

# Classicism

Andrew Bacon and Cian Dorr

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## 1 Classicism

Many of the most central questions in philosophy, and beyond, are naturally understood as questions of identity. For example, philosophers (and scientists) have asked such questions as the following:

Is knowing something the same as believing it truly and justifiedly?

Is being morally right the same as maximizing utility?

Is being hot the same as having a high gradient of entropy with respect to energy?

Once we start asking questions like these, we can formulate a range of further questions that seem initially much less gripping:

Is knowing something the same as knowing it and knowing it?

Is being morally right the same as being both morally right and either profitable or not profitable?

Is being hot the same as being not not hot?

Given the meaningfulness of the questions from the first list, it seems we have all these questions left dangling. What is clear is that they need to be approached in some systematic way, rather than one at a time. Unlike the questions on the first list, each of which raises distinctive issues proprietary to some subfield of philosophy or science, it seems reasonable to seek a general framework for theorizing about identity that settles the questions on the second list in one fell swoop.

The most straightforward such view is *Booleanism*, according to which—intuitively speaking—the propositions, properties and relations of any given type form a Boolean algebra under the operations of conjunction, disjunction and negation. This implies a positive answer to each of the questions on the second list.

Some might think that Booleanism is obviously false, because of putative counterexamples involving attitude reports. For example, Booleanism implies that to be rich is to be either rich and either happy or famous, or rich and not happy; but one might argue that this is false, on the grounds that someone without much logical acumen

could want to be rich without wanting to be either rich and either happy or famous, or rich and not happy. But any attempt to use judgments about propositional attitudes to argue against identities is fraught with difficulties, since everyone must somehow resist the argument from the tempting premise that one could want to visit Hesperus without wanting to visit Phosphorus to the false conclusion that Hesperus isn't Phosphorus. And once these kinds of objections are excluded, Booleanism has many attractions. It is a strong theory which settles a wide range of questions that seem in need of settling. It is also very simple (as we will see later when we consider some ways of axiomatizing it), and thus provides a good explanation of the many cases where substitution of Boolean equivalents is truth-preserving even under some non-truth-functional operator (such as a counterfactual conditional). While opponents of Booleanism have pointed to putative counterexamples, they have struggled to provide a comparably systematic and consistent theory which *predicts* the alleged counterexamples, as opposed to merely accommodating them. Booleanism thus sets the bar for what a simple and predictive theory addressing questions of higher-order identity should look like. This makes the task of investigating views compatible with Booleanism a particularly important component of the broader project of mapping out the space of views concerning the grain of reality (see Fritz 2017: and the introduction to this volume).

In this paper, we will contribute to this project by formulating, defending, and exploring an extremely natural strengthening of Booleanism which we call *Classicism*.<sup>1</sup> Classicism goes beyond Booleanism by adding identities involving identity and the quantifiers that are analogous to Booleanism's characteristic principles concerning conjunction, disjunction and negation. Part 1 of this paper will bolster Classicism's claim to naturalness by presenting several different axiomatizations of the theory. Part 2 maps out two directions in which Classicism can be further strengthened. One direction, which we might think of as the direction of coarseness, has as its endpoint the "Extensionalist" thesis that coextensiveness suffices for identity. Although Extensionalism itself seems to us to face decisive counterexamples, there are several interesting principles entailed by Extensionalism which look like attractive additions to Classicism. The other, less familiar, direction is the direction of fineness, in which identities whose truth value is left open by Classicism are settled negatively. The most extreme version of this idea, which we call *Maximalist Classicism*, adds all of the distinctness claims (in the language of quantifiers and truth-functional connectives) that are compatible with Classicism. This view strikes us as attractively strong and non-arbitrary, in a domain where the avoidance

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<sup>1</sup>We choose the name 'Classicism' for the theory since it stands to classical (higher-order) logic as Booleanism stands to Boolean propositional logic. It's the same system as HE+Modalized Functionality in Bacon 2018a. It is also related to the *intuitionistic* system of higher-order modal logic from Awodey, Kishida, and Kotzsch 2014.

of arbitrariness seems particularly urgent. Part 3 develops a model theory which is sound and complete for Classicism, heuristically helpful, and can be used to establish the consistency of many of the theoretical packages we discuss (including Maximalist Classicism).

Even committed opponents of Booleanism will have something to learn from this investigation. Many such theorists will be either be able to define, or be willing to take as primitive, a connective expressing some notion of “logical equivalence”—some relation less demanding than identity but more demanding than coextensiveness, obeying the analogues of the Boolean identities. If so, they will be able to find an unintended interpretation of our formalism under which they will accept Booleanism, and will be able to raise the question of whether they should also accept Classicism, and the various further strengthenings we will consider, under the same unintended interpretation.<sup>2</sup> For example, in the theories of Goodman (2018) and Dorr (2016), the role of logical equivalence can be played by the relation of being two propositions whose disjunction is identical to their conjunction. Likewise, in the object-language theory suggested by certain versions of truthmaker semantics (Fine 2017), one can define a notion of “classical equivalence” which plays a similar role.<sup>3</sup> And of course, anyone who can make sense of metaphysical necessity has the option of reinterpreting all our uses of identity connectives in terms of necessary coextensiveness. Under this reinterpretation, some of the views we will consider may seem unfamiliar, or even wild. Any view fine-grained enough to distinguish some false identity proposition from some contradiction will correspond, under the reinterpretation, to a view on which certain metaphysically contingent propositions are possibly necessary. This is ruled out by the most widely accepted logic for metaphysical necessity. But the reasons for the orthodoxy of this logic seems to rest more on a dubiously literal-minded attitude to the possible worlds model theory than on any argument.<sup>4</sup> So we think that even under this interpretation, Classicism and the extensions we will discuss are worthy views with much to recommend them.

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<sup>2</sup>The unintended interpretation can be specified by defining, for each type, a notion of *hereditary logical equivalence*: intuitively, entities are hereditarily logically equivalent when they produce hereditarily logically equivalent results when applied to hereditarily logically equivalent arguments. We can then reinterpret ‘=’ as ‘hereditarily logically equivalent’, and reinterpret all quantifiers as restricted to entities that are hereditarily logically equivalent to themselves. (For a formally analogous definition of hereditary *coextensiveness*, see Gandy 1956 and Dorr 2016: n. 106.)

<sup>3</sup> $p$  is classically equivalent to  $q$  iff  $(p \wedge \top) \vee \perp = (q \wedge \top) \vee \perp$ , where  $\top$  is defined as  $\forall p(p \vee \neg p)$  and  $\perp$  is  $\neg \top$ . Thanks to Ethan Russo for showing that this works.

<sup>4</sup>For discussion of some more interesting arguments, see Williamson 1996, Bacon 2018a: §5.2–5.4, and Dorr, Hawthorne, and Yli-Vakkuri 2021: §4.2 and §8.3.

## 1.1 Higher-order logic

We will be theorizing in a higher-order language  $\mathcal{L}$  in which the syntactic role of any given expression, or “term”, is captured by assigning it a unique “type”. (We take both terms and types to be strings of symbols.) Our type system will be  $R$ , defined to be the smallest set that includes the letter ‘ $e$ ’ (the “type of individuals”) and ‘ $t$ ’ (the “type of propositions”), and is such that whenever  $\sigma$  and  $\tau$  are in it and  $\tau$  is distinct from  $e$ ,  $\lceil(\sigma \rightarrow \tau)\rceil$  (the type of operations that make type- $\tau$  things out of type- $\sigma$  things) is in it. We call types in  $R$  distinct from  $e$  *relational* types. Terms of type  $t$  are called *formulae*; when they don’t have any free variables, they are called *sentences*. Terms of any other type are called *predicates*. In writing types, we omit parentheses associating to the right, e.g. writing  $e \rightarrow e \rightarrow t$  for  $(e \rightarrow (e \rightarrow t))$ . Terms of type  $e$  are called *singular terms*.

We will also sometimes consider languages using the larger type system  $F$ , which is the smallest set containing ‘ $e$ ’ and ‘ $t$ ’ and containing  $(\sigma \rightarrow \tau)$  whenever it contains  $\sigma$  and  $\tau$ , even when  $\tau$  is  $e$ :  $F$  thus contains types like  $e \rightarrow e$  and  $(e \rightarrow e) \rightarrow t$  which are not in  $R$ .

*Terms* can be simple or complex. *Simple* terms come in two varieties, namely variables and constants. Each variable and constant has a fixed type, and there are infinitely many variables with each type (which we indicate with a superscript when it is not clear from the context). *Complex* terms can be formed in two ways. First: when  $A$  is a term of type  $\sigma \rightarrow \tau$ , and  $B$  is a term of type  $\sigma$ ,  $(AB)$  is a term of type  $\tau$ . Second: when  $v$  is a variable of type  $\sigma$ , and  $A$  is a term of type  $\tau$ ,  $(\lambda v.A)$  is a term of type  $\sigma \rightarrow \tau$ . In writing terms, parentheses can be omitted associating to the left, and the parentheses around lambda terms include as much as possible; thus  $\lambda x.ABC$  abbreviates  $(\lambda x.((AB)C))$ .

The languages we are interested in will all include some logical constants, including truth-functional operators and quantifiers. The question which to treat as primitive and which as defined is relatively unimportant for our purposes: there is a version of Classicism for each sufficiently rich choice of primitives.<sup>5</sup> But for concreteness, we will choose the following logical constants:

- Truth functional connectives  $\wedge, \vee$  of type  $t \rightarrow t \rightarrow t$ , and  $\neg$  of type  $t \rightarrow t$ .
- For each type  $\sigma$ , quantifiers  $\forall_\sigma$  and  $\exists_\sigma$  of type  $(\sigma \rightarrow t) \rightarrow t$ .

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<sup>5</sup>This is not to say that there might not be any dialectical significance to a given choice of primitives. Someone might, for instance, accept the version of Classicism stated in terms of  $\wedge$  and  $\neg$ , but reject the version of Classicism stated in terms of  $\neg$  and  $\vee$ , on account of a having a non-Boolean theory of disjunction. By contrast Classicism, when formulated with a submaximal basis of logical constants, should be understood as taking the other connectives to be defined out of the primitive logical operations in the usual manner.

$\neg_t := \neg$	$\neg_{\sigma \rightarrow \tau} := \lambda X^{\sigma \rightarrow \tau} . \lambda z^\sigma . \neg_\tau X z$
$\wedge_t := \wedge$	$\wedge_{\sigma \rightarrow \tau} := \lambda X^{\sigma \rightarrow \tau} Y^{\sigma \rightarrow \tau} z^\sigma . X z \wedge_\tau Y z$
$\vee_t := \vee$	$\vee_{\sigma \rightarrow \tau} := \lambda X^{\sigma \rightarrow \tau} Y^{\sigma \rightarrow \tau} z^\sigma . X z \vee_\tau Y z$
$\rightarrow := \lambda p q . \neg p \vee q$	$\leftrightarrow := \lambda p q . (\neg p \vee q) \wedge (\neg q \vee p)$
$\top := \forall p p \vee \neg \forall p p$	$\perp := \forall p p \wedge \neg \forall p p$
$\square := \lambda p . p =_t p \vee \neg p$	$\leq_\tau := \lambda X^\tau Y^\tau . Y =_\tau X \vee_\tau Y$

Figure 1. Metalinguistic abbreviations

- For each type  $\sigma$ , an identity predicate  $=_\sigma$  of type  $\sigma \rightarrow \sigma \rightarrow t$ .

We write  $A \wedge B$  instead of  $((\wedge A)B)$ , and similarly for other terms of types  $\sigma \rightarrow \sigma \rightarrow \tau$ . When  $P$  is a formula and  $v$  is a variable of type  $\sigma$ ,  $\forall v P$  abbreviates  $(\forall_\sigma (\lambda v . A))$ . Other abbreviations are listed in Figure 1: we put type subscripts on logical constants to lift them to properties and relations (e.g.  $\neg_{e \rightarrow t}$ Wise for ‘not wise’).  $\rightarrow$  and  $\leftrightarrow$  are the material biconditional and biconditional; the significance of  $\square$  and  $\leq$  will be discussed later.

We use  $\mathcal{L}$  for the pure language with only the above logical constants, and  $\mathcal{L}(\Sigma)$  for the language that adds non-logical constants from a typed collection  $\Sigma$ .

When providing English glosses on sentences in this formal language, we will make free use of words like ‘individual’, ‘property’, ‘relation’, and ‘proposition’. For example we will gloss  $\forall Z (Zx \rightarrow Zy)$  as ‘y has every property that x has’. This practice should not be taken as providing our official translation manual from the higher-order language into English. Rather, like Prior (1971) and Williamson (2003), our attitude is that the higher-order language can be made intelligible in a way that doesn’t rely on that particular translation into English, and is perhaps independent of any translation into English.

A *theory* is just a set of formulae of some  $\mathcal{L}(\Sigma)$ . For ease of axiomatization, we work with theories whose members include open formulae as well as sentences. But it is only *sentences* that can be said in a non-artificial sense to be true or false; and we can call a theory true just in case every sentence in it is true.

All the theories we will be considering are extensions of a basic higher-order classical logic that we call H. An axiomatization of H is given in Figure 2. The axiomatization consists of principles governing the truth-functional connectives (propositional logic), principles governing the quantifiers and identity at each type (obtained by generalizing standard axiomatizations of first-order logic), and principles governing the behaviour of  $\lambda$ . (In the latter,  $\Phi[A]$  stands for any formula containing

an occurrence of a term  $A$ , possibly with free variables bound by  $\Phi$ , and  $\Phi[B]$  is the result of replacing this occurrence with the term  $B$ .) Note that in the statement of the rules and axiom-schemes, the symbol ‘ $\vdash$ ’ just means ‘is a member of the theory in question’, so that Figure 2 is a list of ten properties of theories.<sup>6</sup> Any theory having these ten properties we call a **H-theory**. All the theories we will be considering later on will be H-theories, which means that they not only contain H but are also closed under the inference rules MP, Gen, and Inst.<sup>7</sup>

Against the background of H, our rather large set of logical constants could be shrunk in various ways. The theorems of H that don’t contain  $\exists$  can be axiomatized by just closing all the  $\exists$ -free instances of the axioms (none of which are instances of EG) under MP and Gen; similarly for  $\forall$ , closing under MP and Inst. We can drop  $=$  and all the axioms involving it without affecting the set of  $=$ -free theorems. We can drop  $\wedge$  so long as we replace the instances of  $\beta$  and  $\eta$  with versions involving a variant of  $\leftrightarrow$  defined just in terms of  $\neg$  and  $\vee$ . And we can drop  $\vee$  so long as we do something similar, and close now not under MP (which is trivial) but under *conjunctive syllogism*: if  $\vdash \neg(P \wedge Q)$  and  $\vdash P$  then  $\vdash \neg Q$ .

One noteworthy theorem of H is

**Existence**  $\exists x(x = x)$

This is a schema, since  $x$  may be a variable of any type. Informally: Existence says that there is something of every type. It follows (by PC and MP) from the Reflexivity instance  $y = y$ , the  $\beta$ -instance  $(\lambda x.x = x)y \leftrightarrow y = y$ , and the EG-instance  $(\lambda x.x = x)y \rightarrow \exists x(x = x)$ . The fact that H implies Existence is not much of an objection to its *truth*, since instances of Existence are not very controversial (only nihilists would deny them). Nevertheless, it is worth knowing that there is a natural, mild weakening  $H^-$  of H that avoids having all instances of Existence as a theorem by restricting when we are allowed to use open formulae in the derivation of a closed theorem.<sup>8</sup>

<sup>6</sup>It is worth noting a few other ways of axiomatizing H or any other H-theory. First of all, PC could be replaced with any of the well-known collection of axiom-schemas whose closure under MP yields PC. Second, EG and Inst could both be dropped in favour of the axiom schema  $\exists v P \leftrightarrow \neg \forall v \neg P$ , or alternatively UI and Gen could both be dropped in favour of  $\forall v P \leftrightarrow \neg \exists v \neg P$ . Third, we could divide the work of Gen (the ‘‘Hilbert-Ackermann Generalization Rule’’) between a simpler generalization rule (if  $\vdash P$ , then  $\vdash \forall v P$ ) and a new axiom scheme  $\vdash \forall v (P \rightarrow Q) \rightarrow (P \rightarrow \forall v Q)$ , where  $v$  is not free in  $P$ ; similarly for EG and Inst.

<sup>7</sup>H is not the only candidate for the label ‘classical higher-order logic’: one might also use that label for the weaker logic  $H_0$  which eliminates  $\eta$  and replaces  $\beta$  with the much weaker ‘‘Extensional  $\beta$ ’’ schema whose instances are just formulae of the form  $(\lambda \vec{v}.P)\vec{A} \leftrightarrow P[\vec{v} \mapsto \vec{A}]$ . For more discussion of  $H_0$  see Dorr, Hawthorne, and Yli-Vakkuri 2021: ch. 1 and Bacon and Zeng 2021; for a philosophical defence of the  $\beta$  axiom, see Dorr 2016: §5.

<sup>8</sup>We define  $H^-$  in terms of a family of theories  $H_{\vec{v}}^-$ , where  $V$  is a set of variables. These are

**PC:**  $\vdash P$  whenever  $P$  is a tautology (substitution instance of a theorem of classical propositional logic).

**UI:**  $\vdash \forall_{\sigma} F \rightarrow FA$  (where  $A$  is of some type  $\sigma$  and  $F$  is a term of type  $\sigma \rightarrow t$ ).

**EG:**  $\vdash FA \rightarrow \exists_{\sigma} F$  (where  $A$  is of some type  $\sigma$  and  $F$  is a term of type  $\sigma \rightarrow t$ ).

**Ref:**  $\vdash A =_{\sigma} A$

**LL:**  $\vdash (A = B) \rightarrow (FA \rightarrow FB)$

**$\beta$ :**  $\vdash \Phi[(\lambda v.A)B] \leftrightarrow \Phi[A[B/v]]$ , where  $A[B/v]$  is the result of replacing every free occurrence of  $v$  in  $A$  with  $B$  (so long as this can be done without any free variable in  $B$  becoming bound).

**$\eta$ :**  $\vdash \Phi[\lambda v.(Fv)] \leftrightarrow \Phi[F]$ , where  $v$  is not free in  $F$ .

**MP:** If  $\vdash P$  and  $\vdash P \rightarrow Q$ , then  $\vdash Q$ .

**Gen:** If  $\vdash P \rightarrow Q$ , and  $v$  does not occur free in  $P$ ,  $\vdash P \rightarrow \forall v Q$ .

**Inst:** If  $\vdash P \rightarrow Q$ , and  $v$  does not occur free in  $Q$ ,  $\vdash \exists v P \rightarrow Q$ .

Figure 2. Axiomatization of H



However, the only type for which this weaker logic  $H^-$  fails to prove Existence is  $e$ . In every  $R$ -type other than  $e$ , we can construct a closed term using only logical constants, and derive Existence using EG from the Ref-instance involving that term, so that the new limits on the use of free variables are not relevant. Moreover, if we add  $\exists x^e(x = x)$  to  $H^-$  and close under MP, we get back  $H$ .<sup>9</sup>

## 1.2 Booleanism

According to the ‘Booleanist’ worldview it is possible to substitute Boolean equivalents *salve veritate* (see, for instance, Bacon 2018a.) We will thus take **Booleanism** to be the result of adding the following schema to  $H$  (and closing under its rules):

**Tautological Substitution**  $\Phi[P] \rightarrow \Phi[Q]$ , where  $P$  and  $Q$  are equivalent formulae in propositional logic.<sup>10</sup>

As with the  $\beta$  and  $\eta$  axioms,  $\Phi[P]$  and  $\Phi[Q]$  are formulae that differ by the replacement of an occurrence of  $P$  with one of  $Q$ . Equivalently (following Dorr 2016: §7), we can define Booleanism to be the smallest  $H$ -theory containing all instances of the following schema:

**Tautological Equivalence**  $(\lambda \vec{v}.P) = (\lambda \vec{v}.Q)$ , whenever  $P$  and  $Q$  are equivalent in propositional logic.

(Here,  $\lambda \vec{v}$  is short for  $\lambda v_1. \dots \lambda v_n.$ , for some  $n$  variables  $v_1, \dots, v_n$  with  $n \geq 0$ .) Every instance of Tautological Equivalence can be derived from the instance  $(\lambda \vec{v}.P) = (\lambda \vec{v}.P) \rightarrow (\lambda \vec{v}.P) = (\lambda \vec{v}.Q)$  of Tautological Substitution together with the Ref-instance  $(\lambda \vec{v}.P) = (\lambda \vec{v}.P)$ . Conversely, any instance of Tautological Substitution

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defined inductively as follows:

- (i)  $H_V^-$  contains all instances of PC, UI, EG, Ref, LL,  $\beta$ , and  $\eta$  with free variables in  $V$ .
- (ii) Whenever  $H_V^-$  contains both  $P \rightarrow Q$  and  $P$ , it contains  $Q$ .
- (iii) Whenever  $H_V^-$  contains  $P \rightarrow Q$ ,  $H_{V-\{v\}}^-$  contains  $P \rightarrow (\forall v.Q)$  if  $v$  is not free in  $P$ , and contains  $(\exists v P) \rightarrow Q$  if  $v$  is not free in  $Q$ .

A formula  $P$  belongs to  $H^-$  just in case it belongs to  $H_V^-$ , where  $V$  is the set of variables free in  $P$ . The formula  $\exists x^e x = x$  is in  $H_V^-$  for every nonempty  $V$ , but it is not in  $H_\emptyset^-$  and hence not in  $H^-$ .

<sup>9</sup>In type system  $F$ , there are other types besides  $e$ —for example,  $t \rightarrow e$  and  $(e \rightarrow e) \rightarrow e$ —in which there are no closed terms without nonlogical constants. The  $F$ -version of  $H^-$  also fails to prove the instances of Existence for these types. However, adding the single instance  $\exists x^e(x = x)$  will also make those instances provable.

<sup>10</sup>This replacement may occur even in the scope of  $\lambda$ -terms and may involve variable capture: e.g., an instance is  $(\lambda p.p) = (\lambda p.p) \rightarrow (\lambda p.p) = (\lambda p.\neg\neg p)$ , which implies that double negation is the identity operation on type  $t$ .

<b>Commutativity-<math>\wedge</math></b>	$(\lambda pq.p \wedge q) = (\lambda pq.q \wedge p)$
<b>Commutativity-<math>\vee</math></b>	$(\lambda pq.p \vee q) = (\lambda pq.q \vee p)$
<b>Distribution-<math>\wedge\vee</math></b>	$(\lambda pqr.p \wedge (q \vee r)) = (\lambda pqr.(p \wedge q) \vee (p \wedge r))$
<b>Distribution-<math>\vee\wedge</math></b>	$(\lambda pqr.p \vee (q \wedge r)) = (\lambda pqr.(p \vee q) \wedge (p \vee r))$
<b>Dissolution-<math>\wedge\vee</math></b>	$(\lambda pq.p \wedge (q \vee \neg q)) = (\lambda pq.p)$
<b>Dissolution-<math>\vee\wedge</math></b>	$(\lambda pq.p \vee (q \wedge \neg q)) = (\lambda pq.p)$

Figure 3. The Boolean Identities

can be derived from Tautological Equivalence using  $\beta$  to extract the formulae to be substituted.<sup>11</sup>

Tautological Substitution and Tautological Equivalence are axiom-schemas with infinitely many instances. It is not a trivial matter to tell whether a given formula is an instance (though it is a *decidable* question, by the decidability theorem for classical propositional logic). But we can equally well characterize Booleanism as the smallest H-theory containing the six individual axioms listed in Figure 3, the “Boolean Identities”. Many other similar lists of axioms could be given, corresponding to different equivalent definitions of Boolean algebras in mathematics.

One noteworthy consequence of Booleanism is that conjunction and disjunction are “interdefinable”, in the sense that both of the following identities are true:

$$\begin{aligned} \wedge\text{-duality} & \quad (\wedge) = (\lambda pq.\neg((\neg p) \vee (\neg q))) \\ \vee\text{-duality} & \quad (\vee) = (\lambda pq.\neg((\neg p) \wedge (\neg q))) \end{aligned}$$

Given the truth of these identities, there is a good sense in which nothing would have been lost if we treated only one of  $\wedge$  and  $\vee$  as a constant, treating the other when convenient as a metalinguistic abbreviation.<sup>12</sup>

<sup>11</sup>Let  $\vec{v}$  be the variables free in either of  $P$  or  $Q$ . Then  $\vdash (\lambda X.\Phi[X\vec{v}]) (\lambda \vec{v}.P) \rightarrow (\lambda X.\Phi[X\vec{v}]) (\lambda \vec{v}.Q)$  by Tautological Equivalence, Ref, and LL. But the two sides of this conditional are  $\beta$ -equivalent respectively to  $\Phi[P]$  and  $\Phi[Q]$ , so  $\vdash \Phi[P] \rightarrow \Phi[Q]$ .

<sup>12</sup>For any sentence  $P$  we accept involving one of the logical constants,  $\wedge$ -duality and  $\vee$ -duality let us find a sentence  $P'$  not involving that constant such that Booleanism implies  $P = P'$ , and thus that the fact we express using  $P$  can also be expressed by  $P'$ . But  $\wedge$ -duality and  $\vee$ -duality are controversial. Their conjunction has some consequences that contain only one of  $\vee$  and  $\wedge$  and are not theorems of H, so while we are free if we please to treat one of the symbols as a metalinguistic abbreviation in such a way as to make one of them uncontroversial, the truth of the other one will then be non-obvious. In a setting where the truth of Booleanism is up for debate, it is thus helpful to work in a signature containing both connectives, even if one in fact accepts Booleanism and hence

Booleanism implies that every relational type  $\tau$  forms a Boolean algebra with respect to the lifted operations  $\neg_\tau$ ,  $\wedge_\tau$ , and  $\vee_\tau$ . There is a weaker version of “Booleanism” which only requires *propositions* to form a Boolean algebra under conjunction, disjunction, and negation, and has nothing to say about properties and relations. This theory—let’s call it **Propositional Booleanism**—is the smallest H-theory containing all instances of

**Propositional Tautological Equivalence**  $Q = Q'$ , whenever  $Q \leftrightarrow Q'$  is a tautology.

Propositional Booleanism can also be characterized as the smallest H-theory containing each of the following formulae:

$$\begin{array}{ll} p \wedge q = q \wedge p & p \vee q = q \vee p \\ p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r) & p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r) \\ p \wedge (q \vee \neg q) = p & p \vee (q \wedge \neg q) = p \end{array}$$

However, it is hard to imagine why anyone would want to endorse Propositional Booleanism but not Booleanism. All the reasons we are aware of for liking or for not liking the claim that propositions form a Boolean algebra seem to carry over with exactly the same strength to every predicate type.<sup>13</sup>

### 1.3 Classicism

Although the axioms of Booleanism are very natural, they also seem like a somewhat arbitrary fragment of a more general picture. They tell us a lot about the interaction of the truth-functional connectives with identity, but are silent about the interaction of the other logical constants—the quantifiers and identity—with identity. For example, while Booleanism implies the identity of any two instances of the law of excluded middle ( $p \vee \neg p = q \vee \neg q$ ), it does not imply the identity of any two instances of the law of identity (Ref:  $(x = x) = (y = y)$ ). Likewise, while Booleanism implies that conjunction is the dual of disjunction, it does not imply that universal quantification is (in the parallel sense) dual to existential quantification. But there are deep connections between the logic of truth functional operations and the logic of identity and quantification: it is hard to conceive of a motivation for a view that accepts the former identities but not the latter ones.

The natural extension of Booleanism to the remaining logical constants is not hard to identify. It’s just a matter of generalizing Tautological Substitution or Tau-

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$\wedge$ -duality and  $\vee$ -duality.

<sup>13</sup>Booleanism can be derived from Propositional Booleanism using the Functionality principle, discussed in Section 1.4 below.

tological Equivalence to give the classical logic of higher-order quantification and identity the same status that these schemas give classical propositional logic. And the obvious thing to mean by “the classical logic of higher-order quantification and identity” is the theory H introduced in the previous section. So we are led to the following generalizations of Tautological Substitution and Tautological Equivalence:

**Logical Substitution**  $\Phi[P] \rightarrow \Phi[Q]$ , whenever  $P$  and  $Q$  are equivalent in H.

**Logical Equivalence**  $(\lambda\vec{v}.P) = (\lambda\vec{v}.Q)$ , whenever  $P$  and  $Q$  are equivalent in H.

These are interderivable for the same reason as Tautological Substitution and Tautological Equivalence. We will dub the smallest H-theory containing all instances of these schemas *Classicism*, or C for short.<sup>14</sup>

(It will sometimes be useful to use the following alternative formulation of Logical Equivalence:

**Logical  $\zeta$ -Equivalence**  $F = G$ , whenever  $F\vec{v}$  and  $G\vec{v}$  are equivalent in H and none of  $\vec{v}$  is free in  $F$  or  $G$ .

Given the availability of  $\eta$ -conversion, this comes to the same thing as Logical Equivalence; it also makes the  $\eta$  axiom-scheme redundant.<sup>15</sup>)

Unlike the other schemas we have considered so far, Logical Substitution and Logical Equivalence are not decidable. But there are also natural decidable axiomatizations of Classicism. By contrast with Booleanism, there is no hope of characterizing Classicism with a *finite* collection of axioms, since the instances of Logical Equivalence that are not already theorems of H include all of our infinitely many

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<sup>14</sup>Cashing out “the classical logic of higher-order quantification and identity” as H might seem rather tendentious. After all, H goes beyond propositional logic not just by adding axioms and rules governing the quantifiers and identity, but by adding the axiom-schemes  $\beta$  and  $\eta$ , which are not specifically about any logical constants. But as it turns out, it doesn’t matter. We are about to consider an alternative axiomatization of Classicism which makes do with a small selection of instances of Logical Equivalence, and the biconditionals  $P \leftrightarrow Q$  that generate these instances are good candidates to belong to any fragment of H that might be considered a better candidate for the label ‘the classical logic of higher-order quantification and identity’. For example they are in the theory  $H_0$  discussed in footnote 7. They also belong to the “existentially neutral” logic  $H^-$  discussed in footnote 8. Thus the smallest H-theory containing the weakening of Logical Equivalence that requires  $H^- \vdash P \rightarrow Q$  is the same as the smallest H-theory containing Logical Equivalence. The smallest  $H^-$ -theory containing that weaker schema is just barely weaker: if we add  $\exists x^e(x = x)$  and close under MP, we get Classicism back again.

<sup>15</sup>By  $\beta$ ,  $(\lambda X^{\sigma \rightarrow \tau}.X)Y^{\sigma \rightarrow \tau}z^\sigma$  and  $(\lambda Xv.Xv)Yz$  are equivalent in H. Thus  $(\lambda X.X) = (\lambda Xv.Xv)$  is an instance of  $\zeta$ -equivalence. But when  $v$  is not free in  $F$ ,  $\Phi[(\lambda v.Fv)]$  is  $\beta$ -equivalent to  $(\lambda Z.\Phi[Z(F)])(\lambda X.X)$ , which by LL and the above identity is equivalent to  $(\lambda Z.\Phi[Z(F)])(\lambda Xv.Xv)$ , which is  $\beta$ -equivalent to  $\Phi[\lambda v.Fv]$ , thus recovering the relevant instance of  $\eta$ .

<b>The Identity Identity</b>	$(\lambda yz. y =_{\sigma} z) = (\lambda yz. \forall X. Xy \leftrightarrow Xz)$
<b>Absorption-<math>\forall\forall</math></b>	$(\lambda Xy. Xy \vee \forall_{\sigma} X) = (\lambda Xy. Xy)$
<b>Distribution-<math>\forall\forall</math></b>	$(\lambda Xp. p \vee \forall_{\sigma} X) = (\lambda Xp. \forall y. p \vee Xy)$
<b>Absorption-<math>\wedge\exists</math></b>	$(\lambda Xy. Xy \wedge \exists_{\sigma} X) = (\lambda Xy. Xy)$
<b>Distribution-<math>\wedge\exists</math></b>	$(\lambda Xp. p \wedge \exists_{\sigma} X) = (\lambda Xp. \exists y. p \wedge Xy)$

Figure 4. The Classicist Identities

logical constants (the quantifiers  $\forall_{\sigma}$  and  $\exists_{\sigma}$  and identity predicate  $=_{\sigma}$  for each type  $\sigma$ ). But we can do the next best thing, namely have a small finite list of axioms for each logical constant. One particularly simple axiomatization of this sort is given in Figure 4. It comprises one closed identity for every identity predicate  $=_{\sigma}$ , and two closed identities for every quantifier  $\forall_{\sigma}$  or  $\exists_{\sigma}$ . All of these identities are easily seen to be instances of Logical Equivalence. For example, the biconditional  $Xy \leftrightarrow Xy \vee \forall_{\sigma} X$ , needed to prove Absorption- $\forall\forall$  from Logical Equivalence, is a theorem of H because it is tautologically equivalent to the UI-instance  $\forall_{\sigma} X \rightarrow Xy$ . Appendix A proves that the Quantifier Identities and Identity Identity are sufficient to recover the remaining instances of Logical Equivalence. The proof works by using the identities to show that each axiom of H is identical to  $\top$  (using the Boolean Identities for PC, the Absorption identities for UI and EG, and the Identity Identity for Ref and LL), and then showing that the rules of proof preserve identity to  $\top$  (using the Boolean Identities for MP and the Distribution identities for Gen and Inst). In the same sense in which Booleanism implies that  $\wedge$  and  $\vee$  are “interdefinable”, Classicism implies that the same is true for  $\forall$  and  $\exists$ . That is, it proves the following two identities:

$$\forall\text{-duality} \quad \forall_{\sigma} = \lambda X. \neg(\exists_{\sigma}(\neg_{\sigma \rightarrow t} X))$$

$$\exists\text{-duality} \quad \exists_{\sigma} = \lambda X. \neg(\forall_{\sigma}(\neg_{\sigma \rightarrow t} X))$$

If Classicism is true we would thus lose no expressive power in dropping one of the connectives from our official signature. However this might be unhelpful for the purposes of debating opponents of Classicism, some of whom might reject one or both of  $\forall$ -def and  $\exists$ -def. (For example one could imagine someone who accepts the Classicist Identities for  $\forall$ , but has some strange alternative take on  $\exists$ .) Similarly, the Identity Identity is already of the right form to license eliminating the primitive identity predicates in favour of the quantifiers. Although this particular identification might well be accepted even by philosophers who reject the other Classicist

or Boolean Identities, there are (as we will mention in Section 1.5) some important objections to Classicism which might motivate rejecting the Identity Identity. So again, for dialectical purposes, it will be helpful to keep the identity predicates as logical constants.

Here we have focused on identity, however related axiomatizations of Classicism in terms of *entailment* are also possible, and helpful in illuminating the distinctive logical roles of the quantifiers. These are explored in Appendix B.

#### 1.4 Axiomatizations with new rules

One might wonder whether Classicism is itself just a somewhat arbitrary fragment of a more general picture, as we earlier claimed to be the case for Booleanism. Why only accept identities corresponding to biconditionals provable from H, when we now have a stronger theory, C, which proves further biconditionals, for which we might also accept the corresponding identities? This motivates a, putatively stronger, theory which generalizes Logical Equivalence by including the identities corresponding to biconditionals provable in C; indeed a series of theories, each adding the identities corresponding to biconditionals provable in its predecessor. The union of all these theories will be closed under the *rule of equivalence*:

**Equivalence** If  $\vdash P \leftrightarrow Q$  then  $\vdash (\lambda\vec{v}.P) = (\lambda\vec{v}.Q)$ .

or in an alternate form,

**$\zeta$ -Equivalence** If  $\vdash F\vec{v} \leftrightarrow G\vec{v}$  then  $\vdash F = G$ , where none of  $\vec{v}$  is free in  $F$  or  $G$ .

But the picture of a series of stronger and stronger theories is completely wrong, since as it turns out, C is *already* closed under the rule of equivalence. Even though the only identities we added as axioms corresponded to biconditionals provable in H, the theorems provable from these axioms also include all identities corresponding to biconditionals provable in C. Indeed, since any H-theory closed under Equivalence must evidently contain every instance of Logical Equivalence, C can be characterized as the smallest H-theory closed under Equivalence (or  $\zeta$ -Equivalence). The availability of this axiomatization further bolsters our case for the centrality and naturalness of C as the endpoint of the theoretical impulse that initially inspires Booleanism.<sup>16</sup>

We can also divide the job of the rule of equivalence up between two inference rules, namely

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<sup>16</sup>If we start with any theory that satisfies all the closure conditions other than  $\eta$  from the axiomatization of H given in Figure 2, closing under  $\zeta$ -Equivalence is equivalent to adding  $\eta$  and closing under Equivalence. One nice feature of  $\zeta$ -Equivalence is that if we start with the very weak logic  $H_0$  mentioned in footnote 7 and close under  $\zeta$ -Equivalence, we still get Classicism (including the full-strength  $\beta$  axiom and the  $\eta$  axiom): see Bacon and Zeng 2021.

**Propositional Equivalence** If  $\vdash P \leftrightarrow Q$  then  $\vdash P = Q$

together with either of the following:

$\xi$  If  $\vdash A = B$  then  $\vdash (\lambda v.A) = (\lambda v.B)$ .

$\zeta$  If  $\vdash Fv = Gv$  then  $\vdash F = G$ , where  $v$  is not free in  $F$  or  $G$ .

Any H-theory closed under Propositional Equivalence and  $\xi$  or  $\zeta$  must evidently be closed under Equivalence, so the smallest H-theory closed under Propositional Equivalence and  $\xi/\zeta$  includes C. C is closed under the rule of propositional equivalence, which is just the  $n = 0$  special case of the rule of equivalence, and can also be shown to be closed under  $\xi$  and  $\zeta$ .<sup>17</sup>

One can think of Equivalence as a “rule” counterpart of the following, much stronger, axiom-scheme, telling us that relations are individuated by their extensions:

**Extensionality**  $\forall \vec{z}(X\vec{z} \leftrightarrow Y\vec{z}) \rightarrow X = Y$

Similarly, Propositional Equivalence and  $\zeta$  can be thought of, respectively, as “rule” counterparts of the following axiom-schemes:

**The Fregean Axiom**  $(p \leftrightarrow q) \rightarrow p = q$

**Functionality**  $\forall z(Xz = Yz) \rightarrow X = Y$

Neither Extensionality, Functionality, nor the Fregean Axiom is a theorem of Classicism (as we will confirm in part 3), so the axioms really are strengthenings of the corresponding rules.

Let *Extensionalism* be the smallest H-theory containing Extensionality; or equivalently, the smallest H-theory containing the Fregean Axiom and Functionality; or

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<sup>17</sup>To see that C is closed under  $\xi$ , remember that for  $(\lambda v.A) = (\lambda v.B)$  to even be well-formed in our type system  $R$ ,  $A$  and  $B$  must be terms of some type of the form  $\sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow t$  (for some  $n > 0$ ). Let the type of  $v$  be  $\sigma_0$ , and choose distinct variables  $\vec{u}$  of types  $\sigma_1 \dots \sigma_n$  that are not free in  $A$  or  $B$ . Then if  $\vdash A = B$ , we have  $\vdash A\vec{u} \leftrightarrow B\vec{u}$  by Ref and LL, which implies  $(\lambda v\vec{u}.A\vec{u}) = (\lambda v\vec{u}.B\vec{u})$  by Equiv, which implies  $(\lambda v.A) = (\lambda v.B)$  by  $\eta$ . (Note that this proof depends crucially on the fact that we our type system is  $R$ . In the more general type system  $F$ , we conjecture it is still true that the smallest H-theory containing Logical Equivalence is closed under  $\zeta$ , but the argument that this is the case is more involved.)

It is worth noting that H is already closed under  $\zeta$ . This follows from the following fact: whenever  $A = B$  is a theorem of H,  $A$  and  $B$  are  $\beta\eta$ -equivalent terms; so in particular, if  $Fv = Gv$  is a theorem and  $v$  is not free in  $F$  or  $G$ ,  $Fv$  and  $Gv$  are  $\beta\eta$ -equivalent, which implies that  $F$  and  $G$  are, so that  $F = G$  is also a theorem of H. This fact can be proved using model-theoretic techniques developed in Fritz, Lederman, and Uzquiano 2021: see the remark about ‘DISTINCTNESS $_{\sim\alpha,\beta,\eta}$ ’ on p. 15 of that paper.

equivalently again, the smallest extension of  $C$  containing the Fregean Axiom.<sup>18</sup> Extensionalism occupies an important position on the map of H-theories: it, and theories that include it, are in a natural sense *maximally coarse-grained*. But—with due deference to the distinguished historical roster of Extensionalists, starting with Frege (1879)—we take Extensionalism to be decisively refuted by arguments such as the following. Although  $\text{snow is white} \leftrightarrow \text{snow is either white or not white}$ ,  $\text{snow is white} \neq \text{snow is either white or not white}$ , since it is necessary that snow is either white or not white, but not necessary that snow is white. (There are many different senses of ‘necessary’ for which this argument is sound.)<sup>19</sup>

Classicism can thus be approached not only “from below”, by starting with H or Booleanism and considering natural strengthenings, but “from above”, by starting with Extensionalism and considering natural weakenings, specifically those that replace axioms with corresponding rules.

## 1.5 Classicism and modal logic

Recall that  $\Box$  abbreviates  $\lambda p.(p = (p \vee \neg p))$ : observe that given Booleanism, any tautology could be substituted for the formula  $p \vee \neg p$ .<sup>20</sup> The choice to use a necessity symbol for this operator is appropriate here. Booleanism already includes all instances of the following schemas:

$$\begin{array}{ll} \mathbf{K} & \Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q) \\ \mathbf{T} & \Box P \rightarrow P \end{array}$$

Classicism, unlike Booleanism, is also closed under the rule

$$\mathbf{Necessitation} \quad \text{If } \vdash P \text{ then } \vdash \Box P$$

<sup>18</sup>The implication from the Fregean Axiom to Functionality in  $C$  follows from the fact that  $C$  includes Modalized Functionality (see Section 1.5). This depends on the fact that we are working in the type system  $R$ ; in type system  $F$ , the smallest extension of H containing Extensionality is still the same as the smallest extension of  $C$  containing the Fregean Axiom, but does not include the instances of Functionality for types ending in  $e$ .

<sup>19</sup>Church’s ‘simple theory of types’ (Church 1940) contains Functionality (his axiom 10); he considers the Fregean Axiom but elects not to add it, although it features in the more complicated, Frege-inspired system in Church 1951. Henkin (1950) does add the Fregean Axiom, and it is standard in the systems used for higher-order formalization of mathematics (Benzmüller and Miller 2014). Gandy (1956) reports Turing as expressing suspicion of Extensionalism, in terms that have thankfully become obsolete.

<sup>20</sup>Church (1951) discusses essentially this definition of  $\Box$  in the context of his ‘Alternative 2’; see also Cresswell 1965.



(This follows from Classicism's being closed under Propositional Equivalence, since if  $\vdash P$ ,  $\vdash P \leftrightarrow (P \vee \neg P)$  by propositional logic.) Thus, in the setting of Classicism, the propositional logic of  $\Box$  is a normal modal logic. Classicism also goes beyond Booleanism by including all instances of the schema:

$$4 \quad \Box P \rightarrow \Box \Box P$$

Indeed, Bacon (2018a) shows that the set of propositional modal formulas in  $\Box$  derivable from Classicism is exactly S4: the smallest set of formulae containing all tautologies and instances of K, T, and 4, and closed under MP and Nec.<sup>21</sup>

Classicism includes the following modal weakening of Extensionality, which can perform many of the same argumentative roles as that principle:

$$\text{Intensionality} \quad \Box \forall \vec{z} (X \vec{z} \leftrightarrow Y \vec{z}) \rightarrow X = Y$$

To see that this is a theorem of Classicism, consider the following instance of Logical Equivalence:

$$\lambda \vec{z}. (X \vec{z} \wedge \forall \vec{z}. (X \vec{z} \leftrightarrow Y \vec{z})) = \lambda \vec{z}. (Y \vec{z} \wedge \forall \vec{z}. (X \vec{z} \leftrightarrow Y \vec{z}))$$

This implies

$$(\forall \vec{z}. (X \vec{z} \leftrightarrow Y \vec{z})) = \top \rightarrow (\lambda \vec{z}. X \vec{z} \wedge \top) = (\lambda \vec{z}. Y \vec{z} \wedge \top)$$

which in turn implies Intensionality, by Booleanism and  $\eta$ -conversion. Conversely, the combination of Intensionality with Propositional Equivalence or Necessitation implies every instance of Logical Equivalence, and thus serves as another possible axiomatization of Classicism.

Just as Extensionality is equivalent to the conjunction of Functionality and the Fregean Axiom, Intensionality is equivalent (given that  $\Box$  behaves as a normal

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<sup>21</sup>We may interpret the language of propositional modal logic with propositional constants  $\Sigma$  in  $\mathcal{L}(\Sigma)$  in the straightforward way. (Interpreting the letters as themselves,  $\wedge$  for  $\wedge$ , etc., and most importantly, interpreting the  $\Box$  of modal logic with the defined operator  $\lambda p.p = \top$ .) One of the results in Bacon 2018a implies that any transitive reflexive Kripke model of the modal language can be extended to a higher-order model of Classicism, that makes exactly the same sentences true (modulo the translation). Because S4 is complete for transitive reflexive Kripke models, it follows that Classicism cannot prove any propositional modal formulas not already proven from S4. The soundness of S4 under this interpretation in Classicism is also spelled out there. Cresswell (1965) already shows how once  $\Box$  is defined in terms of identity, the principles of S4 can be derived from some minimal principles about propositional identity. See also Suszko 1975 and Wiredu 1979.

modal operator) to the conjunction of the following two axioms:

$$\begin{array}{ll} \text{Modalized Fregean Axiom} & \Box(p \leftrightarrow q) \rightarrow p = q \\ \text{Modalized Functionality} & \Box\forall z(Xz = Yz) \rightarrow X = Y^{22} \end{array}$$

Classicism can be axiomatized by the combination of Modalized Functionality and Propositional Equivalence.<sup>23</sup>

Each of these three principles partially articulates the idea that propositions, properties and relations are “individuated by necessary equivalence”, in the present sense of ‘necessary’. The Modalized Fregean Axiom (which is already a theorem of Booleanism) states the identity of necessarily equivalent propositions; Intensionality, of necessarily coextensive relations; and Modalized Functionality, of necessarily co-functional operations.

A widely discussed version of this thesis about individuation, associated with philosophers like Lewis (1986: §1.5) and Stalnaker (1984: ch. 1), holds that *metaphysically* necessary coextensiveness suffices for identity. Note, however, it is not at all clear that metaphysical necessity should be identified with  $\Box$ . Classicists who take the two statuses to be distinct need not accept the Lewis-Stalnaker view, although proponents of that view will themselves accept the identity of metaphysical necessity and  $\Box$  (since they accept their metaphysically necessary coextensiveness). Moreover, while Lewis and Stalnaker take metaphysical necessity to obey the strong modal logic S5, it is consistent with Classicism that many theorems of S5 fail for  $\Box$ . So there are a range of views compatible with Classicism which diverge significantly from the Lewis-Stalnaker picture, and will call for a set of modelling tools substantially different from the most familiar versions of the possible worlds framework.

Indeed, versions of the thought that necessary equivalence suffices for identity

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<sup>22</sup>To derive Modalized Functionality from Intensionality, remember that for an instance of Modalized Functionality to be well formed, the variables  $X$  and  $Y$  must both be of some type  $\sigma_0 \rightarrow \dots \rightarrow \sigma_n \rightarrow t$ . If  $\Box\forall z(Xz = Yz)$  we have  $\Box\forall z\bar{u}(Xz\bar{u} \leftrightarrow Yz\bar{u})$ , which implies  $(\lambda z\bar{u}.Xz\bar{u}) = (\lambda z\bar{u}.Yz\bar{u})$  by Intensionality; this  $\eta$ -reduces to  $X = Y$ .

<sup>23</sup>This axiomatization is in Bacon 2018a. Myhill (1958) reconstructs ‘Alternative 2’ from Church 1951 using Necessitation and Functionality. The above remarks all assume the type system  $R$ . The situation is somewhat different in type system  $F$ . In this setting, there are new instances of Modalized Functionality involving types ending in  $e$ , which are *not* theorems of the smallest H-theory containing all instances of Logical Equivalence. This shouldn’t be too surprising: we have motivated Classicism by the thought that logical equivalence suffices for identity, where logical equivalence is a relation between sentences—i.e. provability of the biconditional in H—not singular terms. The model theories we will be developing in part 3 can be generalized naturally to  $F$ , but the logic of the relevant classes of models is what we might call “Strong Classicism”, which includes all the instances of Modalized Functionality rather than just those that can be derived from Logical Equivalence.

can be articulated for different notions of ‘necessity’: the narrower the necessity in question the stronger and more contentious the thesis. Given Classicism,  $\Box$  can be shown to be the *broadest* (most demanding) necessity, given reasonable purely logical definitions of ‘necessity operator’ and ‘at least as broad as’.<sup>24</sup> Accordingly, we shall pronounce  $\Box$  as ‘it is broadly necessary that’. Viewed in this light, Intensionalism captures the kernel of the thought that properties and relations are individuated by necessary equivalence that is common to all its different versions.

Another controversial consequence of Classicism is the Converse Barcan Formula (Barcan 1946):<sup>25</sup>

$$\mathbf{CBF} \quad \Box \forall x P \rightarrow \forall x \Box P$$

To see why CBF is controversial, observe that it implies:

$$\mathbf{Broad\ Necessitism} \quad \forall x \Box \exists y (y = x)$$

(Taking  $P$  in CBF to be  $\exists y (y = x)$ , we get a conditional whose consequent is Broad Necessitism and whose antecedent,  $\Box \forall x \exists y (y = x)$ , is an uncontroversial theorem of C, since  $\forall x \exists y (y = x)$  is a theorem of H and C is closed under necessitation.) Broad Necessitism says that everything is *broadly* necessarily identical to something; since broad necessity entails every other form of necessity, it follows that nothing could have failed to be something, in any ordinary sense of ‘could’. Many philosophers—“contingentists”, in the terminology of Williamson 2013—have taken this to be false, indeed obviously false.<sup>26</sup> We disagree, but a full defence of this implication of Classicism would take us too far afield (see Williamson 2013, Goodman 2016, Fine 2016, Dorr, Hawthorne, and Yli-Vakkuri 2021). Here, we will content ourselves with noting that many contingentists (e.g. Fine 1977*b*) have been happy to help themselves, either as primitives or as the result of some kind of honest toil, to so-called “outer” or “possibilist” quantifiers  $\Pi$  and  $\Sigma$ , for which they are happy to accept the analogues of CBF and Broad Necessitism. It isn’t obvious how much is really at stake in the debate between those who are willing to accept CBF as written above and those who reject it but accept the analogue with  $\Pi$  instead of  $\forall$ : even though  $\Pi$  seems to behave logically as a quantifier and to entail

<sup>24</sup>See Bacon 2018*a* for details, and Dorr, Hawthorne, and Yli-Vakkuri 2021: ch. 8 and Bacon unpublished for further discussion. The result is quite robust with respect to different precisifications of ‘necessity operator’ and ‘at least as broad as’.

<sup>25</sup>To prove it in Classicism one uses the fact that it is closed under Necessitation and Gen, and contains the K schema. Applying Necessitation to an instance of UI we get  $\Box(\forall x P \rightarrow P)$ . K lets us distribute the necessity,  $\Box \forall x P \rightarrow \Box P$ , and Gen yields  $\Box \forall x P \rightarrow \forall x \Box P$ .

<sup>26</sup>See, for example, Kripke 1963, Fine 1977*a*, Fritz and Goodman 2016, Stalnaker 2012, Plantinga 1974, Menzel 1990.

$\forall$ , proponents of this view refuse for some reason to say that  $\Pi$  is the *unrestricted* universal quantifier (of the relevant type) and  $\forall$  is some restriction of it. Anyhow, we invite contingentists who can make sense of these quantifiers to reinterpret all our uses of  $\forall$  and  $\exists$  in the relevant way.

## 2 Extensions of Classicism

In this part of the paper, we will map out some theories that strengthen Classicism. Section 1.4 already discussed one important strengthening, namely Extensionalism, the result of adding the axiom scheme Extensionality (or the combination of Functionality and the Fregean Axiom) to Classicism (or H). But there are several interesting theories that are stronger than Classicism, but weaker than Extensionalism, and which are not subject to the kinds of counterexamples that make us find Extensionalism to be of merely historical and mathematical interest. Sections 2.1–2.3 will explore some of these theories. Sections 2.4–2.6 will then turn to some quite different ways of strengthening Classicism in a fine-grained direction.

### 2.1 Towards Extensionalism I: coarse-grainedness principles

One thing we might consider adding to Classicism is the principle of Functionality from Section 2.1:

**Functionality**  $\forall z(Xz = Yz) \rightarrow X = Y$

So long as we don't also add the Fregean Axiom, this will not allow us to infer that coextensive properties are identical. Rather, it captures the idea that properties and relations are completely determined by their applicative behaviour with respect to their arguments.

In gauging the plausibility of Functionality, it is useful to note a couple of equivalents:

**Proposition 2.1.** Functionality is equivalent in Classicism to each of the following schemas:

**BF**  $\forall x \Box P \rightarrow \Box \forall x P$

**Tractarianism**  $\forall x(p \leq Fx) \rightarrow p \leq \forall x Fx$ <sup>27</sup>

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<sup>27</sup>To derive Tractarianism from Functionality, suppose  $\forall x(p \leq Fx)$ ; then  $F = \lambda x.(Fx \vee p)$  by Functionality, so  $\forall x Fx = \forall x(Fx \vee p) = \forall x Fx \vee p$  by Distribution- $\forall\forall$ . To derive BF from Tractarianism, just plug in  $\top$  for  $p$ . And to derive Functionality from BF, note that  $\forall z(Xz = Yz)$

BF is the Barcan Formula, taken an axiom in the quantified modal logic of Barcan 1946. Tractarianism says that the universal generalization of a property behaves like the conjunctions of all its instances, i.e. the propositions that predicate that property. Instantiation already tells us that it entails all the instances; Tractarianism adds that it is entailed by anything that entails all the instances, just as a conjunction entails its conjuncts and is entailed by anything that entails all of its conjuncts, as captured by the standard introduction and elimination rules for conjunction. (Such an assimilation of quantification to infinitary conjunction is propounded by Wittgenstein in the *Tractatus*: Wittgenstein 1961: §6.0001; see Proops 2017.) Many have objected to this idea on the grounds that if there could be new objects, distinct from all the objects there already are, there is nothing to stop there from being a proposition,  $p$ , and a property,  $F$ , such that  $p$  entails  $Fx$  for each actual object  $x$ , but is compatible with there being new possible things that aren't  $F$ , and thus is a counterexample to Tractarianism.<sup>28</sup> The informal picture often associated with BF, and thus Functionality and Tractarianism, is that there cannot be anything new.

A further strengthening is to place  $\Box$  in front of any of Functionality, BF or Tractarianism; this results in the system HFE outlined in Bacon 2018a. The fact that this actually strengthens these principles (as we will show in appendix Appendix D) suggests we have to be careful about the intuitive gloss on Functionality as ‘there cannot be anything new’, since given S4, it seems that any claim that was properly glossed like that should be necessary if true. Once we look at models where these principles are contingently true, we will see more reasons to think the slogan should really be associated with the necessitated versions of these schemes.

One might think that principles at the level of generality of BF, Functionality, and Tractarianism should be assumed to be necessary if true. Perhaps this attitude is correct as far as metaphysical necessity is concerned. But it's hard to see why the mere *generality* of a proposition  $p$  should generate any presumption that  $p$ , if true, is *broadly* necessary, i.e., identical to  $p \vee \neg p$ . If the corresponding attitude to metaphysical necessity is appropriate, then perhaps we should take this as an argument for the distinctness of broad necessity and metaphysical necessity, rather than an argument against the view that general principles like BF and Functionality are true but not broadly necessary.

Another noteworthy consequence of Extensionalism that is not a theorem of Classicism is the necessity of distinctness:

$$\text{ND} \quad x \neq_{\sigma} y \rightarrow \Box(x \neq_{\sigma} y)$$

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implies  $\forall z \Box(Xz \leftrightarrow Yz)$  by LL, which implies  $\Box \forall z(Xz \leftrightarrow Yz)$  by BF, which implies  $X = Y$  by Intensionalism.)

<sup>28</sup>The argument against Tractarianism in Russell (1918–9: lecture 5) can be construed this way if we take Russell's ‘it is a further fact that’ to imply ‘not entailed by’.

By contrast, the necessity of *identity* is already a theorem of Classicism:

$$\mathbf{NI} \quad x =_{\sigma} y \rightarrow \Box(x =_{\sigma} y)$$

This follows, by a well-known argument that seems to have been first been discovered by Quine (see Burgess 2014), from the LL-instance  $x = y \rightarrow (\lambda z. \Box(x = z))x \rightarrow (\lambda z. \Box(x = z))y$ , together with  $\Box(x = x)$ , the necessitation of a Ref-instance.

ND also has some more familiar equivalents:

**Proposition 2.2.** ND is equivalent in Classicism to each of:

$$\mathbf{5} \quad \Diamond p \rightarrow \Box \Diamond p$$

$$\mathbf{B} \quad p \rightarrow \Box \Diamond p$$

To derive 5 from the type- $t$  instance of ND, substitute  $\top$  for  $q$  and  $\neg P$  for  $p$ . To derive B from 5, use the T axiom in the dual form  $p \rightarrow \Diamond p$ . And to complete the circle of entailments, we can derive ND (for any type) from B, using an argument due to Prior (1963: 206–7): suppose  $x \neq y$ ; then  $\Box \neg \Box(x = y)$  by B; but  $\Box(x = y \rightarrow \Box(x = y))$  by the necessitation of NI; so  $\Box(x \neq y)$  by K.

Just as in the case of Functionality, these principles do not imply their own necessitations (i.e. the necessitations of their universal closures) in Classicism. But it is hard to think of a principled reason for accepting, say, ND that would not extend to its necessitated analogue:

$$\Box \mathbf{ND} \quad \Box \forall x \forall y (x \neq_{\sigma} y \rightarrow \Box(x \neq_{\sigma} y))$$

We will refer to the result of adding  $\Box \mathbf{ND}$  (or  $\Box \mathbf{5}$  or  $\Box \mathbf{B}$ ) to C as C5, by analogy to the modal system S5. Insofar as any theory in this domain counts as “orthodox”, C5 does.

One might have expected that the idea that distinct things are necessarily distinct would be entirely independent of the question of new things. But this turns out to be wrong:

**Proposition 2.3.** BF (and hence also Functionality and Tractarianism) is a theorem of C5.

The proof of this is essentially due to Prior (1956). Prior (1956) uses S5; Prior (1967: 146) attributes the following simpler proof using B to E.J. Lemmon. Suppose  $\forall x \Box Fx$ . Then  $\Box \Diamond \forall x \Box Fx$  by B. Using CBF we can infer  $\Box \forall x \Diamond \Box Fx$ , and finally, by the necessitation of B,  $\Box \forall x Fx$ .

In the other direction, we have the following novel result:

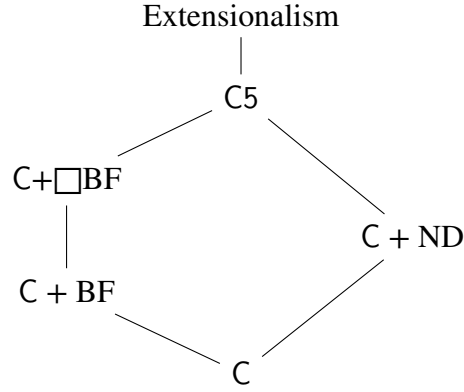


Figure 5. Coarse-grainings of Classicism

**Proposition 2.4.** ND and BF jointly imply  $\Box$ ND in Classicism.

For given ND, we have  $\Box(x = y) \vee \Box(x \neq y)$ . With 4, this implies  $\Box(x = y) \vee \Box\Box(x \neq y)$ , and hence  $\Box(x = y \vee \Box(x \neq y))$ , i.e.  $\Box(x \neq y \rightarrow \Box(x \neq y))$ . By Gen,  $\forall xy\Box(x \neq y \rightarrow \Box(x \neq y))$ , which implies  $\Box$ ND by BF.

So far, then, our map of systems including Classicism is as depicted in Figure 5.

## 2.2 Towards Extensionalism II: lattice-theoretic principles

The hierarchy of strengthenings of Classicism explored in the previous section is particularly important, both philosophically and because of the ways in which the coarser-grained views lend themselves to familiar and simple model theories. This section will survey three other principles inspired by conditions from the theory of Boolean algebras as formulated in classical first-order set theory, a common theoretical framework for modelling propositions. We will see that against the present foundational framework of higher-order logic, the relations between these principles are markedly different.

A “complete” Boolean algebra is one in which every set of elements has a **greatest lower bound**: a lower bound of the set that is  $\geq$  every other lower bound of the set, where being a lower bound of a set means being  $\leq$  every element of the set. Taking this as inspiration, consider the following principle:

**Boolean Completeness**

$$\forall X^{\tau \rightarrow t} \exists y^{\tau} (\text{GLB}_{\tau} yX)$$

where:

$$\begin{aligned}\text{GLB}_\tau &:= \lambda y^\tau X^{\tau \rightarrow t} . \forall z^\tau (\text{LB}_\tau z X \leftrightarrow z \leq_\tau y) \\ \text{LB}_\tau &:= \lambda y^\tau X^{\tau \rightarrow t} . \forall z^\tau (X z \rightarrow y \leq_\tau z)\end{aligned}$$

Boolean Completeness is thus analogous to the claim that each relational type  $\tau$  forms a complete Boolean algebra under entailment, except that quantification into type  $\tau \rightarrow t$  plays the role of quantification over sets.

Boolean Completeness follows from Extensionalism, since Extensionalism implies that for any property  $X$  (of propositions, properties, or relations), *falling under everything  $X$* —that is,  $\lambda \vec{y} . \forall Z (X Z \rightarrow Z \vec{y})$ —is a GLB of the  $X$  things.<sup>29</sup>

The second of our principles also corresponds to a well-known property of Boolean algebras. An *atom* of a Boolean algebra is an element such that the only thing below it is the bottom element; an algebra is *atomic* just in case every element is either the bottom element, an atom, or above at least one atom. This corresponds to a thesis about the propositions, properties, and relations of some type  $\tau$ :

$$\text{Atomicity} \quad \forall x (x \leq \neg_\tau x \vee \exists y (\text{Atom}_\tau y \wedge y \leq_\tau x))$$

where:

$$\text{Atom}_\tau := \lambda y . \forall z ((z \leq_\tau y \wedge z \neq y) \leftrightarrow z \leq_\tau \neg_\tau z)$$

For the special case where  $\tau$  is  $t$  we could equally well have defined the corresponding notion of an atom to be a broadly possible proposition that entails each proposition or its negation. The word ‘world’ would also be a pretty good name for atoms of type  $t$ , since given Atomicity, the broadly possible propositions are exactly those entailed by some atom. Extensionalism implies Atomicity: for example in type  $t$ , the only element other than  $\perp$  is  $\top$ , which is therefore an atom.<sup>30</sup>

If we are calling propositional atoms ‘worlds’, it is natural to use ‘actual world’ to mean ‘true propositional atom’. Obviously there can only be at most one actual

<sup>29</sup>To see that it’s a lower bound of  $X$ , suppose that  $Xu$ . Then by the Fregean Axiom,  $Xu = \top$ , so  $(\lambda \vec{x} . Xu \rightarrow u \vec{x}) = (\lambda \vec{x} . u \vec{x}) = u$ , so  $(\lambda \vec{x} . \forall Z (X Z \rightarrow Z \vec{x})) \leq u$ . To see that it’s a *greatest* lower bound of  $X$ , suppose  $y$  is a lower bound of  $X$ . Then  $\forall \vec{x} (y \vec{x} \rightarrow \forall z (X z \rightarrow z \vec{x}))$ . By Extensionality, this implies that  $y \leq \lambda \vec{x} . \forall Z (X Z \rightarrow Z \vec{x})$ . Note that without Extensionalism, there is no guarantee that *having every  $X$  property* is even a lower bound of the  $X$  properties, let alone a greatest lower bound. For example, even though *being president* is a widely-discussed property, *having every widely-discussed property* plausibly fails to entail *being president*.

<sup>30</sup>More generally, where  $\tau$  is  $\sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow t$ , Extensionalism implies  $\forall \vec{y} (\text{Atom}_\tau (\lambda \vec{z} . \bigwedge_i z_i = y_i))$ . Moreover,  $x \leq_\tau \neg_\tau x$  is equivalent given Extensionalism to  $\neg \exists \vec{y} (x \vec{y})$ ; if this is false, we have  $(\lambda \vec{z} . \bigwedge_i z_i = y_i) \leq_\tau x$ .



world, since any two atoms are incompatible. Call the claim that there is an actual world,

**Actuality**  $\exists p(p \wedge \forall q(q \rightarrow p \leq q))$

Equivalently: any property of propositions all of whose instances are true has a true lower bound.<sup>31</sup> Extensionalism obviously entails Actuality and provides a witness, namely  $\top$ .

In C5, the three principles we have just introduced are intimately related to each other:

**Proposition 2.5.** Actuality is equivalent to Boolean Completeness in C5.

**Proposition 2.6.** Atomicity is equivalent to  $\Box$ Boolean Completeness (and hence also to  $\Box$ Actuality) in C5.

We prove these claims below.<sup>32</sup>

For those used to using the theory of Boolean algebras to guide their reasoning about propositions, this should be surprising, for one can easily construct atomic Boolean algebras that are not complete, and complete Boolean algebras that are not atomic. These facts illustrate the danger of using the theory of arbitrary Boolean algebras in guiding one's theorizing about propositions.

Without the assumption of C5, there are more surprises for this way of thinking. On the model of propositions as a Boolean algebra, worlds—i.e. propositions that settle the truths—are atoms, including the actual world. Given that Atomicity implies that every proposition is the disjunction (LUB) of the atoms that entail it, one might expect Atomicity to imply Actuality. But surprisingly, this does not follow in Classicism: in Appendix D, we show that Classicism is consistent with the hypothesis that although every truth is entailed by some atom, every atom is false. Similarly, if there is a conjunction (GLB) of all the truths (as Boolean Completeness guarantees), one might expect that it would witness the truth of Actuality. But in fact, in general there is no obvious reason why the GLB of some truths should be true (or even possible), so Boolean Completeness does not obviously imply Actual-

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<sup>31</sup>Note that Actuality implies the analogous generalization about arbitrary predicate types:

**Actual Profile**  $\forall \vec{x} \exists Y(Y \vec{x} \wedge \forall Z(Z \vec{x} \rightarrow Y \leq Z))$

For suppose that  $w$  is a witness to Actuality; then for a given choice of  $\vec{x}$ ,  $\lambda \vec{y}.(w \wedge \bigwedge_i x_i = y_i)$  is a witness to Actual Profile.

<sup>32</sup>These results imply Gallin's 1975: 85 result that  $\Box$ Actuality (his At) is equivalent to Atomicity (his At<sub>2</sub>) given principles tantamount to those of C5.

ity.<sup>33</sup> In the other direction, one might have thought that if Actuality is *necessary*, then every possible proposition must be entailed by an atom, namely the proposition that would have been the actual world if it had been true. However, this reasoning forgets the fact that without propositional BF, we cannot import a merely possibly existing world into actuality, and that without ND, that even if an actually existing proposition is possibly a world it needn't actually be, since it might actually be decomposable into a disjunction of stronger consistent propositions that would have been identical had it been true. In Appendix D we show that  $\Box$ Actuality does not imply Atomicity even given BF, by constructing models in which the latter situation occurs (an atomless proposition is possibly atomic).

We postpone the proof of Proposition 2.5 to the next section. The right-to-left direction of Proposition 2.6 can be established by first showing that, given  $\Box$ ND, everything that is possibly an atom is in fact an atom. Suppose  $\Diamond$  Atom  $y$ : possibly, everything is either entailed by or inconsistent with  $y$ . By BF (which follows from  $\Box$ ND by Proposition 2.3), everything is either possibly entailed by or possibly inconsistent with  $y$ . But by ND, anything possibly entailed by  $y$  is entailed by  $y$ , and anything possibly inconsistent with  $y$  is inconsistent with  $y$ , so  $y$  is in fact an atom. Now suppose  $\Box$ Actuality and  $p \neq \perp$ . Then  $p$  is compatible with there being an actual world. Then by BF, there is a proposition  $w$  such that it is possible that  $p$  be true while  $w$  is an actual world; but then  $w$  must in fact be a world, so we have the desired result that there is a world compatible with  $p$ .

For the left-to-right direction of Proposition 2.6, we actually need only BF rather than the full strength of C5:

**Proposition 2.7.** Atomicity and BF jointly imply  $\Box$ Actuality.

For given BF, every atom entails that *it* entails every truth (and hence that it is a true atom, and hence there *is* a true atom), i.e.

$$\text{Atom } w \rightarrow w \leq \forall q(q \rightarrow w \leq q)$$

By Tractarianism (equivalent to BF), the consequent is equivalent to  $\forall q(w \leq (q \rightarrow w \leq q))$ . But this is true whenever  $w$  is an atom, since when  $w \leq q$ ,  $q \rightarrow (w \leq q)$  is  $\top$  and hence entailed by everything, while when  $w \leq \neg q$ ,  $w \leq (q \rightarrow p)$  for any  $p$ . Thus, if every proposition other than  $\perp$  is compatible with some atom, every proposition other than  $\perp$  must be compatible with Actuality, so the negation of Actuality must be identical to  $\perp$ , in which case the necessitation of Actuality is true.

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<sup>33</sup>We conjecture that the combination of Boolean Completeness and the negation of Actuality is consistent in Classicism, although we do not have a proof.

Goodsell and Yli-Vakkuri (unpublished) show that C5+Atomicity has a consequence worthy of special attention:

**No Pure Contingency**  $P \rightarrow \Box P$ , where  $P$  is closed and contains no non-logical constants.

Equivalently,  $\forall \vec{x} Q \rightarrow \Box Q[\vec{a}/\vec{x}]$ , where  $Q$  contains no nonlogical constants and has free variables in  $\vec{x}$ .<sup>34</sup> No Pure Contingency can also be consistently combined with C5 and the denial of Atomicity (indeed with Atomlessness), and with many other combinations of the principles we have discussed in this and the preceding section. Moreover, it has a certain plausibility. It can be derived from the combination of an account of the status of *logical truth* in the spirit of Bolzano (2004), Tarski (1959), and Williamson (2003), on which closed sentences with only logical constants are automatically logically true if true at all, together with the natural idea that the necessitation of any logical truth is itself a logical truth. However this is not by itself a strong argument for No Pure Contingency, since “logical truth” is a term of art, and a rather vexed one: insofar as one doubted No Pure Contingency, one should suspect that the argument just conflates two different interpretations of ‘logical truth’.

### 2.3 Towards Extensionalism III: comprehension principles

Some formulations of second and higher-order logic take as primitive a comprehension schema along the lines of

$$\exists X \forall y (Xy \leftrightarrow P).$$

Since our present system has  $\lambda$ -terms, this is in fact a theorem: the existential quantification is witnessed by the term  $\lambda y.P$ .<sup>35</sup> However Classicism is neutral about certain other comprehension-style principles, which this section will survey.

Our first principle requires some preliminary motivation and definitions. Let a *persistent* property, relation, or proposition be one that entails its own necessitation:

$$\text{Persistent} := \lambda Y.(Y \leq (\lambda \vec{z}.\Box Y \vec{z}))$$

The modal behaviour of a persistent property is a bit like that of a *set* or *plurality*, according to standard modal set theory/plural logic. Any member of a set is necessarily a member of that set, and any one of some things is necessarily one of those things; similarly, any instance of a persistent property is necessarily an instance.

<sup>34</sup>This is one direction of the biconditional ‘Logical Necessity’ schema discussed in Bacon 2020. The other direction will be discussed in Section 2.5 below.

<sup>35</sup> $\beta$  ensures that, for any  $y$ ,  $(\lambda y.P)y \leftrightarrow P$ .

But the standard view of sets and pluralities goes further than this, by ruling out the possibility of a set or plurality acquiring any *new* members beyond those that it in fact has. If C5 fails, persistent properties need not behave like this: for example, when  $a, b, c$  are distinct but possibly  $a = b \wedge a \neq c$ , *being identical to  $c$  or such that  $a = b$*  is a persistent property that in fact has only one instance but could have at least two. For the modal behaviour of a property to be really analogous to that of a set or plurality, it must be not only persistent but *inextensible*, i.e. necessarily such as to entail any property had necessarily by all of its instances:

$$\text{Inextensible} := \lambda Y. \Box \forall X. (\forall \vec{z}. (Y \vec{z} \rightarrow \Box X \vec{z}) \rightarrow Y \leq X)$$

In the above example,  $\lambda x. (x = c \vee a = b)$  is not inextensible, since it fails to entail  $\lambda x. x = c$ , even though that property is necessary to its one and only instance. The property of being a member of a given set or being one of some things, by contrast, would normally be thought of as inextensible: if each of some things is necessarily  $F$ , then necessarily anything that is one of them (belongs to the set of them) is  $F$ .<sup>36</sup>

Define a *rigid* property (or relation or proposition) as one that is both persistent and inextensible. We can simplify this as follows:

$$\text{Rigid}_{\sigma_1, \dots, \sigma_n} := \lambda Y. (\Box \forall X. (\forall \vec{z}. Y \vec{z} \rightarrow \Box X \vec{z}) \leftrightarrow Y \leq X)$$

The principle we want to consider says that every property (or relation or proposition) is coextensive with a rigid one:

$$\text{Rigid Comprehension} \quad \forall X. \exists Y. (\text{Rigid}(Y) \wedge \forall \vec{z}. (X \vec{z} \leftrightarrow Y \vec{z}))$$

This can be thought of on the model of a comprehension principle for pluralities or sets, according to which every property is coextensive with a plurality or set.<sup>37</sup> This assumption is both natural in itself, and needed for the regimentation of some natural-language modal claims not ostensibly about pluralities or sets, for example the most salient reading of ‘Mary could have had all John’s favourite properties’ (see Dorr, Hawthorne, and Yli-Vakkuri 2021: §1.4). Rigid Comprehension also helps to provide a natural account of the prevalent use of extensionalist reasoning

<sup>36</sup>See Linnebo 2013 and Dorr, Hawthorne, and Yli-Vakkuri 2021: §1.5. An inextensible property is one for which BF necessarily holds for the quantifiers restricted by it. In C5, persistence entails inextensibility (see Dorr, Hawthorne, and Yli-Vakkuri 2021: propositions C4 and C5).

<sup>37</sup>Myhill (1958) uses such a comprehension principle in a richer type system with special types for “propositional functions in extension”. Gallin (1975: 77) has a principle of “Extensional Comprehension” like Rigid Comprehension except that his rigidity predicate  $R_n$  means ‘persistent and having a persistent negation’. Given BF, rigidity in Gallin’s sense entails rigidity in ours, though not vice versa. Without BF, the two statuses are independent, since a rigid property in Gallin’s sense need not be inextensible (it could acquire new instances).

in mathematics (see Church 1940, Myhill 1958). Note that Extensionalism trivially implies Rigid Comprehension, since it entails that everything is rigid.<sup>38</sup>

Like all the principles from the previous section, Rigid Comprehension follows from the claim that there are only finitely many entities of the relevant types. Rigid Comprehension also implies two of those principles:

**Proposition 2.8.** Rigid Comprehension implies Boolean Completeness.

**Proposition 2.9.** Rigid Comprehension implies Actuality.

Both results are straightforward. To show that some property of propositions,  $F$ , has a greatest lower bound one takes a rigid property  $G$ , coextensive with  $F$ , and considers the proposition  $\forall p(Gp \rightarrow p)$  (much as we did in the proof of Boolean Completeness from Extensionalism).<sup>39</sup> Parallel arguments establish the proposition at other relational types. To show that Rigid Comprehension entails Actuality, one takes a rigid property,  $T$ , coextensive with truth (i.e. such that  $\forall p(Tp \leftrightarrow p)$ ) and defines the actual world as  $\forall p(Tp \rightarrow p)$ .<sup>40</sup>

In the setting of C5, Rigid Comprehension not only implies but is equivalent to Actuality:

**Proposition 2.10.** Actuality implies Rigid Comprehension in C5.

To prove this, one first shows that in C5 anything persistent is also inextensible.<sup>41</sup> It then suffices to show that Actuality implies that every property  $F$  is coextensive with a persistent one: if  $w$  is the actual world (i.e. the witness to Actuality), the

<sup>38</sup>If we weaken ‘Rigid’ in Rigid Comprehension to ‘Persistent’, the resulting ‘Persistent Comprehension’ principle is equivalent to Actuality. The implication from it to Actuality can be recovered from the proof of Actuality from Rigid Comprehension below, which does not mention inextensibility. To derive Persistent Comprehension from Actuality, suppose  $w$  witnesses Actuality; then for any  $X$ ,  $\lambda\vec{y}.w \leq X\vec{y}$  is persistent and coextensive with  $X$ . We might also consider replacing ‘Rigid’ with ‘Inextensible’. The resulting principle also follows from Actuality, since if  $w$  witnesses Actuality; then for any  $X$ ,  $\lambda\vec{y}.w \wedge X\vec{y}$  is inextensible and coextensive with  $X$ .

<sup>39</sup>Let  $X$  be of type  $\tau \rightarrow t$ , let  $X^*$  be the rigid property coextensive with  $X$ ; and let  $U$  of type  $t$  be  $\lambda\vec{z}.\forall Y(X^*Y \rightarrow Y\vec{z})$ . To show that  $U$  is a GLB of  $X$ , notice that since  $X$  and  $X^*$  are coextensive, any  $V$  is a lower bound of  $X$  just in case it is a lower bound of  $X^*$ , i.e.  $\forall Y(X^*Y \rightarrow \Box\forall\vec{z}(V\vec{z} \rightarrow Y\vec{z}))$ . By the rigidity of  $X$ , this is true just in case  $\Box\forall Y(X^*Y \rightarrow \forall\vec{z}(V\vec{z} \rightarrow Y\vec{z}))$  which is equivalent to the claim that  $V$  entails  $U$ .

<sup>40</sup> $\forall p(Tp \rightarrow p)$  is true since  $T$  is coextensive with truth. And it entails  $(Tp \rightarrow p)$  for every  $p$ . But when  $p$  is true,  $Tp$  is true, so by the persistence of  $T$ ,  $\Box Tp$ , hence  $(Tp \rightarrow p) = p$ ; thus,  $\forall p(Tp \rightarrow p)$  entails every truth. Note that this reasoning does not rely on the inextensibility of  $T$ .

<sup>41</sup>*Proof:* Suppose  $X$  is persistent and  $\forall\vec{y}(X\vec{y} \rightarrow \Box Z\vec{y})$ , i.e.  $\forall\vec{y}(\neg X\vec{y} \vee \Box Z\vec{y})$ . By B (equivalent to ND), we have  $\forall\vec{y}(\neg X\vec{y} \rightarrow \Box\neg X\vec{y})$ , and hence by the persistence of  $X$ ,  $\forall\vec{y}(\neg X\vec{y} \rightarrow \Box\neg X\vec{y})$ . So we can strengthen our assumption to  $\forall\vec{y}(\Box\neg X\vec{y} \vee \Box Z\vec{y})$ , which implies  $\forall\vec{y}\Box(\neg X\vec{y} \vee Z\vec{y})$ , i.e.  $X \leq Z$ . Using  $\Box$ ND we can necessitate this result.

persistent property in question can be defined as being such that  $w$  entails you are  $F$  (i.e.  $\lambda x.w \leq Fx$ ); this is persistent since entailments are necessary if true, and coextensive with  $F$ , since  $w$  entails only the truths. One can extend this argument to relations straightforwardly.

Combining Propositions 2.8 and 2.10 gives us the right-to-left direction of Proposition 2.5 (stated without proof in the previous section): Actuality implies Boolean Completeness in C5.

Here are two other results involving Rigid Comprehension whose proofs we give in footnotes:

**Proposition 2.11.**  $\Box$ Atomicity, Boolean Completeness, and BF jointly imply Rigid Comprehension.<sup>42</sup>

**Proposition 2.12.** Rigid Comprehension and BF jointly imply  $\Box$ BF.<sup>43</sup>

<sup>42</sup>Let  $X$  be some property,  $F$  be the property of being a haecceity of an  $X$  thing (i.e.  $\lambda Y.\exists z(Xz \wedge Y = \lambda x.(z = x))$ ), and  $X^*$  be the least upper bound of  $F$ . We show that  $X^*$  is coextensive with  $X$  and rigid.

- (i) Every  $X$  is  $X^*$ . Suppose  $Xz$ ; then  $\lambda y.y = z$  is  $F$  and hence entails  $X^*$ , so  $X^*z$ .
- (ii) Every  $X^*$  is  $X$ . Actuality follows from our assumptions by Proposition 2.6, so there is a true world-proposition,  $w$ . Then  $(\lambda y.w \rightarrow Xy)$  is an upper bound of  $F$ , since if  $\lambda y.y = z$  is  $F$ ,  $Xz$  is true and so entailed by  $w$ , which implies that  $\Box\forall y(y = z \rightarrow (w \rightarrow Xy))$ . Since  $X^*$  is a *least* upper bound of  $F$ ,  $X^*$  entails  $\lambda y.(w \rightarrow Xy)$ . So if  $z$  is  $X^*$ , then  $w \rightarrow Xz$  and hence  $Xz$ .
- (iii)  $X^*$  is persistent. If  $z$  is  $X$ , then  $X^*$  is entailed by  $\lambda x.x = z$ , so  $\lambda x.\Box X^*x$  is entailed by  $\lambda x.\Box(x = z)$  and hence also by  $\lambda x.x = z$  (by NI). So  $\lambda x.\Box X^*x$  is an upper bound of  $F$ . Since  $X^*$  is a least upper bound of  $F$ , it follows that  $X^*$  entails  $\lambda x.\Box X^*x$ .
- (iv)  $X^*$  is inextensible. We will show  $\forall Y\forall z\Box(\forall x(X^*x \rightarrow \Box Yx) \rightarrow \Box(X^*z \rightarrow Yz))$  and then appeal to BF. Suppose for contradiction that  $\Diamond(\forall x(X^*x \rightarrow \Box Yx) \wedge \Diamond(X^*z \wedge \neg Yz))$ . Let  $w$  be an atom that entails  $\forall x(X^*x \rightarrow \Box Yx) \wedge \Diamond(X^*z \wedge \neg Yz)$ . So  $w$  entails  $\forall x(X^*x \rightarrow \Box Yx)$  and that there is an atom  $w'$  that entails  $X^*z \wedge \neg Yz$ . Hence by BF, there is some  $w'$  for which  $w$  entails:

$$\forall x(X^*x \rightarrow \Box Yx) \wedge \text{Atom}(w') \wedge w' \leq (X^*z \wedge \neg Yz).$$

Without loss of generality, we may assume in addition that  $w'$  is the GLB of the propositions  $p$  such that that  $w$  entails that  $p = w'$ . (If it isn't, let  $w''$  be the GLB of the propositions such that  $w \leq (p = w')$ ; then  $w''$  will also be the GLB of the  $p$  such that  $w \leq (p = w')$ .) Now let  $H := \lambda x.(w' \rightarrow Yx)$ . We will prove that  $H$  is an upper bound of  $F$ , and hence entailed by  $X^*$ , which is a contradiction since  $w'$  is possible (since possibly an atom) and entails that  $X^*z$  and  $\neg Yz$ , and hence  $\neg Hz$ . Let  $u$  be any  $X$  thing. We have  $\Box X^*u$  by parts (i) and (iii) above; so  $w$  entails  $X^*u$ . Since  $w$  entails  $\forall x(X^*x \rightarrow \Box Yx)$ ,  $w$  also entails  $\Box Yu$  and thus  $w' \leq Yu$ . Since  $w'$  is the GLB above,  $w'$  must *in fact* entail  $Yu$ . (For suppose not: then  $w' \wedge Yu$  is stronger than  $w'$ , but  $w$  entails  $w' = (w' \wedge Yu)$ .) So  $\Box Hu$ . Thus for any  $u$  that is  $X$ ,  $\Box\forall y(y = u \rightarrow Hu)$ , so  $H$  is an upper bound of  $F$ , and thus entailed by the least upper bound,  $X^*$ .

<sup>43</sup>To prove this, let  $F$  be a rigid property coextensive with self-identity. Since  $F$  is inextensible, we have  $\Box\forall Y(\forall x(Fx \rightarrow \Box Yx) \rightarrow \Box\forall x(Fx \rightarrow Yx))$ . But since everything is  $F$ ,  $\forall x\Box Fx$  by the persistence of  $F$ , so  $\Box\forall x Fx$  by BF. So we can simply the above to  $\Box\forall Y(\forall x\Box Yx \rightarrow \Box\forall x Yx)$ , i.e.  $\Box$ BF.

To state our second comprehension-style principle, define a *functional* binary relation of a given type  $\sigma \rightarrow \tau \rightarrow t$  in the obvious way as one that relates everything of type  $\sigma$  to exactly one thing of type  $\tau$ :

$$\text{Functional}_{\sigma,\tau} := \lambda U^{\sigma \rightarrow \tau \rightarrow t}. \forall x \exists y (Uxy \wedge \forall z (Uxz \rightarrow y = z))$$

Quantification over functional relations provides one natural higher-order way of regimenting the talk of “functions” that comes so naturally to those schooled in standard mathematics. But in the case where  $\tau$  is a relational type, there is another natural way of regimenting informal talk of “functions from type- $\sigma$  things to type- $\tau$  things”, namely as quantification into type  $\sigma \rightarrow \tau$ . (Indeed the use of the word “function” in connection with types of the form  $\sigma \rightarrow \tau$  has deep roots in the history of higher order logic as well as its contemporary use.) Obviously any  $X$  of type  $\sigma \rightarrow \tau$  corresponds to a unique functional relation of type  $\sigma \rightarrow \tau \rightarrow t$ , namely  $\lambda yz.z = Xy$ . The principle we want to consider lets us turn this around, by positing something of type  $\sigma \rightarrow \tau$  corresponding to every functional relation of type  $\sigma \rightarrow \tau \rightarrow t$ :

**Plenitude**  $\quad \forall R^{\sigma \rightarrow \tau \rightarrow t} (\text{Functional}(R) \rightarrow \exists X^{\sigma \rightarrow \tau} \forall y (Ry(Xy)))$

By contrast with Rigid Comprehension, which is arguably deeply rooted in ordinary-language judgements, Plenitude is on shakier philosophical ground, since the relaxed attitude to the word “function” that it licenses might well be dismissed as a confusion.<sup>44</sup> Like all the principles we have considered so far, it follows from Extensionalism: in the case where  $\tau = t$ , one can simply define  $X$  as  $\lambda y.Ry\top$ , the property of being a type- $\sigma$  thing that bears  $R$  to the one true proposition  $\top$ .<sup>45</sup>

Plenitude follows much more obviously from a version of the Axiom of Choice that is *not* a theorem of Extensionalism, although it is a theorem of several influential systems (including those of Church 1940 and Henkin 1950). Let’s call this:

**Functional Choice**  $\quad \forall R^{\sigma \rightarrow \tau \rightarrow t} (\text{Serial}(R) \rightarrow \exists X^{\sigma \rightarrow \tau} \forall y (Ry(Xy)))$

where

$$\text{Serial} := \lambda R. \forall x \exists y Rxy$$

Given the central role of Choice in large parts of mathematics, Functional Choice might seem to provide the basis for a strong argument for Plenitude. But for the

<sup>44</sup>Plenitude, and further reasons for not taking it to be obviously true, are discussed in Dorr 2016: §6. It also occurs as an axiom—the ‘Typed Comprehension Schema’—in Walsh 2016.

<sup>45</sup>Since  $Rx\top = \top$  whenever  $Rx\top$  and  $Rx\perp = \perp$  whenever  $Rx\perp$ ,  $Rx(Rx\top)$  in either case, so the existential quantification in Plenitude is witnessed by  $\lambda x.Rx\top$ . More generally, when  $\tau$  is  $\vec{\sigma} \rightarrow t$ , Plenitude will be witnessed by  $\lambda x\vec{y}.\exists Z (RxZ \wedge Z\vec{y})$ .

purposes of formalizing Choice-based mathematics, the following weaker Choice axiom will do perfectly fine:

**Relational Choice**  $\forall R^{\sigma \rightarrow \tau \rightarrow t}(\text{Serial}(R) \rightarrow \exists S^{\sigma \rightarrow \tau \rightarrow t}(\text{Functional}(S) \wedge \forall xy(Sxy \rightarrow Rxy)))$

In fact, Functional Choice is easily seen to be equivalent to the conjunction of Relational Choice and Plenitude. We will not further discuss Relational Choice here since we are primarily concerned with principles which follow from Extensionalism.<sup>46</sup> As far as we know, it and its negation are consistent with all consistent combinations of the principles on our list.<sup>47</sup>

Apart from being interesting in its own right, Plenitude also serves as a useful bridge between several of the other principles we have discussed. Here is one important fact:

**Proposition 2.13.** Plenitude implies ND.

For if  $x \neq_{\sigma} y$ , there is a functional relation that maps  $x$  to  $\top$  and everything else to  $\perp$  (namely,  $\lambda zw.(x = z \wedge w = \top) \vee (x \neq z \wedge w = \perp)$ ), so Plenitude implies that there is a  $Z$  of type  $\sigma \rightarrow t$  such that  $Zx = \top$  and  $Zy = \perp$ . Since  $\Box(\perp \neq \top)$ , it follows that  $\Box(Zx \neq Zy)$ , and hence that  $\Box(x \neq y)$ .

We can also use Plenitude to prove the leftover left-to-right direction of Proposition 2.5, as a consequence of the following facts:

**Proposition 2.14.** Boolean Completeness implies Plenitude in C5.

**Proposition 2.15.** Plenitude implies Actuality.

For the former, suppose  $R$  is a functional relation between propositions (the general case is proved similarly). The trick to obtaining the required  $X$  of type  $t \rightarrow t$  is to construct it as a limit from below: take the least upper bound of the properties  $Z$

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<sup>46</sup>The fact that Relational Choice does not follow from Extensionalism can be verified by working in a model of ZF without choice, constructing a full and functional model where the type- $e$  domain is the domain of some serial relation (i.e. set of ordered pairs) with no functional subrelation.

<sup>47</sup>The other principle that occurs in Church's and Henkin's type theories that does not follow from Extensionalism is an axiom of infinity, guaranteeing the existence of infinitely many individuals. This too seems to be consistent with all consistent combinations of the principles discussed in this part of the paper; indeed, we can have an axiom of infinity for every type including  $t$ , which is obviously inconsistent with Extensionalism. The *denial* of infinity for type  $t$  implies both type- $t$  Atomicity and Actuality.



such that for any  $p$ ,  $Zp$  entails the proposition to which  $p$  bears  $R$ .<sup>48</sup> For the latter, consider the functional relation that maps every truth to itself and every falsehood to the tautology:  $\lambda pq.(p \wedge p = q) \vee (\neg p \wedge p = \top)$ . By Plenitude, there exists a  $Z$  of type  $t \rightarrow t$  such that  $\forall pRp(Zp)$ : i.e.,  $Zp = p$  whenever  $p$  is true and  $Zp = \top$  whenever  $p$  is false. Thus the proposition  $\forall pZp$  (that everything is  $Z$ ) is true. Moreover this proposition entails  $Zp$  for every  $p$ , and thus entails  $p$  whenever  $p$  is true; so it is a true atom.

We can also prove a variant of Proposition 2.14 in which Boolean Completeness is strengthened to Rigid Comprehension, while  $\Box$ ND is weakened to ND:

**Proposition 2.16.** Rigid Comprehension and ND jointly imply Plenitude.

We put the proof in a footnote.<sup>49</sup>

The results we have proven imply that as regards views about the five principles that took centre stage in this and the previous section, there are just two combinations strictly between C5 and Extensionalism, namely C5 + Actuality (= C5 + Boolean Completeness = C5 + Rigid Comprehension = C5 + Plenitude), and the stronger C5+Atomicity (= C5 +  $\Box$ Actuality = C5 +  $\Box$ Boolean Completeness = C5 +  $\Box$ Rigid Comprehension = C +  $\Box$ Plenitude). Appendix D shows that all these inclusions are strict, so the map is as in Figure 6.

<sup>48</sup>Suppose  $R$  of type  $\sigma \rightarrow t \rightarrow t$  is functional. Let  $F_R : (\sigma \rightarrow t) \rightarrow t$  be  $\lambda X^{\sigma \rightarrow t}.\forall y \forall p(Ryp \rightarrow Xy \leq p)$ . By Boolean Completeness,  $F_R$  has a least upper bound: an operation  $G_R : \sigma \rightarrow t$  such that (i) whenever  $F_RX$ ,  $X \leq_{\sigma \rightarrow t} G_R$ , and (ii) whenever  $\forall X(F_RX \rightarrow X \leq Y)$ ,  $G_R \leq Y$ . We will show that for any given  $a$ ,  $Ra(G_Ra)$ , so that  $G_R$  witnesses the truth of Plenitude. Fix  $a$ , and let  $p_a$  be the proposition such that  $Rap_a$ , so we want to show that  $p_a = G_Ra$ .

We first show that  $p_a \leq G_Ra$ . Let  $H_a : \sigma \rightarrow t$  be  $\lambda x.x = a \wedge p_a$ . By ND,  $\forall x(x \neq a \rightarrow (H_ax = \perp))$ , and hence  $\forall xp((x \neq a \wedge Rxp) \rightarrow (H_ax \leq p))$ ; meanwhile  $H_aa = p_a$ , and so  $\forall xp((x = a \wedge Rxp) \rightarrow (H_ax \leq p))$ . Putting these facts together, we have that  $\forall xp(Rxp \rightarrow (H_ax \leq p))$ , i.e.  $F_RH_a$ . Since  $G_R$  is the least upper bound of all the  $F_R$  operations, we can conclude that  $H_a \leq G_R$ , and hence  $p_a = H_aa \leq G_Ra$ .

It remains to show that  $G_Ra \leq p_a$ . Let  $J_a : \sigma \rightarrow t$  be  $\lambda x.x \neq a \vee p_a$ . By ND,  $\forall x(x \neq a \rightarrow (x \neq a = \top))$ , hence  $\forall x(x \neq a \rightarrow (J_ax = \top))$ , and hence  $\forall X(F_RX \rightarrow \forall x(x \neq a \rightarrow (Xx \leq J_ax)))$ . We also have  $\forall X(F_RX \rightarrow \forall x(x = a \rightarrow (Xx \leq J_ax)))$ , since  $J_aa = p_a$  and if  $F_RX$  and  $Xa$ ,  $X \leq p_a$ . Putting these facts together we have  $\forall X(F_RX \rightarrow \forall x(Xx \leq J_ax))$ . By BF (which follows from  $\Box$ ND), this implies  $\forall X(F_RX \rightarrow X \leq J_a)$ : i.e.  $J_a$  is an upper bound of the  $F_R$  operations. Hence  $G_R \leq J_a$ , and so  $G_Ra \leq J_aa = p_a$ .

<sup>49</sup>Suppose that  $R$  is a functional relation of type  $\sigma \rightarrow t \rightarrow t$ . (The general case for type  $\sigma \rightarrow \tau \rightarrow t$  is analogous.) Let  $R^*$  be a rigid relation coextensive with  $R$ , and let  $Z$  be  $\lambda y^\sigma.\forall p(R^*yp \rightarrow p)$ . Suppose  $Rxq$ . Then  $R^*xq$ , so  $\Box R^*xq$  by the persistence of  $R^*$ , hence  $q = R^*xq \rightarrow q$ . But then  $Zx \leq q$ , since  $Zx \leq R^*xq \rightarrow q$ . Also, since  $R^*$  is functional (since coextensive with  $R$ ),  $\forall yp(R^*yp \rightarrow (y \neq x \vee p = q))$ . By ND (and NI),  $\forall yp(R^*yp \rightarrow \Box(y \neq x \vee p = q))$ ; by the inextensibility of  $R^*$ , this implies  $\Box \forall yp(R^*yp \rightarrow (y \neq x \vee p = q))$ , hence  $\Box \forall p(R^*xp \rightarrow p = q)$  and thus  $\Box(q \rightarrow \forall p(R^*xp \rightarrow p))$ , i.e.,  $q \leq Zx$ . Hence  $q = Zx$ , and so  $Rx(Zx)$ .

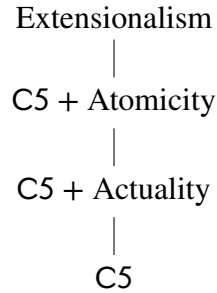


Figure 6. Between C5 and Extensionalism

For systems not including C5, by contrast, the map of possible combinations of our principles is far more complicated. We have not been able to identify any interesting logical relationships between the principles (and their negations and necessitations) other than those we have already stated. Appendix D establishes the consistency of some of these packages, but falls short of a truly systematic exploration of the rather large set of possible distributions of the statuses of necessary truth, contingent truth, contingent falsity, and necessary falsity over the principles we have considered whose consistency is not ruled out by any of our stated results of this section. We are hopeful that the model-theoretic techniques introduced in this appendix will also prove useful for establishing the consistency of some of the other combinations.

## 2.4 Finer-grained strengthenings: Maximalist Classicism

Extensionalism is intuitively an extremely coarse-grained view in higher-order logic. One way to make this precise is to say that theory  $T_1$  is coarser-grained than theory  $T_2$  just in case  $T_2$  contains every closed identity claim (i.e. every sentence of the form  $A = B$ ) that  $T_1$  contains, and  $T_2$  contains every closed distinctness claim (i.e. sentence of the form  $A \neq B$ ) that  $T_1$  contains.<sup>50</sup> Given this definition, Extensionalism does indeed count as a maximally coarse-grained theory. For in Extensionalism every identity  $A = B$  is equivalent to the distinctness claim  $(A = B) \neq \perp$ , and every distinctness claim  $A \neq B$  is equivalent to the identity claim  $(A \neq B) = \top$ . You cannot consistently add a new identity without also adding a distinctness claim, and you cannot subtract a distinctness claim without also subtracting an identity claim (since that identity claim entails the distinctness claim in H). However, Extensionalism is not in this sense the *unique* maximally coarse-grained extension of Classicism, since the above reasoning applies to any strengthening of Extensionalism.<sup>51</sup>

<sup>50</sup>Note that this is not a total order.

<sup>51</sup>E.g. by adding the claim that there are exactly three things of type  $e$ .

It is also interesting to explore extensions of Classicism that are more fine-grained than it. Indeed, one might wonder if there are *maximally* fine-grained extensions of C. The answer turns out to be yes. In fact, by contrast with the case of coarse-grainedness, there is an extension of C at least as fine-grained as each and every consistent extension of C. The weakest such theory we call *Maximalist Classicism*. It is the result of extending C with every distinctness claim that is consistent with C. Clearly, so long as Maximalist Classicism is consistent, it is at least as fine-grained as any consistent extension of C. What is not obvious is that Maximalist Classicism is consistent, i.e., that the set of closed distinctness claims that are *individually* consistent with C are *jointly* consistent with C. After all, this isn't true as regards identity claims: C is consistent both with  $\forall x^e y^e (x = y)$  (the proposition that there is exactly one individual) being identical to  $\top$  and with it being identical to  $\perp$ , but obviously not with the conjunction of these claims. The proof of the consistency of Maximalist Classicism is a central application of the model theoretic techniques we will introduce in Part 3.

Maximalist Classicism is not a recursively axiomatizable theory.<sup>52</sup> But if we don't mind talking about axiom schemas whose instances aren't recursively enumerable, we can obviously axiomatize Maximalist Classicism by the schema:

**Distinctness**  $A \neq B$ , where  $A = B$  is closed and not a theorem of Classicism.

Or, equivalently, we could use the schema:

**Possibility**  $\diamond A$ , where  $A$  is closed and consistent with Classicism.

The failure of Maximalist Classicism to be recursively axiomatizable makes it quite hard to apply theoretical considerations such as simplicity to it. In one sense of 'simplicity', such theories might be considered very unsimple. On the other hand, we have given an extremely simple *characterization* of it, as the maximally fine-grained extension of Classicism. At any rate, it occupies an important position in the space of extensions of Classicism, making it eminently worthy of serious engagement.<sup>53</sup>

<sup>52</sup>If it were, C would have to be decidable, since we could enumerate the non-theorems of C by enumerating the theorems of Maximalist Classicism of the form  $\diamond \neg A$ , and stripping off the initial  $\diamond \neg$ . But C is not decidable, for the same reason first-order logic is not.

<sup>53</sup>Those who like the impulse behind Maximalist Classicism should be interested in the project of finding strong, recursively axiomatizable fragments of Maximalist Classicism. One strategy is to pick some way of encoding " $P$ " is consistent in C' as a sentence of higher-orderese,  $\text{Con} \ulcorner P \urcorner$ , in which case one can formula a decidable axiom-schema  $\text{Con} \ulcorner P \urcorner \rightarrow \diamond P$  (where  $P$  is closed). One can then derive any instance of Possibility from the corresponding consistency assumption. By Gödel's second incompleteness theorem,  $\text{Con} \ulcorner P \urcorner$  will never be a consequence of C, but it will often be derivable from further well-motivated claims. One such well-motivated claim is  $\diamond \exists R^{e \rightarrow e \rightarrow t} \text{ZFC}(R)$ , where  $\text{ZFC}(\epsilon)$  is the conjunction of the nine axioms of second-order ZFC. Notice that this claim is

While Distinctness and Possibility are obviously equivalent given Classicism, the former is perhaps more effective for bringing out the appeal of Maximalist Classicism: the thought is that when it comes to question of identity—at least questions that can be formulated as closed sentences of higher-order logic—there are no surprises. It would be *surprising* if, for instance, the proposition that there are three individuals were the same as the proposition that there are four individuals; Distinctness, by contrast, ensures the only true identities are those forced on us by logic (i.e. by Logical Equivalence). Any identities beyond these would reflect an aspect of the natures of the logical constants, and the way that they fit together, that standard classical logic tells us nothing about. By contrast, Distinctness can be thought of as saying that all there is to the natures of the logical constants are their logical roles, as captured in the usual logical rules (i.e. those of H).

Maximalist Classicism feels more tendentious when stated in terms of Possibility. Given our earlier results about the non-theorems of Classicism, instances of Possibility include the broad possibility of Functionality, the Fregean Axiom, Boolean Completeness, Atomicity, and so on, as well as their negations. They also include the broad possibility of many claims we have not discussed, such as higher-order renditions of contentious set theoretic principles like the continuum hypothesis. This will feel alien to many metaphysicians, who are used to thinking that when it comes to claims that are sufficiently general, whatever is true is necessarily true and whatever is false is necessarily false. But of course, proponents of Maximalist Classicism could accept this impulse as far as *metaphysical* necessity is concerned, in which case they should take this to be another reason to deny that metaphysical necessity is identical to broad necessity. Indeed, there is another way of talking that metaphysicians often slip into that fits very naturally with Maximalist Classicism, namely working with an operator ‘it is logically necessary that’, or ‘it is logically possible that’, in a way that takes for granted that you can go back and forth between the metalinguistic status of logical truth for sentences, and object language formulations in terms of operators. Although there is a lot about this practice to be suspicious of (see Bacon 2018*b*: ch. 4 and Dorr, Hawthorne, and Yli-Vakkuri 2021: ch. 8), the fact that Maximalist Classicism is consistent means that there is a real vision in the vicinity that is not merely a use-mention fallacy.

Maximalist Classicism belongs to a broader family of theories in a similar spirit. For any theory  $T$  in higher order logic, we can define the *maximalization* of  $T$ ,  $\text{Max } T$ , to be the result of adding to it all closed sentences  $A \neq B$  which are consistent with  $T$ . Many theories other than C have consistent maximalizations, which

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also an instance of Possibility (assuming ZFC( $\in$ ) is consistent). The theory comprising this claim together with the aforementioned schema can prove everything that can be proved in ZFC to be in Maximalist Classicism. So in practice there isn’t much difference between being “committed to” Maximalist Classicism and being “committed to” this particular fragment.

offer interesting alternatives to Maximalist Classicism.<sup>54</sup> For example, we could consider the maximalist versions of the results of adding various combinations of the principles considered in Sections 2.1–2.3 to C, or the maximalizations of theories weaker than C such as H. So long as  $T$  can prove that  $\Box$  has a reasonable modal logic (including the necessity of identity),  $\text{Max } T$  will be equivalent to the result of adding  $\Diamond P$  for every closed  $P$  consistent with  $T$ , and will thus support the naïve practice of talking about “logical necessity” as an operator in the same way as Maximalist Classicism. However, not all consistent theories have consistent maximalizations. For example, the maximalization of C5 is inconsistent, since for many choices of  $A$ —for example ‘there are exactly three individuals’—C5 entails  $\Box A \vee \Box \neg A$  but does not entail either of its disjuncts, so that  $\text{Max } C5$  entails both their negations.<sup>55</sup> Much of what we say about Maximalist Classicism below will also apply to these alternative maximalized theories.

## 2.5 Finer-grained strengthenings: non-logical constants and fundamentality

In the previous section we were considering Maximalist Classicism as a theory in the pure language  $\mathcal{L}$ . There is an analogue of this theory for any  $\mathcal{L}(\Sigma)$  extending  $\mathcal{L}$  with non-logical constants,  $\Sigma$ . However, for many choices of non-logical constants, such a view seems deeply implausible. For example, if  $\Sigma$  includes predicates like ‘bachelor’ and ‘married’, then the version of Possibility for that signature will include the claim that it is broadly possible for there to be a married bachelor, i.e.  $\Diamond \exists x(\text{Bachelor } x \wedge \text{Married } x)$ . And if it also has a constant meaning ‘man’, then the version of Distinctness for  $\mathcal{L}(\Sigma)$  will also have the claim that it is not the case that to be a bachelor is to be an unmarried man—i.e.  $\text{Bachelor} \neq \lambda x.(\text{Man } x \wedge \neg \text{Married } x)$ . This seems misguided to us—surely natural languages very often provide us with simple expressions that refer to entities that can also be referred to with more complex expressions.<sup>56</sup>

However, there is considerable attraction to the idea that if all of the non-logical constants in some signature  $\Sigma$  denoted distinct *fundamental* entities, then the version of Maximalist Classicism for  $\mathcal{L}(\Sigma)$  would be true. This theory imposes a controversial but defensible constraint on the fundamental entities denoted by constants in  $\Sigma$ :

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<sup>54</sup>Fritz, Lederman, and Uzquiano (2021) prove the consistency of the maximalizations of  $H_0$  (see footnote 7 above) and of H.

<sup>55</sup>The property of having a consistent maximilization is related to the property of *coherence* in modal logics (see Meyer 1971), and is studied more generally in the context of higher-order theories in Bacon unpublished. In Appendix E, we discuss a construction that can be used to show the consistency of maximilizations of several extensions of C.

<sup>56</sup>There is a radical view worth engaging with that denies all higher-order identity claims where the terms flanking the identity symbol are closed and structurally non-isomorphic. But this view also denies many of the theorems of Classicism, and so is not relevant in the present context.

that any ‘logically consistent’ thing we can say about them corresponds to a way for them to broadly possibly be (see the discussion in (Bacon 2020: sec. 2)). The thought that fundamental entities are in some demanding sense “independent” of one another has been a guiding idea for a broad range of theorists, especially in the “Humean” tradition. Some ways of cashing out this vision take us extremely close to the Logical Maximalist point of view: for example, Dorr and Hawthorne (2013: 14) discuss a view they call ‘combinatorialism’, according to which, ‘in an appropriate language in which all predicates express perfectly natural properties, the only sentences that express metaphysically necessary propositions are the logical truths.’ While the meaning of ‘logical truth’ here is up for grabs, *being a theorem of Classicism* certainly looks like a principled way of filling in the idea, and one that answers to the Humean impulse that generates it.<sup>57</sup> Of course, even those who vehemently reject this sort of combinatorialist thinking so far as metaphysical necessity is concerned might still accept it for *broad* necessity. Indeed, expressions of anti-Humeanism often assume something like Maximalist Classicism, by treating failures of the metaphysical-possibility version of Possibility as establishing that metaphysical possibility is a more demanding status than “logical possibility” (see, e.g., Wilson 2010).

Maximalist Classicism in  $\mathcal{L}(\Sigma)$  is equivalent to the combination of “Pure Maximalist Classicism”—i.e. Maximalist Classicism for  $\mathcal{L}$ —with the following schema (from Bacon 2020: §4):

**Separated Structure**  $Fc = Gc \rightarrow F = G$ , where  $c$  is a non-logical constant and  $F$  and  $G$  are closed terms not including  $c$ .

Even by itself, Separated Structure is completely implausible for languages with arbitrary non-logical constants. But with the assumption that the non-logical con-

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<sup>57</sup>An alternative interpretation of the slogan would cash out “logical truth” in the manner of Williamson (2013) (derived from Tarski 1959 and Bolzano 2004), such that logical truth coincides with plain truth when it comes to closed sentences involving only logical vocabulary. That interpretation suggests the following weakening of Possibility:

**Witnessed Possibility**  $\exists \vec{x} P \rightarrow \Diamond P[\vec{c}/\vec{x}]$

where  $P$  is a formula with no nonlogical constants and only the variables  $\vec{x}$  free,  $\vec{c}$  are distinct non-logical constants, and  $P[\vec{c}/\vec{x}]$  is the closed formula that results when these constants are substituted for the free variables in  $P$ . Witnessed Possibility, unlike Possibility, is consistent with No Pure Contingency (see Section 2.2). No Pure Contingency is in fact equivalent to the converse of Witnessed Possibility; the combination of the two is equivalent to the ‘Logical Necessity’ schema from Bacon 2020. Note that much of the exploration in that paper concerns consequences of the direction of Logical Necessity equivalent to Witnessed Possibility, which follows from Maximalist Classicism; indeed most of the paper concerns the more abstract feature of “stability” which is common to Maximalist Classicism and Logical Combinatorialism.

stants denote distinct fundamental entities, it is an appealing principle in its own right, even for those that reject not only Maximalist Classicism, but Classicism. It can be thought of as offering an important grain of truth in the “structured” picture of propositions shown to be inconsistent by the Russell-Myhill paradox (Russell 1903: App. B), (Myhill 1958), (Dorr 2016: §6), (Goodman 2017).

To see that Separated Structure follows from Maximalist Classicism in  $\mathcal{L}(\Sigma)$ , note that since  $C$  is closed under uniform substitution and the  $\zeta$  rule, if  $Fc = Gc$  is a theorem of  $C$  so is  $Fx = Gx$  and hence also  $F = G$ .<sup>58</sup> If  $Fc = Gc$  is not a theorem of  $C$ , then its negation, and hence also the conditional, is a theorem of Maximalist Classicism. To see that Maximalist Classicism (for the given signature) follows from Pure Maximalist Classicism and Separated Structure, we can begin by showing Separated Structure to be equivalent to the schema

$$Fc_1 \dots c_n = Gc_1 \dots c_n \rightarrow F = G$$

where  $F$  and  $G$  are closed and contain *no* non-logical constants, and  $c_1 \dots c_n$  are any distinct non-logical constants. (This can be shown by an obvious induction on the number of nonlogical constants in  $F$  and  $G$ .) So, suppose that  $A \neq B$  is consistent with  $C$ . Enumerating the non-logical constants in  $A$  and  $B$  as  $c_1 \dots c_n$ , we can, using  $\beta$ , show that  $A = Fc_1 \dots c_n$  and  $B = Gc_1 \dots c_n$  for some closed pure terms  $F$  and  $G$ .  $F \neq G$  must also be consistent with  $C$ , so Pure Maximalist Classicism implies  $F \neq G$ . Hence, by the above equivalent of Separated Structure,  $A = Fc_1 \dots c_n \neq Gc_1 \dots c_n = B$ .

Note that the above reasoning applies equally well to maximalizations of many other theories. So long as  $T$  is closed under uniform substitution and  $\zeta$ ,  $\text{Max } T$  will be equivalent to the combination of Separated Structure with the purely logical instances of Distinctness for  $T$ .

The conviction that certain specific properties, relations, and objects are fundamental might motivate someone to accept Maximalist Classicism for a signature that includes constants for those entities. Disagreements about which entities are fundamental will lead to disagreements about which instances of Possibility to accept, but those sympathetic to the Humean idea can at least agree that, whatever language turns out to be fundamental, all instances of Possibility will be true in that language.

Rather than formulate the idea in this metalinguistic way, we could introduce predicates into object language for talking about the status of fundamentality, in the form of a predicate ‘ $\text{Fun}_\sigma$ ’ (of type  $\sigma \rightarrow t$ ) for each type  $\sigma$ . For a sequence of variables  $\vec{v}$  of types  $\sigma_1, \dots, \sigma_n$ , let  $\text{Fun } \vec{v}$  abbreviate the claim that all of  $\vec{v}$  are fundamental and

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<sup>58</sup>The fact that  $a$  doesn’t appear in  $F$  or  $G$  is crucial here, since only in that case is  $(Fc)[x/c]$  the same as  $Fx$ , and similarly for  $G$ .

(when of the same type) distinct:

$$\text{Fun } \vec{v} := \bigwedge_{i \leq n} \text{Fun}_{\sigma_i} v_i \wedge \bigwedge_{i < j \leq n: \sigma_i = \sigma_j} v_i \neq v_j$$

In this language, we can capture the combinatorialist thought with the following schema:

**Fundamental Possibility**  $\text{Fun } \vec{x} \rightarrow \Diamond P$ , where  $P$  is any formula consistent with Classicism with no nonlogical constants and whose free variables are among  $\vec{x}$ .<sup>59</sup>

One noteworthy consequence of Fundamental Possibility is that the denotations of the logical constants are not themselves fundamental: for example, since the formula  $\exists p(p \wedge Xp)$  is consistent in  $\mathbf{C}$ , an instance of Fundamental Possibility is  $\text{Fun } X \rightarrow \Diamond \exists p(p \wedge Xp)$ . Instantiating  $X$  with  $\neg$  gives  $\text{Fun}(\neg) \rightarrow \Diamond \exists p(p \wedge \neg p)$ , which implies  $\neg \text{Fun}(\neg)$  given  $\mathbf{C}$ . The vision thus stands in contrast with the picture we find in Sider (2011), where certain logical constants are supposed to have exactly the same fundamentality-theoretic status as, e.g., certain predicates needed for physics. The maximalist picture goes more naturally with an alternative picture (Bacon 2020, Dorr 2016) on which fundamentality must be distinguished from a different but also metaphysically important status of “Purity” (or “Logicality”), which the denotations of closed terms containing only logical constants all have, but nothing fundamental has.

<sup>59</sup>One worrisome consequence of Fundamental Possibility is that if a binary relation is fundamental, its converse is not fundamental. Since the formulae  $R \neq (\lambda xy.Ryx)$  and  $S \neq (\lambda xy.Ryx)$  are both consistent in  $\mathbf{C}$ , the following are both instances of Fundamental Possibility:

$$\begin{aligned} & \text{Fun } R \rightarrow \Diamond(R \neq (\lambda xy.Ryx)) \\ & (\text{Fun } R \wedge \text{Fun } S \wedge R \neq S) \rightarrow \Diamond(S \neq (\lambda xy.Ryx)) \end{aligned}$$

By the necessity of identity, the possibility operators in the consequents are redundant, and the conjunction of both formulae is in fact equivalent to  $\text{Fun } R \rightarrow \neg \text{Fun}(\lambda xy.Ryx)$ . This consequence is rather alarming: it conflicts with the plausible idea that a relation and its converse are ‘metaphysically on a par’. Bacon (2019) takes this to suggest we should eschew the ideology of ‘fundamentality’, and instead theorize in terms of a polyadic notion of ‘cofundamentality’. Alternatively, we can weaken Fundamental Possibility in such a way as to avoid the above consequence, by strengthening the definition of  $\text{Fun } \vec{v}$  to include, alongside the conjuncts expressing the *distinctness* of distinct  $v_i$  of the same type, further conjuncts requiring other kinds of “logical independence” among distinct  $v_i$ , including  $v_i \neq (\lambda xy.v_jyx)$  when  $v_i$  and  $v_j$  are of type  $\sigma \rightarrow \sigma \rightarrow \tau$ . (Dorr (2016: §9) suggests, in a non-Classical setting, a picture where fundamental entities come in clusters given by certain kinds of “interdefinability” operations.)



## 2.6 Finer-grained strengthenings: beyond Maximalist Classicism

Although Maximalist Classicism is maximally *fine-grained* in our technical sense, it is far from being maximally *strong*. We will conclude our discussion by mentioning various extensions of it that strike us as interesting and attractive, although we have almost no proofs of consistency.

First: we could consider adding some of the further principles discussed in Sections 2.1–2.3 to Maximalist Classicism. (This is different from adding the principles and *then* maximalizing, which also yields interesting theories.) Clearly we cannot consistently add ND, since for any closed  $A$  and  $B$  such that  $A \neq B$  and  $A = B$  are both consistent in Classicism, Maximalist Classicism implies  $A \neq B \wedge \Diamond(A = B)$ . Nor can we add the *necessitations* of any of the other principles. But for all this tells us, we might be able to consistently add non-necessitated BF, Boolean Completeness, Actuality, Atomicity, or Rigid Comprehension principles. In fact Zach Goodsell (p.c.) has shown that Rigid Comprehension is inconsistent with Maximalist Classicism and the construction in Appendix E show that it is consistent with Actualit); the consistency of the other combinations remains to be investigated.

Second: observe that Maximalist Classicism does not rule out the odd hypothesis that there are two magic individuals  $a$  and  $b$  such that  $a \neq b$  entails every true proposition. All kinds of divergences from actuality are broadly possible, but in order to diverge in any other way, the first thing we have to do is to identify  $a$  and  $b$ . This doesn't feel very much in keeping with the "combinatorialist" spirit that motivates Maximalist Classicism, so it is natural to look for strengthenings that rule it out.

One strategy is to make a special provision for type  $e$ , by strengthening Possibility to

$$\text{Possibility+} \quad \left( \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \right) \rightarrow \Diamond P$$

where  $P$  has only free variables  $\vec{x}$ , all of type  $e$ .<sup>60</sup> This seems attractive, on some ways of thinking about what's special about type  $e$ . The proof we give of the consistency of Possibility extends easily to Possibility+.<sup>61</sup> However, Possibility+ doesn't go as far as we might wish, since it is consistent with the same sort of  $a \neq b$  phenomenon arising in, e.g., type  $e \rightarrow t$ .

To explore a different strategy for strengthening Possibility, let's say that a pro-

<sup>60</sup>This could be derived from Fundamental Possibility (see previous section) together with the thesis that every individual is fundamental.

<sup>61</sup>Indeed, it shows that Possibility is consistent with there being only one individual.

position is  $\neq$ -*necessary* just in case it is entailed by the possibility of each truth:

$$\Box_{\neq} := \lambda p. \exists q (q \wedge \Box(\Diamond q \rightarrow p))$$

Dorr, Hawthorne, and Yli-Vakkuri (2021: appendix D) show that ND holds (in every type) for  $\Box_{\neq}$ , and moreover that on several natural definitions of “ND-respecting necessity operation”,  $\Box_{\neq}$  is equivalent to *having every ND-respecting necessity operation*. Intuitively,  $\Box_{\neq}$  is the strongest restriction of  $\Box$  that respects ND. So we can try to capture the thought that lots of things should be able to happen without any distinct entities of any type having to become identical using a schema whose instances are of the form  $\Diamond_{\neq} P$ , for some appropriately wide range of values of  $P$ . The most obvious idea would be to strengthen Possibility by replacing  $\Diamond$  with  $\Diamond_{\neq}$ . But this is clearly inconsistent: the Fregean Axiom (according to which there are only two propositions) is consistent with Classicism, but the result of applying  $\Diamond_{\neq}$  to the Fregean Axiom implies the truth of the Fregean Axiom, since if  $p, q, r$  were three distinct propositions,  $p \neq q \wedge p \neq r \wedge q \neq r$  (which is inconsistent with the Fregean Axiom) would be a  $\neq$ -necessary truth. More generally, whenever some closed  $P$  is consistent with Classicism but not with Maximalist Classicism,  $\Diamond_{\neq} P$  will also be inconsistent with Maximalist Classicism. So the furthest we could hope to go in this direction is to supplement Maximalist Classicism with the following schema:

**Strong Possibility**  $\Diamond_{\neq} P$ , where  $P$  is closed and consistent with Maximalist Classicism.

Note that adding Strong Possibility to Classicism makes Possibility redundant. For if  $P$  is closed and consistent with Classicism,  $\Diamond P$  is a theorem of, and hence (by our consistency result) consistent with Maximalist Classicism, so  $\Diamond_{\neq} \Diamond P$  is an instance of Strong Possibility; but  $\Diamond_{\neq} \Diamond P$  implies  $\Diamond \Diamond P$  and hence  $\Diamond P$ .

We do not know whether Strong Possibility is consistent.

### 3 Model theory for Classicism

In studying systems of higher-order logic, including Classicism, model theory is a crucial tool. A model for a given higher-order language is a mathematical construct according to which we can assign “denotations” to the terms of that language, and ultimately truth values to its formulas. A class of models,  $C$ , is *sound* for a logic,  $T$  (understood as a set of formulae), just in case every member of  $T$  is true in every model in  $C$ , and *complete* for  $T$  just in case every formula true in every model in  $C$  is in  $T$ . Soundness theorems are particularly useful for proving consistency results, but the enterprise also has considerable heuristic value in generating intuitions about the metaphysical worldviews these theories are describing.

Our guiding idea in the search for a useful notion of model for Classicism will be a deep parallel between Extensionalism and Classicism. We began to notice this already in Section 1.4 where we saw that, where Extensionalism could be axiomatized by certain material conditionals (Extensionality or the combination of the Fregean Axiom with Functionality), turning those material conditionals into rules of proof delivered parallel characterizations of Classicism. Our model theory will be based on similar parallels. In model theoretic terms, a material conditional corresponds to an inference rule preserving truth over a single model, and a rule of proof to the preservation of truth-in-all-models from a certain class, or ‘category’ of models. First, in Section 3.1, we present a general model theory for  $H$  and see that models of this sort satisfying a certain natural “extensionality” condition characterize Extensionalism. Models that satisfy this condition can be simplified into a familiar form due to Henkin, which we discuss in Section 3.2. In Section 3.3, we look at categories of models of  $H$  and formulate a condition on such categories which we call “intensionality”, which is in a natural sense a generalization of to the extensionality condition on single models, and stands to the rule Equivalence as extensionality stands to the Extensionality axiom. With this condition in hand, we go on in Section 3.4 to define a simpler class of models that stand to Classicism as Henkin models stand to Extensionalism. In Section 3.5 and Section 3.6, we will construct some simple examples of these “action models”, use them to verify some of the consistency claims we made in part 2, and explain how they relate to and generalize existing notions of model for Classicism.

### 3.1 BBK-models

Benzmüller, Brown, and Kohlhasse (2004) provide a concept of model that they show to be sound and complete for  $H$ . We will need a few preliminary definitions. A **typed collection**  $C$  is a function that maps each type  $\sigma$  to a set  $C^\sigma$ . When  $C$  and  $D$  are typed collections, a **mapping** from  $C$  to  $D$  is a function  $h$  that maps each type  $\sigma$  to a function  $h^\sigma$  from  $C^\sigma$  to  $D^\sigma$ . A **variable assignment**  $g$  for a typed collection  $C$  is a mapping to  $C$  from some typed collection of variables; a variable assignment is **adequate for** a term if it is defined on all variables free in that term. A model consists of (i) a domain,  $\mathbf{M}^\sigma$ , for each type  $\sigma$ , from which the interpretations of terms of that type are drawn and the quantifiers of that type range, (ii) an interpretation function  $\llbracket \cdot \rrbracket_{\mathbf{M}}$  (we drop the subscript when convenient), mapping terms to their interpretations relative to variable assignments, and (iii) a specification of which elements of  $\mathbf{M}^t$  (the propositions) are true and false:

**Definition 3.1.** A **BBK-model** for a signature  $\Sigma$  is a triple  $\mathbf{M} = \langle \mathbf{M}^t, \llbracket \cdot \rrbracket_{\mathbf{M}}, \text{val}_{\mathbf{M}} \rangle$ , where:

- (i)  $\mathbf{M}$  is a typed collection of nonempty sets.
- (ii)  $\llbracket \cdot \rrbracket_{\mathbf{M}}$  is a function that maps each type- $\sigma$  term  $A$  of  $\mathcal{L}(\Sigma)$  and variable assignment  $g$  for  $\mathbf{M}$  that is adequate for  $A$  to an element  $\llbracket A \rrbracket^g$  of  $\mathbf{M}^\sigma$ , such that the following constraints hold whenever the given assignments are adequate for the given terms:
- a.  $\llbracket v \rrbracket^g = g(v)$
  - b. If  $\llbracket A \rrbracket^g = \llbracket C \rrbracket^h$  and  $\llbracket B \rrbracket^g = \llbracket D \rrbracket^h$  then  $\llbracket AB \rrbracket^g = \llbracket CD \rrbracket^h$ .
  - c.  $\llbracket A \rrbracket^g = \llbracket A \rrbracket^h$  when  $g$  and  $h$  agree on all variables free in  $A$ .
  - d.  $\llbracket A \rrbracket^g = \llbracket B \rrbracket^g$  when  $A$  and  $B$  are  $\beta\eta$ -equivalent.<sup>62</sup>
- (iii)  $\text{val}_{\mathbf{M}}$  (the ‘valuation’) is a function from  $\mathbf{M}^t$  to  $\{0, 1\}$ , subject to the following constraints, where ‘ $\mathbf{M}, g \Vdash P$ ’ means ‘ $\text{val}_{\mathbf{M}}\llbracket P \rrbracket^g = 1$ ’, and  $g[v \mapsto \mathbf{a}]$  is the function that agrees with  $g$  on variables other than  $v$  and maps  $v$  to  $\mathbf{a}$ :
- a.  $\mathbf{M}, g \Vdash \neg P$  iff  $\mathbf{M}, g \not\Vdash P$ .
  - b.  $\mathbf{M}, g \Vdash P \wedge Q$  iff  $\mathbf{M}, g \Vdash P$  and  $\mathbf{M}, g \Vdash Q$ .
  - c.  $\mathbf{M}, g \Vdash P \vee Q$  iff  $\mathbf{M}, g \Vdash P$  or  $\mathbf{M}, g \Vdash Q$ .
  - d.  $\mathbf{M}, g \Vdash \forall_\sigma F$  iff  $\mathbf{M}, g[v \mapsto \mathbf{a}] \Vdash Fv$  for every  $\mathbf{a} \in A^\sigma$  ( $v$  not free in  $F$ ).
  - e.  $\mathbf{M}, g \Vdash \exists_\sigma F$  iff  $\mathbf{M}, g[v \mapsto \mathbf{a}] \Vdash Fv$  for some  $\mathbf{a} \in A^\sigma$  ( $v$  not free in  $F$ ).
  - f.  $\mathbf{M}, g \Vdash A = B$  iff  $\llbracket A \rrbracket_{\mathbf{M}}^g = \llbracket B \rrbracket_{\mathbf{M}}^g$ .<sup>63</sup>

Note that by clause (c),  $\llbracket A \rrbracket^g$  is independent of  $g$  when  $A$  is closed; in this case we just write  $\llbracket A \rrbracket$ .  $P$  *holds* in  $\mathbf{M}$  iff  $\mathbf{M}, g \Vdash P$  for all  $g$  adequate for  $P$ ; the *theory of* a class of models is the set of all formulae that hold in all of them.

The point of these definitions arises from the following key theorem:

**Theorem 3.2.** The class of BBK-models is sound and complete for H: for any signature  $\Sigma$ , every theorem of H in  $\mathcal{L}(\Sigma)$  holds in every BBK-model, and every set of sentences consistent in H holds in some BBK-model. Moreover, if  $\Sigma$  is countable, soundness and completeness also holds for models in which each domain  $\mathbf{M}^\sigma$  is a subset of some given countable set, say  $\mathbb{N}$ .

<sup>62</sup>In the most general notion of model explored in Benzmüller, Brown, and Kohlhase 2004,  $\eta$  isn’t baked in. And indeed, one could even be more general by dropping clause (d) altogether in favour of a further constraint on the valuation, if one wanted models of logics like  $H_0$  not including  $\beta$ , as Muskens (2007) does.

<sup>63</sup>Given clause (b), we only need the special cases of these conditions where  $P, Q, F, A$ , and  $B$  are variables.

Soundness is just a matter of checking that each axiom of  $H$  holds in every model and that the conclusions of MP, Gen, and Inst hold if the premises do. We sketch the proof of completeness in a footnote.<sup>64</sup>

Theorem 3.2 automatically yields soundness and completeness theorems for all manner of theories extending  $H$ : whenever  $T$  extends  $H$ , the class of all BBK-models in which all theorems of  $T$  hold is sound and complete for  $T$ . This is not the most *useful* sort of soundness and completeness result, since the definition of BBK-model does not suggest any methods for constructing BBK-models. However, these automatic results can provide the basis for more useful soundness and completeness theorems where the models are characterized in a more intrinsic, concrete, compositional way. Our goal is to do this for Classicism. But we will begin by seeing how it can be done for the much stronger theory Extensionalism (see Section 1.4), which will provide a helpful starting point for the generalization to Classicism.

Certain elements of a BBK model can be associated with functions determined by how those elements apply to their arguments, and with extensions determined by the truth values of the results of those applications. These notions let us distinguish some special classes of BBK models: the functional and functionally full models, and the extensional and extensionally full models.

**Definition 3.3.** Where  $\mathbf{M}$  is a BBK-model:

<sup>64</sup>To prove completeness, we must first show that any  $H$ -consistent set of formulae  $T$  in  $\mathcal{L}(\Sigma)$  can be extended to a consistent set of formulae  $T^+$  in an expanded language  $\mathcal{L}(\Sigma^+)$ , where  $T^+$  is both negation complete ( $\neg A \in T^+$  whenever  $A \notin T^+$ ) and witness-complete (whenever  $\exists F \in T^+$ ,  $Fc \in T^+$  for some constant  $c$ ). The proof of this fact, “Henkin’s Lemma”, is exactly the same as the corresponding proof for first-order logic—indeed the original version of this result by Henkin (1950) was in a higher order setting. Given a consistent, negation-complete, witness-complete  $T^+$ , we form a BBK-model  $\mathbf{M}_{T^+}$  for  $\mathcal{L}(\Sigma^+)$  as follows. Each domain  $\mathbf{M}_{T^+}^\sigma$  is the set of equivalence classes of closed type- $\sigma$  terms of  $\mathcal{L}(\Sigma^+)$  under the equivalence relation  $\approx_{T^+}$ , where  $A \approx_{T^+} B$  iff  $T^+ \vdash A = B$ .  $\llbracket A \rrbracket_{\mathbf{M}_{T^+}}^\sigma$  is the equivalence class of all the closed terms that can be derived from  $A$  by replacing every free occurrence of any variable  $v$  with any member of  $gv$ . And for  $\mathbf{p} \in \mathbf{M}_{T^+}^t$ ,  $\text{val}_{\mathbf{M}_{T^+}}(\mathbf{p}) = 1$  iff  $P \in T$  for any (or equivalently, all)  $P \in \mathbf{p}$ . It remains to show that  $\mathbf{M}_{T^+}$  is indeed a BBK-model in which every member of  $T$  holds. The properties of substitution secure that  $\llbracket \cdot \rrbracket_{\mathbf{M}_{T^+}}$  meets constraints (ii.a–c), while the fact that  $T^+$  is closed under  $\beta\eta$ -equivalences secures (d). The consistency and negation-completeness of  $T^+$  and the PC-rules for  $\wedge$  and  $\vee$  guarantee that  $\text{val}_{\mathbf{M}_{T^+}}$  is well-behaved with respect to  $\neg$ ,  $\wedge$ , and  $\vee$ ; the witness-completeness of  $T^+$  takes care of one direction of the biconditionals for  $\forall$  and  $\exists$ ; the fact that  $T^+$  contains every instance of UI and EG takes care of the other directions; finally, the fact that  $T^+$  contains Ref and LL yields the biconditional for  $=$ .

If  $\mathcal{L}$  is countable, we can set things up so that  $\mathcal{L}^+$  is also countable, in which case the domain of  $\mathbf{M}_{T^+}$  in each type is countable as well. The proof in Benzmüller, Brown, and Kohlhasse 2004 establishes the completeness of BBK-models for a certain *cut-free sequent calculus*, which requires a proof substantially more complicated than the proof we have sketched here, which is essentially due to Henkin (1950).

- The *applicative behaviour*,  $\text{app}_{\mathbf{M}} \mathbf{d}$ , of an element  $\mathbf{d} \in \mathbf{M}^{\sigma \rightarrow \tau}$ , is the function  $\mathbf{a} \mapsto \llbracket X y \rrbracket^{[X \mapsto \mathbf{d}, y \mapsto \mathbf{a}]}$  from  $\mathbf{M}^{\sigma}$  to  $\mathbf{M}^{\tau}$ .<sup>65</sup>
- The *extension*,  $\text{ext}_{\mathbf{M}} \mathbf{d}$ , of an element  $\mathbf{d} \in \mathbf{M}^{\sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow t}$  is the set

$$\{\langle \mathbf{a}_1, \dots, \mathbf{a}_n \rangle \in \mathbf{M}^{\sigma_1} \times \dots \times \mathbf{M}^{\sigma_n} \mid \mathbf{M}, [X \mapsto \mathbf{d}, y_i \mapsto \mathbf{a}_i] \Vdash X \vec{y}\}^{66}$$

**Definition 3.4.** A BBK-model  $\mathbf{M}$  is:

- *functional* iff its applicative behaviour functions are injective for all  $\sigma, \tau$ : that is, for any  $\mathbf{d} \neq \mathbf{d}'$  in  $\mathbf{M}^{\sigma \rightarrow \tau}$ , there is some  $\mathbf{a} \in \mathbf{M}^{\sigma}$  such that  $\text{app}_{\mathbf{M}} \mathbf{d}(\mathbf{a}) \neq \text{app}_{\mathbf{M}} \mathbf{d}'(\mathbf{a})$ .
- *extensional* iff its extension functions are injective: whenever  $\text{ext}_{\mathbf{M}} \mathbf{a} = \text{ext}_{\mathbf{M}} \mathbf{b}$ ,  $\mathbf{a} = \mathbf{b}$ .
- *Fregean* iff  $\text{val}_{\mathbf{M}}$  is injective, or equivalently, iff  $\mathbf{M}^t$  has exactly two elements.<sup>67</sup>

These three properties of models bear a special relationship to the Functionality schema ( $\forall z(Xz = Yz) \rightarrow X = Y$ ), Extensionality schema ( $\forall \vec{z}(X\vec{z} \leftrightarrow Y\vec{z}) \rightarrow X = Y$ ), and the Fregean Axiom ( $(p \leftrightarrow q) \rightarrow p = q$ ): it is easy to check that a model is functional (extensional, Fregean) iff all instances of the Functionality schema (Extensionality schema, Fregean Axiom) hold in it. (See Section 1.4 for discussion of these principles.) Extensionality is thus equivalent to the combination of functionality and Fregeanness. And as a consequence of Theorem 3.2, we have:

**Theorem 3.5.** The class of extensional BBK-models is sound and complete for Extensionalism.

It is also useful to have special terminology for BBK-models in which the applicative or extension maps are *surjective*. A model  $\mathbf{M}$  is:

- *functionally full* iff  $\text{app}_{\mathbf{M}}$  is surjective on each domain: every function from  $\mathbf{M}^{\sigma}$  to  $\mathbf{M}^{\tau}$  is the applicative behaviour of some element of  $\mathbf{M}^{\sigma \rightarrow \tau}$ .

<sup>65</sup>We use the symbol  $\mapsto$  to denote functions: ' $\mathbf{a} \mapsto \llbracket X y \rrbracket^{[X \mapsto \mathbf{d}, y \mapsto \mathbf{a}]}$ ' means 'the function whose value for any  $\mathbf{a}$  is  $\llbracket X y \rrbracket^{[X \mapsto \mathbf{d}, y \mapsto \mathbf{a}]}$ '. By clauses (b) and (c), it doesn't matter which variables we pick to be  $X$  and  $y$ . Benzmüller, Brown, and Kohlhasse (2004) treat the application map as a separate ingredient in the definition of "model", but since it can be recovered from  $\llbracket \cdot \rrbracket$  we omit it.

<sup>66</sup>Muskens (2007) treats the extension map as a primitive ingredient in the definition of "model", but since it can be recovered from  $\text{val}_{\mathbf{M}}$  (equivalent to the extension map for type  $t$ ), we take only the latter as primitive. ' $[X \mapsto \mathbf{d}, y_i \mapsto \mathbf{a}_i]$ ' means 'the assignment that maps the variable  $X$  to  $\mathbf{d}$ , maps each  $y_i$  to  $\mathbf{a}_i$ , and is undefined on all other variables'.

<sup>67</sup>It must have at least two elements, since no element of type  $t$  can have the same truth value as its negation.

- **extensionally full** iff  $\text{ext}_{\mathbf{M}}$  is surjective on each domain: every subset  $X$  of  $\mathbf{M}^{\sigma_1} \times \dots \times \mathbf{M}^{\sigma_n}$  is the extension of some element of  $\mathbf{M}^{\sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow t}$ .

Functional fullness implies, though it is not implied by, extensional fullness. Analogous to the ways in which the Extensionality and Functionality schemas characterise the eponymous properties of BBK-models, one might hope to find some axioms which characterise functional or extensional fullness. But no such axioms exist: Gödel’s first incompleteness theorem implies that neither of the properties is captured by any recursively enumerable axiom-scheme.<sup>68</sup>

### 3.2 Henkin models

Every extensional (functional and Fregean) BBK-model is equivalent to a model where the elements of a given functional type are simply *identical* to their applicative behaviours, and the elements of propositional type are simply identical to their truth values. The operation of turning certain kinds of models into more “concrete” ones will be important later in analogous settings, so we shall present it in some detail. The relevant concrete models are known as *Henkin* models, (after Henkin 1950). In a Henkin model  $\mathbf{H}$ ,  $\mathbf{H}^e$  can still be any nonempty set, but  $\mathbf{H}^t$  must be  $\{0, 1\}$ , and  $\mathbf{H}^{\sigma \rightarrow \tau}$  must be some subset of  $(\mathbf{H}^\tau)^{\mathbf{H}^\sigma}$ .

Working with this more concrete kind of model has one very significant advantage. BBK-models that are not concrete have few practical uses because they are not constructed compositionally from the interpretations of the constants;  $\llbracket \cdot \rrbracket$  is a function defined on all terms of the language that must satisfy some highly non-trivial constraints. By contrast, to specify a Henkin model one need only specify the interpretations of the non-logical constants; this interpretation extends uniquely to an interpretation of all the terms. However, we do need to ensure that the domains are sufficiently full that we will be able to provide an interpretation for every term relative to every variable assignment. To capture this, we first define a notion of *premodel* whose domains need not be sufficiently full; then recursively define a *partial* interpretation function for any premodel; and finally define a *model* to be a premodel whose interpretation function is full. Spelling this out, we get the following.

**Definition 3.6.** (i) A **Henkin premodel** for a signature  $\Sigma$  is an ordered pair  $\mathbf{H} = \langle \mathbf{H}, \mathcal{I} \rangle$ , where  $\mathbf{H}^e$  is a nonempty set,  $\mathbf{H}^t = \{0, 1\}$ , and  $\mathbf{H}^{\sigma \rightarrow \tau} \subseteq (\mathbf{H}^\tau)^{\mathbf{H}^\sigma}$ , and  $\mathcal{I}$  is a function that takes each nonlogical constant  $c : \sigma$  in  $\Sigma$  to an element of  $\mathbf{H}^\sigma$ .

(ii) When  $\mathbf{H}$  is any Henkin premodel for  $\mathcal{L}$ ,  $\llbracket \cdot \rrbracket_{\mathbf{H}}$  is the partial function that takes a type- $\sigma$  term  $A$  and an assignment function  $g$  for  $\mathbf{H}$  to something of the right sort

<sup>68</sup>For a bit more detail, see Dorr 2016: n. 106.

to be in  $\mathbf{H}^\sigma$ , in accordance with the following clauses:

$$\begin{array}{ll}
\llbracket AB \rrbracket^g = \llbracket A \rrbracket^g(\llbracket B \rrbracket^g) & \llbracket \lambda v. A \rrbracket^g = \mathbf{a} \mapsto \llbracket A \rrbracket^{g[v \mapsto \mathbf{a}]} \\
\llbracket c \rrbracket^g = \mathcal{I}(c) & \llbracket v \rrbracket^g = g(v) \\
\llbracket \neg \rrbracket^g = n \mapsto 1 - n & \llbracket =_\sigma \rrbracket^g = \mathbf{a} \mapsto \left( \mathbf{b} \mapsto \begin{cases} 1 & \text{if } \mathbf{a} = \mathbf{b} \\ 0 & \text{otherwise} \end{cases} \right) \\
\llbracket \wedge \rrbracket^g = n \mapsto (m \mapsto \min\{n, m\}) & \llbracket \vee \rrbracket^g = n \mapsto (m \mapsto \max\{n, m\}) \\
\llbracket \forall_\sigma \rrbracket^g = \mathbf{d} \mapsto \min\{\mathbf{d}(\mathbf{a}) \mid \mathbf{a} \in \mathbf{H}^\sigma\} & \llbracket \exists_\sigma \rrbracket^g = \mathbf{d} \mapsto \max\{\mathbf{d}(\mathbf{a}) \mid \mathbf{a} \in \mathbf{H}^\sigma\}
\end{array}$$

(The first clause means that  $\llbracket AB \rrbracket^g$  exists so long as  $\llbracket A \rrbracket^g$  and  $\llbracket B \rrbracket^g$  exist, and is in that case equal to  $\llbracket A \rrbracket^g(\llbracket B \rrbracket^g)$ .)

(iii) A **Henkin model** for  $\mathcal{L}$  is a Henkin premodel  $\mathbf{H}$  for  $\mathcal{L}$  such that  $\llbracket A \rrbracket_{\mathbf{H}}^g$  exists and is in  $\mathbf{H}^\sigma$  for every type- $\sigma$  term  $A$  and assignment function  $g$  adequate for  $A$ .<sup>69</sup>

(iv) A formula  $P$  **holds** in  $\mathbf{H}$  on  $g$ —in symbols,  $\mathbf{H}, g \vdash P$ —iff  $\llbracket P \rrbracket^g = 1$ .  $P$  holds in  $\mathbf{H}$  iff  $\mathbf{H}, g \Vdash P$  for every  $g$  adequate for  $P$ .

Henkin (1950) (as corrected by Andrews 1972) establishes that the class of Henkin models is sound and complete for Extensionalism. To set the stage for our discussion of Classicism below, we observe that this fact can be derived as a corollary of Theorem 3.2. The derivation uses the following two results.

**Proposition 3.7.** Every Henkin model,  $\mathbf{H}$ , is an extensional BBK-model, with its defined interpretation function  $\llbracket \cdot \rrbracket_{\mathbf{H}}$  and the identity on  $\{0, 1\}$  as valuation.

**Proposition 3.8.** For every extensional BBK-model  $\mathbf{M}$ , there is a Henkin model  $\mathbf{H}_{\mathbf{M}}$  in which the same formulae hold.

Verifying Proposition 3.7 boils down to checking that when  $A$  and  $B$  are  $\beta\eta$ -equivalent terms,  $\llbracket A \rrbracket_{\mathbf{H}}^g = \llbracket B \rrbracket_{\mathbf{H}}^g$  whenever defined. For Proposition 3.8, the idea is to construct each domain  $\mathbf{H}_{\mathbf{M}}^\sigma$  as the range of an injective function  $f^\sigma$  on  $\mathbf{M}^\sigma$ , where  $f^e$  is just the identity;  $f^t$  is  $\text{val}_{\mathbf{M}}$ ; and  $f^{\sigma \rightarrow \tau}$  is defined by systemically replacing each element  $\mathbf{d}$  with  $\text{app}_{\mathbf{M}} \mathbf{d}$ .<sup>70</sup>

By Proposition 3.7 and the soundness part of Theorem 3.5, every theorem of Extensionalism holds in every Henkin model. And by 3.8 and the completeness

<sup>69</sup>More “intrinsic” ways of expressing this condition are known. For example, it can be shown (see, e.g. Bacon unpublished) that Henkin premodel  $\mathbf{H}$  is a Henkin model so long as (i) the domains are closed under application; (ii) the denotations of the logical constants (as given above) all belong to the domain of the appropriate type, and (iii) for any type  $\sigma$  and relational types  $\rho, \tau$ ,  $\mathbf{H}^{(\sigma \rightarrow \rho \rightarrow \tau) \rightarrow (\sigma \rightarrow \rho) \rightarrow \sigma \rightarrow \tau}$  contains the function  $S_{\sigma, \rho, \tau} := \mathbf{d} \mapsto (\mathbf{b} \mapsto (\mathbf{a} \mapsto \mathbf{d}(\mathbf{a})(\mathbf{b}(\mathbf{a}))))$  and  $\mathbf{H}^{\tau \rightarrow \sigma \rightarrow \tau}$  contains the function  $K_{\sigma, \tau} := \mathbf{a} \mapsto (\mathbf{b} \mapsto \mathbf{a})$ .

<sup>70</sup>Appendix C proves a more general result that has Proposition 3.8 as a special case.



part of Theorem 3.5, every formula consistent with Extensionalism holds on some assignment in some Henkin model. Thus:

**Theorem 3.9.** The class of Henkin models is sound and complete for Extensionalism.

### 3.3 Categories of BBK-models

To do for Classicism what Henkin did for Extensionalism, we will consider properties of *collections* of BBK-models analogous to the properties of individual BBK-models that make for the truth of Extensionalism. It will turn out that the properties of interest are properties not of mere collections of BBK-models but of *categories* of BBK-models—collections of models with a specified collection of homomorphisms between them. So, the first thing we will need is an appropriate notion of homomorphism for BBK-models (for a given signature). As usual in model theory, a homomorphism is a mapping that preserves interpretations. More carefully, a homomorphism  $h$  from  $\mathbf{M}$  to  $\mathbf{N}$  is a typed family of functions  $h$ , where  $h^\sigma : \mathbf{M}^\sigma \rightarrow \mathbf{N}^\sigma$  and for any term  $A$  and assignment function  $g$  for  $\mathbf{M}$  that is adequate for  $A$ ,

$$h^\sigma \llbracket A \rrbracket_{\mathbf{M}}^g = \llbracket A \rrbracket_{\mathbf{N}}^{h \circ g}$$

Here,  $h \circ g$  is the assignment function for  $\mathbf{N}$  that maps each type- $\sigma$  variable  $v$  to  $h^\sigma(gv)$ .<sup>71</sup>

Evidently, the composition of any two homomorphisms is a homomorphism, and the identity mapping that maps every element of every domain of a model to itself is a homomorphism from that model to itself. This means that if we take any class of BBK-models and any class of homomorphisms between those models, so long as the latter class is closed under composition and contains all the identity homomorphisms on the models, they will form a *category* according to the standard definition: a class of “objects” (here, the models) and a class of “arrows” (here, the homomorphisms) together with a pair of mappings *src* and *trg* that assigns each arrow a unique “source” and “target” object; a mapping  $\circ$  (here, function-composition) that takes any arrows  $f$  and  $g$  where the source of  $g$  is the target of  $f$  to an arrow  $g \circ f$  which shares a source with  $f$  and a target with  $g$ ; and a mapping  $1$  that takes every object  $A$

<sup>71</sup>Note that the valuation functions  $\text{val}_{\mathbf{M}}$  and  $\text{val}_{\mathbf{N}}$  play no role in this definition: models that are isomorphic (in the sense that there are mutually inverse homomorphisms between them) can thus make different sentences true. Given this there is a natural sense in which homomorphisms relate what BBK call “structures” (BBK-models minus valuations) rather than models. Nevertheless, we will think of the source and target of homomorphisms as models, and indeed take these to be “built in”, so that for each homomorphism, there is a unique model that is its source and another unique model that is its target.

to an arrow  $1_A$  with source and target  $A$ , such that  $h \circ (g \circ f) = (h \circ g) \circ f$ ,  $f \circ 1_A = f$  and  $1_A \circ f = f$ .

Given a BBK-model  $\mathbf{M}$  and a category  $\mathcal{C}$  to which it belongs, we can think of the set of homomorphisms in  $\mathcal{C}$  with source  $\mathbf{M}$ , which we will call  $\mathbf{M}^P$ , as playing a role similar to that of possible worlds in the standard semantics for modal logic. Any element of  $\mathbf{M}^t$  can be assigned not just a truth-value but a “truth value profile”—the subset of  $\mathbf{M}^P$  comprising those homomorphisms that map it to a truth. Likewise, an element of a relational type can be assigned not just an *extension* but an *intension*, which we get by looking at the extensions of the results of transporting it using the homomorphisms, and each element of a functional type can be assigned an *applicative behaviour profile*, by looking at the applicative behaviours of the results of transporting it using the homomorphisms.

**Definition 3.10.** Where  $\mathcal{C}$  is a category of BBK-models and  $\mathbf{M}$  is an object of  $\mathcal{C}$ ,

- The *applicative behaviour profile*  $\text{app}_{\mathbf{M}}^{\mathcal{C}} \mathbf{d}$  of any  $\mathbf{d} \in \mathbf{M}^{\sigma \rightarrow \tau}$  is the function such that for any pair  $\langle h, \mathbf{a} \rangle$  where for some  $\mathbf{N}$ ,  $h : \mathbf{M} \rightarrow \mathbf{N}$  and  $\mathbf{a} \in \mathbf{N}^{\sigma}$ ,

$$\text{app}_{\mathbf{M}}^{\mathcal{C}} \mathbf{d} \langle h, \mathbf{a} \rangle = \text{app}_{\mathbf{N}}(h^{\sigma \rightarrow \tau} \mathbf{d}) \mathbf{a}$$

- The *intension*  $\text{int}_{\mathbf{M}}^{\mathcal{C}} \mathbf{d}$  of any  $\mathbf{d} \in \mathbf{M}^{\sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow t}$  is the set of all  $n + 1$ -tuples  $\langle h, \mathbf{a}_1, \dots, \mathbf{a}_n \rangle$  such that for some  $\mathbf{N}$ ,  $h : \mathbf{M} \rightarrow \mathbf{N}$  and  $\langle \mathbf{a}_1, \dots, \mathbf{a}_n \rangle \in \text{ext}_{\mathbf{N}}(h^{\sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow t} \mathbf{d})$
- The *truth value profile*  $\text{val}_{\mathbf{M}}^{\mathcal{C}} \mathbf{p}$  of any  $\mathbf{p} \in \mathbf{M}^t$  is the set of all arrows with source  $\mathbf{M}$  that map  $\mathbf{p}$  to a truth:

$$\bigcup_{\mathbf{N}} \{h : \mathbf{M} \rightarrow \mathbf{N} \mid \text{val}_{\mathbf{N}}(h^t \mathbf{p}) = 1\}$$

These notions generalize the analogous operations  $\text{app}$ ,  $\text{ext}$ , and  $\text{val}$ , for single BBK-models, in the sense that they end up being equivalent on a category consisting of a single BBK-model with the identity homomorphism.

We can then consider some special categories of BBK-models in which these functions are injective.

**Definition 3.11.** A category  $\mathcal{C}$  of BBK-models is

- *quasi-Fregean* iff  $\text{val}_{\mathbf{M}}^{\mathcal{C}}$  is injective for each  $\mathbf{M}$  in  $\mathcal{C}$ .
- *quasi-functional* iff  $\text{app}_{\mathbf{M}}^{\mathcal{C}}$  is injective for each  $\mathbf{M}$  in  $\mathcal{C}$ .
- *intensional* iff  $\text{int}_{\mathbf{M}}^{\mathcal{C}}$  is injective for each  $\mathbf{M}$  in  $\mathcal{C}$ .

Informally, the first condition corresponds to the idea that propositions are individuated by their truth-values across modal space, the second to the idea that operations are individuated by their applicative behaviour across modal space, and the last condition to the idea that relations are individuated by their extensions across modal space. These are similarly generalizations of the notions of Fregeanness, functionality, and extensionality, with which they coincide in a one-object category with just the identity homomorphism. Moreover, in analogy with the discussion of extensional models, we can see that intensionality is equivalent to the combination of quasi-functionality and quasi-Fregeanness.<sup>72</sup>

And as with our discussion of Extensionalism we will see that there is a close correspondence between these conditions and the rule corresponding to the Fregean Axiom, Functionality and Extensionality, namely Propositional Equivalence,  $\zeta$ , and Equivalence. This correspondence lets us prove the following key result:

**Theorem 3.12.** The class of BBK-models that belong to an intensional category of BBK-models is sound and complete for Classicism.

For the soundness direction, the central observation is that the theory of any quasi-Fregean category of BBK-models (the set of formulae that hold in all of them) is closed under Propositional Equivalence. If  $P \leftrightarrow Q$  is true on all assignments in every model in  $\mathcal{C}$ , there's no way for a homomorphism to pull apart the truth values of the interpretations of  $P$  and  $Q$  in a model, which given quasi-Fregeanness means that denotations must be identical. Likewise, the theory of any quasi-functional category of BBK models is closed under  $\zeta$ . For if  $Fx = Gx$  is true on all assignments in every model in  $\mathcal{C}$ , then there is no way for a homomorphism to pull apart the applicative behaviours of the interpretations of  $F$  and  $G$  in any model on any assignment, which given quasi-functionality, forces those denotations to be identical. For the completeness direction, the fact we need is that for any theory  $T$  closed under both Propositional Equivalence and  $\zeta$ —and hence in particular Classicism—the category of *all* models of  $T$  and all homomorphisms between these models is quasi-Fregean and quasi-functional.<sup>73</sup>

<sup>72</sup>Quasi-Fregeanness is just the  $n = 0$  special case of intensionality; given quasi-functionality, we can extend this by induction on  $n$ .

<sup>73</sup>For Propositional Equivalence, suppose we have a model  $\mathbf{M}$  of  $T$  with  $\mathbf{p}, \mathbf{q} \in \mathbf{M}^t$  such that  $\mathbf{p} \neq \mathbf{q}$ . Consider the expanded language  $\mathcal{L}_{\mathbf{M}}$  in which every element of  $\mathbf{M}^\sigma$  is a constant of type  $\sigma$ . Let  $\mathbf{M}^+$  be the model derived from  $\mathbf{M}$  by extending its interpretation function to terms of  $\mathcal{L}_{\mathbf{M}}$ , with the new constants interpreted as denoting themselves. (More carefully: for any term  $A$  of  $\mathcal{L}_{\mathbf{M}}$ , we find a term  $A'$  by replacing each new constant  $\mathbf{a}$  with a distinct variable  $v_{\mathbf{a}}$  that doesn't already occur in  $A$ , and set  $\llbracket A \rrbracket_{\mathbf{M}^+}^g = \llbracket A' \rrbracket_{\mathbf{M}}^{g[v_{\mathbf{a}} \mapsto \mathbf{a}]}$ .) Let  $T^+$  be the result of adding to  $T$  all closed identities in  $\mathcal{L}_{\mathbf{M}}$  that are true in  $\mathbf{M}^+$ , along with  $\neg(\mathbf{p} \leftrightarrow \mathbf{q})$ .  $T^+$  must be consistent. For if it were inconsistent there would be a finite collection of identities,  $A_1, \dots, A_n$  in  $T^+$  such that  $T \vdash A_1 \wedge \dots \wedge A_n \rightarrow (\mathbf{p} \leftrightarrow \mathbf{q})$ .

Indeed, we can sharpen the proof of Theorem 3.12 to show that for any infinite set—e.g.  $\mathbb{N}$ —the category of BBK-models of Classicism whose domains are all subsets of this set is also quasi-Fregean and quasi-functional, so we also have a soundness and completeness theorem for BBK models that belong to categories that are constrained in this way. Note that, unlike the category of all BBK models of Classicism, such categories are *small*—i.e. there is a set of objects and arrows. This strengthening of the theorem will be useful in the next section.

In Section 3.1 we also discussed “functional fullness” and “extensional fullness” conditions on BBK-models, defined by the surjectiveness of the applicative behaviour mapping  $\text{app}_{\mathbf{M}}$  and extension mapping  $\text{ext}_{\mathbf{M}}$ . These also have analogues at the level of categories; however, we need to think a little about what sets it would make sense to require the relevant functions  $\text{app}_{\mathbf{M}}^C$  and  $\text{int}_{\mathbf{M}}^C$  to be surjective to. For any category of BBK-models  $C$  and model  $\mathbf{M}$  in  $C$ , define:

- $\mathbf{M}^{[\sigma_1, \dots, \sigma_n]}$  to be the powerset of the set of all  $n+1$ -tuples  $\langle h, \mathbf{a}_1, \dots, \mathbf{a}_n \rangle$ , where for some  $\mathbf{N}$ ,  $h : \mathbf{M} \rightarrow \mathbf{N}$  and each  $\mathbf{a}_i \in \mathbf{N}^{\sigma_i}$ .
- $\mathbf{M}^{\sigma \Rightarrow \tau}$  to be the set of all functions  $\alpha$  which take an ordered pair  $\langle h, \mathbf{a} \rangle$  such that for some  $\mathbf{N}$ ,  $h : \mathbf{M} \rightarrow \mathbf{N}$  and  $\mathbf{a} \in \mathbf{N}^\sigma$ , and yield an element of  $\mathbf{N}^\tau$ , and are “well-behaved” in the following sense: for any  $h : \mathbf{M} \rightarrow \mathbf{N}$ ,  $i : \mathbf{N} \rightarrow \mathbf{O}$ , and  $\mathbf{a} \in \mathbf{N}^\sigma$ ,  $\alpha \langle i \circ h, i^\sigma \mathbf{a} \rangle = i^\tau (\alpha \langle h, \mathbf{a} \rangle)$ .

We can then define a category of  $C$  of BBK-models to be:

- *intensionally full* iff  $\text{int}_{\mathbf{M}}^C$  is a surjection from each  $\mathbf{M}^{\sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow t}$  to  $\mathbf{M}^{[\sigma_1, \dots, \sigma_n]}$ .
- *quasi-functionally full* iff  $\text{app}_{\mathbf{M}}^C$  is a surjection from each  $\mathbf{M}^{\sigma \rightarrow \tau}$  to  $\mathbf{M}^{\sigma \Rightarrow \tau}$ .

To motivate the first of these definitions, note that by the definition of  $\text{app}_{\mathbf{M}}^C$ ,  $\text{app}_{\mathbf{M}}^C \mathbf{d}$  must always obey the well-behavedness condition, since homomorphisms must commute with application.

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But since  $T$  is a H-theory closed under Propositional Equivalence, it is closed under Necessitation, so  $T \vdash \Box A_1 \wedge \dots \wedge \Box A_n \rightarrow \Box(\mathbf{p} \leftrightarrow \mathbf{q})$ . But since  $A_1, \dots, A_n$  are identities, they imply their own necessitations, so  $T^+ \vdash \Box(\mathbf{p} \leftrightarrow \mathbf{q})$  and thus also  $T^+ \vdash \mathbf{p} = \mathbf{q}$ , which is impossible since  $\mathbf{M}^+$  is a model of  $T^+$  in which  $\mathbf{p} = \mathbf{q}$  is false. So by Theorem 3.2, there must be a model  $\mathbf{N}^+$  of  $T^+$ . Let  $\mathbf{N}$  be the model for  $\mathcal{L}$  obtained by restricting  $\mathbf{N}^+$ 's interpretation function to  $\mathcal{L}$ , and let  $h$  be the mapping that sends each element  $\mathbf{a}$  of  $\mathbf{M}^{+\sigma}$  to  $\llbracket \mathbf{a} \rrbracket_{\mathbf{N}^+}$  (i.e., the interpretation of  $\mathbf{a}$ , considered as a constant of  $\mathcal{L}_{\mathbf{M}}$ , in  $\mathbf{N}^+$ ). It is easy to see that  $h$  is a homomorphism from  $\mathbf{M}$  to  $\mathbf{N}$ . So, as desired, we get a homomorphism in our category that maps  $\mathbf{p}$  and  $\mathbf{q}$  to elements with different truth values.

To finish the proof, it suffices to show that if Modalized Functionality (which can be derived from Propositional Equivalence and  $\zeta$ ) holds in a quasi-Fregean category, the category must also be quasi-functional. This is straightforward, since Modalized Functionality is in effect just the object-language version of quasi-functionality.

In a single BBK-model, the  $n = 0$  case of extensional fullness is automatically satisfied (since each truth-value is guaranteed to be had by some type- $t$  element). By contrast, the  $n = 0$  case of intensional fullness is non-trivial. We'll call a category of BBK-models that satisfies this condition **propositionally full**: every member of  $\mathbf{M}^P$  (i.e., set of homomorphisms with source  $\mathbf{M}$ ) is the truth-value profile of some element of  $\mathbf{M}^t$ .<sup>74</sup>

### 3.4 Action models

We saw in Section 3.2 that every extensional BBK-model is isomorphic to a “concrete” Henkin model, with a compositionally defined interpretation function, in which propositions are identical to their truth values, and elements of functional type are identical to their applicative behaviours. Our strategy in this section will be to find similarly “concrete” and compositional models for each intensional category of BBK-models, in which propositions are identical to their truth-value *profiles* (which specify their truth value under each homomorphism) and elements of functional type are identical to their applicative behaviour *profiles* (which specify their applicative behaviour under each homomorphism). The main upshot of this, apart from introducing a workable notion of model for proving consistency results, is that we will be able to transfer our soundness and completeness theorems for intensional categories of BBK-models to our more concrete models, just as we did for extensional BBK-models and Henkin models.

The key concept we will need in order to carry out this strategy is that of an *action* on some category (a.k.a. a functor from that category to **Set**). This will play roughly the role of a “domain” that was being played by mere sets in Henkin models.

**Definition 3.13.** An *action* on a category  $C$  is a function  $-^*$  that associates each object  $A$  of  $C$  with a set  $A^*$ , and each arrow  $h : A \rightarrow B$  of  $C$  with a function  $h^* : A^* \rightarrow B^*$ , in such a way that

- $i^* \circ h^* = (i \circ h)^*$  for any composable arrows  $i$  and  $h$
- $1_A^*$  is the identity function on  $A^*$ .

**Example 3.14.** Where  $C$  is a category of BBK-models, each type  $\sigma$  determines an action  $-^\sigma$  of applying the superscript to a model or homomorphism, respectively:

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<sup>74</sup>The connection between these properties is a little more intricate than in the extensional case. Whereas functional fullness implied extensional fullness, quasi-functional fullness does not imply intensional fullness (since the latter does whereas the former does not imply propositional fullness). However, the conjunction of quasi-functional and propositional fullness does imply intensional fullness. Meanwhile, in intensional categories, intensional fullness coincides with the combination of quasi-functional and propositional fullness.

- $-\sigma$  takes a model  $M$  in the category to the set  $M^\sigma$
- $-\sigma$  takes each homomorphism  $h : \mathbf{M} \rightarrow \mathbf{N}$  in  $C$  to the function  $h^\sigma : \mathbf{M}^\sigma \rightarrow \mathbf{N}^\sigma$ .

As another example, we can treat all possible truth-value profiles as an action:

**Example 3.15.** For any category  $C$ , the *powerset action* on  $C$  is the action  $-^P$  where for any object  $A$ ,  $A^P$  is the powerset of the set of all arrows with source  $A$ , and for any arrow  $h : A \rightarrow B$  and set of arrows  $X \in A^P$ ,  $h^P X$  is the set of all arrows with source  $B$  which yield a member of  $X$  when composed with  $h$ , i.e.

$$\bigcup_C \{i : B \rightarrow C \mid i \circ h \in X\}.$$

(This is sometimes called the result of ‘dividing’  $X$  by  $h$ .)

The truth-value profile of any element of  $\mathbf{M}^t$  is an element of  $\mathbf{M}^P$ . Moreover, the action of any homomorphism of BBK-models on the propositional elements induces an action on their truth-value profiles: the truth-value profile of  $h^t \mathbf{p}$  is the set of homomorphisms  $i$  such that  $i^t(h^t \mathbf{p})$  is true; i.e. the set of  $i$  such that  $i \circ h$  belongs to the truth-value profile of  $\mathbf{p}$ . So the induced action of  $h$  on a truth-value profile  $X$  can be stated intrinsically in terms of  $X$ , and is just  $h^P X$  as defined above.

We can likewise treat all possible applicative behaviour profiles as an action:

**Example 3.16.** Suppose  $-^*$  and  $-\dagger$  are actions on  $C$ . Then the *exponential action* is the action  $-\overset{*}{\Rightarrow} \dagger$  on  $C$  such that:

- For every object  $A$ ,  $A^{\overset{*}{\Rightarrow} \dagger}$  is the set of all functions  $\alpha$  whose domain is the set of all pairs  $\langle h, x \rangle$ , where for some object  $B$ ,  $h : A \rightarrow B$  and  $x \in B^*$ ; which map each such  $\langle h, x \rangle$  to a member of  $B^\dagger$ ; and which are well-behaved in the following sense:  $i^\dagger(\alpha \langle h, x \rangle) = \alpha \langle i \circ h, i^* x \rangle$  for any  $h : A \rightarrow B$ ,  $i : B \rightarrow C$ , and  $x \in B^*$ .
- For every arrow  $h : A \rightarrow B$ ,  $h^{\overset{*}{\Rightarrow} \dagger}$  is the function such that for any  $\alpha \in A^{\overset{*}{\Rightarrow} \dagger}$ ,  $i : B \rightarrow C$ , and  $x \in C^*$ ,  $(h^{\overset{*}{\Rightarrow} \dagger} \alpha) \langle i, x \rangle = \alpha \langle i \circ h, x \rangle$ .

The applicative behaviour profile of any element of  $\mathbf{M}^{\sigma \rightarrow \tau}$  is an element of  $\mathbf{M}^{\overset{\sigma}{\Rightarrow} \tau}$ . Moreover, the action of any homomorphism of BBK-models on elements of functional type induces an action on their applicative behaviour profiles: the applicative behaviour profile of  $h^{\sigma \rightarrow \tau} \mathbf{d}$  maps  $\langle i, \mathbf{a} \rangle$  to  $i^{\sigma \rightarrow \tau}(h^{\sigma \rightarrow \tau} \mathbf{d})(\mathbf{a})$ . So the induced action of  $h$  on an applicative behaviour profile  $\alpha$  can be stated intrinsically in terms of  $\alpha$  and is just  $h^{\overset{\sigma}{\Rightarrow} \tau} \alpha$  as defined above.

In general the truth-value profiles of  $\mathbf{M}^t$  will be a subset of  $\mathbf{M}^P$  and the applicative behaviour profiles of elements of  $\mathbf{M}^{\sigma \rightarrow \tau}$  a subset of  $\mathbf{M}^{\sigma \Rightarrow \tau}$ . However,  $-^t$  and  $-\sigma \rightarrow \tau$  will determine *subactions* of  $-^P$  and  $-\sigma \Rightarrow \tau$ :

**Definition 3.17.** One action  $-^*$  is a *subaction* of another action  $-^\dagger$  iff  $A^* \subseteq A^\dagger$  for every object  $A$  and  $h^*(x) = h^\dagger(x)$  for every  $h : A \rightarrow B$  and  $x \in A^*$ .

With these concepts under our belt, we can finally introduce the promised analogue of Henkin models for Classicism. As before, we can start with a notion of premodel; define a *partial* notion of interpretation for premodels; and then define a model to be a premodel which is “sufficiently full” in the sense that its interpretation function is total. Each premodel, and thus model, is built on an arbitrary *rooted category*: an ordered pair  $\langle C, W_0 \rangle$ , where  $W_0$  is an object of  $C$  with an arrow to every other object of  $C$ .<sup>75</sup>

**Definition 3.18.** (i) An *action premodel*  $\mathbf{A}$  for a signature  $\Sigma$  is a tuple  $\langle C, W_0, -, \mathcal{I} \rangle$ , where  $\langle C, W_0 \rangle$  is any rooted category, and for each type  $\sigma$ ,  $-\sigma$  is an action of  $C$ , such that:

1.  $-^e$  is any action of  $C$  such that  $W^e$  is nonempty for every object  $W$ ;
2.  $-^t$  is a subaction of  $-^P$  (the powerset action on  $C$ );
3.  $-\sigma \rightarrow \tau$  is a subaction of  $-\sigma \Rightarrow \tau$  (the exponential action from  $-\sigma$  to  $-\tau$ ); and
4. For every nonlogical constant  $c : \sigma$  in  $\Sigma$ ,  $\mathcal{I}(c) \in W_0^\sigma$ .

The notion of interpretation function for an action premodel will be a bit different from the interpretation functions we have been working with up to now. These interpretation functions require not just a term  $A$  and an assignment function  $g$ , but an arrow  $h$  whose source is the base object  $W_0$ .  $g$  is an assignment function for  $h$ 's target, and the output of the interpretation function is an element of the target's domain in the appropriate type.

**Definition 3.19.** When  $\mathbf{A}$  is an action premodel for  $\mathcal{L}$ , the *interpretation function* of  $\mathbf{A}$  is a partial function  $\llbracket \cdot \rrbracket$  that takes an arrow  $h : W_0 \rightarrow W$ ; a term  $A$  of some type  $\sigma$ ; an assignment function  $g$  for  $W$  adequate for  $A$  that maps each type- $\tau$  variable to an element of  $W^\tau$ ; and returns something of the right sort to belong to  $W^\sigma$ , such that the following conditions hold whenever  $\llbracket \cdot \rrbracket$  is defined for all relevant arguments:

$$\llbracket c \rrbracket_h^g = h(\mathcal{I}c)$$

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<sup>75</sup>Disallowing objects without an arrow from  $W_0$  is just a convenience, since they would make no difference if they were present.

$$\begin{aligned}
\llbracket v \rrbracket_h^g &= gv \\
\llbracket \neg \rrbracket_h^g &= \langle i, \mathbf{p} \rangle \mapsto \bigcup_V \{j : \text{trg } i \rightarrow V\} \setminus \mathbf{p} \\
\llbracket \wedge \rrbracket_h^g &= \langle i, \mathbf{p} \rangle \mapsto (\langle j, \mathbf{q} \rangle \mapsto j^i \mathbf{p} \cap \mathbf{q}) \\
\llbracket \vee \rrbracket_h^g &= \langle i, \mathbf{p} \rangle \mapsto (\langle j, \mathbf{q} \rangle \mapsto j^i \mathbf{p} \cup \mathbf{q}) \\
\llbracket \forall_\sigma \rrbracket_h^g &= \langle i, \alpha \rangle \mapsto \bigcup_V \{j : \text{trg } i \rightarrow V \mid 1_V \in \alpha \langle j, \mathbf{a} \rangle \text{ for every } \mathbf{a} \in V^\sigma\} \\
\llbracket \exists_\sigma \rrbracket_h^g &= \langle i, \alpha \rangle \mapsto \bigcup_V \{j : \text{trg } i \rightarrow V \mid 1_V \in \alpha \langle j, \mathbf{a} \rangle \text{ for some } \mathbf{a} \in V^\sigma\} \\
\llbracket =_\sigma \rrbracket_h^g &= \langle i, \mathbf{a} \rangle \mapsto (\langle j, \mathbf{b} \rangle \mapsto \{k \mid k^\sigma(j^\sigma \mathbf{a}) = k^\sigma \mathbf{b}\}) \\
\llbracket AB \rrbracket_h^g &= \llbracket A \rrbracket_h^g \langle 1_{\text{trg } h}, \llbracket B \rrbracket_h^g \rangle \\
\llbracket \lambda v. A \rrbracket_h^g &= \langle i, \mathbf{a} \rangle \mapsto \llbracket A \rrbracket_{i \circ h}^{(i \circ g)[v \mapsto \mathbf{a}]}
\end{aligned}$$

For now, we won't try to justify these clauses; we will soon provide a way of restating their essential effect in a more familiar-looking format.<sup>76</sup>

**Definition 3.20.** An action premodel  $\mathbf{A}$  is an *action model* iff for every type- $\sigma$   $\mathcal{L}$ -term  $A$ , object  $W$ , arrow  $h : W_0 \rightarrow W$ , and assignment  $g$  for  $W$  adequate for  $A$ :  $\llbracket A \rrbracket_h^g$  exists and belongs to  $W^\sigma$ .<sup>77</sup>

When  $\mathbf{A}$  is an action model,  $h : W_0 \rightarrow W$  and  $g$  is an assignment function for  $W$ , formula  $P$  **holds in  $\mathbf{A}$  on  $h, g$**  ( $\mathbf{A}, h, g \Vdash P$ ) iff  $1_W \in \llbracket P \rrbracket_h^g$ .  $P$  **holds in  $\mathbf{A}$  on  $g$**  ( $\mathbf{A}, g \Vdash P$ ) iff  $\mathbf{A}, 1_{W_0}, g \Vdash P$ .  $P$  **holds in  $\mathbf{A}$**  ( $\mathbf{A} \Vdash P$ ) iff  $\mathbf{A}, g \Vdash P$  whenever  $g$  is an assignment for  $W_0$  adequate for  $P$ .  $P$  is **valid in** a class  $X$  of action models ( $X \Vdash P$ ) iff  $P$  holds in every model in  $X$ .

<sup>76</sup>One potentially puzzling feature is that the logical constants all denote functions defined on ordered pairs that are indifferent to the identity of the first element of the pair (a homomorphism). This is to be expected: in a category of BBK-models, any homomorphisms  $h, i : \mathbf{M} \rightarrow \mathbf{N}$  must agree on  $\llbracket c \rrbracket_{\mathbf{M}}$  for any logical constant  $c$ , so  $\text{app}_{\mathbf{M}}^c \llbracket c \rrbracket_{\mathbf{M}} \langle h, \mathbf{a} \rangle = \text{app}_{\mathbf{M}}^c \llbracket c \rrbracket_{\mathbf{M}} \langle i, \mathbf{a} \rangle$  for any  $\mathbf{a}$  in the appropriate domain of  $\mathbf{N}$ .

<sup>77</sup>Analogous to the fact about Henkin models reported in footnote 69, we can also give a more “intrinsic” version of the sufficient fullness condition: an action premodel  $\mathbf{A}$  is an action model iff (i) the domains are closed under application:  $\alpha \langle 1_W, \mathbf{a} \rangle \in W^\tau$  when  $\alpha \in W^{\sigma \rightarrow \tau}$  and  $\mathbf{a} \in W^\sigma$ ; (ii) the denotations of all the logical constants (as specified above) all belong to the appropriate domains; and (iii) for any type  $\sigma$ , relational types  $\rho, \tau$ , and object  $W$ ,  $W^{(\sigma \rightarrow \rho \rightarrow \tau) \rightarrow (\sigma \rightarrow \rho) \rightarrow \sigma \rightarrow \tau}$  and  $W^{\tau \rightarrow \sigma \rightarrow \tau}$  respectively contain the following functions:

$$\begin{aligned}
S_{\sigma, \rho, \tau} &:= \langle i, \alpha \rangle \mapsto (\langle j, \beta \rangle \mapsto (\langle k, \mathbf{a} \rangle \mapsto \alpha \langle k \circ j, \mathbf{a} \rangle \langle 1_{\text{trg } k}, \beta \langle k, \mathbf{a} \rangle \rangle)) \\
K_{\sigma, \tau} &:= \langle i, \mathbf{b} \rangle \mapsto (\langle j, \mathbf{a} \rangle \mapsto j^\tau \mathbf{b})
\end{aligned}$$



Using this notation, we can parlay the interpretations of the logical constants into the following more helpful form. We can show that when  $\mathbf{A}$  is an action model,  $h : W_0 \rightarrow W$ ,  $g$  is an assignment for  $W$  adequate for the relevant formula, and  $v$  is a variable of type  $\sigma$ :

$$\begin{aligned}
\mathbf{A}, h, g \Vdash \neg P & \quad \text{iff } \mathbf{A}, h, g \not\Vdash P \\
\mathbf{A}, h, g \Vdash P \wedge Q & \quad \text{iff } \mathbf{A}, h, g \Vdash P \text{ and } \mathbf{A}, h, g \Vdash Q \\
\mathbf{A}, h, g \Vdash P \vee Q & \quad \text{iff } \mathbf{A}, h, g \Vdash P \text{ or } \mathbf{A}, h, g \Vdash Q \\
\mathbf{A}, h, g \Vdash \forall v P & \quad \text{iff } \mathbf{A}, h, g[v \mapsto \mathbf{a}] \Vdash P \text{ for all } \mathbf{a} \in W^\sigma \\
\mathbf{A}, h, g \Vdash \exists v P & \quad \text{iff } \mathbf{A}, h, g[v \mapsto \mathbf{a}] \Vdash P \text{ for some } \mathbf{a} \in W^\sigma \\
\mathbf{A}, h, g \Vdash A = B & \quad \text{iff } \llbracket A \rrbracket_h^g = \llbracket B \rrbracket_h^g
\end{aligned}$$

and in consequence,

$$\mathbf{A}, h, g \Vdash \Box P \quad \text{iff } \mathbf{A}, i \circ h, i \circ g \Vdash P \text{ for all } i \text{ with source } W.$$

The theorem that action models are sound and complete for Classicism is proved in Appendix C; here we just sketch the main ideas. For the soundness part, the important fact is

**Proposition 3.21.** Any action model can be turned into a BBK-model in which the same formulae hold, and in which every instance of Logical Equivalence holds.

The idea is to let the domains at each type  $\sigma$  be given by  $W_0^\sigma$ , the interpretation function given by  $\llbracket \cdot \rrbracket_{1_{W_0}}$ , and  $\text{val } \mathbf{p} = 1$  iff  $1_{W_0} \in \mathbf{p}$ . The BBK clauses for the connectives and quantifiers follow immediately from the above biconditionals involving  $\Vdash$ ; the only non-trivial aspect is verifying condition (d) (that  $\beta\eta$ -equivalent terms have the same denotation).

For the completeness part, the important fact is

**Proposition 3.22.** For every small, intensional category of BBK-models  $\mathcal{C}$ , and model  $\mathbf{M}_0$  in  $\mathcal{C}$ , there is an action model  $\mathbf{A}_M^{\mathcal{C}}$  in which the same formulae hold.

The idea here is that the rooted category of  $\mathbf{A}_M^{\mathcal{C}}$  will be  $\langle \mathcal{C}, \mathbf{M}_0 \rangle$  (if necessary throwing away any models in  $\mathcal{C}$  without homomorphisms from  $\mathbf{M}_0$ ). We construct the domains of each object by leaving individuals alone, replacing propositional elements with their truth-value profiles, and iteratively replacing elements of functional type with their applicative behaviour profiles. Note that we could derive Propositions 3.7 and 3.8, relating extensional BBK-models to Henkin models, from special cases of Propositions 3.21 and 3.22, applied to categories with one object with its identity arrow.

By Proposition 3.21 and the soundness of BBK models for H, every theorem of Classicism holds in every action model. And by Proposition 3.22 and the completeness part of Theorem 3.12, every formula consistent with Classicism holds on some assignment in some action model. Thus:

**Theorem 3.23.** The class of action models is sound and complete for Classicism.

### 3.5 Exploring action models

One special kind of action model in which we can already do a lot consists of those in which the base category has only one object. A category with only one object is called a *monoid*, and an action of a monoid is called an *M-set*. Bacon (2019) discusses one-object action models under the label ‘*M-set models*’. Every instance of the schema No Pure Contingency from Section 2.2—i.e.  $P \rightarrow \Box P$  for closed  $P$  with no nonlogical constants—is true in any *M-set model*, since when  $P$  contains no nonlogical constants,  $\llbracket P \rrbracket_h^g = \llbracket P \rrbracket_{h'}^g$  for any arrows  $h$  and  $h'$  with the same target, and when  $P$  contains no free variables,  $\llbracket P \rrbracket_h^g = \llbracket P \rrbracket_h^{g'}$  for any assignment functions  $g$  and  $g'$ .

We can already use *full M-set models* (i.e. those where  $W_0^t = W_0^P$  and  $W_0^{\sigma \rightarrow \tau} = W_0^{\sigma \rightarrow \tau}$ ) to show that Classicism is consistent with failures of ND and BF in arbitrary types, using the following facts:

**Proposition 3.24.** (i)  $\text{ND}_\sigma$  holds in an action model  $\mathbf{A}$  iff  $h^\sigma$  is injective for every arrow  $h$  from  $\mathbf{A}$ 's base object.<sup>78</sup>

(ii) If  $h^\sigma$  is surjective for every arrow  $h$  with source  $W_0$ , then  $\text{BF}_\sigma$  holds in  $\mathbf{A}$ .<sup>79</sup>

(iii) If  $\mathbf{A}$  is quasi-functionally full and  $\text{BF}_\sigma$  holds in  $\mathbf{A}$ , then every  $h^\sigma$  with source  $W_0$  is surjective.<sup>80</sup>

<sup>78</sup>*Proof:*  $\mathbf{A}, 1_{W_0} \Vdash \forall xy(x \neq y \rightarrow \Box x \neq y)$  iff  $\mathbf{A}, 1_{W_0}, g \Vdash \Box x \neq y$  for all  $g$  such that  $gx \neq gy$ , iff  $\mathbf{A}, h, h \circ g \Vdash x \neq y$  for all such  $g$ , all objects  $W$ , and all  $h : W_0 \rightarrow W$ , iff  $h(gx) \neq h(gy)$  for all such  $h$  and  $g$ , iff  $h\mathbf{a} \neq h\mathbf{b}$  for all such  $h$  and all  $\mathbf{a}, \mathbf{b}$  in  $W_0^\sigma$ .

<sup>79</sup>*Proof:* Suppose every  $h^\sigma$  with source  $W_0$  is surjective, and  $\mathbf{A}, 1_{W_0}, g \Vdash \forall y \Box P$ . Let  $V$  be any object,  $\mathbf{a} \in V^\sigma$ , and  $h : W_0 \rightarrow V$ . Then there is some  $\mathbf{b}$  such that  $\mathbf{a} = h\mathbf{b}$ , and hence  $(h \circ g)[y \mapsto \mathbf{a}] = h \circ (g[y \mapsto \mathbf{b}])$ . But since  $1_{W_0}, g[y \mapsto \mathbf{b}] \Vdash \Box P$ ,  $h, h \circ (g[y \mapsto \mathbf{b}]) \Vdash P$ ; i.e.,  $h, (h \circ g)[y \mapsto \mathbf{a}] \Vdash P$ . Since  $\mathbf{a}$  was arbitrary, it follows that  $h, h \circ g \Vdash \forall y P$ . And since  $h$  was arbitrary, this implies  $1_{W_0}, g \Vdash \Box \forall y P$ .

<sup>80</sup>*Proof:* Suppose  $\mathbf{A}$  is quasi-functionally full. Then  $W_0^{\sigma \rightarrow t}$  contains the function  $\alpha$  such that for any  $i : W_0 \rightarrow U$  and  $\mathbf{b} \in U^\sigma$ ,  $\alpha\langle i, \mathbf{b} \rangle = \{k \mid k^\sigma \mathbf{b} = k^\sigma (i^\sigma \mathbf{a}) \text{ for some } \mathbf{a} \in W_0^\sigma\}$ . Note that for every  $\mathbf{a} \in W_0^\sigma$ ,  $\alpha\langle 1_{W_0}, \mathbf{a} \rangle$  is the set of all arrows with source  $W_0$ . Thus  $\mathbf{A}, 1_{W_0}, [X \mapsto \alpha] \Vdash \forall y \Box Xy$ . So, if  $\text{BF}_\sigma$  is true in  $\mathbf{A}$ , we have  $\mathbf{A}, 1_{W_0}, [X \mapsto \alpha] \Vdash \Box \forall y Xy$ , hence  $\mathbf{A}, h, [X \mapsto h^{\sigma \rightarrow t} \alpha] \Vdash \forall y Xy$  for every arrow  $h : W_0 \rightarrow V$ , and hence  $1_V \in h^t \alpha\langle 1_V, \mathbf{b} \rangle$  for every  $h : W_0 \rightarrow V$  and  $\mathbf{b} \in V^\sigma$ . Given the definition of  $\alpha$ , this means that for every such  $\mathbf{b}$ , there is some  $\mathbf{a} \in W_0^\sigma$  that  $\mathbf{b} = h^\sigma \mathbf{a}$ : in other words,  $h^\sigma$  is surjective.

Consider a base category with a single object  $W_0$  and two arrows,  $1 = 1_{W_0}$  and  $k$ , with  $k \circ k = k$ . If we choose an action for type  $e$ —e.g. just have  $W_0^e$  be a singleton—this, together with an interpretation of any nonlogical constants, uniquely determines a full action model. Its propositional domain  $W^t$  contains four propositions,  $\emptyset, \{1\}, \{k\}$  and  $\{1, k\}$ .  $1^t$  is of course the identity function;  $k^t \emptyset = k^t \{1\} = \emptyset$ ,  $k^t \{k\} = k^t \{1, k\} = \{1, k\}$ . Thus  $\text{ND}_t$  and  $\text{BF}_t$  both fail: the former since  $k^t \emptyset = k^t \{1\}$ ; the latter, since the model is quasi-functionally full and  $\{1\}$  is not in the range of  $k^t$ .

By contrast, if we change the base category to have  $k \circ k = 1$ , we get a model in which  $k^t$  is injective:  $k^t \emptyset = \emptyset$ ,  $k^t \{1, k\} = \{1, k\}$ ,  $k^t \{1\} = \{k\}$ ,  $k^t \{k\} = \{1\}$ . Thus  $\Box \text{ND}$  holds. Since there are four propositions, the Fregean Axiom is false, establishing the (already well-known) fact that C5 is weaker than Extensionalism.

For an M-set model with BF but not ND, we can consider a full model where the base category is the monoid of all *surjective* functions on some infinite set  $X$ , and choose  $-^e$  in such a way that each  $h^e$  is also surjective. We can show (see Bacon 2020: proposition A.3) that in that case  $h^\sigma$  must also be surjective for every  $\sigma$ , which is sufficient for the truth of  $\text{BF}_\sigma$  for every type  $\sigma$ ; but  $\text{ND}_t$  still fails, since when  $h$  is not injective  $h^t \{1\} = \{i \mid i \circ h = 1\} = \emptyset = h^t \emptyset$ , so  $h^t$  is not injective.

To model failures of No Pure Contingency, we can turn to categories with multiple objects. For example, consider a full model based on a category with two objects  $W_0$  and  $W_1$  and three arrows  $1_{W_0}$ ,  $1_{W_1}$ , and  $k : W_0 \rightarrow W_1$ . Then the Fregean Axiom is false (since there are four propositions  $\emptyset, \{1_{W_0}\}, \{k\}, \{1_{W_0}, k\}$ ), but it is not necessarily false, since it holds relative to  $k$ . For another example, consider a full model based on the category with two objects  $W_0$  and  $W_1$  and three non-identity arrows  $h : W_0 \rightarrow W_1$ ,  $j : W_1 \rightarrow W_0$ , and  $k : W_1 \rightarrow W_1$  (as well as the identity arrows  $1_{W_0}$  and  $1_{W_1}$ ), with  $k \circ k = h \circ j = k$ .  $W_0^t$  is the four-membered powerset of  $\{1_{W_0}, h\}$ ;  $W_1^t$  is the eight-membered powerset of  $\{1_{W_1}, j, k\}$ .  $\text{ND}_t$  is false at  $W_1$ , since  $j^t \{1_{W_1}\} = j^t \{k\} = \{h\}$ . To show that  $\text{ND}_t$  is true at  $W_0$ , it suffices to show that  $h$  (the only non-identity arrow with source  $W_0$ ) acts injectively on  $W_0^t$ . This is true, since  $h^t \emptyset = \emptyset$ ,  $h^t \{1_{W_0}\} = \{j\}$ ,  $h^t \{h\} = \{j, k\}$ , and  $h^t \{1_{W_0}, h\} = \{1_{W_1}, j, k\}$ . So  $\text{ND}_t$  is true in the model while  $\Box \text{ND}_t$  is false.<sup>81</sup>

Another case of special interest is that of action models where the base category is a *preorder category*—one with at most one arrow having any given source and target. Any set with a transitive and reflexive relation  $R$  can be turned into a preorder category, by counting each ordered pair in  $R$  as an arrow from its first element to its second element. In action models based on preorder categories, the analogy

<sup>81</sup>Note that while No Pure Contingency fails here, we do (unlike in the previous example) have the weaker schema  $P \rightarrow \Box \Diamond P$  for all closed pure  $P$ . This will hold in any action model where every object has an arrow from every other object.

between objects in the category and worlds in an S4 Kripke model becomes much closer; the objects with an arrow from a given object work like the worlds accessible from a world in a Kripke model with a reflexive and transitive accessibility relation. In contrast with the most familiar way of developing Kripke models for quantified modal logics including CBF (Cresswell and Hughes 1996: ch. 15), there is no requirement that the domain of an accessible world is a *subset* of the domain of the accessing world. The role of identity across domains is played instead by the “transition functions”,  $h^\sigma$ , which provide elements in the domain of one world corresponding to elements in the domain of another. In the case of type  $e$ , we could thus recover the standard treatment of expanding domains by identifying the transition maps with the inclusion mappings from a set to a superset. However this forces the truth of  $\text{ND}_e$ , and more generally, failures of ND at any type require non-injective transition maps. What one can do, if one is determined to have the type- $\sigma$  domains of all worlds be subsets of one big domain, is to associate each world  $W$  with a partial equivalence relation  $\sim_W^\sigma$  on that domain (i.e. a reflexive and symmetric relation): failures of injectivity in the transition function from  $W$  to  $W'$  correspond to the case where two things are related to themselves but not to each other at  $W$ , and are related to each other at  $W'$ , and failures of surjectivity correspond to the case where something is related to itself at  $W'$  but not at  $W$ . For the details of these alternative “Expanding Modalized Domain Models”, see Bacon 2018a. There is a natural recipe for transforming action models based on preorder categories into expanding domain models, and vice versa.<sup>82</sup>

In one way we would lose nothing by confining our attention to the class of action models based on preorders: this class is also sound and complete for Classicism. This follows from the fact that there is a procedure that “unravels” any action model  $\mathbf{A}$  based on an arbitrary category into a new model  $\mathbf{A}^*$  based on a preorder category, in which exactly the same formulae are true. The objects (worlds) of the preorder are composable finite sequences of arrows of the old category, starting from the base world: one such sequence is accessible from another (i.e. has an arrow to it) iff it is an initial segment of it. We can think of each such sequence as a copy of the old object that is the target of its final arrow, and there is a natural way of reading off domains in each type for every sequence from the domains of that object. The interpretations of terms in the old model map straightforwardly into the new model and the mapping preserves truth. However, the models output by this unravelling procedure are neither propositionally nor functionally full (except in trivial cases).

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<sup>82</sup>The models in Bacon 2018a correspond to full preorder action models. The basic idea behind the correspondence is this: any action  $-^*$  of a preorder  $\mathcal{P}$  corresponds to a “modalized domain”  $\langle D^*, \sim^* \rangle$  where the elements of  $D^*$  are what we might call “modal worms”—maximal partial functions  $f$  that map each world  $W$  to an elements of  $W^*$  in such a way that whenever  $h : W \rightarrow V$ ,  $h^*(fW) = fV$ —and  $f \sim_W^* g$  iff  $fW = gW$ .

And indeed, the logic of propositionally and functionally full action models based on preorders is a strict strengthening of the logic of all propositionally and functionally full action models. For instance, for every  $n$ , the following sentence belongs to the logic of propositionally full action models based on preorders:

$$\Box \exists x y (x \neq y \wedge \Diamond x = y) \rightarrow \exists p_1 \dots p_n \left( \bigwedge_{i \neq j} p_i \neq p_j \right)$$

For to make the antecedent true, every world must see a world that does not see it back, and so there must be infinitely many worlds, which in a propositionally full model, means that the propositional domain at the base world must be infinite. By contrast, we already saw a one-object action model that makes the antecedent true, that has only two arrows, and four propositions. Allowing for multiple arrows between objects thus affords us extra flexibility in constructing models: full models are easy to construct, whereas checking that non-full action premodels meet the “sufficient fullness” condition is tricky.

Nevertheless, even the logic of all full action models is still rather strong. It may be shown, by appeal to Gödel’s incompleteness theorems, that it is not recursively axiomatizable.<sup>83</sup> More importantly for our purposes, this logic also includes several of the more controversial principles surveyed in Section 2.2. Atomicity and Actuality and their necessitations are true in every propositionally full action model, for the obvious reason: the propositional domain of any object contains all the singletons of arrows with that object as source. And Rigid Comprehension and its necessitation are true in every intensionally full action model, since for every subset  $X$  of  $W_0^\sigma$ , the element of  $\alpha_X$  of  $W_0^{\sigma \rightarrow t}$  defined by  $\alpha_X \langle h, \mathbf{a} \rangle = \{i \mid i^\sigma \mathbf{a} = i^\sigma (h^\sigma \mathbf{b}) \text{ for some } \mathbf{b} \in X\}$  is coextensive with  $X$ , persistent, and inextensible (the case of polyadic rigid relations is similar).<sup>84</sup> So to explore the consistency of packages in which some of these principles are false, or at least possibly false, we will need ways of constructing non-full models. Appendix D develops one method of constructing such models which can be used to verify the consistency of many combinations of the controversial principles and their necessitations.

<sup>83</sup>There is a computable mapping from the language of arithmetic to that of pure higher-order logic that maps all the arithmetical truths the validities in this class of models, and all the arithmetical false to invalidities.

<sup>84</sup>One further limitation of full models is worth noting: if BF is true (in a given type), then so is  $\Box$ BF. For in a full model,  $W_0^{\sigma \rightarrow t}$  contains the function  $\alpha$  defined by  $\alpha \langle h, \mathbf{a} \rangle = \{i \mid i^\sigma x = i^\sigma (h^\sigma \mathbf{b}) \text{ for some } \mathbf{b} \in W_0^\sigma\}$ .  $\forall y \Box X y$  is true on the assignment that maps  $X$  to this  $\alpha$ , so by BF, so is  $\Box \forall y X y$ . This means that for every  $h : W_0 \rightarrow W$ ,  $1_W \in \alpha \langle h, \mathbf{b} \rangle$  for every  $\mathbf{b} \in W_0^\sigma$ ; i.e.  $h^\sigma$  is surjective. But if  $h^\sigma$  is surjective for every  $h$  with source  $W_0$ ,  $i^\sigma$  must be surjective for every arrow  $i$  in the base category, and hence  $\text{BF}_\sigma$  must hold at every object, and thus  $\Box \text{BF}_\sigma$  must hold at  $W_0$ .

### 3.6 The consistency of Maximalist Classicism

Recall that for any theory  $T$  extending Classicism (in a given  $\mathcal{L}(\Sigma)$ ),  $\text{Max } T$ , the maximalization of  $T$ , is the result of adding  $\Diamond P$  to  $T$  for every closed  $\mathcal{L}$ -formula  $P$  consistent with  $T$ . In this section we will see how action models can be used to prove the consistency of the maximalizations of Classicism and many other theories extending Classicism.

A crucial concept will be that of a *truncation* of an action model by an arrow  $h : W_0 \rightarrow V$ . Informally a truncation is what you get by treating  $V$  as your new base world, and throwing away objects with no arrow from  $V$ .<sup>85</sup>

**Definition 3.25.** When  $\mathbf{A} = \langle C, W_0, -, \mathcal{I} \rangle$  is an action premodel and  $h : W_0 \rightarrow V$ , the *truncation* of  $\mathbf{A}$  by  $h$  is the action premodel  $\mathbf{A}_h = \langle C', V, -', \mathcal{I}' \rangle$  whose base category  $C'$  contains all the arrows of  $C$  with an arrow from  $V$  and all arrows between them; whose action  $-^{\sigma'}$  in each type is just the restriction of  $-^{\sigma}$  to  $C'$ , and where for each nonlogical constant  $c$ ,  $\mathcal{I}'c = h(\mathcal{I}c)$ .

It is easy to show (by induction on the complexity of terms) that for any term  $A$ ,  $\llbracket A \rrbracket_{\mathbf{A}_h, i}^g = \llbracket A \rrbracket_{\mathbf{A}, i \circ h}^g$ : thus  $\mathbf{A}_h$  is in fact an action model, not just a premodel. Moreover, since the truth of  $\Diamond P$  for a closed sentence  $P$  amounts to its being true under some arrow (i.e.  $1_{\text{trg } h} \in \llbracket P \rrbracket_h$ ), we can use this fact to show that for a closed sentence  $P$ :

$\Diamond P$  holds in an action model iff  $P$  holds in one of its truncations.

To show the consistency of  $\text{Max } T$ , it is thus sufficient to find an action model  $\mathbf{A}$  of  $T$  such that  $T$  is complete with respect to the set of all truncations of  $\mathbf{A}$ . Indeed, the converse is also true: any action model of  $\text{Max } T$  must be such that  $T$  is complete with respect to its truncations, since if  $P$  is consistent with  $T$ ,  $\Diamond P$  holds in the model, and thus  $P$  must hold in one of its truncations.<sup>86</sup>

We can thus establish the consistency of Maximalist Classicism by finding an action model such that Classicism is complete with respect to its truncations. In Appendix E, we will establish a stronger result which implies this:

**Theorem 3.26.** Every set  $X$  of action models whose base categories are disjoint has a *coalesced sum*—an action model such that every member of  $X$  is among its truncations.

The informal idea behind this construction is this: lay out the rooted categories corresponding to the action models in  $X$  side by side and add a new object,  $W_0$ , at the

<sup>85</sup>It is related to the notion of a generated submodel from a world from modal logic.

<sup>86</sup>If  $T$  is closed under necessitation, then truncations of models of  $T$  will also be models of  $T$ , so  $T$  will be sound as well as complete with respect to the the truncations of a model of  $\text{Max } T$ .

bottom, with one new arrow from  $W_0$  to the root object of each of the old categories. (Since we are making a category, we will also have to add an identity arrow for  $W_0$  and enough additional new arrows to ensure closure under composition.) Now you have a big rooted category containing each of the rooted categories from  $X$  as truncations. This is turned into a big action model by using the models in  $X$  to determine the domains of the old objects, and making the domains of  $W_0$  “as full as possible”. The intensions of the interpretations of relational nonlogical constants are chosen so that their extensions relative to non-identity arrows are given by the old models; their actual extensions may be chosen freely.

Theorem 3.26 gives us what we need, since as we pointed out in Section 3.4,  $\mathbf{C}$  is not only sound and complete with respect to the proper class of *all* action models, but also with respect to various *sets* of action models. And given any set of action models for which  $\mathbf{C}$  is complete, we can easily turn it into a set whose base categories are disjoint just by replacing each base category with an isomorphic copy.

Theorem 3.26 has other interesting consequences. In any category, a *weakly initial* object is any object that has at least one arrow to every other object of that category. If a category  $\mathcal{C}$  of BBK-models of a certain signature  $\Sigma$  contains a weakly initial object  $\mathbf{M}$ , then  $\mathbf{M}$  must be a model of  $\text{Max Th } \mathcal{C}$  (where  $\text{Th } \mathcal{C}$  is the theory of  $\mathcal{C}$ ), since the existence of a homomorphism  $h : \mathbf{M} \rightarrow \mathbf{N}$  for BBK-models  $\mathbf{M}$  and  $\mathbf{N}$  means that  $\diamond P$  holds in  $\mathbf{M}$  whenever  $P$  is closed and  $P$  holds in  $\mathbf{N}$ . Thus in particular,  $\text{Max } \mathcal{C}$  would have to hold in any weakly initial object in the category of all BBK-models of  $\mathcal{C}$ . But models of  $\text{Max } \mathcal{C}$  don’t *have* to be weakly initial in this category, so the consistency of  $\text{Max } \mathcal{C}$  doesn’t immediately imply the existence of such a weakly initial object. Nevertheless, using Theorem 3.26, we can show that there is such an object (at least when  $\mathcal{L}$  is countable).

The additional element we need is a version of the downward Löwenheim-Skolem theorem for BBK-models: for every BBK-model  $\mathbf{M}$  for a countable signature  $\Sigma$ , there is a BBK-model  $\mathbf{M}^\downarrow$  for  $\mathcal{L}$  in which all the domains are countable such that there is a truth-preserving homomorphism  $h : \mathbf{M}^\downarrow \rightarrow \mathbf{M}$ . (“Truth-preserving” in the sense that  $\mathbf{M}, h \circ g \Vdash P$  whenever  $\mathbf{M}^\downarrow, g \Vdash P$ .) The proof of this uses a similar technique to Theorem 3.2 (the completeness theorem for BBK-models). Starting with our BBK-model  $\mathbf{M}$  for  $\mathcal{L}$ , we can extend  $\mathcal{L}$  to a larger (but still countable) language  $\mathcal{L}^+$ , and simultaneously extend  $\mathbf{M}$  to a model  $\mathbf{M}^+$  for  $\mathcal{L}^+$ , in such a way that whenever a sentence  $\exists F$  of  $\mathcal{L}^+$  is true in  $\mathbf{M}^+$ ,  $FA$  is also true for some closed term  $A$  of  $\mathcal{L}^+$ . We can then make a new model  $\mathbf{M}^\downarrow$  for  $\mathcal{L}$  by throwing away all the elements of the domains of  $\mathbf{M}^+$  that are not denoted by any closed term of  $\mathcal{L}^+$ . The identity function on each domain is a homomorphism from  $\mathbf{M}^\downarrow$  to  $\mathbf{M}$ . And since  $\mathcal{L}^+$  still only has countably many terms in each type,  $\mathbf{M}^\downarrow$  is countable in every type.

We can replace all the elements of the domains of  $\mathbf{M}^\downarrow$  with natural numbers in some arbitrary way to get a homomorphism to  $\mathbf{M}$  from a BBK $\mathbb{N}$ -model (a BBK-

model where all domains are subsets of  $\mathbb{N}$ ). So for each BBK-model  $\mathbf{M}$  of some theory  $T$  (in a countable signature), there is a BBK $\mathbb{N}$  model of  $T$ ,  $\mathbf{M}^\downarrow$ , and an injective homomorphism  $h : \mathbf{M}^\downarrow \rightarrow \mathbf{M}$ . We also know that when  $T$  includes Classicism, each BBK-model (and thus each BBK $\mathbb{N}$  model) of  $T$  is (BBK-)isomorphic to an action model. So pick a set  $K$  big enough to index the BBK $\mathbb{N}$  models, and choose for each BBK $\mathbb{N}$  model  $\mathbf{N}_k$  a corresponding action model  $\mathbf{A}_k$ , in such a way that the base categories of any two of these action models are disjoint. We can then apply Theorem 3.26 to show that there exists a coalesced sum  $\mathbf{A}$  of all these action models. We can consider this  $\mathbf{A}$  as a BBK-model: it has a homomorphism to every BBK $\mathbb{N}$  model, since it has a homomorphism to each of its truncations and each BBK $\mathbb{N}$ -model is isomorphic to one of its truncations. But every BBK-model of  $\mathbf{C}$  has a homomorphism from some BBK $\mathbb{N}$ -model; composing this with the homomorphism to that model from  $\mathbf{A}$ , we can deduce that  $\mathbf{A}$  (considered as a BBK-model) is weakly initial in the category of BBK-models of  $T$ . In particular, there is a weakly initial object in the category of BBK-models of Classicism for any countable signature.

The existence of a weakly initial object in the category of BBK-models of some theory  $T$  (e.g., Classicism for  $\mathcal{L}(\Sigma)$ ) leaves several questions open:

- (i) Is there an *initial* object in the category of all BBK-models of  $T$ —i.e., an object with *exactly one* homomorphism into each BBK-model of  $T$ ?
- (ii) Is there an object in the category of all BBK-models of  $T$  with an *injective* homomorphism into every weakly initial object in the category?

If the answer to the first question is ‘yes’, the answer to the second question must also be ‘yes’. For suppose  $\mathbf{M}$  is initial and  $\mathbf{N}$  is weakly initial. Then there must be a homomorphism  $h : \mathbf{M} \rightarrow \mathbf{N}$  and a homomorphism  $i : \mathbf{N} \rightarrow \mathbf{M}$ ; moreover since  $1_{\mathbf{M}}$  is the only homomorphism from  $\mathbf{M} \rightarrow \mathbf{M}$ , we must have  $i \circ h = 1_{\mathbf{M}}$  which implies that  $h$  is injective. If the answer to the second question is ‘yes’, then then the “Strong Possibility” schema for  $T$ —whose instances are  $\diamond_{\neq} P$  for all closed  $P$  consistent with  $T$ —is consistent, since the existence of an injective homomorphism from  $\mathbf{M}$  to  $\mathbf{N}$  means that whenever  $P$  is closed and true in  $\mathbf{N}$ ,  $\diamond_{\neq} P$  is true in  $\mathbf{M}$ . Unfortunately, the models we construct in our proof of Theorem 3.26 are generally very large; their homomorphisms to their truncations are very far from being injective. So establishing a positive answer to either of the above questions would, at least, require a fairly extensive modification of the model-construction technique used in the proof of Theorem 3.26.

We hope that action models will be a useful tool for the investigation of these and many other open questions concerning the space of consistent extensions of Classicism.



## Appendices

### Appendix A Closure of Classicism under Equivalence

This appendix will show that the theory that results from adding the Boolean and Classicist Identities to H is closed under the rule of equivalence. Since any H-theory closed under Equivalence must contain every instance of Logical Equivalence, and the Boolean and Classicist Identities are all  $\beta\eta$ -equivalent to instances of Logical Equivalence, it follows that Classicism can be characterised either as the smallest H-theory closed under Equivalence, the smallest H-theory containing every instance of Logical Equivalence, or the smallest H-theory containing the Boolean and Classicist Identities.

In what follows,  $\vdash$  denotes provability from H + the Boolean and Classicist Identities.

**Proposition A.1.**  $\vdash \Box \forall x \top$

*Proof.*

$$\begin{aligned}
 \vdash \forall y \top &= \forall y (Xy \vee \top) && \text{(Booleanism)} \\
 &= (\lambda X p. \forall y (Xy \vee p)) X \top && (\beta) \\
 &= (\lambda X p. \forall X \vee p) X \top && \text{(Distribution-}\forall\vee\text{)} \\
 &= \forall X \vee \top && (\beta) \\
 &= \top && \text{(Booleanism)} \quad \square
 \end{aligned}$$

**Proposition A.2.** For any formula  $P$  and variables  $\vec{v}$ , if  $\vdash P$  then  $\vdash (\lambda \vec{v}. P) = (\lambda \vec{v}. \top)$ .

*Proof.* By induction (“on the length of proofs”). *Base cases:*

(i)  $P$  is an instance of PC. Then  $(\lambda \vec{v}. P) = (\lambda \vec{v}. \top)$  follows from the Boolean identities.

(ii)  $P$  is an instance  $\forall F \rightarrow FA$  of UI. Then:

$$\begin{aligned}
 \vdash (\lambda \vec{v}. P) &= (\lambda \vec{v}. \forall F \rightarrow (\lambda X y. Xy) FA) && (\beta) \\
 &= (\lambda \vec{v}. \forall F \rightarrow (\lambda X y. Xy \vee \forall X) FA) && \text{(Absorption-}\forall\vee\text{)} \\
 &= (\lambda \vec{v}. \forall F \rightarrow (FA \vee \forall F)) && (\beta) \\
 &= (\lambda \vec{v}. \top) && \text{(Booleanism)}
 \end{aligned}$$

(iii)  $P$  is an instance  $FA \rightarrow \exists F$  of EG. Similar to (ii), using Absorption- $\wedge\exists$

(iv)  $P$  is an instance  $A = A$  of Ref. Then:

$$\begin{aligned}
\vdash (\lambda \vec{v}.P) &= (\lambda \vec{v}.(\lambda yz.y = z)AA) && (\beta) \\
&= (\lambda \vec{v}.(\lambda yz.\forall X(Xy \leftrightarrow Xz))AA) && (\text{Identity Identity}) \\
&= (\lambda \vec{v}.\forall X(XA \leftrightarrow XA)) && (\beta) \\
&= (\lambda \vec{v}.\forall X\top) && (\text{Booleanism}) \\
&= (\lambda \vec{v}.\top) && (\text{Proposition A.1})
\end{aligned}$$

(vi)  $P$  is an instance  $A = B \rightarrow FA \rightarrow FB$  of LL. Similar to (iv).

(vii)  $P$  is an instance  $\Phi[(\lambda v.A)B] \leftrightarrow \Phi[A[B/v]]$  of  $\beta$ . Then

$$\begin{aligned}
\vdash (\lambda \vec{v}.P) &= (\lambda \vec{v}.\Phi[A[B/v]] \leftrightarrow \Phi[A[B/v]]) && (\beta) \\
&= (\lambda \vec{v}.\top) && (\text{Booleanism})
\end{aligned}$$

(viii)  $P$  is an instance  $\Phi[\lambda v.Fv] \leftrightarrow \Phi[F]$  of  $\eta$ . Similar to (vii).

(ix)  $P$  is a closed identity  $A = B$  that is one of the Boolean or Classicist Identities. Then  $\vdash (\lambda \vec{v}.A = B) = (\lambda \vec{v}.A = A)$  by the relevant identity, Ref, and LL, so  $\vdash (\lambda \vec{v}.A = B) = (\lambda \vec{v}.\top)$  by part (iv) above.

*Inductive steps:*

(i)  $P$  follows by MP from some previously proved  $Q$  and  $Q \rightarrow P$ . By the induction hypothesis,  $\vdash (\lambda \vec{v}.Q) = (\lambda \vec{v}.\top)$  and  $\vdash (\lambda \vec{v}.Q \rightarrow P) = (\lambda \vec{v}.\top)$ . Then we can appeal to the Boolean identities and  $\beta$  to derive that:

$$\begin{aligned}
\vdash (\lambda \vec{v}.P) &= (\lambda \vec{v}.P \vee (Q \wedge (Q \rightarrow P))) && (\text{Booleanism}) \\
&= (\lambda \vec{v}.P \vee ((\lambda \vec{v}.Q)\vec{v} \wedge (\lambda \vec{v}.Q \rightarrow P)\vec{v})) && (\beta) \\
&= (\lambda \vec{v}.P \vee ((\lambda \vec{v}.\top)\vec{v} \wedge (\lambda \vec{v}.\top)\vec{v})) && (\text{IH}) \\
&= (\lambda \vec{v}.P \vee (\top \wedge \top)) && (\beta) \\
&= (\lambda \vec{v}.\top) && (\text{Booleanism})
\end{aligned}$$

(ii)  $P$  is of the form  $P' \rightarrow \forall uQ$  and follows by Gen from some previously proved  $P' \rightarrow Q$ . By the induction hypothesis,  $\vdash (\lambda \vec{v}u.P' \rightarrow Q) = (\lambda \vec{v}u.\top)$ . So we have:

$$\begin{aligned}
\vdash (\lambda \vec{v}.P) &= (\lambda \vec{v}.\forall uQ \vee \neg P') && (\text{Booleanism}) \\
&= (\lambda \vec{v}.\lambda Xp.\forall X \vee p)(\lambda u.Q)(\neg P') && (\beta) \\
&= (\lambda \vec{v}.\lambda Xp.\forall u(Xu \vee p))(\lambda u.Q)(\neg P') && (\text{Distribution-}\forall\forall) \\
&= (\lambda \vec{v}.\forall u(Q \vee \neg P')) && (\beta) \\
&= (\lambda \vec{v}.\forall u((\lambda \vec{v}u.Q \vee \neg P')\vec{v}u)) && (\beta) \\
&= (\lambda \vec{v}.\forall u((\lambda \vec{v}u.P' \rightarrow Q)\vec{v}u)) && (\text{Booleanism})
\end{aligned}$$

<b>Monotonicity-<math>\forall</math></b>	$\lambda X^{\sigma \rightarrow t} Y^{\sigma \rightarrow t} . \forall_{\sigma} X \leq \lambda XY . \forall_{\sigma} (X \vee_{\sigma \rightarrow t} Y)$
<b>Instantiation</b>	$\lambda X^{\sigma \rightarrow t} y^{\sigma} . \forall_{\sigma} X \leq \lambda X . X$
<b>Vacuity-<math>\forall</math></b>	$\lambda p . p \leq \lambda p . \forall x^{\sigma} p$
<b>Monotonicity-<math>\exists</math></b>	$\lambda X^{\sigma \rightarrow t} Y^{\sigma \rightarrow t} . \exists_{\sigma} (X \wedge_{\sigma \rightarrow t} Y) \leq \lambda XY . \exists_{\sigma} X$
<b>Generalization</b>	$\lambda X^{\sigma \rightarrow t} . X \leq \lambda X y^{\sigma} . \exists_{\sigma} X$
<b>Vacuity-<math>\exists</math></b>	$\lambda p . \exists x^{\sigma} p \leq \lambda p . p$

Figure 7. The Adjunctive Entailments

$$\begin{aligned}
&= (\lambda \vec{v} . \forall u ((\lambda \vec{v} u . \top) \vec{v} u)) && \text{(IH)} \\
&= (\lambda \vec{v} . \forall u \top) && (\beta) \\
&= (\lambda \vec{v} . \top) && \text{(Proposition A.1)}
\end{aligned}$$

(iii)  $P$  is of the form  $(\exists v P') \rightarrow Q$  and follows from some previously proved  $P' \rightarrow Q$  by Inst. Similar to (ii) using Distribution- $\wedge \exists$ .  $\square$

**Proposition A.3.** Classicism is closed under Equivalence.

*Proof.* Suppose  $\vdash A \leftrightarrow B$ ; then  $\vdash (\lambda \vec{v} . A \leftrightarrow B) = (\lambda \vec{v} . \top)$  by the previous lemma. Then:

$$\begin{aligned}
\vdash (\lambda \vec{v} . A) &= (\lambda \vec{v} . (B \wedge (A \leftrightarrow B)) \vee (\neg B \wedge \neg(A \leftrightarrow B))) && \text{(Booleanism)} \\
&= (\lambda \vec{v} . (B \wedge (\lambda \vec{v} . A \leftrightarrow B) \vec{v}) \vee (\neg B \wedge \neg(\lambda \vec{v} . A \leftrightarrow B) \vec{v})) && (\beta) \\
&= (\lambda \vec{v} . (B \wedge (\lambda \vec{v} . \top) \vec{v}) \vee (\neg B \wedge \neg(\lambda \vec{v} . \top) \vec{v})) && \text{(Proposition A.2)} \\
&= (\lambda \vec{v} . (B \wedge \top) \vee (\neg B \wedge \neg \top)) && (\beta) \\
&= (\lambda \vec{v} . B) && \text{(Booleanism)} \square
\end{aligned}$$

## Appendix B Axiomatizations in terms of entailment

This appendix will discuss a couple of other axiomatizations of Classicism which give a central role to the entailment relations  $\leq_{\tau}$ . Recall (from Figure 1) that  $\leq_{\tau}$  is short for  $\lambda XY . Y = Y \vee_{\tau} X$ . Booleanism proves that  $\leq$  is reflexive, transitive, and antisymmetric in each type, and that it is identical to  $\lambda XY . X = X \wedge_{\tau} Y$ . The first axiomatization we'll discuss is given by adding the schemas in Figure 7, along with the Identity Identity, to an axiomatization of Booleanism. Deriving these from the Classicist Identities (Figure 4) is straightforward. Note that  $\forall$ -Instantiation and  $\exists$ -Generalization just rewrite Absorption- $\forall \forall$  and Absorption- $\wedge \exists$  using  $\leq$ . Vacuity- $\forall$

and Vacuity- $\exists$ , meanwhile, can be derived from Distribution- $\forall\forall$  and Distribution- $\wedge\exists$  by instantiating  $X$  with  $(\lambda x.\perp)$  and  $(\lambda x.\top)$ , respectively.

To preserve the duality of the axioms, we stated Monotonicity axioms using  $\vee$  for  $\forall$  and  $\wedge$  for  $\exists$ , but we could just as well have used the same connective in both cases. These axioms imply that if  $X \leq_{\sigma \rightarrow t} Y$  (i.e.  $Y = X \vee_{\sigma \rightarrow t} Y$ ), then  $\forall_{\sigma} X \leq \forall_{\sigma} Y$  and  $\exists_{\sigma} X \leq \exists_{\sigma} Y$ . To see what's going on with the remaining Adjunctive Entailments, we can suggestively rewrite ( $\beta$ -equivalents of) them using the following abbreviations:

$$\begin{aligned} I_{\tau} &:= \lambda x^{\tau}.x \\ K_{\sigma,\tau} &:= \lambda x^{\sigma} y^{\tau}.x \\ A \circ B &:= \lambda x.A(Bx) \end{aligned}$$

The relevant four Adjunctive Entailments can now be rewritten as follows:

$$\begin{array}{ll} \text{Instantiation} & K_{t,\sigma} \circ \forall_{\sigma} \leq I_{\sigma \rightarrow t} \\ \text{Vacuity-}\forall & I_t \leq \forall_{\sigma} \circ K_{t,\sigma} \\ \text{Generalization} & I_{\sigma \rightarrow t} \leq K_{t,\sigma} \circ \exists_{\sigma} \\ \text{Vacuity-}\exists & \exists_{\sigma} \circ K_{t,\sigma} \leq I_t \end{array}$$

In the theory of partial orders, when we have two partially ordered sets,  $\langle X, \leq_1 \rangle$  and  $\langle Y, \leq_2 \rangle$ , a function  $f : X \rightarrow Y$  is *monotonic* just in case whenever  $x \leq_1 x'$ ,  $fx \leq_2 fx'$ . When  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$ , we say that  $f$  is a *right adjoint* of  $g$ , and  $g$  a *left adjoint* of  $f$ , just in case both are monotonic and:

- (i)  $x \leq_1 g(fx)$  for every  $x \in X$
- (ii)  $f(gy) \leq_2 y$  for every  $y \in Y$

Using common notational shorthands, (i) and (ii) can be rewritten respectively as  $1_X \leq_1 g \circ f$  and  $f \circ g \leq_2 1_Y$ , mirroring the pair of Vacuity- $\forall$  and Instantiation, or Generalization and Vacuity- $\exists$ .<sup>87</sup> The Adjunctive Entailments can thus be summed up by saying that universal and existential quantifiers in type  $(\sigma \rightarrow t) \rightarrow t$  are respectively a right-adjoint and a left-adjoint of the  $K$  combinator in type  $t \rightarrow (\sigma \rightarrow t)$ . Note that H already implies that the  $K$  combinator is monotonic: if  $p \leq q$ , then  $q = p \vee q$ , so  $\lambda x.q = \lambda x.(p \vee q) = (\lambda x.p) \vee_{\sigma \rightarrow t} (\lambda x.q)$ ; i.e.,  $K_{\sigma,t}p \leq K_{\sigma,t}q$ .<sup>88</sup>

<sup>87</sup>Here  $1_Z$  stands for the identity function on the set  $Z$ , and where  $h$  and  $h'$  are functions from some set  $Z$  to a partial order,  $h \leq h'$  means that  $h(z) \leq h'(z)$  for all  $z \in Z$ .

<sup>88</sup>More generally, for any type  $\tau$  (ending in  $t$ ), the above axioms entail that the “lifted” quantifiers  $\forall_{\sigma,\tau}$  (defined by  $\forall_{\sigma,t} := \forall_{\sigma}$  and  $\forall_{\gamma,\sigma \rightarrow \tau} := \lambda X^{\gamma \rightarrow (\sigma \rightarrow \tau)} y^{\sigma}.\forall_{\gamma,\tau}(\lambda z'.Xzy)$ ) are right adjoints of  $K_{\tau,\sigma}$ ; similarly the lifted existential quantifiers are left adjoints. By contrast, if we only had the quantified versions of the axioms—e.g.  $\forall p(p \leq \forall xp)$  instead of  $(\lambda p.p) \leq (\lambda p.\forall xp)$ —we would not be able to

Our definition of ‘ $f$  is a right adjoint of  $g$ ’ is easily seen to be equivalent to the following: for every  $x \in X$  and  $y \in Y$ ,  $x \leq_1 g(y)$  iff  $f(x) \leq_2 y$ .<sup>89</sup> This biconditional definition of adjointness suggests yet another axiomatization of Classicism, which adds the following biconditionals to Booleanism (together with the Identity Identity):

$$\text{Adjunction-}\forall \quad ((\lambda v_0 \vec{v}.Q) \leq (\lambda v_0 \vec{v}.P)) \leftrightarrow ((\lambda \vec{v}.Q) \leq (\lambda \vec{v}.\forall v_0 P))$$

$$\text{Adjunction-}\exists \quad ((\lambda v_0 \vec{v}.P) \leq (\lambda v_0 \vec{v}.Q)) \leftrightarrow ((\lambda \vec{v}.\exists v_0 P) \leq (\lambda \vec{v}.Q))$$

where in each case  $v_0$  is not free in  $Q$ .<sup>90</sup> Dorr (2014) shows how these principles can be regarded as capturing the “validity” of the standard natural deduction quantifier rules for  $\forall$  and  $\exists$ , in a certain natural sense of “validity” on which linguistic facts about validity turn on nonlinguistic facts about entailment.

To derive the left-to-right direction of Adjunction- $\forall$  from the Adjunctive Entailments, note that we have  $(\lambda \vec{v}.Q) \leq (\lambda \vec{v}.\forall v_0 Q)$  by Vacuity- $\forall$ , which given the left-hand side, Monotonicity- $\forall$ , and the transitivity of entailment gives  $(\lambda \vec{v}.Q) \leq (\lambda \vec{v}.\forall v_0.P)$ . To derive the right-to-left direction of Adjunction- $\forall$ , note that we have  $((\lambda v_0 \vec{v}.\forall v_0 P) \leq (\lambda v_0 \vec{v}.P))$  by Instantiation, which given the right-hand-side, the monotonicity of the K-combinator, and the transitivity of entailment gives  $((\lambda v_0 \vec{v}.Q) \leq (\lambda v_0 \vec{v}.P))$ . In the other direction, the Adjunctive Entailments for  $\forall$  follow from the following three instances of Adjunction- $\forall$ :

$$(i) \quad ((\lambda y X.\forall y X y) \leq (\lambda y X.X y)) \leftrightarrow ((\lambda X.\forall y X y) \leq (\lambda X.\forall y X y))$$

$$(ii) \quad ((\lambda x p.p) \leq (\lambda x p.p)) \leftrightarrow ((\lambda p.p) \leq (\lambda p.\forall x p))$$

$$(iii) \quad ((\lambda z XY.\forall z X z) \leq (\lambda z XY.X z \vee Y z)) \leftrightarrow ((\lambda XY.\forall z X z) \leq (\lambda XY.\forall z (X z \vee Y z)))$$

---

derive this generalization (unless we rely on some new axioms or rules to be discussed below). This situation, where failures of functionality motivate replacing quantified identities involving functions with identities between functions, instantiates a pattern that is pervasive in category theory. In this context, standard set theoretic definitions involving functions understood set theoretically can be turned into definitions that make sense in some more general category, by first formulating them in a way that doesn’t directly involve quantifying over members of the set, and consequently doesn’t make any strong functionality assumptions.

<sup>89</sup>If  $x \leq_1 g(y)$ , then  $f(x) \leq_2 f(g(y))$  by the monotonicity of  $f$ , so  $f(x) \leq_2 y$  by (ii) and the transitivity of  $\leq_2$ ; if  $f(x) \leq_2 y$ , then  $g(f(x)) \leq_1 g(y)$  by the monotonicity of  $g$ , so  $x \leq_1 g(y)$  by (i) and the transitivity of  $\leq_1$ . In the other direction, (i) and (ii) follow immediately from the reflexivity of  $\leq_1$  and  $\leq_2$  respectively, while the monotonicity of  $f$  and  $g$  follows from their transitivity.

<sup>90</sup>Dorr (2016: note 59) states essentially these biconditionals, describing them rather inscrutably, as the ‘natural analogues of Booleanism for the quantifiers’. The ‘Adjunction’ principle in Goodman 2016 is strictly weaker: it is equivalent to the special case of Adjunction- $\forall$  where  $\vec{v}$  is empty.

The right side of (i) and the left side of (ii) are both consequences of the reflexivity of  $\leq$ , and their other sides are  $\beta\eta$ -equivalent to Instantiation and Vacuity- $\forall$  respectively. Meanwhile, the left side of (iii) follows from Instantiation (given Booleanism), and its right side is  $\beta\eta$ -equivalent to Monotonicity- $\forall$ . Parallel derivations can be given for Adjunction- $\exists$ .

## Appendix C Soundness and completeness of action models for Classicism

This appendix will prove the following two results stated in Section 3.4, which together with Theorem 3.12 imply the soundness and completeness of action models for Classicism.

**3.21.** Any action model can be turned into a BBK-model which makes the same formulae true, and in which every instance of Logical Equivalence is true.

**3.22.** For every small, intensional category of BBK-models  $\mathcal{C}$ , and model  $\mathbf{M}_0$  in  $\mathcal{C}$  there is an action model  $\mathbf{A}_M^{\mathcal{C}}$  in which the same formulae hold.

For Proposition 3.21, suppose  $\mathbf{A}$  is an action model with base object  $W_0$  and  $h : W_0 \rightarrow V$ . Then we will construct a BBK-model  $\mathbf{M}_A^h$  by setting each type- $\sigma$  domain to be  $V^\sigma$  for each type  $\sigma$ ,  $\llbracket A \rrbracket^g = \llbracket A \rrbracket_{\mathbf{A},h}^g$  for every term, and  $\text{val } \mathbf{p} = 1$  iff  $1_V \in \mathbf{p}$ . Looking at the definitions of BBK-model and action model, it is easy to check that  $\mathbf{M}_A^h$  obeys all the conditions to be a BBK-model apart from condition (d) (that  $\llbracket A \rrbracket^g = \llbracket B \rrbracket^g$  when  $A$  and  $B$  are  $\beta\eta$ -equivalent). To establish this, we must first show that our interpretation functions are well-behaved in a few other ways.

We begin with the following fundamental fact about the interpretation functions:

**Proposition C.1.** In any action premodel, when  $h : W_0 \rightarrow W_1$  and  $i : W_1 \rightarrow W_2$ ,  $g$  is an assignment for  $W_1$ , and  $A$  is a type- $\sigma$  term such that  $\llbracket A \rrbracket_h^g$  is defined,

$$\llbracket A \rrbracket_{i \circ h}^{i \circ g} = i^\sigma \llbracket A \rrbracket_h^g$$

*Proof.* By induction on the complexity of terms. It is immediate for variables and nonlogical constants. For the logical constants, the claim follows from the fact that they do not care about the first co-ordinate of their argument (the arrow). (For example,  $i^{t \rightarrow t'} \llbracket \wedge \rrbracket_h^g \langle k, \mathbf{p} \rangle = \llbracket \wedge \rrbracket_h^g \langle k \circ i, \mathbf{p} \rangle = \langle j, \mathbf{q} \rangle \mapsto j' \mathbf{p} \cap \mathbf{q} = \llbracket \wedge \rrbracket_{i \circ h}^{i \circ g} \langle k, \mathbf{p} \rangle$ .) For an abstraction  $\lambda v. A$ , where  $v$  is of type  $\sigma$  and  $A$  of type  $\tau$ ,  $(i^{\sigma \rightarrow \tau} \llbracket \lambda v. A \rrbracket_h^g) \langle j, \mathbf{a} \rangle = \llbracket \lambda v. A \rrbracket_h^g \langle j \circ i, \mathbf{a} \rangle = \llbracket A \rrbracket_{j \circ i \circ h}^{(j \circ i \circ g) \mapsto \mathbf{a}} = \llbracket \lambda v. A \rrbracket_{i \circ h}^{i \circ g} \langle j, \mathbf{a} \rangle$ . For an application  $AB$  where  $A$  is of type  $\sigma$  and  $B$  of type  $\tau$ , we have  $\llbracket AB \rrbracket_{i \circ h}^{i \circ g} = \llbracket A \rrbracket_{i \circ h}^{i \circ g} \langle 1_{W_2}, \llbracket B \rrbracket_{i \circ h}^{i \circ g} \rangle = (i^{\sigma \rightarrow \tau} \llbracket A \rrbracket_h^g) \langle 1_{W_2}, i^\sigma \llbracket B \rrbracket_h^g \rangle$  (by the induction hypothesis)  $= \llbracket A \rrbracket_h^g \langle i, i^\sigma \llbracket B \rrbracket_h^g \rangle$  (since  $i^{\sigma \rightarrow \tau}$  acts by division)  $= i^\tau (\llbracket A \rrbracket_h^g \langle 1_{W_1}, \llbracket B \rrbracket_h^g \rangle)$  (since  $\llbracket A \rrbracket_h^g \in W_1^{\sigma \Rightarrow \tau}$ )  $= i^\tau \llbracket AB \rrbracket_h^g$ .  $\square$

**Proposition C.2.** Suppose that in an action model  $\mathbf{A}$ ,  $\llbracket A \rrbracket_h^g = \llbracket B \rrbracket_h^g$  for any  $g$  and  $h$  for which both sides are defined. Then  $\llbracket \Phi[A] \rrbracket_h^g = \llbracket \Phi[B] \rrbracket_h^g$  whenever  $\Phi[A]$  and  $\Phi[B]$  are terms that differ only by the replacement of an occurrence of  $A$  for one of  $B$ .

*Proof.* By induction on the construction of  $\Phi[A]$  from  $A$ . □

**Proposition C.3.**  $\llbracket A[B/v] \rrbracket_h^g = \llbracket A \rrbracket_h^{g[v \mapsto \llbracket B \rrbracket_h^g]}$  whenever both sides are defined.

*Proof.* By induction on the complexity of  $A$ . □

**Proposition C.4.**  $\llbracket (\lambda v.A)B \rrbracket_h^g = \llbracket A[B/v] \rrbracket_h^g$ .

*Proof.* When  $h : W_0 \rightarrow V$ ,  $\llbracket (\lambda v.A)B \rrbracket_h^g = \llbracket \lambda v.A \rrbracket_h^g \langle 1_V, \llbracket B \rrbracket_h^g \rangle = \llbracket A \rrbracket_h^{g[v \mapsto \llbracket B \rrbracket_h^g]} = \llbracket A[B/v] \rrbracket_h^g$  by Proposition C.3. □

**Proposition C.5.**  $\llbracket \lambda v.Av \rrbracket_h^g = \llbracket A \rrbracket_h^g$  when  $v$  is not free in  $A$ .

*Proof.* For any  $i$  composable with  $h$  and  $\mathbf{a}$  in  $i$ 's target's domain of the appropriate type,

$$\begin{aligned}
\llbracket \lambda v.Av \rrbracket_h^g \langle i, \mathbf{a} \rangle &= \llbracket Av \rrbracket_{ioh}^{(iog)[v \mapsto \mathbf{a}]} && \text{by the clause for abstraction} \\
&= \llbracket A \rrbracket_{ioh}^{(iog)[v \mapsto \mathbf{a}]} \langle 1_{trg\ i}, \llbracket v \rrbracket_{ioh}^{(iog)[v \mapsto \mathbf{a}]} \rangle && \text{by the clause for application} \\
&= \llbracket A \rrbracket_{ioh}^{iog} \langle 1_{trg\ i}, \mathbf{a} \rangle && \text{since } v \text{ isn't free in } A \\
&= (i^{\sigma \rightarrow \tau} \llbracket A \rrbracket_h^g) \langle 1_{trg\ i}, \mathbf{a} \rangle && \text{by Proposition C.1} \\
&= \llbracket A \rrbracket_h^g \langle i, \mathbf{a} \rangle && \text{by the definition of } i^{\sigma \rightarrow \tau} \quad \square
\end{aligned}$$

**Proposition C.6.** When  $A$  and  $B$  are  $\beta\eta$ -equivalent,  $\llbracket A \rrbracket_h^g = \llbracket B \rrbracket_h^g$  whenever  $g$  is adequate for both.

*Proof.* By Propositions C.2, C.4 and C.5. □

This completes the proof that  $\mathbf{M}_A^h$  is always a BBK-model. By construction,  $\mathbf{M}_A^h, g \Vdash P$  iff  $\mathbf{A}, h, g \Vdash P$ . We can also show that every instance of Logical Equivalence holds in every action model (relative to every  $h, g$ ):

**Proposition C.7.** If  $H \vdash P \leftrightarrow Q$  then  $\mathbf{A}, h, g \Vdash (\lambda \vec{v}.P) = (\lambda \vec{v}.Q)$  for all  $h, g, \vec{v}$ .

*Proof.* Suppose  $H \vdash P \leftrightarrow Q$ . Then by what we have just proved, for any  $h$  from  $\mathbf{A}$ 's base object  $W_0$  to some  $V$ ,  $\mathbf{A}, h, g \Vdash P$  iff  $\mathbf{A}, h, g \Vdash Q$ . Hence  $\llbracket P \rrbracket_h^g = \llbracket Q \rrbracket_h^g$  for all  $h$  and  $g$ , and so by Proposition C.2,  $\llbracket \lambda \vec{v}.P \rrbracket_h^g = \llbracket \lambda \vec{v}.Q \rrbracket_h^g$  for all  $h$  and  $g$ , so  $\mathbf{A}, h, g \Vdash (\lambda \vec{v}.P) = (\lambda \vec{v}.Q)$  for all  $h$  and  $g$ . □

It follows that class of all action models is sound for Classicism.

Turning to Proposition 3.22, suppose  $\mathcal{C}$  is a small, intensional category of BBK-models, and  $\mathbf{M}_0$  is an object in it. We will use these to construct an action model  $\mathbf{A}_{\mathbf{M}_0}^{\mathcal{C}}$  and show that the same formulae hold in it. First we truncate  $\mathcal{C}$  by throwing away any models with no homomorphism from  $\mathbf{M}_0$ . Then we choose  $\mathbf{A}_{\mathbf{M}_0}^{\mathcal{C}}$ 's base category to be  $\mathcal{C}$  and its base object to be  $\mathbf{M}_0$ . For each  $\mathbf{M}$  in  $\mathcal{C}$  and each type  $\sigma$ , we let  $\mathbf{M}$ 's type- $\sigma$  domain (considered now as an object in  $\mathbf{A}_{\mathbf{M}_0}^{\mathcal{C}}$ ) be the range of the function  $f_{\mathbf{M}}^{\sigma}$  whose domain is  $\mathbf{M}^{\sigma}$  ( $\mathbf{M}$ 's built-in type- $\sigma$  domain), defined recursively as follows.

$$\begin{aligned} f_{\mathbf{M}}^e \mathbf{a} &= \mathbf{a} \\ f_{\mathbf{M}}^t \mathbf{p} &= \text{val}_{\mathbf{M}}^{\mathcal{C}} \mathbf{p} \\ f_{\mathbf{M}}^{\sigma \rightarrow \tau} \mathbf{d} &= \langle h, \mathbf{a} \rangle \mapsto f_{\text{irg } h}^{\tau} (@_{\mathbf{M}}^{\mathcal{C}} \mathbf{d} \langle h, (f_{\text{irg } h}^{\sigma})^{-1} \mathbf{a} \rangle) \end{aligned}$$

To establish the legitimacy of this definition, we must simultaneously prove that each  $f_{\mathbf{M}}^{\sigma}$  is injective. The first clause automatically secure this for type  $e$ ; the quasi-Fregeanness of  $\mathcal{C}$  secures it for type  $t$ , and the quasi-functionality of  $\mathcal{C}$  guarantees it for types  $\sigma \rightarrow \tau$ .

$\mathbf{A}_{\mathbf{M}_0}^{\mathcal{C}}$  so defined is automatically an action premodel. Moreover, we can show by induction on the complexity of formulae that for any term  $A$  of type  $\sigma$ , any  $h : \mathbf{M}_0 \rightarrow \mathbf{M}$ , and any assignment function  $g$  for  $\mathbf{M}$  adequate for  $A$ ,

$$\llbracket A \rrbracket_h^{f_{\mathbf{M}}^{\sigma} \circ g} = f_{\mathbf{M}}^{\sigma} \llbracket A \rrbracket_{\mathbf{M}}^g$$

Since each  $f_{\mathbf{M}}^{\sigma}$  is a bijection, every assignment function  $g$  for  $\mathbf{M}$  considered as an object of  $\mathbf{A}_{\mathbf{M}_0}^{\mathcal{C}}$  is identical to  $f_{\mathbf{M}}^{\sigma} \circ f_{\mathbf{M}}^{-1} \circ g$ , so we can infer that

$$\llbracket A \rrbracket_h^g = f_{\mathbf{M}}^{\sigma} \llbracket A \rrbracket_{\mathbf{M}}^{f_{\mathbf{M}}^{-1} \circ g}$$

It follows that  $\llbracket A \rrbracket_h^g$  is always well defined when  $g$  is adequate for  $A$ , i.e. that  $\mathbf{A}_{\mathbf{M}_0}^{\mathcal{C}}$  is an action model. Moreover, the mapping preserves truth value: for any formula  $P$  and assignment  $g$  for  $\mathbf{M}_0$  adequate for  $P$ ,  $\text{val}_{\mathbf{M}_0} \mathbf{p} = 1$  iff  $1_{\mathbf{M}_0} \in f_{\mathbf{M}_0}^t \mathbf{p}$ , hence

$$\text{val}_{\mathbf{M}_0} \llbracket P \rrbracket_{\mathbf{M}_0}^g = 1 \text{ iff } 1_{\mathbf{M}_0} \in \llbracket P \rrbracket_{1_{\mathbf{M}_0}}^{f_{\mathbf{M}_0}^t \circ g}$$

or in other words,

$$\mathbf{M}_0, g \Vdash P \text{ iff } \mathbf{A}_{\mathbf{M}_0}^{\mathcal{C}}, 1_{\mathbf{M}_0}, f_{\mathbf{M}_0}^t \circ g \Vdash P.$$



Since  $f_{\mathbf{M}_0}$  is bijective, it follows that  $P$  holds in  $\mathbf{M}$  iff it holds in  $\mathbf{A}_{\mathbf{M}_0}^C$ .

One noteworthy feature of this construction is that since any two homomorphisms from one BBK-model to another must agree on the interpretations of all non-logical constants, the generated action models  $\mathbf{A}_{\mathbf{M}_0}^C$  will always obey the following condition of *non-logical harmony*:  $h(\mathcal{I}c) = i(\mathcal{I}c)$  whenever  $h, i : W_0 \rightarrow V$  and  $c$  is a nonlogical constant. In a non-logically harmonious model, all that matters about an arrow as far as the interpretation function is concerned is its target: i.e. when  $h, i : W_0 \rightarrow V$ ,  $\llbracket A \rrbracket_h^g = \llbracket A \rrbracket_i^g$ .<sup>91</sup> Our proof of Proposition 3.22 shows that Classicism is also complete for non-logically harmonious action models. But we find the more general notion of action model more intuitive, in that any model for in which  $\exists xFx$  is true can be extended into a model for a larger signature in which  $Fc$  is true for some new constant, and also more useful, in that it allows one to construct smaller models of certain theories involving nonlogical constants.

## Appendix D Consistency results using non-full action models

This appendix will introduce a technique for defining certain non-full action models, and use it to prove the consistency of certain packages of principles from Part 2.

We start with some definitions.

**Definition D.1.** When  $-^\dagger$  is an action on a category  $C$ , an *ideal* of  $-^\dagger$  is an action  $-^\ddagger$  on  $C$  such that for every object  $A$  and arrow  $h$ , (i)  $X \subseteq A^\dagger$  whenever  $X \in A^\ddagger$ ; (ii)  $h^\ddagger X = \{h^\dagger x \mid x \in X\}$ ; (iii)  $\emptyset \in A^\ddagger$ ; (iv)  $X \in A^\ddagger$  whenever  $X \subseteq Y$  and  $Y \in A^\ddagger$ , and (iv)  $X \cup Y \in A^\ddagger$  whenever  $X \in A^\ddagger$  and  $Y \in A^\ddagger$ .

For example, we can define one ideal on any  $-^\dagger$  by taking  $A^\ddagger$  to be the set of all finite subsets of  $A^\dagger$ , or alternatively the set of all subsets of  $A^\dagger$ .

**Definition D.2.** Suppose  $-^*$  and  $-^\dagger$  are actions on  $C$ ,  $A$  is an object of  $C$ ,  $X \subseteq A^\dagger$  and  $y \in A^*$ . Then  $y$  is *pinned down* by  $X$  iff for any object  $B$  and arrows  $h, i : A \rightarrow B$ , if  $h^\dagger x = i^\dagger x$  for all  $x \in X$ , then  $h^* y = i^* y$ . If  $-^\ddagger$  is an ideal of  $-^\dagger$ ,  $y$  is pinned down by  $-^\ddagger$  iff  $y$  is pinned down by some  $X \in A^\ddagger$ . And  $-^*$  is pinned down by  $-^\ddagger$  iff for every object  $A$ , every  $x \in A^*$  is pinned down by  $-^\ddagger$ .

**Definition D.3.** When  $\mathbf{A} = \langle C, W_0, -, \mathcal{I} \rangle$  is an action premodel and  $-^\ddagger$  is an ideal of some action  $-^\dagger$  on  $C$ ,  $\mathbf{A}$  is *ideally full* (with ideal  $-^\ddagger$ ) iff

- (i)  $-^e$  is pinned down by  $-^\ddagger$ .

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<sup>91</sup>This is shown by a straightforward induction on the complexity of terms: the logical constants all have this property, since they denote functions whose value on a given pair  $\langle h, \mathbf{a} \rangle$  does not depend on  $h$ .

(ii) For each  $W$ ,  $W^\dagger = \{\mathbf{p} \in W^{\mathcal{P}} \mid \mathbf{p} \text{ is pinned down by } -^\ddagger\}$ .

(iii) For each  $W$ ,  $W^{\sigma \rightarrow \tau} = \{\alpha \in W^{\sigma \Rightarrow \tau} \mid \alpha \text{ is pinned down by } -^\ddagger\}$ .

Conditions (ii) and (iii) can equivalently be stated in terms of *intensions* (see Section 3.3):  $I \in W^{\lceil \bar{\sigma} \rceil}$  is the intension of an element of  $W^{\bar{\sigma} \rightarrow \dagger}$  iff there is some  $X \in W^\ddagger$  such that whenever  $h^\dagger$  with  $i^\dagger$  on every element of  $X$  and  $\langle i, \vec{x} \rangle \in I$ ,  $\langle h, \vec{x} \rangle \in I$ . In particular, for any  $R \subseteq W^{\sigma_1} \times \cdots \times W^{\sigma_n}$ ,  $\{\langle h, \vec{x} \rangle \mid \vec{x} \in R\}$  is the intension of an element of  $W^{\bar{\sigma} \rightarrow \dagger}$  (pinned down by  $\emptyset$ ) with extension  $R$ ; so any ideally full action premodel is extensionally full.

The prefix ‘pre-’ is actually unnecessary, thanks to:

**Proposition D.4.** If  $\mathbf{A}$  is an ideally full action premodel then  $\mathbf{A}$  is an action model.

*Proof.* We use the alternative version of the “sufficient fullness” condition from footnote 77. Since the denotation of each logical constant relative to any  $h : W_0 \rightarrow V$  is a function  $\alpha$  in some  $V^{\sigma \Rightarrow \tau}$  with the property that  $\alpha \langle i, \mathbf{a} \rangle = \alpha \langle j, \mathbf{a} \rangle$  whenever  $i, j : V \rightarrow U$  and  $\mathbf{a} \in U^\sigma$ ,  $i^{\sigma \Rightarrow \tau} \alpha = j^{\sigma \Rightarrow \tau} \alpha$  for any two such parallel  $i, j$ , so  $\alpha$  is pinned down by  $\emptyset \in V^\ddagger$ . Similarly for the  $S$  and  $K$  combinators. So all we need to show is that when  $\alpha \in W^{\sigma \Rightarrow \tau}$  and  $\mathbf{b} \in V^\sigma$  are both pinned down by  $-^\ddagger$  and  $h : W \rightarrow V$ ,  $\alpha \langle h, \mathbf{b} \rangle$  is also pinned down by  $-^\ddagger$ . Let  $X \in W^\ddagger$  pin down  $\alpha$  and  $Y \in V^\ddagger$  pin down  $\mathbf{b}$ . We will show that  $h^\ddagger X \cup Y$ —which belongs to  $V^\ddagger$  since  $V^\ddagger$  is closed under finite unions—pins down  $\alpha \langle h, \mathbf{b} \rangle$ . Suppose  $i, j : V \rightarrow U$  agree on  $h^\ddagger X \cup Y$ . Then they agree on  $Y$ , so  $i^\sigma \mathbf{b} = j^\sigma \mathbf{b}$ . And they agree on  $h^\ddagger X$ , which implies that  $i \circ h$  and  $j \circ h$  agree on  $X$ , which implies that  $(i \circ h)^{\sigma \rightarrow \tau} \alpha = (j \circ h)^{\sigma \rightarrow \tau} \alpha$ . Hence,  $i^\tau (\alpha \langle h, \mathbf{b} \rangle) = \alpha \langle i \circ h, i^\tau \mathbf{b} \rangle = ((i \circ h)^{\sigma \rightarrow \tau} \alpha) \langle 1_U, i^\tau \mathbf{b} \rangle = ((j \circ h)^{\sigma \rightarrow \tau} \alpha) \langle 1_U, j^\sigma \mathbf{b} \rangle = \alpha \langle j \circ h, j^\sigma \mathbf{b} \rangle = j^\tau (\alpha \langle h, \mathbf{b} \rangle)$ .  $\square$

The payoff of all this is that we have a way of building non-full action models to verify the consistency of various packages of claims that cannot hold in full models.

**Proposition D.5.** The following combinations are consistent, in Classicism, with failures of Boolean Completeness (and hence also Rigid Comprehension).<sup>92</sup>:

1. ND and BF (and hence  $\neg$ Actuality and  $\neg$ Atomicity, by Propositions 2.5 and 2.6).
2.  $\neg$ ND,  $\neg$ Atomicity,  $\neg$ Actuality, and BF.

<sup>92</sup>Here, the negation symbol just means that the schema in question fails in some type. In fact, in all the models below, if Atomicity fails it fails in type  $t$ , and hence in every type, and if ND or BF fail they fail in type  $e$ , and hence in every type. The failures of Boolean Comprehension we have identified in our models are in type  $e \rightarrow t$ , though we conjecture that there are also failures in type  $t$  (and hence in every type).

3.  $\neg$ ND,  $\neg$ Atomicity,  $\neg$ Actuality, and  $\neg$ BF.
4.  $\neg$ ND,  $\neg$ Atomicity, Actuality, and  $\neg$ BF.
5.  $\neg$ ND,  $\neg$ Atomicity, Actuality, and BF.
6.  $\neg$ ND, Atomicity,  $\neg$ Actuality, and  $\neg$ BF.
7.  $\neg$ ND, Atomicity, Actuality, and  $\neg$ BF.
8.  $\neg$ ND, Atomicity, Actuality, and BF.<sup>93</sup>

To construct an ideally full model, it suffices to specify an underlying category  $\mathcal{C}$ , an action  $-^\dagger$ , an ideal  $-^\ddagger$  of  $-^\dagger$ , and an action  $-^e$  pinned down by  $-^\ddagger$ . In the following examples, we will choose  $\mathcal{C}$  to be a category of sets and functions, and  $-^\dagger = -^e$  to be the identity action on  $\mathcal{C}$  (i.e.  $W^\dagger = W^e = W$  and  $h^\dagger = h^e = h$ ).  $W^\ddagger$  will in each case be the set of all finite subsets of  $W (= W^\dagger)$ .

Part 1. Let's consider what the action model constructed in this way will look like where  $\mathcal{C}$  is the permutation group on the natural numbers  $\mathbb{N}$ : i.e., the category with just one object  $W_0$ , namely  $\mathbb{N}$ , and whose arrows are all the bijections  $\mathbb{N} \rightarrow \mathbb{N}$ .

- $W_0^e = \mathbb{N}$ ;  $h^e = h$  for every permutation  $h$ .
- $W_0^t$  is the set of all  $\mathbf{p} \subseteq \mathbb{N}^{\mathbb{N}}$  which are pinned down by some finite  $X \subseteq \mathbb{N}$ . Or equivalently: for any arrows  $h$  and  $i$  such that  $hx = ix$  for all  $x \in X$ ,  $h \in \mathbf{p}$  iff  $i \in \mathbf{p}$ .
- $W_0^{\sigma \rightarrow \tau}$  is the set of all  $\alpha \in W_0^{\sigma \Rightarrow \tau}$  which are pinned down by some finite  $X \subseteq \mathbb{N}$ . That is: whenever  $hx = ix$  for all  $x \in X$ ,  $\alpha\langle h, \mathbf{b} \rangle = \alpha\langle i, \mathbf{b} \rangle$  for all  $\mathbf{b} \in W_0^\sigma$ .

The intuition for this model is that there are infinitely many individuals, all playing distinct qualitative roles, and over which these roles can be redistributed in any way. Each permutation  $h$  represents the possibility where each individual  $n$  plays the role actually played by  $hn$ . But the only propositions and properties are ones that are “about” some finite collection of individuals, and thus indifferent to the question how the qualitative roles are distributed over individuals not in that collection.<sup>94</sup>

ND and BF hold in this model: since every arrow has an inverse, its action in each type must be bijective. To see that Actuality fails (which implies by Propositions

<sup>93</sup>We have not been able to settle the consistency of the combination of BF,  $\neg$ Actuality, and Atomicity, either with or without Boolean Comprehension.

<sup>94</sup>See footnote ?? below for rigorous definitions of the operations that convert applicative behaviour profiles to intensions and back again.

2.5 and 2.6 that Atomicity and Boolean Comprehension also fail), take any  $\mathbf{p} \in W_0^t$  that is true (i.e. contains the identity permutation  $1_{\mathbb{N}}$ ).  $\mathbf{p}$  is pinned down by some finite  $X \subseteq \mathbb{N}$ . Choose  $n \notin X$ , and let  $\mathbf{q} = \{h \in \mathbf{p} : hn = n\}$ .  $\mathbf{q}$  is pinned down by  $X \cup \{n\}$ , and thus also belongs to  $W_0^t$ .  $\mathbf{q}$  is also true, since it contains  $1_{\mathbb{N}}$ .  $\langle \mathbf{q}, \mathbf{p} \rangle$  is in the extension of  $\llbracket \leq \rrbracket$ , since  $\mathbf{q}$  is a subset of  $\mathbf{p}$ . But  $\mathbf{q} \neq \mathbf{p}$ . For suppose  $h \in \mathbf{q}$ , and let  $h'$  be the function that agrees with  $h$  except that  $h'n = n + 1$ . Then  $h' \notin \mathbf{q}$  since  $h'n \neq n$ , but  $h' \in \mathbf{p}$  since  $h'$  and  $h$  agree on  $X$ . Indeed by applying the same reasoning to an arrow other than  $1_{\mathbb{N}}$ , we can show that the following claim holds in the model:

$$\text{Atomlessness} \quad \forall p(\diamond p \rightarrow \exists q(\diamond q \wedge q \leq p \wedge q \neq p))$$

Before we proceed to Part 2, it will be useful to establish a sufficient condition for BF to hold in ideally full action models.

**Proposition D.6.** If  $\mathbf{A}$  is an ideally full action model such that for all  $h : W_0 \rightarrow V$ ,  $h^e$  and  $h^\ddagger$  are both surjective, BF holds in  $\mathbf{A}$  in every type.

*Proof.* By Proposition 3.24, it is sufficient to show that each  $h^\sigma : W_0 \rightarrow V$  is surjective. This is given for type  $e$ . For any other type  $\vec{\sigma} \rightarrow t$ , suppose  $h : W_0 \rightarrow V$  and  $\mathbf{b}$  is an element of  $V^{\vec{\sigma} \rightarrow t}$ , pinned down by  $Y \in V^\ddagger$ , with intension  $I_{\mathbf{b}}$ . Choose  $X \in W_0^\ddagger$  such that  $h^\ddagger X = Y$ , and let  $I_{\mathbf{a}} = \{\langle i, \vec{x} \rangle \mid i \text{ agrees on } X \text{ with } j \circ h \text{ for some } j \text{ such that } \langle j, \vec{x} \rangle \in I_{\mathbf{b}}\}$ . Since  $I_{\mathbf{a}}$  is insensitive to differences outside  $X$ , it is the intension of an element  $\mathbf{a} \in W_0^{\vec{\sigma} \rightarrow t}$ . Moreover,  $h^{\vec{\sigma} \rightarrow t} \mathbf{a} = \mathbf{b}$ , since  $\langle i, \vec{x} \rangle$  is in the intension of  $h^{\vec{\sigma} \rightarrow t} \mathbf{a}$  iff  $\langle i \circ h, \vec{x} \rangle \in I_{\mathbf{a}}$ , iff  $i \circ h$  agrees on  $X$  with  $j \circ h$  for some  $j$  such that  $\langle j, \vec{x} \rangle \in I_{\mathbf{b}}$ , iff  $i$  agrees on  $Y$  with  $j$  for some  $j$  such that  $\langle j, \vec{x} \rangle \in I_{\mathbf{b}}$  (since  $h$  surjectively maps  $X$  to  $Y$ ), iff  $\langle i, \vec{x} \rangle \in I_{\mathbf{b}}$  (since  $\mathbf{b}$  is pinned down by  $Y$ ).  $\square$

Note that any surjection from  $A$  to  $B$  induces a surjection from finite subsets of  $A$  to finite subsets of  $B$ . Thus in the models we are currently considering (where  $W^\ddagger$  is the set of finite subsets of  $W^\dagger = W^e = W$ ), Proposition D.6 means that BF holds whenever every arrow from  $W_0$  is a surjection.

This suggests a natural strategy for Part 2: simply expand the underlying monoid  $\mathcal{C}$  to include all surjections on  $\mathbb{N}$ . ND will now fail thanks to the non-injective arrows; but BF still holds by Proposition D.6, and Atomlessness holds for the same reason as before, so Actuality and Atomicity still fail.

Boolean Completeness does in fact fail in this model, but its failure is easier to show if we restrict the monoid to the *monotonic* surjections: those for which  $hn \leq hm$  whenever  $m \leq n$ . This does not disrupt BF or Atomlessness. Boolean Completeness can be seen to fail, in type  $e \rightarrow t$ . By extensional fullness, there is an element  $E \in W_0^{(e \rightarrow t) \rightarrow t}$  whose extension is the set of haecceities of even numbers—i.e. the denotations of  $\lambda x.x = y$  relative to assignments that map  $y$  to an even number.

We claim that  $E$  lacks a LUB. Equivalently, there is no strongest *persistent* property whose extension includes every even number.<sup>95</sup> The extension of a persistent property at any arrow must include the image under that arrow of its actual extension. Thus for any finite  $X$ , the strongest persistent property pinned down by  $X$  whose extension includes all the even numbers is the one whose extension, at an arbitrary arrow  $h$ , is  $\{n \mid n = gm \text{ for some even } m \text{ and some } g \text{ that agrees with } h \text{ on } X\}$ . When  $X$  is  $\{0, \dots, n\}$  for even  $n$ , this set is  $\{h0, h2, \dots, h(n-2), hn, hn+1, hn+2, \dots\}$ . So, the strongest such property pinned down by  $\{0, \dots, n+2\}$  is strictly stronger than (i.e., has a smaller extension at some arrows than) the strongest one pinned down by  $\{0, \dots, n\}$ . It follows that there is no strongest such property.

For part 3, we need to make BF fail too. We might expect that we could do this by basing the model on the monoid of *all* functions  $\mathbb{N} \rightarrow \mathbb{N}$ . But BF turns out to hold here as well—surprisingly, since intuitively the function mapping every number to 0 represents a possibility where new individuals play all the qualitative roles except that of 0. Recall that if  $h : \mathbb{N} \rightarrow \mathbb{N}$  is surjective, it has a right inverse  $i$  such that  $h \circ i = 1_{\mathbb{N}}$ , so that  $h^\sigma$  must be surjective for every  $\sigma$ . Now, observe that for any finite  $X \subseteq \mathbb{N}$  and any  $h : \mathbb{N} \rightarrow \mathbb{N}$ , there is a surjective function that agrees with  $h$  on  $X$ . Suppose for contradiction that there is an assignment  $g$  on which  $\forall y \Box Xy$  is true but  $\Box \forall y Xy$  is false. Let  $\alpha = gX$ . By the first assumption,  $\alpha \langle 1_{\mathbb{N}}, \mathbf{a} \rangle = \mathbb{N}^{\mathbb{N}}$  for all  $\mathbf{a} \in W_0^\sigma$ ; by the second, there are  $h, i$  and  $\mathbf{a}$  such that  $i \notin \alpha \langle h, \mathbf{a} \rangle$ .  $\alpha$  must be pinned down by some finite  $X \subseteq \mathbb{N}$ . Let  $j$  be surjective and agree with  $h$  on  $X$ ; then  $\alpha \langle j, \mathbf{a} \rangle = \alpha \langle h, \mathbf{a} \rangle$ , so  $i \notin \alpha \langle j, \mathbf{a} \rangle$ . But because  $j^\sigma$  is surjective, there is some  $\mathbf{b}$  such that  $j^\sigma \mathbf{b} = \mathbf{a}$ , and hence  $\alpha \langle j, \mathbf{a} \rangle = j^t(\alpha \langle 1_{\mathbb{N}}, \mathbf{b} \rangle) = j^t(\mathbb{N}^{\mathbb{N}}) = \mathbb{N}^{\mathbb{N}}$ : contradiction.

However, if we restrict the monoid to include only *monotonic* functions, BF comes out false. Let  $\alpha = \langle h, n \rangle \mapsto \{i \mid in \neq 0\}$ . This is pinned down by  $\emptyset$  since it is indifferent to its  $h$  argument: intuitively, it is the qualitative property of being positive. Consider the interpretations of  $\forall y \Box (Xz \rightarrow Xy)$  and  $\Box \forall y (Xz \rightarrow Xy)$  on the assignment  $[X \mapsto \alpha, z \mapsto 0]$ . The latter is false, since  $\forall y Xy$  denotes  $\emptyset$  on this assignment while  $Xz$  denotes the set of arrows that map 0 to something positive. But the former is true, since by monotonicity, any arrow that maps 0 to a positive number maps every number to a positive number. Atomlessness still holds for the same reason as before. And Boolean Completeness fails for the same reason as in Part 2.<sup>96</sup>

<sup>95</sup>Since every persistent property whose extension includes the even numbers is entailed by every  $E$  property, if  $X$  were the LUB of  $E$  it would entail every such property. It would also be persistent itself, since  $\lambda y. \Box Xy$  would also be an upper bound of  $E$ .

<sup>96</sup>Indeed, even the qualitative property *being a haecceity* ( $\lambda X^{e \rightarrow t}. \exists y (X = \lambda z. z = y)$ ) lacks a LUB in this model. The same is true in all the models below where BF fails. By contrast, the truth of BF means that  $\lambda x. x = x$  is a LUB of this property, which is why we needed to consider the more complicated example *being the haecceity of an even number* in Part 2.

Part 4. To get Actuality to hold, we can restrict the previous monoid to include only those monotonic functions  $h$  such that either  $h0 = h1$  or  $h = 1_{\mathbb{N}}$ . Actuality now holds: since  $1_{\mathbb{N}}$  is the only arrow that doesn't collapse 0 and 1,  $\{1_{\mathbb{N}}\}$  is pinned down by  $\{0, 1\}$ . But the actual world is the *only* world in this model: the restriction of Atomlessness to false propositions holds, and thus Atomicity fails. BF and Boolean Completeness fail too, for the same reasons as in Part 3.<sup>97</sup>

Part 5. As we might expect, we can restore BF alongside Actuality by further restricting the monoid to include only the *surjective* monotonic functions  $h$  such that either  $h0 = h1$  or  $h = 1_{\mathbb{N}}$ . BF now holds by Proposition D.6. Everything else is the same as before.

Part 6. To secure Atomicity without Actuality or BF, we can modify the model in Part 3 by letting the monoid contain just  $1_{\mathbb{N}}$  together with the functions  $g_n$ , where  $g_n m = \min\{m, n\}$ . BF fails since any arrow that maps 1 to 0 maps everything to 0. Actuality fails as before: the strongest true proposition pinned down by  $\{0, \dots, n\}$  is  $\{1_{\mathbb{N}}\} \cup \{g_m \mid m \geq n\}$ , so these propositions become ever stronger as  $n$  increases. Boolean Completeness also fails as before: the strongest persistent property with extension  $\mathbb{N}$  pinned down by  $\{0, \dots, n\}$  is the one whose extension at  $g_m$  is  $\{0, \dots, m\}$  if  $m < n$  and  $\mathbb{N}$  otherwise, so these properties also become ever stronger as  $n$  increases. But Atomicity now holds. For by the failure of Actuality, every nonempty proposition contains some  $g_n$ . When  $n > 0$ ,  $\{g_n\}$  is pinned down by  $\{n - 1, n, n\}$ , since it is  $\{h \mid h(n - 1) \neq h(n) \text{ and } h(n) = h(n + 1)\}$ ;  $\{g_0\}$  is pinned down by  $\{0, 1\}$ , since it is  $\{h \mid h0 = h1\}$ . Thus every  $\{g_n\}$  belongs to  $W_0^t$ .

Part 7. To secure both Atomicity and Actuality without BF, let the monoid contain, for each  $n$  that is a power of 2, the function  $f_n$  where  $f_n m$  is the greatest multiple of  $n \leq m$ . When  $n$  is a positive power of 2, the singleton  $\{f_n\}$  is  $\{h \mid h(n - 2) = h(n - 1) \neq hn\}$ , which is pinned down by  $\{n - 2, n - 1, n\}$  and the singleton  $\{f_1\} = \{1_{\mathbb{N}}\}$  is  $\{h \mid h0 \neq h1\}$ , which is pinned down by  $\{0, 1\}$ , so Actuality and Atomicity both hold. BF still fails, since every arrow that maps 1 to 0 maps everything to an even number. And Boolean Completeness also fails: when  $n$  is a power of 2, the strongest persistent property with extension  $\mathbb{N}$  pinned down by  $\{0, \dots, n\}$  is the one whose extension at  $f_m$  is the set of all multiples of  $\min\{m, n\}$ , so these properties become ever stronger as  $n$  increases.

Part 8. Finally, we can secure all three principles with a monoid containing just the functions  $k_n$  where  $k_n m = 0$  when  $m \leq n$  and  $k_n m = m - n$  otherwise. BF now holds since all the arrows are surjective.  $\{k_0\} = \{1_{\mathbb{N}}\} = \{h \mid h0 \neq h1\}$  and  $\{k_n\}$  is  $\{h \mid h(n - 1) = hn \neq h(n + 1)\}$  for positive  $n$ , so Actuality and Atomicity also hold. And Boolean Completeness still fails: the strongest persistent property pinned down

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<sup>97</sup>Many thanks to Christopher Sun for correcting a mistake in an earlier discussion of this combination.

by  $\{0, \dots, n\}$  whose extension includes all evens is the property whose extension at  $h_m$  is the set of even numbers if  $m$  is  $< n$  and even, the set of odd numbers together with 0 if  $m$  is  $< n$  and odd, and  $\mathbb{N}$  if  $m \geq n$ .

All of the above models were based on monoids, meaning that No Pure Contingency holds in them. To model cases where some of the principles are only contingently true or false, we can turn to multi-object models. For example, for any of the above models, we can adjoin a second object  $W_1 = \{0\}$ , with a single arrow from  $W_0$  to  $W_1$  (namely, the function from  $\mathbb{N}$  to  $\{0\}$ ) and no arrows from  $W_1$  to  $W_0$ . Then all the same principles (from among ND, BF, Atomicity, Actuality, and Boolean Completeness) hold as in the original model, but we also have  $\Diamond(\Box\text{ND} \wedge \text{Atomicity})$ . A wide range of other distributions of necessity, contingent truth, contingent falsehood, and impossibility over the principles can be modelled in a similar way.

One particularly interesting result is that we can have BF without  $\Box\text{BF}$ . For this, we can use a two-object model  $W_0 = \mathbb{N}$  and  $W_1 = \{0\}$ , with *all* functions from  $W_i$  to  $W_j$  as arrows. As before,  $W_i^\ddagger$  is the set of all finite subsets of  $W_i$ . (Thus  $W_1^\ddagger = \{\emptyset, \{0\}\}$ .) BF is false at  $W_1$ , since  $\forall y \Box x = y$  holds on the assignment  $[x \mapsto 0]$  but  $\Box \forall y x = y$  does not. But BF holds at  $W_0$ , for the same reason that we found it to hold in Part 3 above when we considered the monoid of all functions on  $\mathbb{N}$ : for any function  $f$  and finite set  $X$ , there is a surjection that agrees with  $f$  on  $X$ .

One other claim from Section 2.2 which we still haven't justified is that Actuality doesn't imply Atomicity in C5. We can show this using a generalization of the notion of an ideally full action model. Consider a category with two objects  $W_0$  and  $W_1$  which are both copies of  $\mathbb{N}$ , where the arrows between any pair of objects correspond to the permutations of  $\mathbb{N}$  (and composition is function-composition). In constructing  $-^t$  we impose a new constraint: for a set of functions to belong to  $W_i^t$ , it must not only be pinned down by some finite subset of  $W_i$ , but must also be such that whenever it contains a function  $h : W_i \rightarrow W_0$ , it also contains  $g \circ h$  for every permutation  $g : W_0 \rightarrow W_0$ . Intuitively,  $W_0$  represents a state of affairs where all individuals are qualitatively indiscernible, while  $W_1$  represents a state of affairs where each individual plays a unique qualitative role. In higher types we proceed as before:  $W_i^{\sigma \rightarrow \tau}$  contains exactly those  $\alpha \in W_i^{\sigma \Rightarrow \tau}$  that are pinned down by some finite set. In this model, the smallest set in  $W_0^t$  that contains  $1_{W_0}$  is the set of all arrows  $W_0 \rightarrow W_0$ , so this set witnesses the truth of Actuality at  $W_0$ . But Actuality is false at  $W_1$  for the usual reason, which means that  $\Box\text{Actuality}$  (and hence Atomicity) is false at  $W_0$ .

We are optimistic that more could be done using ideally full models as well as generalizations like the one above. For example, we have not tried very hard to find models where one or both of Boolean Completeness and Rigid Comprehension hold

although one or both of Atomicity and Actuality still fail. We hope that the results presented here will prompt others to make a more systematic exploration.

## Appendix E Coalesced sums and Maximalist Classicism

This appendix will prove the following theorem, whose significance was explained in Section 3.6:

**3.26.** For any set of action models with disjoint base categories, there is an action model such that every member of that set is among its truncations.

We first introduce an operation for combining an arbitrary set of non-overlapping rooted categories into a new category, their “coalesced sum”. Intuitively, the coalesced sum is the smallest category containing all the objects and arrows of the input categories, one new object  $W_0$ , and one new arrow from  $W_0$  to the base world of each input category, without making any unnecessary identifications.

**Definition E.1.** Given a disjoint set of rooted categories  $\langle C_k, W_k \rangle$  for  $k \in K$ , their *coalesced sum*,  $\nabla_{k \in K} \langle C_k, W_k \rangle$ , is the minimal category that includes all the objects and arrows of each  $C_k$ ; one new object  $W_0$ ; and a new distinguished arrow  $k$  from  $W_0$  to each  $W_k$  such that whenever  $h \circ k = h' \circ k$ ,  $h = h'$ .<sup>98</sup>

To prove Theorem 3.26, we define a corresponding operation on sets of action models.

**Definition E.2.** Given a set of action models  $\mathbf{A}_k = \langle C_k, W_k, -, \mathcal{I}_k \rangle$  for  $k \in K$  (for a given signature) whose underlying categories are disjoint, their *coalesced sum*  $\nabla_{k \in K} \mathbf{A}_k$  is an action premodel  $\langle C, W_0, -, \mathcal{I} \rangle$  defined as follows.

- The underlying rooted category  $\langle C, W_0 \rangle$  is  $\nabla_{k \in K} \langle C_k, W_k \rangle$ , the coalesced sum of the underlying categories of the models  $\mathbf{A}_k$ .
- For any type  $\sigma$ ,  $V^\sigma = V_k^\sigma$  for every object  $V$  of  $C_k$ , and  $h^\sigma = h_k^\sigma$  for every arrow  $h$  of  $C_k$ . (That is: the action of each type on the objects and arrows of each  $\mathbf{A}_k$  is just carried over unchanged into  $\nabla_{k \in K} \mathbf{A}_k$ .)
- $W_0^e$  is  $\prod_k W_k^e$  (the Cartesian product of the type- $e$  domains of the  $W_k$ ).

<sup>98</sup>This definition does not actually specify the identity of the new object and new arrows, which can be anything we like. If we want to officially choose, we could require the index set  $K$  to be disjoint from each  $C_k$ , choose  $W_0 = K$ , and for each object  $V$  of  $C_k$ , choose  $\text{Hom}(W_0, V)$  to be the set of ordered pairs  $\{\langle W_0, h \rangle : h \in \text{Hom}(W_k, V)\}$ . In this representation the distinguished arrow from  $W_0$  to  $W_k$  is  $\langle W_0, 1_{W_k} \rangle$  rather than just  $k$ .



- When  $x \in W_0^e$ ,  $k^e x = \pi_k x$  (the projection of  $x$  onto its  $k$ th co-ordinate).
- $W_0^t = \{X \in W_0^P \mid k^P X \in W_k^t \text{ for each } k \in K\}$ .
- $W_0^{\sigma \rightarrow \tau} = \{\alpha \in W_0^{\sigma \Rightarrow \tau} \mid k^{\sigma \Rightarrow \tau} \alpha \in W_k^{\sigma \rightarrow \tau} \text{ for all } k \in K\}$ .
- For a nonlogical constant  $c$  of type  $e$ ,  $\mathcal{I}c = \Pi_k \mathcal{I}_k c$ .
- For a nonlogical constant  $c$  of type  $\sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow t$ , we define  $\mathcal{I}c$  by way of its *intension*:

$$\mathcal{I}c = \text{App}\{\langle h \circ k, \mathbf{a}_1, \dots, \mathbf{a}_n \rangle \mid \langle h, \mathbf{a}_1, \dots, \mathbf{a}_n \rangle \in \text{Int } \mathcal{I}_k c\}$$

Here,  $\text{Int}$  stands for the operation that turns applicative behaviour profiles in  $W^{\sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow t}$  into their corresponding intensions, analogous to the  $\text{int}$  operation from Section 3.3, and  $\text{App}$  stands for the inverse operation turning intensions back into applicative behaviour profiles, analogous to  $\text{app}$  from Section 3.3.<sup>99</sup>

To show that this is a premodel, we need to check that for a nonlogical constant  $c$  of type  $\sigma$ ,  $\mathcal{I}c \in W_0^\sigma$ . For type  $e$  this is immediate. For type  $\tau = \sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow t$ , we need to show that for all  $k \in K$ ,  $k^\tau \mathcal{I}c \in W_k^\tau$ . But this is true:  $k^\tau \mathcal{I}c = k^\tau \text{App}\{\langle h \circ k, \mathbf{a}_1, \dots, \mathbf{a}_n \rangle \mid \langle h, \mathbf{a}_1, \dots, \mathbf{a}_n \rangle \in \text{Int } \mathcal{I}_k c\} = \text{App}\{\langle h, \mathbf{a}_1, \dots, \mathbf{a}_n \rangle \mid \langle h, \mathbf{a}_1, \dots, \mathbf{a}_n \rangle \in \text{Int } \mathcal{I}_k c\} = \text{App Int } \mathcal{I}_k c = \mathcal{I}_k c \in W_k^\tau$ .

Clearly, if  $\nabla_k \mathbf{A}_k$  so defined is an action model (i.e. if its domains are “sufficiently full”), then each of the  $\mathbf{A}_k$  is a truncation of it. So all we need to do to prove the theorem is verify that the sufficient fullness condition is met: i.e. that for any type- $\sigma$  term  $A$  and  $i : W_0 \rightarrow V$ ,  $\llbracket A \rrbracket_i^\sigma$  exists and is in  $V^\sigma$ .

Any arrow from the base object  $W_0$  is either the identity  $1_{W_0}$  or can be written uniquely in the form  $h \circ k$  for some arrow  $h : W_k \rightarrow V$ . In the case of a non-identity arrow  $h \circ k$ , a straightforward induction on the complexity of terms shows that  $\llbracket A \rrbracket_{h \circ k}^\sigma = \llbracket A \rrbracket_{k, h}^\sigma$  for every term  $A$  (where  $\llbracket \cdot \rrbracket_{k, \cdot}^\sigma$  is the interpretation function of  $\mathbf{A}_k$ ), and thus in the domain of  $h$ ’s target. The interesting cases here are those of the nonlogical constants, where we need to check that our chosen denotation in the coalesced sum gets mapped to the constant’s denotation in  $\mathbf{A}_k$ . When  $c$  is of type

<sup>99</sup>For  $X \in W^t$ ,  $\text{App } X = X$  and  $\text{Int } X = \{\langle h \rangle \mid h \in X\}$ . For  $\alpha \in W^{\sigma_1 \Rightarrow \dots \Rightarrow \sigma_n \Rightarrow t}$ ,

$$\text{Int } \alpha = \{\langle h, \mathbf{a}_1, \dots, \mathbf{a}_n \rangle \mid \langle 1_{\text{trg } h}, \mathbf{a}_2, \dots, \mathbf{a}_n \rangle \in \text{Int } \alpha \langle h, \mathbf{a}_1 \rangle\}$$

And when  $X$  is a set of  $n + 1$ -tuples  $\langle h, \mathbf{a}_1, \dots, \mathbf{a}_n \rangle$  (where for some  $V$ ,  $h : W \rightarrow V$  and each  $\mathbf{a}_i \in V^{\sigma_i}$ ),

$$\text{App } X = \langle h, \mathbf{a}_1 \rangle \mapsto \text{App}\{\langle i, \mathbf{a}_2, \dots, \mathbf{a}_n \rangle \mid \langle i \circ h, i^{\sigma_1} \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n \rangle \in X\}.$$

It is readily shown that these operations are inverses and commute with arrows.

$e$ ,  $\llbracket c \rrbracket_{h \circ k}^g = (h \circ k)^\sigma(\mathcal{I}c) = h^\sigma(k^\sigma(\mathcal{I}c)) = h^\sigma(\pi_k(\mathcal{I}c)) = h^\sigma(\mathcal{I}_k c) = \llbracket c \rrbracket_{k,h}^g$ . When  $c$  is of type other than  $e$ ,  $\llbracket c \rrbracket_{h \circ k}^g = (h \circ k)^\sigma(\mathcal{I}c) = h^\sigma(k^\sigma \mathcal{I}c) = h^\sigma(\mathcal{I}_k c) = \llbracket c \rrbracket_{k,h}^g$ . All the other cases of this induction are trivial.

This leaves us with one more thing to check, namely where  $i$  is  $1_{W_0}$ , the identity arrow of the base world. But here everything goes smoothly because we chose the domains of the base world to be “as full as possible”. Again we need an induction on the complexity of  $A$ .

- For a variable  $v$  of type  $\sigma$ ,  $\llbracket v \rrbracket_{1_{W_0}}^g$  is  $g v$ , which belongs to  $W_0^\sigma$  by definition of assignment function.
- For a nonlogical constant  $c$ ,  $\llbracket c \rrbracket_{1_{W_0}}^g$  is  $\mathcal{I}(c)$ , which we already showed was in the domain of  $W_0$  as part of showing that the coalesced sum was a premodel.
- For a logical constant  $A$  of type  $\sigma \rightarrow \tau$ , it suffices to show that  $k^{\sigma \Rightarrow \tau} \llbracket A \rrbracket_{1_{W_0}}^g \in W_k^{\sigma \rightarrow \tau}$  for each  $k \in K$ . But this is trivial: since the logical constants do the same thing relative to each arrow in any model, the result of acting with  $k^{\sigma \Rightarrow \tau}$  will just be the interpretation of the same logical constant in  $\mathbf{A}_k$  and so will automatically belong to  $W_k^{\sigma \rightarrow \tau}$ . E.g., for negation we have that  $k^{t \Rightarrow t} \llbracket \neg \rrbracket_{1_{W_0}}^g \langle i, \mathbf{p} \rangle = \llbracket \neg \rrbracket_{1_{W_0}}^g \langle i \circ k, \mathbf{p} \rangle = (\text{trg } i)^P \setminus \mathbf{p} = \llbracket \neg \rrbracket_{k, 1_{W_k}}^g \langle i, \mathbf{p} \rangle$ , and so  $k^{t \Rightarrow t} \llbracket \neg \rrbracket_{1_{W_0}}^g = \llbracket \neg \rrbracket_{k, 1_{W_k}}^g \in W_k^{t \rightarrow t}$ .
- For any application  $AB$  where  $A$  is of type  $\sigma \rightarrow \tau$  and  $B$  is of type  $\sigma$ , suppose  $\llbracket A \rrbracket_{1_{W_0}}^g \in W_0^{\sigma \rightarrow \tau}$  and  $\llbracket B \rrbracket_{1_{W_0}}^g \in W_0^\sigma$ . Then  $k^\tau \llbracket AB \rrbracket_{1_{W_0}}^g = k^\tau (\llbracket A \rrbracket_{1_{W_0}}^g \langle 1_{W_0}, \llbracket B \rrbracket_{1_{W_0}}^g \rangle) = \llbracket A \rrbracket_{1_{W_0}}^g \langle k, k^\sigma \llbracket B \rrbracket_{1_{W_0}}^g \rangle \in W_k^\tau$ ; thus  $\llbracket AB \rrbracket_{1_{W_0}}^g \in W_0^\tau$ .
- For an abstraction  $\lambda v. A$  where  $v$  is of type  $\sigma$  and  $A$  is of type  $\tau$ , we note that  $\llbracket \lambda v. A \rrbracket_{1_{W_0}}^g$  definitely exists by the induction hypothesis, so we just need to show that acting on it with  $k^{\sigma \Rightarrow \tau}$  gives an element of  $W_k^{\sigma \rightarrow \tau}$ . But for any  $i$  and  $\mathbf{a}$ ,  $(k^{\sigma \Rightarrow \tau} \llbracket \lambda v. A \rrbracket_{1_{W_0}}^g) \langle i, \mathbf{a} \rangle = \llbracket \lambda v. A \rrbracket_{1_{W_0}}^g \langle i \circ k, \mathbf{a} \rangle = \llbracket A \rrbracket_{i \circ k}^{(i \circ k \circ g)[v \rightarrow \mathbf{a}]} = \llbracket A \rrbracket_{k,i}^{(i \circ k \circ g)[v \rightarrow \mathbf{a}]}$  (by the induction hypothesis)  $= \llbracket \lambda v. A \rrbracket_{k, 1_{W_k}}^{k \circ g}$ , which is in  $W_k^{\sigma \rightarrow \tau}$  since  $\mathbf{A}_k$  is an action model.

This concludes the proof.

We made some unforced choices when we constructed the coalesced sum. Most obviously, when choosing how to interpret the non-logical constants in the sum, any suitable element  $\mathbf{a}$  such that  $k^\sigma \mathbf{a} = \mathcal{I}_k c$  for each  $k$  would have done. This fixes the extensions of the non-logical predicates relative to every arrow other than  $1_{W_0}$ , but leaves us free to set the extensions relative to  $1_{W_0}$ —i.e. the “actual” extensions in the

model—however we please. For concreteness, we gave every nonlogical constant of a relational type the empty extension. We also chose a very large domain for type  $e$  at  $W_0$ , namely the Cartesian product of all of the type- $e$  domains of the input models. However any set  $X$  equipped with functions  $k^e : X \rightarrow W_k^e$  whose ranges include all the interpretations of the type- $e$  non-logical constants in the  $W_k$  would have done. If our language doesn't have any nonlogical constants of type  $e$ , we could even have  $X$  be a singleton, while if it has at least one such constant, we could choose it to be the set of type- $e$  constants.<sup>100</sup>

By taking as our input a family of action models for which Classicism is complete and building the type- $e$  domains in this alternative way, we will get a model of the Possibility+ schema discussed in Section 2. However, the alternative construction still gives the base world enormous domains in every type other than  $W_0$ . Indeed, for every arrow  $h$  other than  $1_{W_0}$ ,  $W_0^t$  contains propositions  $\mathbf{p} \neq \mathbf{q}$  such that  $h^t\mathbf{p} = h^t\mathbf{q}$ : for example, take  $\mathbf{p}$  to be the set of all arrows from  $W_0$ , and  $\mathbf{q}$  to be the set of all arrows other than  $1_{W_0}$ . So with this way of constructing the models, we have the principle  $\forall p(p \rightarrow \Box_{\neq} p)$ : nothing can be different in any way without some distinct propositions becoming identical. They are thus as far as can be from being models of Strong Possibility, which requires that all sorts of things—anything compatible with Possibility—can happen without any propositions becoming identical. We do not know whether Strong Possibility is consistent, but any proof of its consistency using the methods of this section would have to involve a major modification of our construction that in some sense keeps the domains of  $W_0$  as small as possible, so that there is no need to identify any elements when we follow any arrow into any model of Max C. This seems difficult to pull off.

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<sup>100</sup>If all of the input models are *non-logically harmonious* (see Appendix C), then we could also build a coalesced sum on a different underlying category which has *exactly* one arrow from its base object  $W_0$  to every object in every input category. The definition of the premodel is just as before. The only thing that needs to be redone is the proof that  $\mathcal{I}c$ , for a nonlogical constant  $c$  of type  $\sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow t$ , does indeed belong to  $W_0^{\sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow t}$  (so that the definition is actually an action premodel).

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