus \{(0,2)\}$. In this model, there are two maximally but incomparable $\mathsf{S4.2}$-necessities, given by $R \setminus \{(0,1)\}$ and $R \setminus \{(0,2)\}$, and so there is no broadest $\mathsf{S4.2}$-necessity.

Proof. Let's fix a W. Assume that $\forall X(\mathrm{S5}\,X \to WX)$ and $\forall YY'(WY \wedge WY' \to W(Y \circ Y'))$. Given that $\mathrm{S5}^*\,X$ and $\mathrm{S5}^*\,Y$, we can derive WX as well as WY. By appealing to $\forall YY'(WY \wedge WY' \to W(Y \circ Y'))$ again, we have $W(X \circ Y)$. □

The definition of $\mathrm{S5}^*$ allows us to prove things about $\mathrm{S5}^*$ by induction: for any W, if W applies to every S5-necessities, and is closed under composition (of things in $\mathrm{S5}^*$), we may conclude that $\mathrm{S5}^*\,X \to WX$ for all X.

Proposition 3.8. *(i)* $\vdash_{\mathsf{TN}} \mathrm{S5}^*\,X \to \mathrm{Nec}\,X$*; and (ii)* $\vdash_{\mathsf{TN}} \mathrm{S5}^*\,X \to L\,\mathrm{S5}^*\,X$.

Proof. (i) It is trivial that all S5-necessities are necessities. Proposition 2.3 tells us that Nec is closed under composition.

(ii) Recall that we have Necessitation and the closure of L. It is trivial that $\mathrm{S5}\,X \to \mathrm{S5}^*\,X$. So by the 4 axiom for L, $\mathrm{S5}\,X \to L\,\mathrm{S5}^*\,X$. Further, according to Proposition 3.7, the property of being L-necessarily an $\mathrm{S5}^*$ is closed under composition. □

Let's define the notion of *reversal* here. It will help us to present our core idea involved in the next proof. Fix a necessity X. A reversal of it is a necessity Y: it can trivialise X in the sense that the composition of X and $\lambda p.(\neg Y \neg p)$ amounts to the truth operator I; more intuitively, the reversal Y brings us back to the actual world from any accessible $X^\diamond$-possibility.[39] For example, the tense operators *it will always be the case that* and *it was always the case that* are reversals of each other.

$$\mathrm{Rev} := \lambda XY.\forall p(p \to X\neg Y \neg p).$$

Proposition 3.9. $\vdash_{\mathsf{TN}} \mathrm{S5}^*\,X \to \exists Y(\mathrm{S5}^*\,Y \wedge L\,\mathrm{Rev}\,XY)$.

Proof. If X is an S5-necessity, it obeys the B axiom, so it's a reversal of itself necessarily. Suppose X_1 and X_2, which belong to $\mathrm{S5}^*$, have reversals Y_1 and Y_2 necessarily, which belong to $\mathrm{S5}^*$ too. From $L\forall p(p \to X_2\neg Y_2 \neg p)$, we can get $L(\neg Y_1 \neg p \to X_2 \neg Y_2 \neg(\neg Y_1 \neg p))$. Since X_2 belongs to $\mathrm{S5}^*$, it is a necessity by Proposition 3.8-(i). So we have $L(\neg Y_1\neg p \to X_2\neg(Y_2 \circ Y_1)\neg p)$. Since X_1 is also a necessity, we then have $X_1\neg Y_1 \neg p \to (X_1 \circ X_2)\neg(Y_2 \circ Y_1)\neg p$. From $L\forall p(p \to X_1\neg Y_1\neg p)$, we have $p \to X_1 \neg Y_1 \neg p$. Therefore, $\mathrm{Rev}(X_1 \circ X_2)(Y_2 \circ Y_1)$. Then by Proposition 3.8-(ii) and the 4 axiom for L, we can conclude that $L\,\mathrm{Rev}(X_1 \circ X_2)(Y_2 \circ Y_1)$. Finally, note that according to Proposition 3.7, $Y_2 \circ Y_1$ belongs to $\mathrm{S5}^*$. □

Proposition 3.10. *(i)* $\vdash_{\mathsf{TN}} \mathrm{S5}\,X \to L_{\mathrm{S5}^*} \leq X$*; (ii)* $\vdash_{\mathsf{TN}} \neg L_{\mathrm{S5}^*}\neg p \to L_{\mathrm{S5}^*}\neg L_{\mathrm{S5}^*}\neg p$*; and (iii)* $\vdash_{\mathsf{TN}} \mathrm{S5}\,L_{\mathrm{S5}^*}$.

Proof. (i) Just recall that $\mathrm{S5}\,X$ implies $\mathrm{S5}^*\,X$.

(ii) Suppose we have $\neg X \neg p$ for some X belonging to $\mathrm{S5}^*$. For all Y in $\mathrm{S5}^*$, it is guaranteed by Proposition 3.9 that it has a reversal Y' in $\mathrm{S5}^*$. Hence, $\neg X\neg p$ implies $Y\neg Y' X \neg p$, which then implies $Y\neg(Y' \circ X)\neg p$. Note that $Y' \circ X$ belongs to $\mathrm{S5}^*$. So we can conclude that $\exists Z(\mathrm{S5}^*\,Z \wedge Y\neg Z \neg p)$. Given Proposition 3.8-(ii), it is not hard to derive such a converse Barcan formula restricted to necessities in $\mathrm{S5}^*$: $\mathrm{Nec}\,Y \to Y\forall Z(\mathrm{S5}^*\,Z \to A) \to \forall Z(\mathrm{S5}^*\,Z \to YA)$. Because Y is indeed a necessity, we have $Y\forall Z(\mathrm{S5}^*\,Z \to A) \to \forall Z(\mathrm{S5}^*\,Z \to YA)$. Replace A with $A \to \exists Z(\mathrm{S5}^*\,Z \wedge A)$. Note that we have $Y\forall Z(\mathrm{S5}^*\,Z \to A \to \exists Z(\mathrm{S5}^*\,Z \wedge A))$. Thus, we can get $\forall Z(\mathrm{S5}^*\,Z \wedge YA \to Y\exists Z(\mathrm{S5}^*\,Z \wedge A))$, which turns out to imply

[39] Recall that we've shown in section 2.2 that X is a necessity only if $X^\diamond$, namely $\lambda p.(\neg X \neg p)$, is the dual possibility.

$\exists Z(\mathrm{S5}^* Z \wedge YA) \to Y\exists Z(\mathrm{S5}^* Z \wedge A)$. Then from $\exists Z(\mathrm{S5}^* Z \wedge Y\neg Z\neg p)$, we are allowed to infer that $Y\exists Z(\mathrm{S5}^* Z \wedge \neg Z\neg p)$.

(iii) By Proposition 3.8-(i) and Mix-and-Match, L_{S5^*} is a K-necessity. It obeys T for I is an $\mathsf{S5}$-necessity. Since we have also proved that L_{S5^*} obeys $\mathsf{5}$, L_{S5^*} is itself an $\mathsf{S5}$-necessity. □

Proposition 3.11. $\vdash_{\mathsf{TN}} \neg L_{\mathrm{S5}}\neg p \to L_{\mathrm{S5}}\neg L_{\mathrm{S5}}\neg p$.

Proof. Suppose that $\neg X\neg p$ for some $\mathsf{S5}$-necessity X. Given an $\mathsf{S5}$-necessity Y, notice that by Proposition 3.10, $\neg X\neg p$ implies $\neg L_{\mathrm{S5}^*}\neg p$ and then implies $L_{\mathrm{S5}^*}\neg L_{\mathrm{S5}^*}\neg p$, which turns out to imply $Y\neg L_{\mathrm{S5}^*}\neg p$; moreover, it's the case that $\mathrm{S5}\, L_{\mathrm{S5}^*}$. So by the $\mathsf{4}$ axiom for L, we have $L\,\mathrm{S5}\, L_{\mathrm{S5}^*} \wedge Y\neg L_{\mathrm{S5}^*}\neg p$, and by Theorem 3.1, we have $Y\,\mathrm{S5}\, L_{\mathrm{S5}^*} \wedge Y\neg L_{\mathrm{S5}^*}\neg p$ and therefore $Y\exists Z(\mathrm{S5}\, Z \wedge Y\neg Z\neg p)$. □

In fact, it can be further shown that if $A \in \mathcal{L}_{\mathrm{P}}^{\Box}$ is not derivable in $\mathsf{S5}$, then $A[L_{\mathrm{S5}}/\Box]$ cannot be derived in TN either (see section 6.1). We believe this conclusion has some philosophical significance. Kripke famously introduced the notion of metaphysical necessity in [20]. There he introduced it as the necessity "in the highest degree". But Kripke, and early commentators, also said many specific things about it which have since become to be taken as constitutive of the notion, for instance facts about the necessity of origins, or that it is governed by a logic of $\mathsf{S5}$. The former idea of necessity in the highest degree can be straightforwardly captured using our notion of broad necessity, L.[40] However, we have taken seriously the idea that there might be notions of necessity broader than metaphysical necessity, and also the idea that the logic of L might not include the $\mathsf{5}$ axiom.[41] The existence of the broadest $\mathsf{S5}$-necessity L_{S5} provides us with a natural fall-back for playing the role of metaphysical necessity, as it appears in post-Kripkean modal metaphysics.

3.3 The pre-lattice of necessities

At the beginning of section 3, we defined the entailment relation $\leq$. In the current subsection, we investigate the logic of $\leq$ over the space of necessity operators. For instance, it is fairly easy to show that $\leq$ is a preorder over necessities: that is, it is a reflexive and transitive order.[42] Given our present assumptions, $\leq$ *cannot* be shown to be a partial order: that is to say, we do not have that if $X \leq Y$ and $Y \leq X$ then $X = Y$. The reason is that our theory is consistent with many very fine-grained conceptions of operators, in which two operators may be necessarily coextensive, in every sense of necessity, but still be distinct — perhaps because they are structured differently. (Later we will consider an axiom, Intensionalism, which forces $\leq$ to be a partial order.) When X and Y are just as broad as each other, we will write $X \sim Y$. $\sim$ is clearly an equivalence relation, given the reflexivity and transitivity of $\leq$. Indeed, modulo $\sim$, we talk as though $\leq$ is partial order, and freely employ lattice-theoretic notions, like meets and joins.

[40] Williamson [36] and Roberts [30] recently put forward an alternative interpretation: according to them, metaphysical necessity should be the broadest *objective* necessity but may not be a broadest necessity. See section 5 for more discussion.

[41] See Bacon [1], §5 for positive arguments according to which it is weaker than $\mathsf{S5}$. Although those arguments are originally run under the assumption of Classicism, they can be smoothly moved into our current grain-neutral setting without any loss of argumentative power, so we won't repeat them here.

[42] Since we have Persistence, Reflexivity can be established by necessitating the trivial truth $\mathrm{Nec}\, X \to \forall p(Xp \to Xp)$ and Transitivity can be established by necessitating the trivial truth $\mathrm{Nec}\, X \wedge \mathrm{Nec}\, Y \wedge \mathrm{Nec}\, Z \to \forall p(Xp \to Yp) \to \forall p(Yp \to Zp) \to \forall p(Xp \to Zp)$.

Given that $\leq$ satisfies the constraints of the familiar mathematical notion of being a preorder, one might wonder what other lattice-theoretic properties it has. For instance, does it have a top and a bottom element? We have already shown that there are necessities that are as broad as any necessity: L (and any other necessity exactly as broad as L). And it is easy to see that there are necessities that are no broader than any necessity: $\lambda p.\top$ (and any other necessity that is exactly as broad as $\lambda p.\top$). (In the case that $\leq$ is a partial order, L and $\lambda p.\top$ are the unique broadest and narrowest necessities respectively.) We might also ask whether the necessities have finite meets and joins under $\leq$, making it a *pre-lattice*. And if so, whether the resulting pre-lattice is distributive. We will answer the former question in the affirmative. The main theorem of this subsection is thus:

Theorem 3.12. *According to* TN*, necessities forms a bounded pre-lattice under* $\leq$*.*

In other words, all of the following principles can be derived within our theory of necessities:

Reflexivity $\mathrm{Nec}\, X \to X \leq X$;

Transitivity $\mathrm{Nec}\, X \wedge \mathrm{Nec}\, Y \wedge \mathrm{Nec}\, Z \to X \leq Y \to Y \leq Z \to X \leq Z$;

Minimum $\exists X(\mathrm{Nec}\, X \wedge \forall Y(\mathrm{Nec}\, Y \to X \leq Y))$;

Maximum $\exists X(\mathrm{Nec}\, X \wedge \forall Y(\mathrm{Nec}\, Y \to Y \leq X))$;

Meets $\mathrm{Nec}\, X \wedge \mathrm{Nec}\, Y \to \exists Z(\mathrm{Nec}\, Z \wedge Z \leq X \wedge Z \leq Y \wedge \forall Z'(\mathrm{Nec}\, Z' \wedge Z' \leq X \wedge Z' \leq Y \to Z' \leq Z))$;

Joins $\mathrm{Nec}\, X \wedge \mathrm{Nec}\, Y \to \exists Z(\mathrm{Nec}\, Z \wedge X \leq Z \wedge Y \leq Z \wedge \forall Z'(\mathrm{Nec}\, Z' \wedge X \leq Z' \wedge Y \leq Z' \to Z \leq Z'))$.

We have described how to get Reflexivity, Transitivity, Minimum and Maximum above. The existence of meets may be established by showing that if X and Y are necessities then the conjunctive operator $\lambda p.(Xp \wedge Yp)$ is a necessity and satisfies the conditions for being a meet:

$$\sqcap := \lambda XY\lambda p.(Xp \wedge Yp).$$

Proposition 3.13.

(i) $\vdash_{\mathsf{TN}} \mathrm{Nec}\, X \wedge \mathrm{Nec}\, Y \to \mathrm{Nec}(X \sqcap Y)$*;*

(ii) $\vdash_{\mathsf{TN}} \mathrm{Nec}\, X \wedge \mathrm{Nec}\, Y \to X \sqcap Y \leq X \wedge X \sqcap Y \leq Y$*;*

(iii) $\vdash_{\mathsf{TN}} \mathrm{Nec}\, X \wedge \mathrm{Nec}\, Y \to \forall Z(\mathrm{Nec}\, Z \wedge Z \leq X \wedge Z \leq Y \to Z \leq X \sqcap Y)$*.*

Proof. (i) Clearly, if X and Y are necessities, then we can get $\forall p(Lp \to LXp \wedge LYp)$, which amounts to $\forall p(Lp \to L(X \sqcap Y)p)$. So by the $\mathsf{4}$ axiom for L, we can infer $L\forall p(Lp \to L(X \sqcap Y)p)$, which amounts to $LN(X \sqcap Y)$, from $\mathrm{Nec}\, X$ and $\mathrm{Nec}\, Y$. For similar reasons, we can derive $LK(X \sqcap Y)$ as long as X and Y are necessities.

(ii) Just observe that $(X \sqcap Y)p \to Xp$ and $(X \sqcap Y)p \to Yp$ are theorems of $\mathsf{H_0}$.

(iii) Note that $(Zp \to Xp) \wedge (Zp \to Yp) \to Zp \to (X \sqcap Y)p$ is a theorem of $\mathsf{H_0}$. □

The meet of two necessities is the obvious generalization of the meet operation on propositions under the entailment ordering: conjunction. One might have naïvely thought that the join of two necessities would be defined similarly as their disjunction, i.e. $\lambda XY\lambda p.(Xp \vee Yp)$.

But this is not so. The disjunction of two necessities need not be closed under modus ponens: for instance p might be X-necessary but not Y-necessary, $p \to q$ might be Y-necessary but not X-necessary, allowing q to be neither X nor Y-necessary. But the join of X and Y will be given by the operator representing the smallest collection containing all X and Y-propositions and closed under modus ponens:

$$\sqcup := \lambda XY\lambda p.\forall Z(\forall q(Xq \vee Yq \to Zq) \wedge KZ \to Zp).$$

Proposition 3.14.

(i) $\vdash_{\mathsf{TN}} \mathrm{Nec}\,X \wedge \mathrm{Nec}\,Y \to \mathrm{Nec}(X \sqcup Y)$*;*

(ii) $\vdash_{\mathsf{TN}} \mathrm{Nec}\,X \wedge \mathrm{Nec}\,Y \to X \leq X \sqcup Y \wedge Y \leq X \sqcup Y$*;*

(iii) $\vdash_{\mathsf{TN}} \mathrm{Nec}\,X \wedge \mathrm{Nec}\,Y \to \forall Z(\mathrm{Nec}\,Z \wedge X \leq Z \wedge Y \leq Z \to X \sqcup Y \leq Z)$.

Proof. (i) Note that both Xp and Yp can imply $(X \sqcup Y)p$. So $L\forall p(Xp \vee Yp \to (X \sqcup Y)p)$ is derivable. If X and Y are necessities, then by the 4 axiom for L, we can derive $L\forall p(Lp \to L(X\sqcup Y)p)$, which amounts to $LN(X\sqcup Y)$, from the conjunction of $L\forall p(Xp\vee Yp \to (X\sqcup Y)p)$ and LNX/LNY. The case of $LK(X \sqcup Y)$ is obvious; we leave the proof as an exercise.

(ii) Just observe that $Xp \to (X \sqcup Y)p$ and $Yp \to (X \sqcup Y)p$ are theorems of H_0.

(iii) Note that $(Xp \to Zp) \wedge (Yp \to Zp) \to Xp \vee Yp \to Zp$ is a theorem of H_0 and Z's being a necessity implies its being closed under modus ponens. □

There is a question that we have not been able to settle: is this ordering distributive?[43]

Distributivity $\mathrm{Nec}\,X \wedge \mathrm{Nec}\,Y \wedge \mathrm{Nec}\,Z \to X \sqcap (Y \sqcup Z) \sim (X \sqcap Y) \sqcup (X \sqcap Z)$.

It is worth emphasizing that this principle is non-trivial even under the assumption of Classicism, even when the operators as a whole form a distributive lattice under $\leq$. The reason is that while the meet of two necessities in the lattice of all operators is the same as their meet in the lattice of necessities, the join of two necessities in the lattice of all operators, namely their disjunction, is in general distinct (indeed $\leq$-lower than) their join in the lattice of necessities.

We do not claim that the above is an exhaustive list of the distinctive features of the lattice of necessities, but feel it is enough to motivate this investigation. Let us end the section by posing a question of completeness. Could there be an equational theory in the operators of $\sqcap$ and $\sqcup$ which is *complete* for the lattice of necessities? More specifically, consider the algebraic language in variables, $\sqcap$ and $\sqcup$. An individual term s in the algebraic theory can be translated into an operator term of higher-order logic by mapping the individual variables $x_1, \ldots, x_n$ in s to operator variables, $X_1, \ldots, X_n$ and translating $\sqcap$ and $\sqcup$ into the expressions by the same name defined above. An equation $s = r$ may then be translated to a corresponding formula of the form $M \sim N$, which may then be prefixed by a string of restricted universal quantifiers, $\forall X_1 \ldots X_n(\mathrm{Nec}\,X_1 \wedge \cdots \wedge \mathrm{Nec}\,X_n \to \ldots)$ to obtain a closed sentence which we'll call $(s = r)^*$. Let the equational theory of necessities be the set of equations $s = r$ such that $(s = r)^*$ is a theorem of TN. Question: can the equational theory of necessities be axiomatized by a finite or recursive set of equations?

[43] In fact, if X, Y and Z are all necessities, it is not difficult to prove $(X \sqcap Y) \sqcup (X \sqcap Z) \leq X \sqcap (Y \sqcup Z)$: Suppose we have $(X \sqcap Y) \sqcup (X \sqcap Z)$. It suffices to show Xp and $\forall Z'(\forall q(Yq \vee Zq \to Z'q) \wedge \forall qr(Z'(q \to r) \to Z'q \to Z'r) \to Z'p)$. To show them, just notice these two theorems of H_0: $(Xp \wedge Yp) \vee (Xp \wedge Zp) \to Xp$ and $\forall q(Yq \vee Zq \to (Xq \wedge Yq) \vee (Xq \vee Zq))$. What we are not able to show is the other direction, and we suspect it doesn't generally hold.

3.4 Relative necessities

Sometimes one sort of necessity is a restriction of another. For instance, it is widely believed that physical necessity is a restriction of metaphysical necessity. By contrast, Kripke is sometimes read as having demonstrated that metaphysical necessity is neither a restriction of a priori truth nor, conversely, that a priori truth is a restriction of metaphysical necessity. A number of authors have tried to provide a general definition of what it means for one necessity to be a restriction of another. Suppose, for example, that a physical necessity is a proposition metaphysically entailed by laws of physics. Following this line of thought, Smiley [31] proposed that being a physical necessity could be analysed in terms of metaphysical necessity and a sentential constant P, denoting the conjunction of the physical laws in the actual world (we use $\Box_{meta}$ and $\Box_{phys}$ for metaphysical necessity and physical necessity respectively):

$$\Box_{phys} := \lambda p.\Box_{meta}(\mathrm{P} \to p).$$

However, Humberstone [18] raised a number of problems for this account.[44] For example, it is widely accepted that the logic of metaphysical necessity is not weaker than S4. But if so, it directly follows that $\Box_{phys}$ defined by Smiley also obeys the 4 axiom no matter what physical laws are.[45] Physical necessity may or may not obey the 4 axiom. Even though it obeys the 4 axiom, this is due to the nature of phyiscal laws, not its being a restriction of metaphysical necessity.[46]

Hale and Leech [16] rightly point out the problem is that Smiley's definition fails to track which propositions are the laws of physics at different worlds, and propose a definition in terms of a property of propositions, Law, which characterises the propositions that are laws of physics, and suggest that

$$\Box_{phys} := \lambda p.\exists q_1 \dots q_n(\mathrm{Law}\, q_1 \wedge \dots \wedge \mathrm{Law}\, q_n \wedge \Box_{meta}(q_1 \wedge \dots \wedge q_n \to p)).$$

But as Roberts [29] emphasizes, this account faces some different problems. A nearly uncontroversial idea in modal philosophy is that if necessity X is a restriction of necessity Y, then it should be (at least) Y-necessary that every Y-necessary proposition is also an X-necessary proposition. Hale and Leech's definition of relative necessity, however, is in conflict with this idea. Just imagine a metaphysical possibility according to which there are no physical laws. At this possibility, $\Box_{phys}$ applies to nothing but $\Box_{meta}$ still applies to something. Consequently, $\exists p(\Box_{meta} p \wedge \neg\Box_{phys} p)$ turns out to be metaphysically possible.

Roberts [29] then put forward a novel account which overcomes all of these problems. But he doesn't work in a grain-neutral picture — his assumption about grain implies Propositional Intensionalism we mentioned in section 1; and he works with a narrower conception of necessity according to which every necessity is closed under infinitary consequence, which goes beyond the minimal assumptions we are making here.[47]

In our theory TN, we can define a natural candidate of $\Box_{phys}$ to be the restriction of $\Box_{meta}$ by Law and prove that it is a necessity. More generally, suppose that we have an

[44] The problems are attributed to Kit Fine in that paper.

[45] Proof: Given the 4 axiom for $\Box_{meta}$, we have $\Box_{meta}(\mathrm{P} \to p) \to \Box_{meta}\Box_{meta}(\mathrm{P} \to p)$ for any p. Since $\Box_{meta}(\mathrm{P} \to p) \to \mathrm{P} \to \Box_{meta}(\mathrm{P} \to p)$ is a tautology, by the rule of necessitation and the K axiom for $\Box_{meta}$, we have $\Box_{meta}\Box_{meta}(\mathrm{P} \to p) \to \Box_{meta}(\mathrm{P} \to \Box_{meta}(\mathrm{P} \to p))$.

[46] In the present context, $\Box_{phys}$ defined by Smiley would not even obey T in every sense of necessity, if there are some possibility in which P is metaphysically necessarily false: for then $\Box_{meta}(\mathrm{P} \to p)$ would be vacuously true whatever p is.

[47] As we briefly discussed in the end of section 2, it is easy to capture such a narrower conception of necessity within our framework: just let $\mathrm{Closed}^{\infty} X$ be a necessary condition for $\mathrm{Nec}\, X$.

operator Y. Then given any necessity X, we may define a restricted necessity X^Y as follows:

$$X^Y := X \sqcup Y.$$

Proposition 3.15. $\vdash_{\mathsf{TN}} \mathrm{Nec}\, X \to \mathrm{Nec}\, X^Y$.

Proof. See the proof of Proposition 3.14-(i). Note that in that proof, we assume the two operators X and Y are both necessities. But the same conclusion can be achieved even if Y is not. □

Since we directly define X^Y as $X \sqcup Y$, one would expect the notion of a restriction of a necessity to be somehow related to the ordering of $\leq$. Here's a nice result:

Proposition 3.16. $\vdash_{\mathsf{TN}} \mathrm{Nec}\, Y \to (X \leq Y \leftrightarrow Y \sim X^Y)$.

Proof. Suppose $X \leq Y$. Note that $X^Y = X \sqcup Y$. We have proved $Y \leq X \sqcup Y$ (see Proposition 3.14-(ii)). Consider the converse direction: Since we have $X \leq Y$, we have $L\forall q(Xq \vee Yp \to Yp)$. Since Y is a necessity, we have LKY. Finally, suppose $Y \sim X \sqcup Y$. We have also proved $X \leq X \sqcup Y$ (see Proposition 3.14-(ii) again). □

As a corollary, all necessities are necessarily coextensive with some restriction of L because L is as broad as all necessities.

To see that our account does provide an appropriate characterization of a restriction of a necessity, we may turn back to the case of $\Box_{phys}$ and $\Box_{meta}$. Now, $\Box_{phys}$ is defined as $\Box_{meta}^{\mathrm{Law}}$. Our definition captures the idea that a physical necessity is a proposition metaphysically entailed by zero or more laws of physics. Suppose B is metaphysically entailed by the conjunction of laws $A_1, \ldots, A_n$. Since $A_1, \ldots, A_n$ are laws, we get $\Box_{phys}A_1 \wedge \cdots \wedge \Box_{phys}A_n$ by our definition of $\Box_{phys}$. According to Proposition 3.15, $\Box_{phys}$ is a necessity and hence closed under modus ponens. So we can derive $\Box_{phys}(A_1 \wedge \cdots \wedge A_n)$. Because we have assumed $\Box_{meta}(A_1 \wedge \cdots \wedge A_n \to B)$, by Proposition 3.16, $\Box_{phys}(A_1 \wedge \cdots \wedge A_n \to B)$ and thus $\Box_{phys}B$. Moreover, our account doesn't suffer from any problems mentioned before. What the logic of $\Box_{phys}$ is remains an open question. And our definition predicts that in a metaphysically possible world where there are no physical laws, $\Box_{phys}$ is just coextensive with $\Box_{meta}$; in general, if necessity Y is a restriction of necessity X, it follows that $X \leq Y$ and therefore $X\forall p(Xp \to Yp)$.

One limit of the current account, as we saw above, is that it only characterizes those physically necessary propositions that are metaphysically entailed by a finite set of laws. Perhaps this is not a real limit — perhaps there are only finitely many laws (at least in the actual world) or the set of laws is compact in the sense that a proposition is $\Box_{meta}$-entailed by it only if the proposition is $\Box_{meta}$-entailed by a finite subset of it. But to provide a sufficient characterization for those who insist there are physical necessities only $\Box_{meta}$-entailed by infinitely many laws, we may redefine the restriction of a necessity as follows:

$$X^Y := \lambda p.\forall Z(\forall q(Xq \vee Yq \to Zq) \wedge \forall q(\forall r(\forall r'(Zr' \to X(r \to r')) \to X(r \to q)) \to Zq) \to Zp).$$

Given this new definition, we can still prove that so long as X is a necessity, X^Y is also a necessity.[48] Now, suppose B is $\Box_{meta}$-entailed by infinitely many laws. This means it is $\Box_{meta}$-entailed by the set of all laws. Recall that we imitate the entailment relation between

[48]The proof of LNX^Y is similar to the proof of Proposition 3.15 and thus the proof of Proposition 3.14-(i). To show LKX^Y, it is crucial to observe that given the closure of X, $\forall r(\forall r'(Zr' \to X(r \to r')) \to X(r \to q)) \to Zq$ implies the closure of Z.

a set of propositions and a single proposition by using propositional operators: p is entailed by a set corresponding to X just in case $\forall q(\forall r(Xr \to q \leq r) \to q \leq p)$. Similarly, we can formulate the idea that B is $\Box_{meta}$-entailed by laws in this way: $\forall q(\forall r(\mathrm{Law}\, r \to \Box_{meta}(q \to r)) \to \Box_{meta}(q \to B))$. Then we can show it follows from our new definition that $\Box_{phys}B$.[49]

However, with this new characterization of a restriction of a necessity, we cannot prove the result stated by Proposition 3.16, so we cannot guarantee that every necessity is necessarily coextensive with a restriction of L. The reason is that not all necessities are closed under infinitary consequence, as we emphasized before, although we can still prove for instance $\mathrm{Closed}^{\infty}\, Y \to (X \leq Y \leftrightarrow Y \sim X^{Y})$. If one wants to insist that every necessity is equivalent to a restriction of L as well as our new characterization at the same time, one can always adopt Roberts' conception of necessity according to which all necessities are closed under infinitary consequence.

3.5 Conservativeness

We have proved many results, and we have claimed to do so without taking on any grain-theoretic commitment. But this latter claim of grain-neutrality is in need of justification. While it is known that one cannot derive, for example, the Boolean identities in $\mathsf{H_0}$, we need some guarantee that one cannot derive them in our stronger theory of necessities. In this section we will in fact show that any theorem of TN that can be stated in the language $\mathcal{L}$ of pure higher-order logic (i.e. not including the primitive Nec) is already a theorem of $\mathsf{H_0}$. That is to say, TN is *conservative* over $\mathsf{H_0}$. So principles of granularity, like $p \wedge q = q \wedge p$, cannot be proven from TN unless they are already theorems of the minimal system $\mathsf{H_0}$.[50]

Lemma 3.17. *TN is interpretable in $\mathsf{H_0}$ via the translation i which replaces* Nec *with* $\lambda X.(\forall p(p \to Xp) \wedge KX)$:

$i : \mathcal{L}^{\mathrm{Nec}} \to \mathcal{L}$

For all $A \in \mathcal{L}^{\mathrm{Nec}}$, $\vdash_{\mathsf{TN}} A$ only if $\vdash_{\mathsf{H_0}} i(A)$.

Proof. We only need to show that given the translation i, all the axioms of TN become theorems of $\mathsf{H_0}$ and the rule Necessitation preserves theoremhood.

Note that $i(L) = \lambda p.\forall X(\forall q(q \to Xq) \wedge KX \to Xp)$ and hence $i(L)A \leftrightarrow A$ is provable in $\mathsf{H_0}$ for all $A \in \mathcal{L}$. So it is an admissible rule of $\mathsf{H_0}$ that if $\vdash A$ then $\vdash i(L)A$. Moreover, to show that i(Necessity) and i(Mix-and-Match) are theorems of $\mathsf{H_0}$, it suffices to prove that the following two statements are theorems of $\mathsf{H_0}$:

(i) $\forall p(p \to Xp) \wedge KX \leftrightarrow \forall p(p \to Xp) \wedge KX$;

[49] Proof: Suppose for any Z, we have (i) $\forall p(\Box_{meta}p \vee \mathrm{Law}\, p \to Zp)$ and (ii) $\forall p(\forall q(\forall r(Zr \to \Box_{meta}(q \to r)) \to \Box_{meta}(q \to p)) \to Zp)$. Suppose further that we have (iii) $\forall q(\forall r(\mathrm{Law}\, r \to \Box_{meta}(q \to r)) \to \Box_{meta}(q \to B))$. Our target is to show ZB. From (ii), we can get $\forall q(\forall r(Zr \to \Box_{meta}(q \to r)) \to \Box_{meta}(q \to B)) \to ZB$. So it suffices to show that (iii) implies $\forall q(\forall r(Zr \to \Box_{meta}(q \to r)) \to \Box_{meta}(q \to B))$. Consequently, it suffices to show that $\forall r(Zr \to \Box_{meta}(q \to r))$ implies $\forall r(\mathrm{Law}\, r \to \Box_{meta}(q \to r))$. Then it turns out to be sufficient to show that $\mathrm{Law}\, r$ implies Zr, which has already been guaranteed by (i).

[50] Conservativity is not the only dimension of grain-neutrality one might demand. For instance, conservativity does not tell us whether TN implies any distinctively grain-theoretic identities involve the new predicate Nec itself. An identity like $\mathrm{Nec} = \lambda X.\neg\neg\, \mathrm{Nec}\, X$, for instance, is distinctive to theories like Classicism, but since it involves Nec, conservativity offers no guarantee as to its unprovability. The stronger requirement is that if TN proves an identity (possibly involving Nec) then that identity is provable in $\mathsf{H_0}$ as formulated in the same language $\mathcal{L}^{\mathrm{Nec}}$. We believe this stronger result is true, but it would take us too far afield to prove it here, as we suspect it would require a model theoretic argument.

(ii) $\forall X(WX \to \forall p(p \to Xp) \land KX) \to \forall p(p \to L_W p) \land KL_W$.

(i) is trivial and the proof of (ii) is immediate. □

Theorem 3.18 (Conservativeness)**.** TN *is conservative over* H_0.

Proof. Let $A \in \mathcal{L}$ and suppose that there is a derivation of A in TN. Given the lemma above, it is easy to see $i(A)$ is derivable in H_0 by induction. But since A belongs to $\mathcal{L}$, $A = i(A)$. □

3.6 Interpretability

The conservativeness result of the last section provided an interpretation of TN in H_0 in which L became equivalent to the truth operator I. More generally, it is possible to interpret TN in any theory augmented with an operator expression governed by a logic of $\mathsf{S4}$ and vice versa (so by using the truth operator I, we obtain our previous result as a special case).

Recall the higher-order language $\mathcal{L}^\Box$ with the operator constant $\Box$. Let $\mathsf{H_0S4} \subseteq \mathcal{L}^\Box$ be the theory $\mathsf{H}_0 \oplus \mathsf{S4}$. Clearly, $\mathsf{H_0S4}$ can be interpreted in TN: since we have shown in section 3.1 that the logic of L is at least $\mathsf{S4}$, we may just interpret $\Box$ as L. Now, let's see the converse direction. Define:

$$\begin{aligned} \text{Logical}^\Box &:= \lambda X.\Box\forall p(\Box p \to \Box Xp); \\ \text{Closed}^\Box &:= \lambda X.\Box(\forall pq(X(p \to q) \to Xp \to Xq)); \\ \text{Nec}^\Box &:= \lambda X.\, \text{Logical}^\Box X \land \text{Closed}^\Box X. \end{aligned}$$

Theorem 3.19. TN *is interpretable in* $\mathsf{H_0S4}$ *via the translation* $i^\Box$ *that replaces* Nec *with* Nec$^\Box$*:*

$$i^\Box : \mathcal{L}^{\text{Nec}} \to \mathcal{L}^\Box$$

For all $A \in \mathcal{L}^{\text{Nec}}$, $\vdash_{\mathsf{TN}} A$ *only if* $\vdash_{\mathsf{H_0S4}} i^\Box(A)$.

Proof. Given that $\Box$ obeys principles of $\mathsf{S4}$, for all $A \in \mathcal{L} \cup \mathcal{L}^\Box$, $i^\Box(L)A \leftrightarrow \Box A$ is provable in $\mathsf{H_0S4}$. Thus, by the rule N of $\mathsf{H_0S4}$, we have the rule that $\vdash A$ only if $\vdash i^\Box(L)A$. Moreover, it is obvious that $i^\Box$(Necessity) is a theorem of $\mathsf{H_0S4}$. To show that $\mathsf{H_0S4}$ proves $i^\Box$(Mix-and-Match), it suffices to show that the following statement is derivable in $\mathsf{H_0S4}$:

$$\Box\forall X(WX \to \text{Logical}^\Box X \land \text{Closed}^\Box X) \to \text{Logical}^\Box L_W \land \text{Closed}^\Box L_W.$$

From $WX \to \text{Logical}^\Box X$, we can infer $\Box p \to WX \to \Box Xp$. Because the logic of $\Box$ is $\mathsf{S4}$, we then have $\forall p(\Box p \to \Box(\lambda p.\forall X(WX \to Xp))p)$. It means from $\Box(WX \to \text{Logical}^\Box X)$, we can infer $\text{Logical}^\Box L_W$. $\text{Closed}^\Box L_W$ can also be easily inferred from $\Box(WX \to \text{Closed}^\Box X)$. □

4 Strengthenings

The theory TN is not only neutral about questions of grain, but is also neutral about many classical debates in the philosophy of modality. The preceding arguments — about the existence and logic of the broadest necessity, on the pre-lattice of necessities, and so on — therefore can be accepted without taking a stance on these questions. Nonetheless, as a metaphysical theory TN is weak. Further axioms about necessities can be added to provide a more fleshed out theory.

Let us consider one extreme position in the philosophy of modality, which we shall call Quineanism:

Quineanism $\mathrm{Nec}\, X \to (p \leftrightarrow q) \to (Xp \leftrightarrow Xq)$.

This axiom says that every necessity is truth-functional. In particular, given that necessities are closed under modus ponens, this leaves only the truth operator and the vacuously true operator: thus every necessities is coextensive with one of these two truth-functional operators. So there is no contingency. (Of course, since this principle is stated in higher-order logic, it would not be acceptable to Quine, but we feel it captures an important aspect of the extensionalist world-view of which Quine is the most salient proponent.) Given Lemma 3.17, it follows from Quineanism that I is coextensive with L and is therefore a broadest necessity.

The simplest way to be a Quinean is to be, what we shall call, a Fregean:

Fregeanism $\forall XY(\forall z_1 \ldots z_n(Xz_1 \ldots z_n \leftrightarrow Yz_1 \ldots z_n) \to X = Y)$,

where the $n = 0$ case tells us that materially equivalent propositions are identical. Unlike Quineanism, Fregeanism is not a principle about necessity (it does not involve the primitive Nec), rather it is a pure principle of granularity. It is easily seen that Fregeanism entails Quineanism (in TN_0).[51] But crucially, Quineanism does *not* entail Fregeanism. In fact, implicit in our proof of the conservativeness result in section 3.5 was an argument that any theory of granularity consistent with H_0 is consistent with Quineanism. This highlights an important issue, namely one can accept a very fine-grained picture of reality — perhaps even some sort structured picture — but still embrace Quine's anti-modal scruples.[52]

Other principles of granularity formulated in the pure language of higher-order logic can be added into our theory of necessities as well. (We will explore some systematic and deep connections between necessity and granularity in section 6.) But we can even use our theory of necessities itself to formulate principles about granularity. For instance, consider the following view:

Intensionalism $\forall XY(L\forall z_1 \ldots z_n(Xz_1 \ldots z_n \leftrightarrow Yz_1 \ldots z_n) \to X = Y)$.

Unlike Fregeanism, Intensionalism is stated using our distinctive primitive predicate Nec (through L). Once we add Intensionalism to TN, the axiom Mix-and-Match will become redundant.[53] More interestingly, the resulting theory is in a certain sense, exactly the same as Classicism: the theorems not involving Nec are exactly the theorems of Classicism, and Nec itself is provably identical to predicate in the language of Classicism (i.e. the language of pure higher-order logic), so even the theorems involving Nec do not extend Classicism in an interesting way. We'll return to this result in section 6.1.

[51] By Fregeanism we can show that $L = I$. Then the axiom Necessity become equivalent to $\mathrm{Nec}\, X \leftrightarrow \forall p(p \to Xp) \wedge KX$, from which Quineanism follows.

[52] There are some theories of granularity that sit less comfortably with Quineanism: for instance one might accept HE or $\mathsf{HE}\zeta$, whilst rejecting the Fregean view that there are only two propositions. Within these theories, one can prove the existence of operators that formally behave like necessities (such as $\lambda p.(p = \top)$), which will not count as necessities by the lights of Quineanism. We view this as a consistent, but highly unattractive position to take; see section 6 for more discussion.

[53] Consider the result of adding Intensionalism to the theory TN_0 and closing under mp, Gen and Necessitation. Suppose that Lp is true. Then we have $L(p \leftrightarrow \top)$. By Intensionalism, p is identical to $\top$. We know that Necessitation allows us to get $LL\top$. So by Leibniz's Law, we also have LLp. This reasoning gives us the $\mathsf{4}$ for L and its necessitated version. We have shown in section 2.3 that Mix-and-Match follows from the latter.

One might wonder if it is possible to strengthen our theory in the opposite direction than Quineanism. For instance, are there any axioms that would force there to be as much contingency as possible? One option in this direction is to adopt a schema of this form:[54]

Logical Necessity $LA(c_1 \dots c_n) \leftrightarrow \forall x_1 \dots x_n A(x_1 \dots x_n)$,

provided A involves no free variables, $c_1, \dots, c_n$ enumerate all the distinct non-logical constants in A, and $A(x_1 \dots x_n)$ denotes the result of replacing them with distinct free variables.[55] Roughly speaking, the principle tells us that the logical predicate $A(x_1 \dots x_n)$ is satisfiable for some $x_1, \dots, x_n$ just in case it is $L^{\diamond}$-possible that $c_1, \dots, c_n$ instantiate this predicate.

The notion of satisfiability involved in the principle Logical Necessity could be replaced by other notions of consistency, for instance, one could consider the schema

Humeanism $\neg L \neg A$,

whenever A is a consistent formula of TN.[56] So long as we are formulating this schema in a fundamental language, where every constant denotes a distinct fundamental entity, this principle goes some way to capturing the Humean maxim that there are no necessary connections between distinct fundamental entities. Unlike Logical Necessity, which is compatible with a coarse-grained theory like Classicism, Humeanism implies a very fine-grained picture of reality. For instance, since TN is conservative over $\mathsf{H_0}$, anything consistent in the latter will be possible. For instance $p \wedge q \neq q \wedge p$ is consistent in $\mathsf{H_0}$, and so its possibility is an instance of Humeanism. But since we can also prove $L(p \wedge q = p \wedge q)$ in TN, we may infer that in fact $p \wedge q$ and $q \wedge p$ are distinct.

A surprising consequence of adding Logical Necessity or Humeanism to our theory TN is that no necessities are fundamental. Consider Logical Necessity, and suppose we're working in a language where every constant denotes a distinct fundamental entity. Assume for reductio that C is a fundamental necessity. Note that $NX \wedge KX$ is a logical predicate, so $\neg L(NC \wedge KC) \leftrightarrow \exists X \neg(NX \wedge KX)$ is an instance of Logical Necessity. Since $\exists X \neg(NX \wedge KX)$ is true,[57] we have $\neg L(NC \wedge KC)$. Then by Necessity, C is not a necessity. A contradiction. The argument involving Humeanism proceeds similarly.

We've discussed several ways to strengthen our theory by saying something more about necessities. Another natural dimension to strengthen the theory is to extend the modal logic of the broadest necessity L. Quineanism indirectly does so — it makes the modal logic of L be Triv, whose characteristic axiom is:

Triv$_L$ $p \leftrightarrow Lp$.

But there is a great number of strengthenings of the modal logic $\mathsf{S4}$ that are less extreme than this one.[58] Any one of these modal principles provides a potential way in which to strengthen the theory we have presented above. Perhaps the most famous such axiom is Brouwer's axiom, B, yielding the logic $\mathsf{S5}$ when added to $\mathsf{S4}$:

B$_L$ $p \rightarrow L \neg L \neg p$.

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[54] See the principle Logical Necessity from [3].

[55] For the purposes of formulating the schemata we count Nec as a logical constant.

[56] We do not know whether Humeanism is consistent.

[57] Consider the operator $\lambda p.\bot$. If it has the property N, then we have $L\top \rightarrow L\bot$ and therefore $L\bot$. But since L is factive, we'll then derive $\bot$.

[58] Indeed, there are continuum many between $\mathsf{S4}$ and Triv; see Fine [10].

This principle could simply be added to our system as a way of strengthening it. But unlike the B axiom of modal logic, the principle B_L is really a shorthand for something stated explicitly in terms of the operator predicate Nec, and therefore B_L so understood states something very non-obvious about the domain of necessity operators. It would be nice to have a more transparent principle directly about necessities that corresponds to B_L. Williamson [36] suggests the principle that every necessity has a reversal, which in our system corresponds to the principle:

Reversal $\mathrm{Nec}\,X \to \exists Y(\mathrm{Nec}\,Y \wedge \mathrm{Rev}\,XY)$.

Recall that we defined the relation Rev in section 3.2 as $\lambda XY.\forall p(p \to X\neg Y\neg p)$. Reversal is far from an obvious principle: while some tense operators, for example, evidently have reversals, it is far from obvious what the reversal of, say, physical necessity is. As it turns out, Reversal and B_L are equivalent.

Proposition 4.1. $\vdash_{\mathsf{TN}} B_L \leftrightarrow$ *Reversal.*

Proof. See the proof of Proposition 4.2 below. □

Note that once B_L (or equivalently Reversal) is added into TN, $\mathrm{CBF}_{\mathrm{Nec}}$ will imply $\mathrm{BF}_{\mathrm{Nec}}$. Of course, one may directly add $\mathrm{BF}_{\mathrm{Nec}}$ to strengthen the theory.

Another well-known modal logic between $\mathsf{S4}$ and Triv is $\mathsf{S4.2}$, the result of extending $\mathsf{S4}$ by adding G:

$\mathbf{G}_L$ $\neg L\neg Lp \to L\neg L\neg p$.

As with B_L, this indirectly imposes a constraint on necessity operators. We can make that constraint on necessities explicit as follows:

Convergence $\mathrm{Nec}\,X \wedge \mathrm{Nec}\,Z \to \exists YU(\mathrm{Nec}\,Y \wedge \mathrm{Nec}\,U \wedge \forall p(\neg X\neg Yp \to Z\neg U\neg p))$.

Proposition 4.2. $\vdash_{\mathsf{TN}} G_L \leftrightarrow$ *Convergence.*

Proof. Suppose we have G_L. Because L is itself a necessity, we have $\exists YU(\mathrm{Nec}\,Y \wedge \mathrm{Nec}\,U \wedge \forall p(\neg L\neg Y \to L\neg U\neg p))$, which amounts to $\exists YU(\mathrm{Nec}\,Y \wedge \mathrm{Nec}\,U \wedge \forall p(\neg L\neg Y \to \forall Z(\mathrm{Nec}\,Z \to Z\neg U\neg p)))$ or equivalently, $\exists YU(\mathrm{Nec}\,Y \wedge \mathrm{Nec}\,U \wedge \forall Z(\mathrm{Nec}\,Z \to \forall p(\neg L\neg Y \to Z\neg U\neg p))$. As we consequence, we have $\forall Z(\mathrm{Nec}\,Z \to \exists YU(\mathrm{Nec}\,Y \wedge \mathrm{Nec}\,U \wedge \forall p(\neg L\neg Yp \to Z\neg U\neg p)))$. Note that $\neg X\neg Yp$ implies $\neg L\neg Yp$ whenever X is a necessity. So we can get $\forall XZ(\mathrm{Nec}\,X \wedge \mathrm{Nec}\,Z \to \exists YU(\mathrm{Nec}\,Y \wedge \mathrm{Nec}\,U \wedge \forall p(\neg X\neg Yp \to Z\neg U\neg p)))$.

Conversely, suppose we have Convergence. Suppose further that $\neg X\neg\forall Y(\mathrm{Nec}\,Y \to Yp)$ for some necessity X. Then it is not difficult to infer that $\forall Y(\mathrm{Nec}\,Y \to \neg X\neg Yp)$ by Persistence. Now, let Z be an arbitrary necessity. According to Convergence, we have $\forall p(\neg X\neg Yp \to Z\neg U\neg p)$ for some necessities Y and U, so by $\mathrm{Nec}\,Y$, $Z\neg U\neg p$ follows. This means we can infer from $\exists X(\mathrm{Nec}\,X \wedge \neg X\neg\forall Y(\mathrm{Nec}\,Y \to Yp))$ that $\forall Z(\mathrm{Nec}\,Z \to \exists U(\mathrm{Nec}\,U \wedge Z\neg U\neg p))$, which then implies $\forall Z(\mathrm{Nec}\,Z \to Z\exists U(\mathrm{Nec}\,U \wedge \neg U\neg p))$ by Persistence again. □

Curiously, adding Reversal or Convergence to TN does not create any more Nec-free consequences: it is also conservative over H_0.[59]

[59] The argument is the same as given in section 3.5, simply check that Reversal and Convergence are also true under the interpretation of Nec provided there.

5 Comparison with other theories

In this section we compare our approach to other theories of necessities. Here we begin with some ideas articulated in Williamson [36]. Related ideas, formulated in the present framework of higher-order logic can be found both in Roberts [30] and Dorr, Hawthorne and Yli-Vakkuri [9], ch. 8.4.[60]

Like our approach, Williamson takes the notion of *being a necessity* as basic, and subjects it to some natural closure conditions. Let's begin with the following two,[61] which he introduces informally as

> The composition of any two necessities is a necessity;
>
> The conjunction of any collection of necessities is a necessity.

Unlike us, Williamson formulates these principles in an algebraic language instead of a higher-order one. However, they have natural analogues in this framework, as other authors mentioned above have shown. Recall that we defined the composition of two operators X and Y as $X \circ Y := \lambda p.XYp$, so the first principle becomes:

Composition $\mathrm{Nec}\, X \to \mathrm{Nec}\, Y \to \mathrm{Nec}(X \circ Y)$.

The formulation of the second principle is somewhat delicate. For finitely many operators $X_1, \ldots, X_n$, we may define their conjunction simply as $\lambda p.(X_1 p \wedge \cdots \wedge X_n p)$. But a collection of necessities might be infinite. So we need a more general notion of conjunction. We know a collection can be represented by a property. Someone may therefore suggest the conjunction of all operators with the property W is just the *greatest lower bound* (henceforth, GBL) of W under the entailment relation:

$$\mathrm{GLB} := \lambda XW.(\forall Y(WY \to X \leq Y) \wedge \forall Z(\forall Y(WY \to Z \leq Y) \to Z \leq X)).$$

However this condition does not suffice for X to count as a conjunction of the W-operators, since an actual greatest lower bound could fail to be a greatest lower bound if there had been new necessities (i.e. necessities which do not actually exist) between X and the W-operators in strength. In this case the things that is in fact the greatest lower bound of the Ws possibly violates the conjunction introduction rule: if there could be an operator Y strictly weaker than X but entailing each member of W, then Y is analogous to the possible existence of a sentence A which entails p_1, p_2, p_3 et cetera, without entailing their conjunction. Thus the notion of a conjunction is strictly *stronger* than that of a greatest lower bound of some propositions. A conjunction, thus, is *necessarily* a greatest lower bound of W, in every sense of necessity.

$$\mathrm{Conj} := \lambda XW.L\, \mathrm{GLB}\, XW.$$

The next problem is that in order to talk about the same collection of operators across different possibilities we need some way to pick out those operators out rigidly. (Indeed, a non-rigid property of operators most likely won't have anything that is necessarily a GBL.) But so long as a property is *rigid*, the existence of its GLB is guaranteed. Here we say that

[60] Although Williamson and Roberts assume HE and $\mathsf{HE}\zeta$ in their works respectively, their ideas concerning necessities can be formulated against the background of a grain-neutral theory like H_0. In fact, Dorr, Hawthorne and Yli-Vakkuri just do so.

[61] Williamson endorses other closure conditions but only the following two are relevant here.

a property W is rigid iff the extension of it doesn't expand or shrink between worlds, which we cash out in terms of the Barcan formula and its converse holding for the quantifiers restricted to W:[62]

$$\text{Persistent} := \lambda W.\forall X(WX \to LWX);$$
$$\text{Inextensible} := \lambda W.\forall U(\forall X(WX \to LUX) \to L\forall X(WX \to UX));$$
$$\text{Rigid} := \lambda W.(\text{Persistent}\, W \wedge \text{Inextensible}\, W).$$

It is fairly easy to show that if W is rigid, then L_W (i.e. $\lambda p.\forall X(WX \to Xp)$) is a GLB of W.[63] Thus, in order to talk about the conjunction of the W operators, we shall require that W be a rigid property of operators in every sense of necessity. Then the second principle listed above may be formulated in this way:

Conjunction $L\,\text{Rigid}\, W \wedge L\forall X(WX \to \text{Nec}\, X) \to \text{Nec}\, L_W$.

Of course, even though W is not necessarily rigid, one may still talk about the conjunction of W in a derivative sense, by assuming there is a necessarily rigid property W' coextensive with W and then regarding the conjunction of W' as the conjunction of W.[64]

From here Williamson and Roberts attempt to define the broadest necessity as follows. They firstly note that by Conjunction, the conjunction of all necessities, C_{Nec} (assuming it exists), is itself a necessity. They then argue that the conjunction of all necessities entails each necessity:[65]

$$\forall X(\text{Nec}\, X \to C_{\text{Nec}} \leq X).$$

Secondly, like us, they show that the 'broadest necessity' so defined satisfies the $\mathsf{4}$ axiom. Since necessities are closed under composition, and C_{Nec} is a necessity, $C_{\text{Nec}} \circ C_{\text{Nec}}$ is a necessity. Since C_{Nec} entails each necessity, C_{Nec} entails $C_{\text{Nec}} \circ C_{\text{Nec}}$, which we are spelling out as $L\forall p(Lp \to LLp)$, the $\mathsf{4}$ axiom.[66]

[62] See Bacon and Dorr [4]. Persistence is also equivalent to the condition that $\forall U(L\forall X(WX \to UX) \to \forall X(WX \to LUX))$, corresponding to the converse Barcan formula.

[63] This can be shown in a pretty weak theory. Just suppose we have $\mathsf{H_0}$ and L obeys the modal logic K — so the background theory is even weaker then $\mathsf{TN_0}$. Fix a rigid property W. By definition, we have $WX \to \forall p(L_W p \to Xp)$. By the rule N and the axiom K for L, we get $LWX \to L_W \leq X$. So given the persistence of W, L_W is a lower bound of W. Next, suppose that $\forall Y(WY \to \forall p(Zp \to Yp))$ for an arbitrary Z. It follows that $\forall p(Zp \to L_W p)$. By N and K again, we can have $L\forall(WY \to \forall p(Zp \to Yp)) \to Z \leq L_W$. Note that by the inextensibility of W, $\forall Y(WY \to Z \leq Y)$ implies $L\forall(WY \to \forall p(Zp \to Yp))$, so L_W is a greatest lower bound of W.

[64] For example Dorr, Hawthorne and Yli-Vakkuri assume in their background theory that every property is coextensive with a necessarily rigid property (see [9], ch. 1.5). But note that our theory TN is neutral about this idea. If W is a property that isn't coextensive with a necessarily rigid one, then, surprisingly, it doesn't really make sense to talk about the conjunction of the Ws. We have no way to even state what it means for the conjunction of the Ws to have no possible failures of the analogues of conjunction elimination and introduction.

[65] Note, however, that entailment in Williamson's framework is being taken as primitive, or at least, taken to fall out of the algebraic structure of propositions. We take it to be a significant advantage of our approach that we can simply define entailment in terms of necessity itself, via L-strict implication. Note also that because Williamson is working in an algebraic framework, he defines operator entailment proposition wise — for each proposition p, $C_{\text{Nec}}p$ entails Xp — so the force of the L in front of $\forall p(C_{\text{Nec}}p \to Xp)$ in our formulation is lost.

[66] The two closure conditions discussed here cannot guarantee that the logic of C_{Nec} is at least $\mathsf{S4}$. More principles are needed. For example, Roberts adds a principle similar to Closure ($\text{Nec}\, X \to \text{Closed}\, X$) to guarantee the K axiom for C_{Nec} and the principle Identity ($\text{Nec}\, I$) to guarantee the T axiom for C_{Nec}. Since Roberts assumes $\mathsf{HE}\zeta$, the rule N for C_{Nec} becomes admissible once he has K. But in a grain-neutral setting, one may directly add this rule. Williamson adds the principle Reversal of section 4 further. We earlier showed that Reversal is equivalent to the Brouwerian principle for L in our theory, and Williamson argues that it implies something similar for C_{Nec} in his framework as well, so for him C_{Nec} satisfies a logic of $\mathsf{S5}$.

Dorr, Hawthorne and Yli-Vakkuri adopt the same definition of C_{Nec}, but they do not claim that the result of the definition is a broadest necessity. They more cautiously call it an "extensionally minimal" necessity (see below).

Recall that our theory of necessities is very liberal concerning what counts as a necessity: any operator that is Logical and Closed. By contrast, the Williamson-Roberts-DHY approach is consistent with a much narrower conception of necessity. It should be emphasized that their project is not necessarily opposed to ours: one could simply view them as theories of two different notions. For instance, Williamson and Roberts are explicit that their are theories of *objective* necessities, which may be a subclass of a broader class of necessities, including epistemic, deontic and vagueness theoretic operators.

However, we think even on a narrower conception of what a necessity is, the two principles identified above are not enough to deliver a broadest necessity in an interesting sense. In fact, it is worth noting that all of the above reasoning concerning C_{Nec}, the conjunction of all the actually existing necessities, can also be carried out in our present theory, without invoking Mix-and-Match.[67] But we believe this is not sufficient for proving the existence of a *broadest* necessity. To be a real broadest necessity, it's not sufficient that you simply be a necessity which entails every other necessity, for this could be true only contingently. Specifically, C_{Nec} will clearly entail all the actually existing necessities, but if there could have been new necessities, then C_{Nec} need not entail them: a conjunction doesn't entail anything not already entailed by the conjuncts.

To circumvent these issues, Roberts entertains a further axiom which says that the property of being a necessity is rigid; in other words, he embraces the conjunction of Persistence (or equivalently CBF_{Nec}) and BF_{Nec} of section 2.3. So there can't be new necessities, avoiding the above problem.[68] (Williamson implicitly imposes the same constraint since in his algebraic framework the domain of necessities is constant.) The persistence of Nec is a theorem of our theory. But why should be accept the assumption that there can't be new necessities? It is natural to think that there could have been. For instance, imagine a possibility with alien physical properties and new laws governing them: one would expect the resulting physical necessity to not exist in the actual world, in virtue of its involving properties that don't actually exist. Even Roberts' preferred background theory of Classicism allows for the possibility of new necessary operators. Indeed, there is a close relationship between Classicism and our theory TN: TN is interpretable in Classicism in the sense that there is a translation from the former into the latter such that the theorems of Classicism include the translates of theorems of TN. By contrast, the translation in question maps the principle BF_{Nec} to a non-theorem of Classicism. (The details of this interpretation are spelled out in the next section.)

Our strategy to guarantee the real broadest necessity without any loss of generality is to endorses the axiom Mix-and-Match. But Mix-and-Match is a very strong closure condition. In what follows, we suggest another closure condition on necessities: a principle

[67] As we just saw, the reasoning relies on Composition and Conjunction. We have shown that Composition is derivable in TN_0 (see Proposition 2.3). Let's turn to Conjunction. Suppose W is a rigid property of necessities. Then by Necessity, it follows that $\forall p(Lp \to \forall X(WX \to LXp))$. Given the rigidity of W, we then have $\forall p(Lp \to L\forall X(WX \to Xp))$ which amounts to NL_W. So by Necessitation and the closure of L, if W is L-necessarily a rigid property of necessities, we have Logical L_W. Moreover, it's easy to see that Closed L_W follows from that W is L-necessarily a property of necessities.

[68] More technically, C_{Nec} must be the conjunction of some W such that $L\,\text{Rigid}\,W \wedge \forall X(WX \leftrightarrow \text{Nec}\,X)$. By $L\,\text{Rigid}\,W$, we have $L\forall X(WX \to C_{\text{Nec}} \leq X)$. If Nec is also inextensible, from $\forall X(\text{Nec}\,X \to WX)$ we can derive $\forall X(\text{Nec}\,X \to LWX)$ and then $L\forall X(\text{Nec}\,X \to WX)$, which will give us $L\forall X(\text{Nec}\,X \to C_{\text{Nec}} \leq X)$. (If Nec is necessarily rigid, we can even directly show that L is the conjunction of Nec, by the proof in note 63.)

strictly between Conjunction and Mix-and-Match in strength. So the resulting view retains the sort of neutrality we have sought in the present investigation; but it is in the same spirit as Williamson and Roberts, because as it is still consistent with narrower conceptions of necessity and doesn't commit you to the liberal conception encoded by principles like Necessity.

Our principle states that whenever W is L-necessarily a persistent property of necessities, the operator *possessing all W necessities* is itself a necessity:

Modalized GLB $L\,\mathrm{Persistent}\,W \wedge L\forall X(WX \to \mathrm{Nec}\,X) \to \mathrm{Nec}\,L_W$.

Notice the principle is a weakening of Mix-and-Match because we have strengthened the antecedent to require that W is necessarily persistent. And it is a strengthening of Conjunction because we have weakened the antecedent by requiring that W is necessarily persistent, but not necessarily rigid.

To explain why Modalized GLB is a natural principle, it's necessary to make a little detour. We motivated the definition of a conjunction from the order theoretical notion of a GLB, where the background theory of mathematical objects is *set theory*. However properties are not extensional, like sets are, and we saw that we needed special assumptions to talk about the GLB (or the conjunction) of the W-entities — for instance, that there exists a (necessarily) rigid property coextensive with W.

Category theory has allowed us to formulate abstract definitions of notions like being a partial order, or being a GLB, in a way that's applicable within other realms of mathematical objects that behave relevantly like sets, but are not necessarily as 'extensional' as sets. Since quantification into predicate and operator position need not be extensional, we believe these generalizations are helpful both for obtaining intuitions about higher-order logic and for constructing models of it.

Of particular interest is the realm of 'modalized' sets. A modalized set is effectively a family of sets indexed by worlds in a transitive reflexive Kripke frame.[69] The elements of a modalized set necessarily persist, in the sense that if you have an element at world w and w' is accessible from w, then that element exists there too.[70] We may informally think of them as necessarily persistent properties: a property such that necessarily if something has it, it necessarily has it. And among these modalized sets are modalized partial orders that roughly stand to the background realm of modalized sets as partial orders stand to sets in set theory: a family of partial orders indexed by worlds, with similar persistence properties. Just as a GLB of a set of elements from a partial order is defined in the realm of sets, one can define the modalized GLB of any modalized set of objects contained in the modalized partial order.

Translating this into the present setting, we may introduce a more general relation between an operator and a property of operators, being the *modalized GLB* of that property. Roughly, the modalized GLB of W is something which is necessarily a lower bound of W, and necessarily as great as anything else that's necessarily a lower bound of W.

$$\mathrm{MGLB} := \lambda XW.L(\forall Y(WY \to X \leq Y) \wedge \forall Z(L\forall Y(WY \to Z \leq Y) \to Z \leq X)).$$

As you can see, it is different from a conjunction in a couple of ways. Firstly, one can take the modalized GLB of any necessarily persistent property of operators, even if it is

[69] We are talking here about the functor category Set^W of functors from a transitive reflexive Kripke frame (W, R) to Set.

[70] Note that the accessibility relation at issue is transitive.

not necessarily rigid. Secondly, you don't need to be, necessarily, a greatest lower bound of the Ws, you need only be, necessarily, a lower bound that is greater than anything that is necessarily a lower bound of the Ws. It is also not an extensional notion: W and W' might be coextensive, yet have different modalized GLBs. Just as we were able to show that for a necessarily rigid W, L_W is a conjunction of W, it is possible to show that if W is necessarily persistent, then L_W is a modalized GLB of W.[71] When understood this way, the principle Modalized GLB just states that necessities are closed under the more general operator of modalized GLB. From a mathematical perspective, we feel the notion of a modalized GLB is far more natural than the notion of conjunction, as a generalization of GLB, and thus the principle Modalized GLB is far more natural than the principle Conjunction.

Now, consider the theory $\mathsf{TN}^- = \mathsf{H_0} \oplus \text{Closure} \oplus \text{Identity} \oplus \text{Persistence} \oplus \text{Composition} \oplus \text{Modalized GLB} \oplus \text{Necessitation}$. It looks like our theory TN in many formal aspects; in particular, the operator L is still a broadest necessity in the interesting sense and it is still obeys principles of $\mathsf{S4}$.

Proposition 5.1. *(i)* $\vdash_{\mathsf{TN}^-} \text{BroadestNec}\, L$*; (ii) according to* TN^-*, the modal logic of* L *contains* $\mathsf{S4}$ *and the modal fragment of* TN^- *contains no non-theorems of* $\mathsf{S4}$*.*

Proof. (i) By Closure, L is closed under modus ponens. By Persistence and Modalized GLB, L is a necessity. It follows from the definition of L that $\text{Nec}\, X \to \forall p(Lp \to Xp)$, so by Necessitation and the closure of L, we have $L\,\text{Nec}\, X \to L \le X$. By Persistence again, $\text{Nec}\, X \to L \le X$, and by Necessitation again, $L\forall X(\text{Nec}\, X \to L \le X)$.

(ii) We have shown the closure of L, and the rule of necessitation is just the rule Necessitation. Provided the result in (i) above, the T axiom for L follows from Identity and the $\mathsf{4}$ axiom follows from Composition. Moreover, it is easy to see that all theorems of TN^- are also derivable in TN. By Corollary 6.5 of section 6.1, no non-theorem of $\mathsf{S4}$ can be derived in the modal fragment of TN^-. $\Box$

However, since Necessity is not a theorem of TN^-, one may take TN^- as theorizing a narrower conception of necessity.

6 Necessity and granularity

In this section, we explore some connections between necessity and granularity. We explained in section 2.1 that to provide a comprehensive theory of necessities in a grain-neutral setting, it is inevitable to take some modal notion(s) as primitive. For example, in our theory TN, we take the predicate Nec, representing the notion of being a necessity, as primitive, and due to the interpretability theorem in section 3.6, it is equivalent to start with a primitive operator expression $\Box$ for the broadest necessity. But once we strengthen the background logic $\mathsf{H_0}$ by adding principles of granularity, we may provide a reductionist account of being a necessity and of the broadest necessity — TN can then be reinterpreted in the resulting theory. In fact, we have already seen an instance in section 4: once we add the principle Fregeanism to $\mathsf{H_0}$, we can get a Quinean interpretation of TN by our conservativeness result of section 3.5. But we also noticed that the Quinean interpretation is a trivial one, since according to it all necessities are truth-functional operators, so the resulting reductionist

[71] Like the proof in note 63, this argument can be run in a pretty weak theory — $\mathsf{H_0}$ plus a logic K for L: Suppose W is necessarily persistent. Since $\text{Persistent}\, W$ implies $\forall Y(WY \to L_W \le Y)$, $L\,\text{Persistent}\, W$ implies $L\forall Y(WY \to L_W \le Y)$. Moreover, since for any Z, $\forall Y(WY \to Z \le Y)$ implies $\forall p(Zp \to L_W p)$, $L\forall Y(WY \to Z \le Y)$ implies $Z \le L_W$, and we therefore have $L(\forall Z(L\forall Y(WY \to Z \le Y) \to Z \le L_W))$.

theory is not very interesting. If we add some more modest constraints of granularity in $\mathsf{H_0}$ however, we may end up with a non-Quinean interpretation of TN. One existing theory of this sort is developed by Bacon [1]. Let's begin with his account.

6.1 Classicism

Bacon operates with a more liberal notion of necessity than we are employing here; for instance, his notion needn't be Closed. Perhaps it is more appropriate to use the term *modality* for that notion.[72] His background theory of higher-order logic is $\mathsf{HE}\zeta$, namely Classicism, which admits rules ensuring that provably equivalent things are identical.

However, it is possible to offer a reductive account of our notion of a Logical and Closed necessity in that theory too. Recall that we write $\Box_\top$ for the operator $\lambda p.(p = \top)$. The reductive definitions can be given as follows:

$$\text{Logical}' := \lambda X.\Box_\top \forall p(\Box_\top p \to \Box_\top Xp);$$
$$\text{Closed}' := \lambda X.\Box_\top \forall pq(X(p \to q) \to Xp \to Xq);$$
$$\text{Nec}' := \lambda X.\, \text{Logical}'\, X \land \text{Closed}'\, X.$$

It is quite easy to see that according to $\mathsf{HE}\zeta$, namely Classicism, the modal logic of $\Box_\top$ is at least $\mathsf{S4}$.[73] Thus, by Theorem 3.19 we have:

Theorem 6.1. TN *has a non-Quinean interpretation in* $\mathsf{HE}\zeta$ *via the translation* j *that replaces* Nec *with* Nec$'$*:*

$j : \mathcal{L}^{\text{Nec}} \to \mathcal{L}$

For all $A \in \mathcal{L}^{\text{Nec}}$, $\vdash_{\mathsf{HE}\zeta} A$ *only if* $\vdash_{\mathsf{TN}} j(A)$.

By this interpretation, $\Box_\top$ turns out to be the broadest necessity.

Of course, Classicism proves a lot of sentences about grain that are translations of non-theorems of TN. But one might conjecture a much tighter connection between TN and Classicism: that once one blurs the distinction between L-necessarily equivalent entities within TN, the theories coincide. We will consider a couple of ways of making this precise.

As a preliminary, we prove an important lemma. The result is also interesting in itself. It says that closing the system $\mathsf{H_0}$ under E and ζ yields $\mathsf{HE}\zeta$ as well. But we have to restrict attention to the theories as formulated in the *relational types*.[74] Let $\mathsf{H_0E}\zeta$ be $\mathsf{H_0} \oplus \mathsf{E} \oplus \zeta$. Then we have:[75]

Proposition 6.2. $\mathsf{H_0E}\zeta = \mathsf{HE}\zeta$ *when they are formulated in the relational types.*

[72] For instance, as we showed in the end of section 2.2, if X is a modality (either a necessity or a possibility), its dual operator $\lambda p.\neg X \neg p$ is also a modality, but this does not hold for necessities.

[73] $\Box_\top$ obeys K: if $(p \to q) = \top$ and $p = \top$, then by Leibniz's Law $(\top \to q) = \top$ and therefore $q = \top$, since q and $\top \to q$ are provably equivalent and, by the rule E, are identical. $\Box_\top$ obeys T: it is obvious that $\Box_\top$ is factive. $\Box_\top$ obeys $\mathsf{4}$: note that $(\top = \top) = \top$ is provable in $\mathsf{HE}\zeta$, so if $p = \top$ then by Leibniz's Law, $(p = \top) = \top$. Finally, the rule N for $\Box_\top$ is admissible in $\mathsf{HE}\zeta$. This is because A is derivable only if $A \leftrightarrow \top$ and hence $A = \top$ are derivable.

[74] Both e and t are relational types; and whenever σ, τ are both relational types and $\tau \neq e$, $(\sigma \to \tau)$ is a relational type.

Since $\mathsf{H_0}$ has no principles governing identities between terms with types ending in e, we cannot even prove $(\lambda x.x)a = a$ where x and a are of type e, and we don't have anything analogous to β_{E} for non-relational types, so we certainly can't recover $\mathsf{HE}\zeta$.

[75] Thanks to Cian Dorr for discussing the proof of this proposition.

Proof. We only show that $\mathsf{HE}\zeta \subseteq \mathsf{H_0E}\zeta$ since the converse direction is trivial. This amounts to showing that all instances of $\beta\eta^*$ mentioned in section 1 are theorems of $\mathsf{H_0E}\zeta$. To get the intended conclusion, it suffices to prove that if M is $\beta\eta$-reducible to M', then $M = M'$ is derivable in $\mathsf{H_0E}\zeta$.[76]

So suppose that M is $\beta\eta$-reducible to M'. By induction on the complexity of M. If M is a variable or a constant, then M' must be the same variable or constant. When M is $\lambda x.N$, either M' is $\lambda x.N'$ for some N' where N is $\beta\eta$-reducible to N' or M is $M'x$ where x is not free in M'. The former case can be easily dealt with by I.H. As to the latter case, we suppose that the of type M' is $\sigma \to \tau \to t$ for brevity. So x is of type σ. Moreover, let y be a variable of type τ not free in M'. Note that $M'xy$ is immediately β-equivalent to both $(\lambda y.M'xy)y$ and $(\lambda xy.M'xy)xy$. Therefore by using β_{E}, E and ζ, we have $M'x = \lambda y.M'xy = (\lambda xy.M'xy)x$. Further, by Leibniz's Law and ζ, we can get $M' = \lambda x.M'x$. When M is N_1N_2, either M' is $N_1'N_2'$ for some N_1' and N_2' where N_1/N_2 is $\beta\eta$-reducible to N_1'/N_2' or N_1 is $\lambda x.N$ for some N and M' is $N[N_2/x]$. Again, the former case can be dealt with by I.H. In the latter case, we suppose the type of N is $\sigma \to t$ for brevity. Let y be a variable of type σ not free in N. Note that $N[N_2/x]y$ is immediately β-equivalent to $(\lambda xy.Ny)N_2y$. So we have $N[N_2/x] = (\lambda xy.Ny)N_2$. According to the last inductive step, $N = \lambda y.Ny$. Hence, we can get $N[N_2/x] = (\lambda x.N)N_2$. □

Now, we can introduce two ways to make the connection between TN and Classicism tighter. One is simply that adding the thesis Intensionalism of section 4 to TN yields a theory such that the Nec-free theorems of it are exactly the theorems of Classicism. Moreover, the sense in which this theory extends Classicism is uninteresting, since one can prove the identity $\text{Nec} = \text{Nec}'$ showing that even TN's new primitive is identical to something already definable in the base language of Classicism. We use $\mathcal{L}_{\mathcal{R}}$ be the language of pure higher-order logic based on relational types and $\mathcal{L}_{\mathcal{R}}^{\text{Nec}}$ the corresponding language equipped with the primitive predicate Nec. Let TNI denote the theory $\mathsf{TN} \oplus$ Intensionalism. Then we have:[77]

Theorem 6.3. *(i) For all $A \in \mathcal{L}_{\mathcal{R}}$, $\vdash_{\mathsf{TNI}} A$ iff $\vdash_{\mathsf{HE}\zeta} A$; (ii) $\vdash_{\mathsf{TNI}} \text{Nec} = \text{Nec}'$, so for all $A \in \mathcal{L}^{\text{Nec}}$, there is a $B \in \mathcal{L}$ such that $\vdash_{\mathsf{TNI}} A \leftrightarrow B$.*

Proof. (i) Given Proposition 6.2, to show that a formula $A \in \mathcal{L}_{\mathcal{R}}$ is derivable in $\mathsf{HE}\zeta$ only if it is derivable in TNI, it suffices to show that E and ζ are both admissible rules of TNI. If $A \leftrightarrow B$ is provable in TNI, so is $L(A \leftrightarrow B)$. Then by Intensionalism, we have $A = B$. Moreover, suppose M and N are of type $\sigma_1 \to \cdots \to \sigma_n \to t$ and $Mx = Nx$ is provable in TNI. Let $y_2, \ldots, y_n$ be distinct variables free in neither M nor N. Note that $L\forall xy_2 \ldots y_n(Mxy_2 \ldots y_n \leftrightarrow Nxy_2 \ldots y_n)$ is also provable. So by Intensionalism again, we have $M = N$.

To show the converse direction, recall the translation function j introduced in Theorem 6.1, which translates all theorems of TN as theorems of $\mathsf{HE}\zeta$. So it suffices to show that j also translates Intensionalism to a theorem of $\mathsf{HE}\zeta$. According to j, j(Intensionalsism)

[76] One term is said to be immediately β/η-reducible to another if they are immediately β/η-equivalent, the former is of the form $(\lambda.M)N/\lambda x.Nx$, and the latter is of the form $M[N/x]/N$. One term is $\beta\eta$-reducible to another if the former can be gotten from the latter by replacing one term with another which is immediately β or η-reducible to it for 1 time.

[77] Note that in the second claim we don't need the restriction that the theories at issue are formulated in the relational types.

is provably equivalent, in $\mathsf{HE}\zeta$, to $\forall XY(\Box_\top \forall x_1 \dots x_n(Xz_1 \dots z_n \leftrightarrow Yz_1 \dots z_n) \to X = Y)$, which is clearly a theorem of $\mathsf{HE}\zeta$.[78]

(ii) Given Intensionalism, to prove $\mathrm{Nec} = \mathrm{Nec}'$ we just need to show that $\mathrm{Nec}\, X \leftrightarrow \mathrm{Nec}'\, X$ is provable in TNI. By Necessity$'$, it suffices to show that $LNX \wedge LKX \leftrightarrow \Box_\top \forall p(\Box_\top p \to \Box_\top Xp) \wedge \Box_\top KX$ is provable, and this claim follows from the observation that $LA \leftrightarrow \Box_\top A$ is provable for all A: if $\Box_\top A$ holds, which means $A = \top$, then since L applies to $\top$, by Leibniz's Law, it applies to A as well; conversely, if we have LA, then we can get $L(A \leftrightarrow \top)$ and therefore $A = \top$ by Intensionalsim. □

Another way of making this connection tighter is to ask for a converse interpretability result, allowing us to interpret Classicism in our theory of necessities. The rough idea is to translate the vocabulary of Classicism in such a way that identity gets reinterpreted as necessary equivalence in TN, and thus the broadest necessity according to Classicism, $\Box_\top$, corresponds to $\lambda p.L(p \leftrightarrow \top)$, which is evidently necessarily equivalent to L. Of course, we didn't take identity or $\Box_\top$ as primitive in our axiomatization of Classicism, rather we defined both in terms of the truth-functional connectives and the higher-order quantifiers. Our strategy, then, will be to reinterpret the quantifiers by restricting them to higher-order entities that preserve necessary equivalence. We can make this precise by introducing a notion, $\approx_\sigma$, within the language $\mathcal{L}_{\mathcal{R}}^{\mathrm{Nec}}$ of TN which simultaneously defines necessary equivalence at each type, and removes operators that do not preserve necessary equivalence:

- $\approx_e := =_e$;
- $\approx_t := \lambda pq.L(p \leftrightarrow q)$;
- $\approx_{\sigma\to\tau} := \lambda XY.L\forall xy(x \approx_\sigma y \to Xx \approx_\tau Yy)$.

Such a relation is symmetric and transitive but not reflexive: it generates a partition of a subcollection of entities. An operator X of type σ preserves necessary equivalence when it stands the relation $\approx_\sigma$ to itself, so we may define our restricted quantifiers as follows:

$$\forall_\sigma^{\approx} := \lambda X.\forall_\sigma x(x \approx_\sigma x \to Xx).$$

We may now establish the following correspondence between Classcism and TN. It states that this reinterpretation of the quantifiers is a *faithful* interpretation of Classicism.

Theorem 6.4. $\mathsf{HE}\zeta$ *has a faithful interpretation in* TN *via the translation* j^* *that replaces each* $\forall_\sigma$ *with* $\forall_\sigma^{\approx}$*:*

$j^* : \mathcal{L}_{\mathcal{R}} \to \mathcal{L}_{\mathcal{R}}^{\mathrm{Nec}}$

For all closed $A \in \mathcal{L}_{\mathcal{R}}$*,* $\vdash_{\mathsf{HE}\zeta} A$ *iff* $\vdash_{\mathsf{TN}} j^*(A)$*.*

Proof. To establish the claim that $\mathsf{HE}\zeta$ is interpretable in TN via j^*, we prove a more general claim for open formulae A. If $\vdash_{\mathsf{HE}\zeta} A$, then $\vdash_{\mathsf{TN}} \bar{x} \approx \bar{x} \to j^*(A)$, where $\bar{x} = x_1, \dots, x_n$ are the variables free in A.

Let's begin with the following two rules corresponding to E and ζ respectively:

$\vdash A \leftrightarrow B$ only if $\vdash A \approx B$;

[78] This is just a version of Property Intensionalism we introduced in section 1. We have proved it in note 14.

$\vdash x \approx y \to Mx \approx Ny$ only if $\vdash M \approx N$.

Clearly, they are admissible in TN. Given these two rules, since TN is also closed under mp as well as Gen, our task is to show that $\bar{x} \approx \bar{x} \to j^*(A)$ is derivable in TN for each axiom A of HEζ.

The case of $\beta\eta$ can be dealt with because we have the previous mentioned rules and Proposition 6.2. Thus, the remaining non-trivial case is that A is an instance of UI, so $\bar{x} \approx \bar{x} \to j^*(A)$ amounts to $\bar{x} \approx \bar{x} \to \forall x(x \approx x \to j^*(F)x) \to j^*(F)j^*(a)$. To prove that this is derivable in TN, it suffices to show by induction that if $\bar{y} = y_1, \ldots, y_m$ enumerate the free variables in a term M, then $\bar{y} \approx \bar{z} \to M \approx M[\bar{z}/\bar{y}]$ is a theorem of TN where $\bar{z} = z_1, \ldots, z_m$.

When M is a variable, the proof is trivial. When M is a logical constant, it is also easy to check that $M \approx M$ is a theorem of TN. When M is the predicate Nec, Nec $\approx$ Nec holds because (i) $X \approx_{t\to t} Y$ amounts to $L\forall pq(L(p \leftrightarrow q) \to L(Xp \leftrightarrow Yq))$ and therefore implies $L\forall p(Xp \leftrightarrow Yp)$ and (ii) every operator necessarily coextensive with a necessity is itself a necessity. When M is N_1N_2, by I.H., we have $\bar{y} \approx \bar{z} \to N_1 \approx N_1[\bar{z}/\bar{y}] \wedge N_2 \approx N_2[\bar{z}/\bar{y}]$. Note that $N_1 \approx N_1[\bar{z}/\bar{y}]$ amounts to $L\forall yy'(y \approx y' \to N_1y \approx N_1[\bar{z}/\bar{y}]y')$, so $N_1 \approx N_1[\bar{z}/\bar{y}] \wedge N_2 \approx N_2[\bar{z}/\bar{y}]$ implies $N_1N_2 \approx (N_1N_2)[\bar{z}/\bar{y}]$. When M is $\lambda x.N$, by I.H., we have $\bar{y} \approx \bar{z} \to x \approx x' \to N \approx (N[\bar{z}/\bar{y}])[x'/x]$. Note that N is β-equivalent to $(\lambda x.N)x$ and $(N[\bar{z}/\bar{y}])[x'/x]$ is β-equivalent to $(\lambda x.N[\bar{z}/\bar{y}])x'$. Moreover, since we now have $\beta\eta$, both $\bar{y} \approx \bar{z} \to x \approx x' \to (\lambda x.N)x \approx N$ and $\bar{y} \approx \bar{z} \to x \approx x' \to (\lambda x.N[\bar{z}/\bar{y}])x' \approx (N[\bar{z}/\bar{y}])[x'/x]$ are derivable. So we can get $\bar{y} \approx \bar{z} \to x \approx x' \to (\lambda x.N)x \approx (\lambda x.N[\bar{z}/\bar{y}])x'$ and therefore $\bar{y} \approx \bar{z} \to \lambda x.N \approx (\lambda x.N)[\bar{z}/\bar{y}]$. (Note that we use the necessity of identity and the 4 axiom for L repeatedly. In model theoretic terms, this result is related to the 'basic lemma' for Kripke logical relations (see Mitchell [22]).)

Conversely, given Theorem 6.1, to show that $\vdash_{\mathsf{TN}} j^*(A)$ only if $\vdash_{\mathsf{HE}\zeta} A$, it suffices to show that for each $M \in \mathcal{L}_{\mathcal{R}}$, $\vdash_{\mathsf{HE}\zeta} M = j(j^*(M))$, where j replaces Nec with Nec$'$. Consider the unique non-trivial case in which M is $\forall_\sigma$. Since $j(j^*(\forall_\sigma))$ is $\lambda X.\forall_\sigma x(j(x \approx_\sigma x) \to Xx)$, let's directly prove that $\vdash_{\mathsf{HE}\zeta} j(N \approx N) = \top$ for all N, by showing that $\vdash_{\mathsf{HE}\zeta} j(N \approx N') \leftrightarrow j(N) = j(N')$.

By induction on the type of N. When N is of type e or type t, it's easy to see that the result holds. When N is of a non-basic relational type $\sigma \to \tau$, $N \approx_{\sigma\to\tau} N'$ amounts to $L\forall_\sigma xx'(x \approx_\sigma x' \to Nx \approx_\tau N'x')$, so $j(N \approx_{\sigma\to\tau} N')$ amounts to $\Box_\top\forall_\sigma xx'(j(x \approx_\sigma x') \to j(Nx \approx_\tau N'x'))$, which is in fact equivalent to $\Box_\top\forall_\sigma xx'(j(x) = j(x') \to j(Nx) = j(N'x'))$ given I.H. Then, it turns out that $j(N \approx_{\sigma\to\tau} N') \leftrightarrow j(N) = j(N')$ is equivalent to the principle Modalized Functionality: $\forall XY(\Box_\top\forall x(Xx = Yx) \to X = Y)$, which is a theorem of HEζ.[79] $\Box$

We promised in previous sections to show that the modal logic of L is cannot be proven to be stronger than S4 in TN, and the modal logic of L_{S5} cannot be proven to be stronger than S5 in TN. Given the interpretability theorem 6.1 established in this section, we can fulfill our promise.

Corollary 6.5. *For all $A \in \mathcal{L}^{\Box}_{\mathrm{P}}$, if $\nvdash_{\mathsf{S4}} A$, then $\nvdash_{\mathsf{TN}} A[L/\Box]$.*

Proof. Suppose that there is some $A \in \mathcal{L}^{\Box}_{\mathrm{P}}$ such that $\nvdash_{\mathsf{S4}} A$. Since A is not derivable in S4, it must be false in some Kripke model $\mathfrak{M}$ with a reflexive and transitive accessibility

[79]We omit the proof for Modalized Functionality because it is very similar to the proof for Property Intensionalism we've given in note 14. In the current setting of relational type theory, HEζ can even be equivalently axiomatized by adding Modalized Functionality to HE.

relation. But given Corollary A.6 in Bacon [1], $\mathfrak{M}$ can always be used to generate a model $\mathcal{M}_{\mathfrak{M}}$ for $\mathsf{HE}\zeta$ falsifying $A[\Box_\top/\Box]$, which means that $A[\Box_\top/\Box]$ cannot be derived in $\mathsf{HE}\zeta$. So by Theorem 6.1, it follows that $(A[\Box_\top/\Box])[L/\Box_\top]$, namely $A[L/\Box]$, is not derivable in TN. $\Box$

To get the result for L_{S5}, the first step is to observe that since TN is interpretable in $\mathsf{HE}\zeta$, it is interpretable in any theory stronger than $\mathsf{HE}\zeta$. In particular, let $\mathsf{HE}\zeta^+ = \mathsf{HE}\zeta \oplus \neg\Box_\top\neg p \to \Box_\top\neg\Box_\top\neg p$. Clearly, TN can be interpreted in $\mathsf{HE}\zeta^+$ via the same translation function j. The next step and also the most crucial step is to show that for every $A \in \mathcal{L}$, $j(L_{\mathrm{S5}})A \leftrightarrow \Box_\top A$ is provable in $\mathsf{HE}\zeta^+$. This is warranted by the fact that $\Box_\top$ is an $\mathsf{S5}$-necessity according to $\mathsf{HE}\zeta^+$. So by a proof similar to the one of Corollary 6.5, we can conclude that the modal logic of L_{S5} cannot be proved to be stronger than $\mathsf{S5}$ in TN.

6.2 Other theories of granularity

We have seen that given a background of Classicism one can offer completely reductive definitions of necessity and the broadest necessity, and moreover, do so in a way that is distinct from the Quinean interpretation of Nec and allows for contingency.

This possibly because, within this theory of granularity, there is only one logical truth, so that the condition of being Logical may be defined reductively. However we believe that non-Quinean reductive definitions of necessity should be possible in a wide range of more fine-grained theories.

Our discussion here will be far from comprehensive, however. We consider a theory T extending H_0 that contains all instances of following schema as theorems, where $\mathrm{Con}(M)$ denotes the set of non-logical constants in M, and $\mathrm{FV}(M)$ the set of free variables:

Excision $((A \wedge C) \vee A = (B \wedge C) \vee B) \to A = B$, provided $\mathrm{Con}(A) = \mathrm{Con}(B)$ and $\mathrm{FV}(A) = \mathrm{FV}(B)$.

And moreover, suppose T is closed under the following rule of proof:

Strong Equivalence If $\vdash A \leftrightarrow B$, then $\vdash A = B$, provided $\mathrm{Con}(A) = \mathrm{Con}(B)$ and $\mathrm{FV}(A) = \mathrm{FV}(B)$.

Classicism satisfies both of these conditions, however many more fine-grained theories do as well. For instance, consider views in which, roughly, propositions may be thought of as ordered pairs of logical contents (e.g. sets of possible worlds) and non-logical contents (e.g. the set of individuals that proposition is about). The theory of agglomerative algebras of Goodman [15] and the theory of Berto [5] have this form. We also suspect that Kit Fine's truthmaker semantics [12] could also fall under this general class of views. Excision effectively states that we can excise redundant non-logical contents: the only way for $(A \wedge C) \vee A$ and $(B \wedge C) \vee B)$ to be identical is if A and B share the same Boolean logical content (in a Boolean algebra this identity only holds when A and B are identical). Moreover, if A and B contain the same free variables and constants, they must have the same non-logical contents and thus be identical. We assume here that logical constants and λ do not contribute non-logical contents; they are not about any individuals for instance. So A and B are identical. The rule of Strong Equivalence can be motivated similarly: if A and B are provably equivalent in the theory, one ought to expect them to have the same logical contents, and if they involve the same non-logical constants and variables, they are alike in non-logical content as well.

We may interpret TN in any theory $T \supseteq \mathsf{H_0}$ satisfying these two properties in such a way that the operator $\Box^* := \lambda p.(p = (p = p))$ turns out to be a broadest necessity. Before we continue, let us note a remarkable property of this operator. Without assuming any logic beyond Leibniz's law and propositional logic, we may show an analogue of the 4 axiom:

Proposition 6.6. $\vdash_T (p = (p = p)) = ((p = (p = p)) = (p = (p = p)))$*, where* T *is a theory containing propositional logic and Leibniz's Law.*

Proof. Suppose that $p = (p = p)$. By Leibniz's law, we may replace all of the ps in this assumption with $p = p$s, getting $(p = p) = ((p = p) = (p = p))$. Again, using Leibniz's law, we may replace the second, fourth and sixth ps with $p = p$s to obtain $(p = (p = p)) = ((p = (p = p)) = (p = (p = p)))$. $\Box$

Notice that if we had $\beta\eta$ our proposition would be equivalent to the 4 axiom for $\Box^*$: $\Box^* p \rightarrow \Box^*\Box^* p$. However, even without $\beta\eta$ we can justify this move using the rule of Strong Equivalence, since by β_{E}, $\Box^* A$ is equivalent to $A = (A = A)$ for all A, and they involve the same free variables (and non-logical constants).

We may also show that $\Box^*$ satisfies other principles of S4.

Lemma 6.7. *According to* T*, the modal logic of* $\Box^*$ *is at least* S4*, where* T *is any extension of* $\mathsf{H_0} \oplus$ *Excision* $\oplus$ *Strong Equivalence.*

Proof. We just showed that $\Box^*$ satisfies the 4 axiom in such a theory. It satisfies the T axiom because the reflexivity of identity is provable, so whenever we have $p = (p = p)$ we can infer p. Moreover, the rule of necessitation is admissible: If A is derivable, so is $A \leftrightarrow (A = A)$. Then by Strong Equivalence, we have $A = (A = A)$.

Let's turn to the K axiom: Suppose $\Box^*(p \rightarrow q)$ and $\Box^* p$. So we have (i) $(p \rightarrow q) = ((p \rightarrow q) = (p \rightarrow q))$ and (ii) $p = (p = p)$. By applying the identity in (ii) and Leibniz's Law to (i), we obtain $((p = p) \rightarrow q) = ((p \rightarrow q) = (p \rightarrow q))$. $(p = p) \rightarrow q$ is provably equivalent to $(q \wedge p) \vee q$, and they involve the same propositional variables, so by Strong Equivalence they are identical. Similarly, $(p \rightarrow q) = (p \rightarrow q)$ is provably equivalent to $((q = q) \wedge p) \vee (q = q)$, and they involve the same propositional variables and are identical, so we may conclude that $((q \wedge p) \vee q) = (((q = q) \wedge p) \vee q = q)$. Finally, by Excision, we obtain $q = (q = q)$, which amounts to $\Box^* q$. $\Box$

We may now interpret Nec in any such theory T as follows:

$$\text{Logical}^* := \lambda X.\Box^*\forall p(\Box^* p \rightarrow \Box^* Xp);$$
$$\text{Closed}^* := \lambda X.\Box^*\forall pq(X(p \rightarrow q) \rightarrow Xp \rightarrow Xq);$$
$$\text{Nec}^* := \lambda X.\,\text{Logical}^*\, X \wedge \text{Closed}^*\, X.$$

Given the lemma above and Theorem 3.19, the following interpretability result is a routine corollary:

Theorem 6.8. TN *has a non-Quinean interpretation in* T *via the translation* h *that replaces* Nec *with* Nec**, where* T *is any extension of* $\mathsf{H_0} \oplus$ *Excision* $\oplus$ *Strong Equivalence:*

$h : \mathcal{L}^{\text{Nec}} \rightarrow \mathcal{L}$

For all $A \in \mathcal{L}^{\text{Nec}}$, $\vdash_T A$ *only if* $\vdash_{\mathsf{TN}} h(A)$.

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