Classicism

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1 Classicism

Many of the most central questions in philosophy, and beyond, are naturally understood as questions of identity. For example, philosophers (and scientists) have asked such questions as the following:

> Is knowing something the same as believing it truly and justifiedly?
>
> Is being morally right the same as maximizing utility?
>
> Is being hot the same as having a high gradient of entropy with respect to energy?

Once we start asking questions like these, we can formulate a range of further questions that seem initially much less gripping.

> Is knowing something the same as knowing it and knowing it?
>
> Is being morally right the same as being both morally right and either profitable or not profitable?
>
> Is being hot the same as being not not hot?

Once we admit the meaningfulness of the questions from the first list, it seems we have all these questions left dangling. What is clear is that they need to be approached in some systematic way, rather than one at a time. Unlike the questions on the first list, each of which raises distinctive issues proprietary to some subfield of philosophy of science, it seems reasonable to seek a general framework for theorizing about identity that settles the questions on the second list in one fell swoop.

The most straightforward such view is *Booleanism*, according to which—intuitively speaking—the propositions, properties and relations of any given type form a Boolean algebra under the operations of conjunction, disjunction and negation. This implies a positive answer to each of the questions on the second list.

Some might think that Booleanism is obviously false, because of putative counterexamples involving attitude reports. For example, Booleanism implies that to be rich is to be either rich and either happy or famous, or rich and not happy, but one might argue that this is false on the grounds that someone without much logical acumen could want to be rich but not want to be either rich and either happy or famous, or rich and not happy. But any attempt to use judgments about propositional attitudes to argue against identities is fraught with difficulties, since one must somehow resist the argument from the tempting premise that one could want to visit Hesperus without wanting to visit Phosphorus to the false conclusion that Hesperus isn't Phosphorus. And once these kinds of objections are excluded, Booleanism has many attractions. It is a strong theory which settles a wide range of questions that seem in need of settling. It is also very simple (as we will see later when we consider some axiomatizations of it), and thus provides a good explanation of the many cases where substitution of Boolean equivalents is truth-preserving even under some non-truth-functional operator (such as a counterfactual conditional). While its opponents have pointed to putative counterexamples, they have struggled to provide a comparably systematic and consistent theory which *predicts* the alleged counterexamples, as opposed to merely accommodating them. Booleanism thus sets the bar for what a simple and predictive theory addressing questions of higher-order identity should look like. This makes the task of investigating views compatible with Booleanism a particularly important component of the broader project of mapping out the space of views concerning the grain of reality (see Fritz 2017, and the introduction to this volume). n this paper, we will contribute to this project by formulating, defending, and exploring an extremely natural strengthening of Booleanism which we call *Classicism*.[1] Classicism goes beyond Booleanism by adding identities involving identity and the quantifiers that are analogous to Booleanism's characteristic principles concerning conjunction, disjunction and negation.

Part 1 of this paper will bolster Classicism's claim to naturalness by presenting a range of different axiomatizations of Classicism. Part 2 will map out two directions in which Classicism can be further strengthened. One direction, which we might think of as the direction of coarseness, has as its endpoint the "Extensionalist" thesis that coextensiveness suffices for identity. Although Extensionalism itself seems to us to be subject to decisive counterexamples, there are several interesting principles entailed by Extensionalism which look like attractive additions to Classicism. The other, less familiar, direction is the direction of fineness, in which identities whose truth value is left open by Classicism are settled negatively. The most extreme version of this idea, which we call *Maximalist Classicism*, adds to Classicism all of the distinctness claims in the language of quantifiers and truth-functional connectives that are compatible with Classicism. This view strikes us as an attractively strong and non-arbitrary, in a domain where the avoidance of arbitrariness seems

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[1] We choose the name 'Classicism' as it stands to classical (higher-order) logic as Booleanism stands to Boolean propositional logic. It's the same system as HE+Modalized Functionality in Bacon 2018a. It is also related to the intuitionistic system of higher-order modal logic from Awodey, Kishida and Kotzsch 2014.

particularly urgent. Part 3 develops a model theory which is sound and complete for Classicism, heuristically helpful, and can be used to establish the consistency of many of the theoretical packages we discuss including Maximalist Classicism.

Even committed opponents of Booleanism will have something to learn from this investigation. Many such theorists will be either be able to define, or be willing to take as primitive, a connective expressing some notion of "logical equivalence"—some relation less demanding than identity but more demanding that coextensivenes, obeying the analogues of the Boolean identities. If so, they will be able to find an unintended interpretation of our formalism under which they will accept Booleanism, and will be able to raise the question of whether they should also accept Classicism, and the various further strengthenings we will consider, under the same unintended interpretation.[2] For example, in the theories of Goodman (n.d.) and Dorr (2016), the role of logical equivalence can be played by the relation of being two propositions whose disjunction is identical to their conjunction. Likewise, in the object-language theory suggested by certain versions of truthmaker semantics (Fine 2017a), one can define a notion of "classical equivalence" which might play a similar role.[3] And of course, anyone who can make sense of metaphysical necessity has the option of reinterpreting all our uses of identity connectives in terms of necessary coextensiveness. Under this reinterpretation, some of the views we will consider may seem unfamiliar, or even wild. Any view fine-grained enough to distinguish some false identity proposition from some contradiction will correspond, under the reinterpretation, to a view on which certain metaphysically contingent propositions are possibly necessary. This is ruled out by the most widely accepted logic for metaphysical necessity. But the reasons for the orthodoxy of this logic seems to rest more on a dubiously literal-minded attitude to the possible worlds model theory than on any argument.[4] So we think that even under this interpretation, both Classicism and its logical maximalist extension are worthy views with much to recommend them.

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[2] The unintended interpretation can be specified by defining, for each type, a notion of *hereditary* logical equivalence: intuitively, two items are hereditarily logically equivalent when they produce hereditarily logically equivalent results when applied to hereditarily logically equivalent arguments. We can then reinterpret '$=$' as 'hereditarily logically equivalent', and reinterpret all quantifiers as restricted to entities that are hereditarily logically equivalent to themselves. (For a formally analogous definition of hereditary *coextensiveness*, see Gandy 1956 and Dorr 2016, n. 106.)

[3] p is classically equivalent to q iff $(p \wedge \top) \vee \bot = (q \wedge \top) \vee \bot$, on a certain definition of $\top$ and $\bot$. Thanks to Ethan Russo for showing that this works.

[4] For discussion of some more interesting arguments, see Williamson 1996, Bacon 2018a, §5.2–5.4, and Dorr, Hawthorne and Yli-Vakkuri n.d., §4.2 and §8.3.

1.1 Background higher order logic

We will be theorizing in a higher-order language in which the syntactic role of any given expression, or "term", is captured by assigning it a unique "type". (We take both terms and types to be strings of symbols.) In general, our type system will be $\boldsymbol{R}$, defined to be the smallest set that includes the letter 'e' (the "type of individuals") and 't' (the "type of propositions"), and is such that whenever σ and τ are in it and τ is distinct from e, $\ulcorner(\sigma \to \tau)\urcorner$ (the type of operations that make type-τ things out of type-σ things) is in it. We call types distinct from e "relational" types. In writing types we omit parentheses associating to the right, e.g. writing $e \to e \to t$ for $(e \to (e \to t))$. Terms of type e are called *singular terms*. Terms of type t are called *formulae*; when they don't have any free variables, they are called *sentences*. Terms of any other type are called *predicates*.

We will also sometimes consider languages using the larger type system $\boldsymbol{F}$, which is the smallest set containing 'e' and 't' and containing $(\sigma \to \tau)$ whenever it contains σ and τ, even when τ is e: $\boldsymbol{F}$ thus contains types like $e \to e$ and $(e \to e) \to t$ which are not in $\boldsymbol{R}$.

Terms can be simple or complex. *Simple* terms come in two varieties, namely variables and constants. Each variable and constant has a fixed type, and there are infinitely many variables with each type (which we indicative with a superscript when it is not clear from the context). *Complex* terms can be formed in two ways. First: when A is a term of type $\sigma \to \tau$, and B is a term of type σ, (AB) is a term of type τ. Second: when v is a variable of type σ, and A is a term of type τ, $(\lambda v.A)$ is a term of type $\sigma \to \tau$. In writing terms, parentheses can be omitted associating to the left, and the parentheses around lambda terms include as much as possible; thus $\lambda x.ABC$ abbreviates $(\lambda x.((AB)C))$.

The languages we are interested in will all include some logical constants, including truth-functional operators and quantifiers. The question which to treat as primitive and which as defined is relatively unimportant for our purposes: there is a version of Classicism for each sufficiently rich choice of primitives.[5] But for concreteness, we will focus on signatures that contain the following logical constants:

- Truth functional connectives $\wedge$, $\vee$ of type $t \to t \to t$, and $\neg$ of type $t \to t$.
- For each type σ, quantifiers $\forall_\sigma$ and $\exists_\sigma$ of type $(\sigma \to t) \to t$.

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[5] This is not to say that there might not be any dialectical significance to a given choice of primitives. Someone might, for instance, accept the version of Classicism stated in terms of $\wedge$ and $\neg$, but reject the version of Classicism stated in terms of $\neg$ and $\vee$, on account of a having a non-Boolean theory of disjunction. By contrast Classicism, when formulated with a submaximal basis of logical constants, should be understood as taking the other connectives to be defined out of the primitive logical operations in the usual manner.

$$\begin{array}{rlrl}
\neg_t & := \neg & \neg_{\sigma\to\tau} & = \lambda X^{\sigma\to\tau}.\lambda z^{\sigma}.\neg_{\tau} Xz \\
\wedge_t & := \wedge & \wedge_{\sigma\to\tau} & := \lambda X^{\sigma\to\tau} Y^{\sigma\to\tau} z^{\sigma}.Xz \wedge_{\tau} Yz \\
\vee_t & := \vee & \vee_{\sigma\to\tau} & := \lambda X^{\sigma\to\tau} Y^{\sigma\to\tau} z^{\sigma}.Xz \vee_{\tau} Yz \\
\forall_{\sigma,t} & := \forall_{\sigma} & \forall_{\gamma,\sigma\to\tau} & := \lambda X^{\gamma\to(\sigma\to\tau)} y^{\sigma}.\forall_{\gamma,\tau}(\lambda z^{\gamma}.Xzy) \\
\exists_{\sigma,t} & := \exists_{\sigma} & \exists_{\gamma,\sigma\to\tau} & := \lambda X^{\gamma\to(\sigma\to\tau)} y^{\sigma}.\exists_{\gamma,\tau}(\lambda z^{\gamma}.Xzy) \\
\to & := \lambda pq.\neg p \vee q & \leftrightarrow & := \lambda pq.(\neg p \vee q) \wedge (\neg q \vee p) \\
\Box & := \lambda p.p =_t p \vee \neg p & \leq_{\tau} & := \lambda X^{\tau} Y^{\tau}.Y =_{\tau} X \vee_{\tau} Y
\end{array}$$

Figure 1. Metalinguistic abbreviations

- For each type σ, an identity predicate $=_\sigma$ of type $\sigma \to \sigma \to t$.

We write $A \wedge B$ instead of $((\wedge A)B)$, and similarly for other terms of types $\sigma \to \sigma \to \tau$. When P is a formula and v is a variable of type σ, $\forall v\, P$ abbreviates $(\forall_\sigma(\lambda v.A))$. Other abbreviations are listed in Figure 1: we put type subscripts on logical constants to lift them to properties and relations (e.g. $\neg_{e\to t}$ Wise for 'not wise'). → and ↔ are the material biconditional and biconditional; the significance of $\Box$ and $\leq$ will be discussed later.

When providing English glosses on sentences in this formal language, we will make free use of words like 'individual', 'property', 'relation', and 'proposition'. For example we will gloss $\forall Z.Zx \to Zy$ as 'y has every property that x has'. This practice should not be taken as providing our official translation manual from the higher-order language into English. Rather, like Prior (1971) and Williamson (2003), our attitude is that the higher-order language can be made intelligible in a way that doesn't rely on that particular translation into English, and is perhaps independent of any translation into English.

A *theory* (in a given higher order language) is just a set of formulae. For ease of axiomatization, we work with theories whose members include open formulae as well as sentences. But it is only *sentences* that can be said in a non-artificial sense to be true or false; and we can call a theory true just in case every sentence in it is true.

All the theories we will be considering are extensions of a fairly weak version of higher-order classical logic that we call H. An axiomatization of H is given in Figure 2. The axiomatization consists of principles governing the truth-functional connectives (propositional logic), principles governing the quantifiers at each type (obtained by generalizing standard axiomatizations of first-order logic), and principles governing the behaviour of λ. (In the latter, $\Phi[A]$ stands for any formula containing an occurrence of a term A, possibly with free variables bound by Φ, and $\Phi[B]$ is the result of replacing this occurrence with the term B.) Note that in the statement of the rules and axiom-schemes, the symbol '$\vdash$' just means 'is a member of the theory in question', so that Figure 2 is a list of ten properties of theories.[6] Any theory having these ten properties we call a H-theory. All the theories we will be considering later on will be H-theories, which means that they not only contain H but are also closed under MP, Gen, and Inst.[7]

Against the background of H, our rather large set of logical constants could be shrunk in various ways. The theorems of H that don't contain $\exists$ can be axiomatized by just closing all the $\exists$-free instances of the axioms (none of which are instances of EG) under MP and Gen; similarly for $\forall$, closing under MP and Inst. We can drop $=$ and all the axioms involving it without affecting the set of $=$-free theorems. We can drop $\wedge$ so long as we replace the instances of β and η with versions involving a variant of $\leftrightarrow$ defined just in terms of $\neg$ and $\vee$. And we can drop $\vee$ so long as we do something similar, and close now not under MP (which is trivial) but under *conjunctive syllogism*: if $\vdash \neg(P \wedge Q)$ and $\vdash P$ then $\vdash \neg Q$.

One noteworthy theorem of H is

Existence $\qquad \exists x.x = x$

This is a schema, since x may be a variable of any type. Informally: **Existence** says that there is something of every type. It follows (by PC and MP) from the Ref-instance $y = y$, the β-instance $(\lambda x.x = x)y \leftrightarrow y = y$, and the EG-instance $(\lambda x.x = x)y \to \exists x.x = x$. The fact that H implies Existence is not much of an objection to its *truth*, since instances of Existence are not very controversial (only nihilists would deny them). Nevertheless, it is worth knowing that there is a natural, mild weakening H^- of H that avoids having all instances of Existence as a theorem by restricting when we are allowed to use open formulae in the derivation of a closed theorem.[8]

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[6] It is worth noting a few other ways of axiomatizing H or any other H-theory. First of all, PC could be replaced with any of the well-known collection of axiom-schemas whose closure under MP yields PC. Second, EG and Inst could both be dropped in favour of the axiom schema $\exists v(P) \leftrightarrow \neg\forall v(\neg P)$, or alternatively UI and Gen could both be dropped in favour of $\forall v(P) \leftrightarrow \neg\exists v(\neg P)$. Third, we could divide the work of Gen (the "Hilbert-Ackermann Generalization Rule") between a simpler generalization rule (if $\vdash P$, then $\vdash \forall v.P$) and a new axiom scheme $\vdash (\forall v.P \to Q) \to (P \to \forall v.Q)$, where v is not free in P; similarly for EG and Inst.

[7] H is not the only candidate for the label 'classical higher-order logic': one might also use that label for the weaker logic H_0 which eliminates the η schema and replaces β with the much weaker "Extensional β" schema whose instances are just formulae of the form $(\lambda \vec{v}.P)\vec{A} \leftrightarrow P[\vec{v} \mapsto \vec{A}]$. For more discussion of H_0 see Dorr, Hawthorne and Yli-Vakkuri n.d., ch. 1 and Bacon and Zeng 2021; for a philosophical defence of the β axiom see Dorr 2016, §5.

[8] We define H^- in terms of a family of theories H^-_V, where V is a set of variables. These are

PC: $\vdash P$ whenever P is a tautology (substitution instance of a theorem of classical propositional logic).

UI: $\vdash \forall_\sigma F \to FA$ (where A is of some type σ and F is a term of type $\sigma \to t$).

EG: $\vdash FA \to \exists_\sigma F$ (where A is of some type σ and F is a term of type $\sigma \to t$).

Ref: $\vdash A =_\sigma A$

LL: $\vdash (A = B) \to (FA \to FB)$

β: $\vdash \Phi[(\lambda v.A)B] \leftrightarrow \Phi[A[v \mapsto B]]$, where $A[v \mapsto B]$ is the result of replacing every free occurrence of v in A with B (so long as this can be done without any free variable in B becoming bound).

η: $\vdash \Phi[\lambda v.(Fv)] \leftrightarrow \Phi[F]$, where v is not free in F.

MP: If $\vdash P$ and $\vdash P \to Q$, then $\vdash Q$.

Gen: If $\vdash P \to Q$, and v does not occur free in P, $\vdash P \to \forall v\, Q$.

Inst: If $\vdash P \to Q$, and v is does not occur free in Q, $\vdash \exists v\, P \to Q$.

Figure 2. Axiomatization of H

However, the only type for which this weaker logic H^- fails to prove Existence is e. In every R-type other than e, we can construct a closed term using only logical constants, and derive Existence using EG from the Ref-instance involving that term, so that the new limits on the use of free variables are not relevant. Moreover, if we add $\exists x^e.x = x$ to H^- and close under MP, we get back H.[9]

1.2 Booleanism

According to the 'Booleanist' worldview it is possible to substitute Boolean equivalents *salve veritate* (see, for instance, Bacon 2018a.) We will thus take *Booleanism* to be the result of adding the following schema to H and closing under its rules:

Tautological Substitution $\Phi[P] \to \Phi[Q]$, where P and Q are equivalent in propositional logic, and $\Phi[P]$ and $\Phi[Q]$ are formulae the differ by the replacement of an occurrence of P with one of Q.[10]

Alternatively (following Dorr 2016, §7), we can define Booleanism to be the smallest H-theory containing all instances of the following schema:

Tautological Equivalence $(\lambda\vec{v}.P) = (\lambda\vec{v}.Q)$, whenever P and Q are equivalent in propositional logic.

Here, $\lambda\vec{v}$ is short for $\lambda v_1. \dots .\lambda v_n.$, for some n variables $v_1, \dots, v_n$ with $n \geq 0$. Every instance of Tautological Equivalence can be derived from the instance $(\lambda\vec{v}.P) = (\lambda\vec{v}.P) \to (\lambda\vec{v}.P) = (\lambda\vec{v}.Q)$ of Tautological Substitution together with the Ref-instance $(\lambda\vec{v}.P) = (\lambda\vec{v}.P)$. Conversely, any instance of Tautological Substitution

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defined inductively as follows:

(i) H^-_V contains all instances of PC, UI, EG, Ref, LL, β, and η with free variables in V.

(ii) Whenever H^-_V contains both $P \to Q$ and P, it contains Q.

(iii) Whenever H^-_V contains $P \to Q$, $\mathsf{H}^-_{V-\{v\}}$ contains $P \to (\forall v.Q)$ if v is not free in P, and contains $(\exists v.P) \to Q$ if v is not free in Q.

A formula P belongs to H^- just in case it belongs to H^-_V, where V is the set of variables free in P. The formula $\exists x^e.x = x$ is in H^-_V for every nonempty V, but it is not in $h(\emptyset)$ and hence not in H^-.

[9] In type system F, there are other types besides e—for example, $t \to e$ and $(e \to e) \to e$—in which there are no closed terms without nonlogical constants. The F-version of H^- also fails to prove the instances of **Existence** for these types. However, adding the single instance $\exists x^e.x = x$ will also make those instances provable.

[10] This replacement may occur even in the scope of λ-terms and may involve variable capture: e.g., an instance is $(\lambda p.p) = (\lambda p.p) \to (\lambda p.p) = (\lambda p.\neg\neg p)$, which implies that double negation is the identity operation on type t.

Commutativity-$\wedge$	$(\lambda pq.p \wedge q) = (\lambda pq.q \wedge p)$
Commutativity-$\vee$	$(\lambda pq.p \vee q) = (\lambda pq.q \vee p)$
Distributivity-$\wedge\vee$	$(\lambda pqr.p \wedge (q \vee r)) = (\lambda pqr.(p \wedge q) \vee (p \wedge r))$
Distributivity-$\vee\wedge$	$(\lambda pqr.p \vee (q \wedge r)) = (\lambda pqr.(p \vee q) \wedge (p \vee r))$
Dissolution-$\wedge\vee$	$(\lambda pq.p \wedge (q \vee \neg q)) = (\lambda pq.p)$
Dissolution-$\vee\wedge$	$(\lambda pq.p \vee (q \wedge \neg q)) = (\lambda pq.p)$

Figure 3. The Boolean Identities

can be derived from Tautological Equivalence using Beta and LL to to extract the formulae to be substituted.[11]

Tautological Equivalence is an axiom-schema with infinitely many instances. It is not a trivial matter to tell whether a given formula is an instance (though it is a *decidable* question, by the decidability theorem for classical propositional logic). But we can equally well characterize Booleanism as the smallest H-theory containing the six individual axioms listed in Figure 3, the "Boolean Identities". Many other similar lists of axioms could be given, corresponding to different equivalent definitions of Boolean algebras in mathematics.

One noteworthy consequence of Booleanism is that conjunction and disjunction are "interdefinable", in the sense that both of the following identities are true:

$\wedge$-duality	$(\wedge) = (\lambda pq.\neg((\neg p) \vee (\neg q)))$
$\vee$-duality	$(\vee) = (\lambda pq.\neg((\neg p) \wedge (\neg q)))$

Given the truth of these identities, there is a good sense in which nothing would have been lost if we had worked in the smaller signature containing only one of $\wedge$ and $\vee$, treating the other when convenient as a metalinguistic abbreviation.[12]

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[11] Let $\vec{v}$ be the variables free in either of P or Q. Then $\vdash (\lambda X.\Phi[X\vec{v}])(\lambda\vec{v}.P) \to (\lambda X.\Phi[X\vec{v}])(\lambda\vec{v}.Q)$ by Tautological Equivalence, Ref, and LL. But the two sides of this conditional are β-equivalent respectively to $\Phi[P]$ and $\Phi[Q]$, so $\vdash \Phi[P] \to \Phi[Q]$.

[12] For any sentence P we accept involving one of the logical constants, $\wedge$-def and $\vee$-def let us find a sentence P' not involving that constant such that Booleanism implies $P = P'$, and thus that the fact we express using P can also be expressed by P'. But $\wedge$-def and $\vee$-def are controversial. Their conjunction has some consequences that contain only one of $\vee$ and $\wedge$ and are not theorems of H, so while we are free if we please to treat one of the symbols as a metalinguistic abbreviation in such a way as to make one of them uncontroversial, the truth of the other one will then be non-obvious. In a setting where the truth of Booleanism is up for debate, it is thus helpful to work in a signature containing both connectives, even if one in fact accepts Booleanism and hence $\wedge$-def and $\vee$-def.

Booleanism implies that every relational type τ forms a Boolean algebra with respect to the lifted operations $\neg_\tau$, $\wedge_\tau$, and $\vee_\tau$. There are weaker versions of 'Booleanism' which only requires *propositions* to form a Boolean algebra under conjunction, disjunction, and negation, and has nothing to say about properties and relations. This theory—let's call it Propositional Booleanism—is the smallest H-theory containing all instances of

Propositional Tautological Equivalence $Q = Q'$, whenever $Q \leftrightarrow Q'$ is a tautology.

Propositional Booleanism can also be characterized as the smallest H-theory containing each of the following formulae (or their universal closures):

$$\begin{aligned} p \wedge q &= q \wedge p & \qquad p \vee q &= q \vee p \\ p \wedge (q \vee r) &= (p \wedge q) \vee (p \wedge r) & \qquad p \vee (q \wedge r) &= (p \vee q) \wedge (p \vee r) \\ p \wedge (q \vee \neg q) &= p & \qquad p \vee (q \wedge \neg q) &= p \end{aligned}$$

However, it is hard to imagine why anyone would want to endorse Propositional Booleanism but not Booleanism. All the reasons we are aware of for liking or for not liking the claim that propositions form a Boolean algebra seem to carry over with exactly the same strength to every predicate type.[13]

1.3 Classicism

Although the axioms of Booleanism are very natural, they also seem like a somewhat arbitrary fragment of a more general picture. They tell us a lot about the interaction of the truth-functional connectives with identity, but are silent about the interaction of the other logical constants—the quantifiers and identity—with identity. For example, while Booleanism implies the identity of any two instances of the law of excluded middle ($p \vee \neg p = q \vee \neg q$) it does not imply the identity of any two instances of the law of identity (Ref: $(x = x) = (y = y)$). Likewise, while Booleanism implies that conjunction is the dual of disjunction, it does not imply that universal quantification is (in the parallel sense) dual to existential quantification. But there are deep connections between the logic of truth functional operations and the logic of identity and quantification: it is hard to conceive of a motivation for a view that accepts the former identities but not the latter ones.

The natural extension of Booleanism to the remaining logical constants is not hard to identify. It's just a matter of generalizing Tautological Substitution or Tautological Equivalence to give the classical logic of higher-order quantification and

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[13] Booleanism can be derived from Propositional Booleanism using the Functionality principle discussed in §1.4 below.

identity the same status that these schemas give classical *propositional* logic. And the obvious thing to mean by "the classical logic of higher-order quantification and identity" is the theory H introduced in the previous section. So we are led to the following generalizations of Tautological Substitution and Tautological Equivalence:

Logical Substitution $\Phi[P] \to \Phi[Q]$, whenever P and Q are equivalent in H and $\Phi[Q]$ results from $\Phi[P]$ by replacing an occurrence of P with one of Q.

Logical Equivalence $(\lambda\vec{v}.P) = (\lambda\vec{v}.Q)$, whenever P and Q are equivalent in H.

These are interderivable for the same reason as Tautological Substitution and Tautological Equivalence. We will dub the smallest H-theory containing all instances of these schemas 'Classicism', or C for short.[14]

Unlike the other schemas we have considered so far, Logical Substitution and Logical Equivalence are not decidable. But there are also natural decidable axiomatizations of Classicism. By contrast with Booleanism, there is no hope of characterization Classicism as the smallest H-theory containing some *finite* collection of axioms, since the instances of Logical Equivalence that are not already theorems of H include all of our infinitely many logical constants (the quantifiers $\forall_\sigma$ and $\exists_\sigma$ and identity predicate $=_\sigma$ for each type σ). But we can do the next best thing, namely have a small finite list of axioms for each logical constant. One particularly simple axiomatization of this sort is given in Figure 4. It comprises one closed identity for every identity predicate $=_\sigma$, and two closed identities for every quantifier $\forall_\sigma$ or $\exists_\sigma$. All of these identities are easily seen to be instances of Logical Equivalence. For example, the biconditional $Xy \leftrightarrow Xy \vee \forall_\sigma X$, needed to prove Absorption-$\vee\forall$ from Logical Equivalence, is a theorem of H because it is truth-functionally equivalent to the-UI instance $\forall_\sigma X \to Xy$. Appendix A proves that the Quantifier Identities and Identity Identity are sufficient to recover the remaining instances of Logical Equivalence. The proof works by using the identities to show that each axiom of H is

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[14] Cashing out "the classical logic of higher-order quantification and identity" as H might seem rather tendentious. After all, H goes beyond classical propositional logic not just by adding axioms and rules governing the quantifiers and identity, but by adding the axiom-schemes β and η, which are not specifically about any of the logical constants. But as it turns out, it doesn't matter. We will shortly be considering some alternative axiomatizations of Classicism which make do with only a small selection of instances of Logical Equivalence, and the biconditionals $P \leftrightarrow Q$ that generate instances look like good candidates to be part of any fragment of H that one might think of as a better candidate of the label "the classical logic of higher-order quantification and identity". Moreover, the relevant biconditionals are theorems of the "existentially neutral" logic H^- as well as of H; thus the smallest H-theory containing every instance of the weakening of Logical Equivalence that requires $\mathsf{H}^- \vdash P \to Q$ is the same as the smallest H-theory containing Logical Equivalence. The smallest H^- theory containing the weaker schema is just minimally weaker: if we add $\exists x^e.x = x$ and close under MP, we get Classicism back again.

The Identity Identity	$(\lambda yz.y =_\sigma z) = (\lambda yz.\forall X.Xy \leftrightarrow Xz)$
Absorption-$\vee\forall$	$(\lambda Xy.Xy \vee \forall_\sigma X) = (\lambda Xy.Xy)$
Dist-$\vee\forall$	$(\lambda Xp.p \vee \forall_\sigma X) = (\lambda Xp.\forall y.p \vee Xy)$
Absorption-$\wedge\exists$	$(\lambda Xy.Xy \wedge \exists_\sigma X) = (\lambda Xy.Xy)$
Dist-$\wedge\exists$	$(\lambda Xp.p \wedge \exists_\sigma X) = (\lambda Xp.\exists y.p \wedge Xy)$

Figure 4. The Classicist Identities

identical to $\top$ (using the Boolean identities for PC, the Absorption identities for UI and EG, and the Identity Identity for Ref and LL), and then showing that the rules of proof preserve identity to $\top$ (using the Boolean identities for MP and the Dist identities for Gen and Inst).

In the same sense in which Booleanism implies that $\wedge$ and $\vee$ are "interdefinable", Classicism implies that the same is true for $\forall$ and $\exists$. That is, it proves the following two identities:

$\forall$-duality	$\forall_\sigma = \lambda X.\neg(\exists_\sigma(\neg_{\sigma\to t} X))$
$\exists$-duality	$\exists_\sigma = \lambda X.\neg(\forall_\sigma(\neg_{\sigma\to t} X))$

If Classicism is true we would thus lose no expressive power in dropping one of the connectives from our official signature. However this might be unhelpful for the purposes of debating opponents of Classicism, some of whom might reject one or both of $\forall$-def and $\exists$-def. (For example one could imagine someone who accepts the Classicist Identities for $\forall$, but has some strange alternative take on $\exists$.) Similarly, the Identity Identity is already of the right form to license eliminating all the identity predicates from the signature in favour of the quantifiers. Although this particular identification might well be accepted even by philosophers who don't accept any of the other Classicist or Boolean identities, there are (as we will mention in §1.5) some important objections to Classicism which might motivate rejecting the Identity Identity. So again, for dialectical purposes, it will be helpful to keep identity in the signature.

Here we have focused on identity, however related axiomatizations of Classicism in terms of entailment are also possible; this is explored in appendix B.

1.4 Axiomatizations with new rules

One might wonder whether Classicism is itself just a somewhat arbitrary fragment of a more general picture, in the way we earlier claimed to be the case for Booleanism. Why only accept identities corresponding to biconditionals provable from H, when we now have a stronger theory C which proves further biconditionals, for which we might also accept the corresponding identities? This motivates a, putatively stronger, theory which includes the identities corresponding to biconditionals provable in C; indeed a series of theories, each adding the identities corresponding to biconditionals provable in its predecessor. The union of all these theories will be closed under the rule:

Equiv+ If $\vdash P \leftrightarrow Q$ then $\vdash (\lambda\vec{v}.P) = (\lambda\vec{v}.Q)$.

But the picture of a series of stronger and stronger theories is completely wrong, since as it turns out, C is *already* closed under Equiv+. Even though the only identities we added as axioms corresponded to biconditionals provable in H, the theorems provable from these axioms also include all identities corresponding to biconditionals provable in C. Indeed, since any H-theory closed under Equiv+ must evidently contain every instance of Logical Equivalence, C can be characterized as the smallest H-theory closed under Equiv+. The availability of this axiomatization provides further adds to our case for the centrality and naturalness of C as the endpoint of the theoretical impulse that initially inspires Booleanism.[15]

We can also divide the job of Equiv+ up between two different inference rules:

Equiv If $\vdash P \leftrightarrow Q$ then $\vdash P = Q$.
ξ If $\vdash A = B$ then $\vdash (\lambda v.A) = (\lambda v.B)$.[16]

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[15] Given η-conversion, closing under Equiv+ is equivalent to closure under the following alternative version of Equiv+:

Strong Equiv+ If $\vdash F\vec{v} \leftrightarrow G\vec{v}$ then $\vdash F = G$ where $\vec{v}$ are not free in F or G.

(See Dorr, Hawthorne and Yli-Vakkuri (n.d., §8.2): note that the version given there accidentally omits the crucial restriction that $\vec{v}$ not be free in F or G, without which the rule is inconsistent.) If we start with any theory that satisfies all the closure conditions other than η from the axiomatization of H given in Figure 2, closing under Strong Equiv+ is equivalent to adding η and closing under Equiv+. By β we have $(\lambda X^{\sigma\to\tau}.X)Y^{\sigma\to\tau}z^{\sigma} \leftrightarrow (\lambda Xv.Xv)Yz$, which we can feed into η-Equiv+ to get $\lambda X.X = \lambda Xv.Xv$, which straightforwardly yields η. A nice feature of **Strong Equiv+** is that if we start with the very weak logic H_0 mentioned in footnote7 and close under Strong Equiv+, we still get Classicism (including the full-strength β axiom and the η axiom): see Bacon and Zeng 2021.

Any H-theory closed under Equiv and ξ must evidently be closed under Equiv+, so the smallest H-theory closed under Equiv and ξ includes C. Since Equiv is just the $n = 0$ special case of Equiv+, C is closed under Equiv.[17]

One can think of Equiv+ as a "rule" counterpart of the following, much stronger, axiom-scheme, telling us that relations are individuated by their extensions:

Extensionality $\forall \vec{z}(X\vec{z} \leftrightarrow Y\vec{z}) \to X = Y$

Similarly, Equiv and ζ can be thought of, respectively, as "rule" counterparts of the following axiom-schemes:

The Fregean Axiom $(p \leftrightarrow q) \to p = q$

Functionality $\forall z(Xz = Yz) \to X = Y$

Neither Extensionality, Functionality, nor the Fregean Axiom is a theorem of Classicism (as we will confirm in part 3), so the axioms really are strengthenings of the corresponding rules.

Let *Extensionalism* be the smallest H-theory containing Extensionality; or, the smallest H-theory containing the Fregean Axiom and Functionality; or equivalently again, the smallest extension of C containing the Fregean Axiom.[18] Extensionalism

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[16] Alternatively we can replace ξ with

ζ If $\vdash Fv = Gv$ then $\vdash F = G$ where v is not free in F or G

In the presence of β and the identity axioms, closing under ζ is equivalent to adding η and closing under ξ.

[17] To see that C is closed under ξ, remember that for $(\lambda v.A) = (\lambda v.B)$ to even be well-formed in our type system, A and B must be terms of some type of the form $\sigma_1 \to \cdots \to \sigma_n \to t$. Let v be of type σ_0. Choose distinct variables $\vec{u}$ of types $\sigma_1 \ldots \sigma_n$ that are not free in A or B; then if $\vdash A = B$, we have $\vdash A\vec{u} \leftrightarrow B\vec{u}$ by Ref and LL, which implies $(\lambda v\vec{u}.A) = (\lambda v\vec{u}.B)$ by Equiv, which implies $(\lambda v.A) = (\lambda v.B)$ by η. Note that this proof depends crucially on the fact that we our type system is R. In the more general type system F, **we think** it is still true that the smallest H-theory containing Logical Equivalence is closed under ζ, but the argument that this is the case is more involved.)

It is worth noting that H is already closed under ζ. This follows from the following fact: whenever $A = B$ is a theorem of H, A and B are $\beta\eta$-equivalent terms; so in particular if $Fv = Gv$ is a theorem where v is not free in F or G, Fv and Gv are $\beta\eta$-equivalent, which implies that F and G are, so $F = G$ is also a theorem of H. This fact can be proved using model-theoretic techniques developed in Fritz, Lederman and Uzquiano forthcoming (see the remark about '$\text{DISTINCTNESS}_{\sim_{\alpha,\beta,\eta}}$' on p. 15 of that paper).

[18] The implication from the Fregean Axiom to Functionality in C follows from the fact that C proves Modalized Functionality (see §1.5). Note that this depends on the fact that we are working in the type system R; in the full functional type system F, the smallest extension of H containing Extensionality is still the same as the smallest extension of C containing the Fregean Axiom, but does not include the instances of Functionality for types ending in e.

occupies an important position on the map of H-theories: it, and theories that include it, are in a natural sense *maximally coarse-grained*. But—with due deference to the distinghished historical roster of adherents of extensionalism, starting with Frege—we take Extensionalism to be decisively refuted by arguments such as the following. Although snow is white ↔ snow is either white or not white, snow is white ≠ snow is either white or not white, since it is necessary that snow is either white or not white, but not necessary that snow is white. (There are many different senses of 'necessary' for which this argument is sound.)[19]

Classicism can thus be approached not only "from below", by starting with H or Booleanism and considering natural strengthenings, but "from above", by starting with Extensionalism and considering natural weakenings, specifically those that replace axioms with corresponding rules.

1.5 Classicism and modal logic

Recall that $\Box$ abbreviates $\lambda p.(p = (p \vee \neg p))$: observe that given Booleanism, any tautology could be substituted for the formula $p \vee \neg p$.[20] The choice to use a necessity symbol for this operator is appropriate here. Booleanism already includes all instances of the following schemas:

K $$\Box(P \to Q) \to (\Box P \to \Box Q)$$

T $$\Box P \to P$$

Classicism, unlike Booleanism, is also closed under the rule

Necessitation If $\vdash P$ then $\vdash \Box P$

(This follows from Classicism's being closed under Equiv, since if $\vdash P$, $\vdash P \leftrightarrow (P \vee \neg P)$.) Thus, in the setting of Classicism, the propositional logic of $\Box$ is a normal modal logic. Classicism also goes beyond Booleanism by including all instances of the schema:

4 $$\Box P \to \Box\Box P$$

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[19] Church's 'simple theory of types' (Church 1940) contains Functionality (axiom 10) but he elects not to add the Fregean Axiom. Henkin (1950) does add the latter, and it is standard in the systems used for higher-order formalization of mathematics. Gandy (1956) reports Turing as expressing suspicion of Extensionalism in terms that have thankfully become obsolete.

[20] Church (1951) discusses this definition of $\Box$ in the context of his 'Alternative 2'; see also Cresswell 1965.

Indeed, it follows from results in Bacon 2018a that the set of propositional modal formulas in $\Box$ derivable from Classicism is exactly the modal logic S4, the smallest set of formulae containing all tautologies and instances of K, T, and 4, and closed under MP and Nec.[21]

Classicism includes the following modal weakening of Extensionality, which can perform many of the same argumentative roles as that principle:

Intensionality $\Box\forall\vec{z}(X\vec{z} \leftrightarrow Y\vec{z}) \rightarrow X = Y$

To see that this is a theorem of Classicism, consider the following instance of Logical Equivalence:

$$\lambda\vec{z}.(X\vec{z} \wedge \forall\vec{z}.(X\vec{z} \leftrightarrow Y\vec{z})) = \lambda\vec{z}.(Y\vec{z} \wedge \forall\vec{z}.(X\vec{z} \leftrightarrow Y\vec{z}))$$

This implies

$$(\forall\vec{z}.(X\vec{z} \leftrightarrow Y\vec{z})) = \top \rightarrow (\lambda\vec{z}.X\vec{z} \wedge \top) = (\lambda\vec{z}.Y\vec{z} \wedge \top)$$

which implies Intensionality, by Booleanism and η-conversion. Note that the combination of Intensionality with Equiv or Necessitation implies every instance of Logical Equivalence, and thus serves as another possible axiomatization of Classicism.

Just as Extensionality is equivalent to the conjunction of Functionality and the Fregean Axiom, Intensionality is equivalent given that $\Box$ behaves as a normal modal operator to the conjunction of the following two axioms:

Modalized Fregean Axiom $\Box(p \leftrightarrow q) \rightarrow p = q$

Modalized Functionality $\Box\forall z(Xz = Yz) \rightarrow X = Y$[22]

Classicism can be axiomatized by the combination of Modalized Functionality with

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[21] We may interpret the language of propositional modal logic in the signature of higher-order logic augmented with type t constants for each propositional letter in the straightforward way. (Interpreting the letters as themselves, $\wedge$ for $\wedge$, etc., and most importantly, interpreting the $\Box$ of modal logic with the defined operator $\lambda p.p = \top$.) One of the results in Bacon 2018a implies that any transitive reflexive Kripke model of the modal language can be extended to a higher-order model of Classicism over this signature, that makes exactly the same sentences true (modulo the translation). Because S4 is complete for transitive reflexive Kripke models, Classicism cannot prove any propositional modal formulas not already proven from S4. The soundness of S4 under in this interpretation in Classicism is also spelled out there. Cresswell (1965) shows how once $\Box$ is defined in terms of identity, the principles of S4 can derived from some minimal principles about propositional identity. See also Suszko 1975 and Wiredu 1979.

[22] To derive Modalized Functionality from Intensionality, remember that for an instance of Modalized Functionality to be well formed, the variables X and Y must both be of some type

the Equiv rule.[23][24]

Each of these three principles partially articulates the idea that propositions, properties and relations are “individuated by necessary equivalence”, in the present sense of ‘necessary’. The Modalized Fregean Axiom (which is already a theorem of Booleanism) states the identity of necessarily equivalent propositions, and Intensionality of necessarily coextensive relations.[25]

A widely discussed version of this thesis about individuation, associated with philosophers like Lewis and Stalnaker, holds that *metaphysically* necessary coextensiveness suffices for identity. Note, however, it is not at all clear that metaphysical necessity should be identified with $\Box$. Classicists who take the two statuses to be distinct need not accept the Lewis-Stalnaker view, although proponents of that view will themselves accept the identity of metaphysical necessity and $\Box$ (since they accept their metaphysically necessary coextensiveness). Moreover, while Lewis and Stalnaker take metaphysical necessity to obey S5, it is consistent with Classicism that many theorems of S5 fail for $\Box$. So there are a range of views compatible with Classicism which diverge significantly from the Lewis-Stalnaker picture, and will call for a set of modelling tools substantially different from the most familiar versions of the possible worlds framework.

Indeed, versions of the thought that necessary equivalence suffices for identity can been articulated for different notions of ‘necessity’: the narrower the necessity in question the stronger and more contentious the thesis. Given Classicism, $\Box$ can be shown to be the *broadest* (most demanding) necessity, given reasonable purely logical definitions of ‘necessity operator’ and ‘at least as broad as’.[26] Accordingly, we

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$\sigma_0 \to \cdots \to \sigma_n \to t$. If $\Box\forall z.(Xz = Yz)$ we have $\Box\forall z\forall u_1 \ldots \forall u_n.(Xz\vec{u} \leftrightarrow Yz\vec{u})$, which implies $(\lambda\vec{u}z.Xz\vec{u}) = (\lambda\vec{u}z.Yz\vec{u})$ by Intensionality; this β-reduces to $(\lambda z.Xz) = (\lambda z.Yz)$, which η-reduces to $X = Y$.

[23] This axiomatization is in Bacon 2018a. Myhill (1958) reconstructs ‘Alternative 2’ from Church 1951 using Necessitation and Functionality.

[24] The above remarks all assume the type system R. The situation is somewhat different in type system F. In this setting, there are new instances of Modalized Functionality involving types ending in e, which are *not* theorems of the smallest H-theory containing all instances of Logical Equivalence. This shouldn’t be too surprising: we have motivated Classicism by the thought that logical equivalence suffices for identity, where logical equivalence is a relation between sentences—i.e. provability of the biconditional in H—not singular terms. The model theories we will be developing in part 3 can be generalized naturally to F, but the logic of the relevant classes of models is what we might call “Strong Classicism”, which includes all the instances of Modalized Functionality rather than just those that can be derived from Logical Equivalence.

[25] Modalized Functionality, by contrast, is concerns a less demanding notion of necessary “equivalence” — necessary *cofunctionality* — which can be applied even to operations that only belong to the non-relational type system F, such as operations of type $e \to e$.

[26] See Bacon 2018a for details, and Dorr, Hawthorne and Yli-Vakkuri n.d., Bacon n.d.(a), ch. 8 for further discussion. The result is fairly robust with respect to different precisifications of ‘necessity

shall pronounce $\Box$ as 'it is broadly necessary that'. Viewed in this light, Intensionalism captures the kernel of the thought that properties and relations are individuated by necessary equivalence that is common to all its different versions.

Another controversial consequence of Classicism is the Converse Barcan Formula (Barcan 1946):[27]

CBF $\Box\forall x\, P \to \forall x \Box P$

To see why CBF is controversial, observe that it implies the following schema:

Broad Necessitism $\forall x \Box \exists y(y = x)$

Take P in CBF to be the formula $\exists y. y = x$, we get a conditional whose consequent is Necessitism and whose antecedent, $\Box\forall x\, \exists y(y = x)$, is an uncontroversial theorem of C (since $\forall x\, \exists y(y = x)$ is a theorem of H and C is closed under necessitation).Broad Necessitism says that everything is *broadly* necessarily identical to something; since broad necessity entails every other form of necessity, it follows that nothing could have failed to be something, in any ordinary sense of 'could'. Many philosophers—"contingentists", in the terminology of Williamson 2013—have taken this to be false, indeed obviously false.[28] We disagree, but a full defence of this implication of Classicism would take us too far afield (see Williamson 2013, Goodman 2016, Fine 2017b, Dorr, Hawthorne and Yli-Vakkuri n.d.). Here, we will content ourselves with noting that many contingentists have been happy to help themselves, either as primitives or as the result of some kind of honest toil, to so-called "outer" or "possibilist" quantifiers Π and Σ, for which they are happy to accept the analogue of CBF and Broad Necessitism. It isn't obvious how much is really at stake in the debate between those who are willing to accept CBF as written above and those who reject it but accept the analogue with Π instead of $\forall$: even though Π seems to behave logically as a quantifier and to entail $\forall$, proponents of this view refuse for some reason to say that Π is the unrestricted universal quantifier and $\forall$ is some restriction of it. Anyhow, we invite contingentists who can make sense of these quantifiers to reinterpret all our uses of $\forall$ and $\exists$ in the relevant way.

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operator', such as whether you build in normality or not.

[27] To prove it in Classicism one uses the fact that it is closed under Necessitation and Gen, and contains the K schema. Applying Necessitation to an instance of UI we get $\Box(\forall x P \to P)$. K lets us distribute the necessity, $\Box\forall x P \to \Box P$, and Gen yields $\Box\forall x P \to \forall x \Box P$.

[28] See, for example, Kripke 1963, Fine 1977, Fritz and Goodman n.d., Stalnaker 2012, Plantinga 1974, Menzel 1990.

2 Extensions of Classicism

In this part of the paper, we will map out some theories that strengthen Classicism. Section 1.4 already discussed one important strengthening, namely Extensionalism, which can be got by adding the axiom scheme Extensionality (or the combination of Functionality and the Fregean Axiom) to Classicism. But there are several interesting theories that are stronger than Classicism, but weaker than Extensionalism, and which are not subject to the kinds of counterexamples that make us find Extensionalism to be of merely historical and mathematical interest. Sections 2.1–2.3 will explore some of these theories. Sections 2.4–2.6 will then turn to some quite different ways of strengthening Classicism in a fine-grained direction.

2.1 Towards Extensionalism I: Coarse-grainedness principles

One thing we might consider adding to Classicism is the principle of Functionality from §2.1:

Functionality $\forall z(Xz = Yz) \to X = Y$

So long as we don't also add the Fregean Axiom, this will not allow us to infer that coextensive properties are identical. Rather, it captures the idea that properties are completely determined by their applicative behaviour with respect to their arguments.

In understanding what Functionality says, it's useful to note a couple of equivalents:

Proposition 2.1. Functionality is equivalent in Classicism to each of the following schemas:

BF $\forall x\Box P \to \Box\forall x P$

Tractarianism $\forall x(p \leq Fx) \to p \leq \forall x Fx.$[29]

BF is the Barcan Formula, taken an axiom in the quantified modal logic of Barcan 1946. Tractarianism says that the universal generalization of a property behaves like the conjunctions of all its instances, i.e. the propositions that predicate that property.

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[29] To derive Tractarianism from Functionality, suppose $\forall x(p \leq Fx)$; then $F = \lambda x.(Fx \vee p)$ by Functionality, so $\forall x Fx = \forall x(Fx \vee p) = \forall x Fx \vee p$ by Dist-∨∀. To derive BF from Tractarianism, just plug in $\top$ for p. And to derive Functionality from BF, note that $\forall x(Fx = Gx)$ implies $\forall x\Box(Fx \leftrightarrow Gx)$ by LL, which implies $\Box\forall x(Fx \leftrightarrow Gx)$ by BF, which implies $F = G$ by Intensionalism.)

Instantiation already tells us that it entails all the instances; Tractarianism adds that it is entailed by anything that entails all the instances, just as a conjunction entails its conjuncts and is entailed by anything that entails all of its conjuncts (see the conjunction and elimination rules for conjunction). (Such an assimilation of quantification with infinitary conjunction is propounded by Wittgenstein in the *Tractatus*: Wittgenstein 1961, p. 6.0001; see Proops 2017.) Many have objected to it on the grounds that if there could be new objects, distinct from all the objects there already are, there's nothing to stop there from being a proposition, p, and a property F, such that p entails Fx for each object, but is compatible with there being possibly new things that aren't F, and thus is a counterexample to Tractarianism.[30] The informal picture often associated with BF, and thus Functionality and Tractarianism, is that there cannot be anything new.

A further strengthening is to place $\Box$ in front of any of Functionality, BF or Tractarianism. This results in the system HFE outlined in Bacon 2018a. The fact that this actually strengthens these principles suggests we have to be careful about the intuitive gloss on Functionality as 'there can't be new things', since given S4, it seems that any claim that was properly glossed like that should be necessary if true. Once we look at models where these principles are contingently true (in Appendix D), we will see more reasons to think the slogan 'there can't be new things' should really be associated with the necessitated versions of these schemes.

One might think that principles at the level of generality of BF and Functionality are necessary if true. Perhaps this attitude is correct as far as metaphysical necessity is concerned. But it's hard to see why the mere generality of P should generate any presumption that P, if true, is *broadly* necessary, i.e. identical to $P \vee \neg P$. If the corresponding attitude to metaphysical necessity is appropriate, then perhaps we should take this as an argument for the distinctness of broad necessity and metaphysical necessity, rather than an argument against the view that general principles like BF and Functionality are true but not broadly necessary.

Another noteworthy consequence of Extensionalism that is not a theorem of Classicism is the necessity of distinctness:

ND $$x \neq_\sigma y \to \Box(x \neq_\sigma y)$$

By contrast, the necessity of *identity* is already a theorem of Classicism:

NI $$x =_\sigma y \to \Box(x =_\sigma y)$$

This follows, by a well-known argument that seems to have been first been dis-

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[30] The argument against Tractarianism in Russell (1924, lecture 5) can be construed this way if we take Russell's 'it is a further fact that' to imply 'not entailed by'.

covered by Quine (see Burgess 2014), from the LL-instance $x = y \to (\lambda z.\Box(x = z))x \to (\lambda z.\Box(x = z))y$, together with $\Box(x = x)$, the necessitation of a Ref-instance.

ND also has some more familiar equivalents:

Proposition 2.2. ND is equivalent in Classicism to each of:

5 $\quad \Diamond p \to \Box\Diamond p$

B $\quad p \to \Box\Diamond p$

To derive 5 from the type-t instance of ND, substitute $\top$ for q and $\neg P$ for p. To derive B from 5, use the T axiom in the dual form $P \to \Diamond P$. And to complete the circle of entailments, we may can derive ND (for any type) from B: suppose $x \neq y$; then $\Box\neg\Box(x = y)$ by B; but $\Box(x = y \to \Box(x = y))$ by the necessitation of NI; so $\Box(x \neq y)$ (see Prior 1963, pp. 206–7).

Just as in the case of Functionality, these principles do not imply their own necessitations (i.e. the necessitations of their universal closures) in Classicism. But it's hard to think of a principled reason for accepting, say, ND that would not extend to its necessitated analogue:

□ND $\quad \Box\forall x(x \neq_\sigma y \to \Box(x \neq_\sigma y))$

We will refer to the result of adding □ND (or □5 or □B) to C C5, by analogy to the modal system S5. Insofar as any theory in this domain counts as "orthodox", C5 does.

One might have expected that the idea that distinct things are necessarily distinct would be entirely independent of the question of new things. But this turns out to be wrong:

Proposition 2.3. BF (and hence also Functionality and Tractarianism) is a theorem of C5.

The proof of this is essentially due to Prior (1956).[31]

In the other direction, we have the following novel result:

Proposition 2.4. ND and BF jointly imply □ND in Classicism.

For given ND, we have $\Box(x = y) \vee \Box(x \neq y)$. With 4, this implies $\Box(x = y) \vee \Box\Box(x \neq y)$, and hence $\Box(x = y \vee \Box(x \neq y))$, i.e. $\Box(x \neq y \to \Box(x \neq y))$. By Gen, $\forall xy\Box(x \neq y \to \Box(x \neq y))$, which implies □ND by BF. So far, then, our map of systems including Classicism is as depicted in Figure 5.

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[31] Prior uses S5; Prior 1967, p. 146 attributes the following simpler proof using B to E.J. Lemmon.

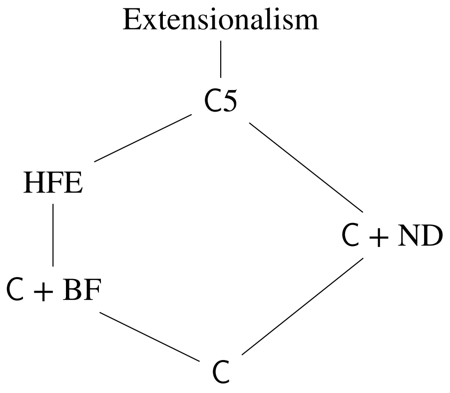

Figure 5. Coarse-grainings of Classicism

2.2 Towards Extensionalism II: Lattice-theoretic principles

The hierarchy of strengthenings of Classicism explored in the previous section is particularly important, both philosophically and because of the ways in which the coarser-grained views lend themselves to familiar and simple model theories. This section will survey three other principles inspired by conditions from the theory of Boolean algebras as formulated in classical first-order set theory, a common theoretical framework for modelling propositions. We will see that against the present foundational framework of higher-order logic, the relations between these principles are markedly different.

A "complete" Boolean algebra is one in which every set of elements has a *greatest lower bound*: a lower bound of the set that is $\geq$ every other lower bound of the set, where being a lower bound of a set means being $\leq$ every element of the set. Taking this as inspiration, consider the following principle:

Boolean Completeness $\forall X^{\tau\to t}.\exists y^{\tau}.\,\mathrm{GLB}_{\tau}\, yX$

where:

$$\mathrm{GLB}_{\tau} := \lambda y^{\tau} X^{\tau\to t}.\forall z^{\tau}.\,\mathrm{LB}_{\tau}\, zX \leftrightarrow z \leq_{\tau} y$$
$$\mathrm{LB}_{\tau} := \lambda z^{\tau} X^{\tau\to t}.\forall y^{\tau}.X y \to y \leq_{\tau} z$$

Boolean Completeness is thus analogous to the claim that each relational type τ forms a complete Boolean algebra under entailment, except that quantification into

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Suppose $\forall x \Box F x$. Then $\Box\Diamond\forall x\Box F x$ by B. Using CBF we can infer $\Box\forall x\Diamond\Box F x$, and finally, by the necessitation of B, $\Box\forall x F x$.

type $\tau \to t$ plays the role of quantification over sets.

To see that Boolean Completeness follows from Extensionalism, it suffices to note that Extensionalism implies that for any property X (of propositions, properties, or relations), *falling under everything* X (that is, $\lambda\vec{y}.\forall Z.XZ \to Z\vec{y}$) is a GLB of the X things. [32]

The second of our principles also corresponds to a well-known property of Boolean algebras. An *atom* of a Boolean algebra is an element such that the only thing below it is the bottom element; an algebra is *atomic* just in case every element is either the bottom element, an atom, or above at least one atom. This corresponds to a thesis about the propositions, properties, and relations of some type τ:

Atomicity $\forall x(x \leq \neg_\tau x \vee \exists y(\mathrm{Atom}_\tau\, y \wedge y \leq_\tau x))$

where:

$$\mathrm{Atom}_\tau := \lambda y.\forall z((z \leq_\tau y \wedge z \neq y) \leftrightarrow z \leq_\tau \neg_\tau z)$$

For the special case where τ is t we could equally well have defined the corresponding notion of an atom to be a broadly possible proposition that entails each proposition or its negation. The word ‘world’ would also be a pretty good name for atoms of type t, since given Atomicity, the broadly possible propositions are exactly those entailed by an atom. Extensionalism implies Atomicity.[33]

If we are calling propositional atoms ‘worlds’, it is natural to use ‘actual world’ to mean ‘true propositional atom’. Obviously there can only be at most one actual world, since any two atoms are incompatible. Call the claim that there is an actual world,

Actuality $\exists p(p \wedge \forall q(q \to p \leq q))$

Equivalently: any property of propositions all of whose instances are true has a true

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[32] To see that it’s a lower bound of X, suppose that Xu. Then given the Fregean Axiom $Xu = \top$, so $(\lambda\vec{x}.Xu \to u\vec{x}) = (\lambda\vec{x}.u\vec{x}) = u$, so $(\lambda\vec{x}.\forall Z.XZ \to Z\vec{x}) \leq u$. To see that it’s a *greatest* lower bound of X, suppose y is a lower bound of X. Then $\forall\vec{x}.y\vec{x} \to (\forall z.Xz \to z\vec{x})$. By Extensionality, this implies that $y \leq (\lambda\vec{x}.\forall Z.XZ \to Z\vec{x})$. Note that without Extensionalism, there is no guarantee that *having every X property* is even a lower bound of the X properties, let alone a greatest lower bound. For example, even though *being president* is a widely-discussed property, *having every widely-discussed property* plausibly fails to entail *being president*.

[33] In type t it provides a witness: we have $\mathrm{Atom}_t \top$. More generally, where τ is $\vec{\sigma} \to t$, Extensionalism implies $\forall\vec{y}.\,\mathrm{Atom}_\tau(\lambda\vec{z}.\vec{z} = \vec{y})$, and moreover $x \leq_\tau \neg_\tau x$ is equivalent given Extensionalism to $\neg\exists\vec{y}.x\vec{y}$; if this is false, we have $(\lambda\vec{z}.\vec{z} = \vec{y}) \leq_\tau x$ and $\mathrm{Atom}_\tau(\lambda\vec{z}.\vec{z} = \vec{y})$.

lower bound.[34] Extensionalism obviously entails Actuality and provides a witness, namely $\top$.

In $\mathsf{C5}$, the three principles we have just introduced are intimately related to each other:

Proposition 2.5. Boolean Completeness is equivalent to Actuality in $\mathsf{C5}$.

Proposition 2.6. Atomicity is equivalent to $\Box$Actuality (and hence also to $\Box$Boolean Completeness) in $\mathsf{C5}$.

For those used to using the theory of Boolean algebras to guide their reasoning about propositions, this should be surprising, for one can easily construct atomic Boolean algebras that are not complete, and complete Boolean algebras that are not atomic. These facts illustrate the danger of using the theory of arbitrary Boolean algebras in guiding one's theorizing about propositions.

Without the assumption of $\mathsf{C5}$, there are more surprises for this way of thinking. On the model of propositions as Boolean algebras worlds—i.e. propositions that settle the truths—are atoms, including the actual world. Given that Atomicity implies that every proposition is the disjunction (LUB) of the atoms that entail it, one might expect Atomicity to imply Actuality. But surprisingly this does not follow in Classicism: in Appendix D we will see that Classicism is consistent with the hypothesis that although every truth is entailed by some atom, every atom is false. Similarly, if there's a conjunction (GLB) of all the truths, as Boolean Completeness guarantees, one might expect that it would witness the truth of Actuality: but in fact, in general there is no obvious reason why the GLB of some truths should be true (or even possible), so Boolean Completeness does not seem to imply Actuality.[35] In the other direction, one might have thought that if, necessarily, there is an actual world, then every possible proposition must be entailed by an atom, namely the proposition that would have been the actual world if it had been true. However, this reasoning forgets the fact that without propositional BF, we cannot import a merely possibly existing world into actuality, and without ND, that even if an actually existing proposition is possibly a world it needn't actually be, since it it might actually be decomposable into stronger consistent propositions that become identical had it been

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[34] Note that Actuality implies the analogous generalization about arbitrary predicate types:

Actual Profile $\forall\vec{x}\exists Y(Y\vec{x} \land \forall Z(Z\vec{x} \to Y \leq Z))$

For suppose that w is a witness to Actuality; then for a given choice of $\vec{x}$, $\lambda\vec{y}.w\land x_1 = y_1\land\cdots\land x_n = y_n$ is a witness to Actual Profile.

[35] We conjecture that the combination of Boolean Completeness and the negation of Actuality is consistent, although we do not have a proof.

a world. In Appendix D we show that even given BF, $\Box$Actuality does not imply Atomicity by constructing models in which the latter situation occurs (an atomless proposition is possible atomic).

Proposition 2.5 will follow from results involving certain other principles to be introduced in the next section. The right-to-left direction of Proposition 2.6 can be established by showing that, given $\Box$ND, everything that is possibly an atom is in fact an atom. Suppose $\Diamond$ Atom y: possibly, everything is either entailed by or inconsistent with y. By BF (which follows from $\Box$ND), everything is either possibly entailed by or possibly inconsistent with y. But by ND, anything possibly entailed by y is entailed by y, and anything possibly inconsistent with y is inconsistent with y, so y is in fact an atom. Now suppose $\Box$Actuality and $p \neq \bot$. Then p is compatible with there being an actual world. Then by BF there is a proposition w such that it is possible that p be true while w is an actual world; but then w must in fact be a world, so we have the desired result that there is a world compatible with p.

For the other direction of Proposition 2.6, we actually need only BF rather than the full strength of C5:

Proposition 2.7. Atomicity and BF jointly imply $\Box$Actuality.

For given BF, every atom entails that *it* entails every truth (and hence that it it is a true atom, and hence there *is* a true atom), i.e.

$$\text{Atom}\, w \to w \leq \forall q(q \to w \leq q)$$

By Tractarianism (equivalent to BF), the consequent is equivalent to $\forall q.w \leq (q \to w \leq q)$. But this is true when w is an atom since when $w \leq q$, in which case $q \to (w \leq q)$ is $\top$ and hence entailed by everything, while when $w \leq \neg q$, $w \leq (q \to p)$ for any p. Thus, if every proposition other than $\bot$ is compatible with some atom, every proposition other than $\bot$ must be compatible with Actuality, so the negation of Actuality must be identical to $\bot$, i.e. the necessitation of Actuality is true.

Goodsell and Yli-Vakkuri (***) show that C5+Atomicity has a consequence that is worthy of special attention:

No Pure Contingency $P \to \Box P$, where P is closed and contains no non-logical constants.

Equivalently, $\forall \vec{x}\, Q \to \Box Q[\vec{a}/\vec{x}]$, where Q contains no nonlogical constants and has free variables in $\vec{x}$.[36] No Pure Contingency can also be consistently combined with C5 and the denial of Atomicity (indeed with Atomlessness), and with many other

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[36] This is one direction of the biconditional ‘Logical Necessity’ schema discussed in Bacon 2020. The other direction will be discussed below.

combinations of the principles we have discussed in this and the preceding section. And it has a certain plausibility. It can be derived from the combination of an account of the status of *logical truth* in the spirit of Bolzano (2004), Tarski (1959), and Williamson (2003), on which closed sentences with only logical constants are automatically logically true if true at all, together with the natural idea that the necessitation of any logical truth is itself a logical truth. However this is not by itself a strong argument for No Pure Contingency, since "logical truth" is a term of art, and a rather vexed one: insofar as one doubted No Pure Contingency one should suspect that the argument just conflates two different interpretations of 'logical truth'.

2.3 Towards Extensionalism III: Comprehension principles

Some formulations of second and higher-order logic take as primitive a comprehension schema along the lines of

$$\exists X\, \forall y(Xy \leftrightarrow P).$$

Since our present system has λ-terms this is in fact a theorem: the existential is witnessed by the term $\lambda y.P$.[37] However Classicism is neutral about certain other comprehension-style principles which this section will survey.

Our first principle requires some preliminary motivation. Let a *persistent* property, relation, or proposition be one that entails its own necessitation:

$$\text{Persistent} \coloneqq \lambda Y.Y \leq (\lambda \vec{z}.\Box Y\vec{z})$$

The modal behaviour of a persistent property is a bit like that of a *set* or *plurality*, according to standard modal set theory/plural logic. Any member of a set is necessarily a member of that set; any one of some things is necessarily one of those things; similarly any instance of a persistent property is necessarily an instance. But the standard view of sets and pluralities goes further than this by ruling out the possibility of a set of plurality acquiring any *new* members beyond those that it in fact has. If C5 fails, persistent properties need not behave like this; for example, when a, b, c are distinct but possibly $a = b \wedge a \neq c$, *being identical to* c *or such that* $a = b$ is a persistent property that in fact has only one instance but could have two. For the modal behaviour of a property to be really analogous to that of a set or plurality, it must be not only persistent but *inextensible*: necessarily such as to entail any property had necessarily by all of its instances:

$$\text{Inextensible} \coloneqq \lambda Y.\Box\forall X(\forall\vec{z}(Y\vec{z} \to \Box X\vec{z}) \to Y \leq X)$$

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[37] β ensures that, for any y, $(\lambda y.P)y \leftrightarrow P$.

In our example, $\lambda x.x = c \vee a = b$ is not inextensible, since it fails to entail $\lambda x.x = c$, even though that property is necessary to its one and only instance. The property of being a member of a given set or being one of some things, by contrast, would normally be thought of as inextensible: if each of some things is necessarily F, then necessarily anything that is one of them (belongs to the set of them) is F.[38]

Define a *rigid* property (or relation or proposition) as one that is both persistent and inextensible. We can simplify this as follows:

$$\mathrm{Rigid}_{\sigma_1,\ldots,\sigma_n} := \lambda Y.\Box\forall X.(\forall\vec{z}.Y\vec{z} \to \Box X\vec{z}) \leftrightarrow Y \leq X$$

The principle of interest says that every property (or relation or proposition) is coextensive with a rigid one:

Rigid Comprehension $\quad \forall X.\exists Y(\mathrm{Rigid}(Y) \wedge \forall\vec{z}(X\vec{z} \leftrightarrow Y\vec{z}))$

This can be thought of on the model of a comprehension principle for pluralities or sets, according to which every property is coextensive with a plurality or set. This assumption is both natural in itself, and needed for the regimentation of some natural-language modal claims not ostensibly about pluralities or sets, for example the most salient reading of ‘Mary could have had all John’s favourite properties’ (see Dorr, Hawthorne and Yli-Vakkuri n.d., §1.4). Rigid Comprehension also helps to provide a natural account of the prevalent use of extensionalist reasoning in mathematics (see Church 1940, Myhill 1958). Extensionalism entails that everything is rigid, and thus trivially implies Rigid Comprehension.[39]

Like all the principles from the previous section, Rigid Comprehension follows from the claim that there are only finitely many entities of the relevant types. Figid Comprehension also implies two of those:

Proposition 2.8. Rigid Comprehension implies Boolean Completeness.

Proposition 2.9. Rigid Comprehension implies Actuality.

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[38] See Linnebo 2013 and Dorr, Hawthorne and Yli-Vakkuri n.d., §1.5. An inextensible property is one for which BF necessarily holds for the quantifiers restricted by it. In C5, persistence entails inextensibility (see Dorr, Hawthorne and Yli-Vakkuri n.d., propositions C4 and C5).

[39] If we weaken ‘Rigid’ in Rigid Comprehension to ‘Persistent’, the resulting ‘Persistent Comprehension’ principle is equivalent to Actuality. The implication from it to Actuality can be recovered from the proof of Actuality from Rigid Comprehension below, which does not mention inextensibility. To derive Persistent Comprehension from Actuality, suppose w witnesses Actuality; then for any X, $\lambda\vec{y}.w \leq X\vec{y}$ is persistent and coextensive with X. We might also consider replacing ‘Rigid’ with ‘Inextensible’. The resulting principle also follows from Actuality, since if w witnesses Actuality; then for any X, $\lambda\vec{y}.w \wedge X\vec{y}$ is inextensible and coextensive with X.

Let's start with Proposition 2.8. To show that some property of propositions, F, has a greatest lower bound one takes a rigid property, G, coextensive with F and considers the proposition $\forall p G p$ (much as we did in the proof of Boolean Completeness from Extensionalism).[40] Parallel arguments establish the proposition at other relational types. To show that Rigid Comprehension entails Actuality, one takes a rigid property, T, coextensive with the truths (i.e. $\lambda p.p$) and defines the actual world as $\forall_t p.Tp$.[41]

In the setting of C5, Rigid Comprehension not only implies but is equivalent to Actuality:

Proposition 2.10. Actuality implies Rigid Comprehension in C5.

To prove this, one first shows that in C5 anything persistent is also inextensible.[42] It then suffices to show that Actuality implies that every property F is coextensive with a persistent one: if w is the actual world (i.e. the witness to Actuality) then the persistent property in question can be defined as being such that w entails you are F (i.e. $\lambda x(w \leq Fx)$); this is persistent since entailments are necessary if true, and coextensive with F, since w entails only the truths. One can extend this argument to relations straightforwardly.

Combining Propositions 2.8 and 2.10 gives us the right-to-left direction of Proposition 2.5 (stated without proof in §2.2): Actuality implies Boolean Completeness in C5.

Here are two other results involving Rigid Comprehension whose proofs we give in footnotes:

Proposition 2.11. $\Box$Atomicity, Boolean Completeness, and BF jointly imply Rigid Comprehension.[43]

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[40] Let X be of type $\tau \rightarrow t$, let X^* be the rigid property coextensive with X; and let U of type t be $\lambda \vec{z}.\forall Y(X^*Y \rightarrow Y\vec{z})$. To show that U is a GLB of X, notice that since X and X^* are coextensive, any V is a lower bound of X just in case it is a lower bound of X^*, i.e. $\forall Y(X^*Y \rightarrow \Box\forall\vec{z}(V\vec{z} \rightarrow Y\vec{z}))$ By the rigidity of X, this is true just in case $\Box\forall Y(X^*Y \rightarrow \forall\vec{z}(V\vec{z} \rightarrow Y\vec{z}))$ which is equivalent to the claim that V entails U.

[41] This proposition is true, since T is coextensive with truth. And by Instantiation it entails $(Tp \rightarrow p)$ for every p. But when p is true, Tp is true, so by the persistence of T, $\Box Tp$ hence $(Tp \rightarrow p) = p$; thus the proposition entails every truth.

[42] Suppose X is persistent and $\forall\vec{y}(X\vec{y} \rightarrow \Box Z\vec{y})$, i.e. $\forall\vec{y}(\neg X\vec{y} \vee \Box Z\vec{y})$. By B (equivalent to ND), we have $\forall\vec{y}(\neg X\vec{y} \rightarrow \Box\neg\Box X\vec{y})$, and hence by the persistence of X, $\forall\vec{y}(\neg X\vec{y} \rightarrow \Box\neg X\vec{y})$. So we can strengthen our assumption to $\forall\vec{y}(\Box\neg X\vec{y} \vee \Box Z\vec{y})$, which implies $\forall\vec{y}\Box(\neg X\vec{y} \vee Z\vec{y})$, i.e. $X \leq Z$. Using $\Box$ND we can necessitate this result.

[43] Let X be some property, F be the property of being a haecceity of an X thing (i.e. $\lambda Y.\exists z(Xz \wedge Y = \lambda x(z = x)))$, and X^* be the least upper bound of F. We show that X^* is coextensive with X and rigid.

(i) Every X is X^*. Suppose z is X; then $\lambda y.y = z$ is F and hence entails X^*, hence Xz.

Proposition 2.12. Rigid Comprehension and BF jointly imply $\Box$BF.[44]

To state our second comprehension-style principle, define a *functional* binary relation of a given type $\sigma \to \tau \to t$ in the obvious way as one that relates everything of type σ to exactly one thing of type τ:

$$\text{Functional}_{\sigma,\tau} \coloneqq \lambda U^{\sigma\to\tau\to t}.\forall x \exists y(Uxy \land \forall z(Uxz \to y = z))$$

Quantification over functional relations provides one natural higher-order way of regimenting the quantification over "functions" that comes so naturally to those schooled in standard mathematics. But in the case where τ is a relational type, there is another natural way of regimenting informal quantification over "functions from type-σ things to type-τ things", namely as quantification into type $\sigma \to \tau$. Indeed the use of the word "function" in connection with types of the form $\sigma \to \tau$ has very deep roots in the history of higher order logic as well as its contemporary use. Obviously any x of type $\sigma \to \tau$ corresponds to a unique functional relation of type $\sigma \to \tau \to t$, namely $\lambda yz.z = xy$. Our principle lets us turn this around by positing something of

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(ii) Every X^* is X. Actuality follows from our assumptions by Proposition 2.6, so there is a true world-proposition, w. Then $(\lambda y.w \to Xy)$ is an upper bound of F, since if $\lambda y.y = z$ is F, Xz is true and so entailed by w, which implies that $\Box\forall y(y = x \to (w \to Xy))$. Since X^* is a least upper bound, X^* entails $\lambda y.(w \to Xy)$: so if z is X^*, then $Xz \lor \neg w$, and since w and Xz.

(iii) X^* is persistent. If z is X, then $\lambda x.x = z$ entails X^*, so $\lambda x.\Box x = z$ entails $\lambda x.\Box X^*$, and by NI, $\lambda x.x = z$ entails $\lambda x.\Box X^*$. So $\Box X^*$ is an upper bound of F. Since X^* is a least upper bound of F, it follows that X^* entails $\Box X^*$.

(iv) X^* is inextensible. We will show $\forall Y \forall z \Box(\forall x(X^*x \to \Box Yx) \to \Box(X^*z \to Yz))$ and then appeal to BF. Suppose for contradiction that $\Diamond(\forall x(X^*x \to \Box Yx) \land \Diamond(X^*z \land \neg Yz))$. Let w be an atom that entails $\forall x(X^*x \to \Box Yx) \land \Diamond(X^*z \land \neg Yz)$. So w entails $\forall x(X^*x \to \Box Yx)$ and that there is an atom w' that entails $X^*z \land \neg Yz$. So by BF there are w and w' such that w entails:

$$\forall x(X^*x \to \Box Yx) \land \text{Atom}(w') \land w' \leq (X^*z \land \neg Yz).$$

Without loss of generality, we may assume in addition that w' is the GLB of the propositions p such that that w entails that $p = w'$. (If it isn't, let w'' be the GLB of the propositions such that $w \leq (p = w')$; then w'' will also be the GLB of the p such that $w \leq (p = w'')$.). Now let $H \coloneqq \lambda x.(w' \to Yx)$. We'll show that H is an upper bound of F, and hence entailed by X^*, which is a contradiction since w' is possible (since possibly an atom) and entails that X^*z and $\neg Yz$, and hence $\neg Hz$. Let u be any X thing. We have $\Box X^*u$ by parts (i) and (iii); so w entails X^*u. Since w entails $\forall x(X^*x \to \Box Yx)$, w also entails that $\Box Yu$ and thus that $w' \leq Yu$. Since w' is the GLB above, w' must *in fact* entail Yu. (For suppose not: then $w' \land Yu$ is stronger than w', but w entails $w' = (w' \land Yu)$.) So $\Box Hu$. Thus for any u that is X, $\Box\forall y(y = u \to Hu)$, so H is an upper bound of F, and thus X^* entails H.

[44] To prove this, let F be a rigid property coextensive with self-identity. But since F is inextensible, we have $\Box\forall Y(\forall x(Fx \to \Box Yx) \to \Box\forall x(Fx \to Yx)$, But since everything is F, $\Box\forall xFx$ by the persistence of F, so $\Box\forall xFx$ by BF, so we can simply the above to $\Box\forall Y(\forall x\Box Yx \to \Box\forall xYx$, i.e. $\Box$BF.

type $\sigma \to \tau$ corresponding to every functional relation:

$$\textbf{Plenitude} \qquad \forall R^{\sigma\to\tau\to t}(\text{Functional}(R) \to \exists Y^{\sigma\to\tau}\,\forall x(Rx(Yx)))$$

To see that the $\tau = t$ case follows from Extensionalism, one defines Y as the property of being a type-σ thing that bears R to something true.[45] By contrast with Rigid Comprehension (which is arguably deeply rooted in ordinary-language judgements), Plenitude is on shakier philosophical ground, since the relaxed attitude to the word "function" that it licenses might well be dismissed as a confusion.[46]

Plenitude follows much more obviously from a version of the Axiom of Choice that is *not* a theorem of Extensionalism, although it is a theorem of several influential systems (including those of Church 1940 and Henkin 1950). Let's call this:

$$\textbf{Functional Choice} \qquad \forall R^{\sigma\to\tau\to t}(\text{Serial}(R) \to \exists Y^{\sigma\to\tau}\forall x(Rx(Yx)))$$

where

$$\text{Serial} \coloneqq \lambda R.\forall x\,\exists y\,Rxy$$

Given the central role of Choice in large parts of mathematics, this might seem to provide the basis for an argument for Plenitude. But for the purposes of formalizing Choice-based mathematics, the following weaker Choice schema will do perfectly fine:

$$\begin{aligned}\textbf{Relational Choice} \quad & \forall R^{\sigma\to\tau\to t}(\text{Serial}(R) \to \\ & \qquad \exists S^{\sigma\to\tau\to t}(\text{Functional}(S) \land \forall xy(Sxy \to Rxy)))\end{aligned}$$

In fact, Functional Choice is easily seen to be equivalent to the conjunction of Relational Choice and Plenitude.

We will not further discuss Relational Choice here since we are primarily concerned with principles which follow from Extensionalism, but as far as we know, it and its negation are consistent with all consistent combinations of the principles on our list.[47]

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[45] Since $Rx\top = \top$ whenever $Rx\top$ and $Rx\top = \bot$ whenever $Rx\bot$, $Rx(Rx\top)$ in either case, so Plenitude is witnessed by $\lambda x.Rx\top$. More generally, when τ is $\vec{\sigma} \to t$, Plenitude will be witnessed by $\lambda x\vec{y}.\exists Z(RxZ \land Z\vec{y})$.

[46] Plenitude, and further reasons for not taking it to be obviously true, are discussed in Dorr 2016, §6. It also occurs as an axiom—the 'Typed Comprehension Schema'—in Walsh 2016.

[47] The other principle we need to add to Extensionalism to get the Simple Theory of Types is an axiom of infinity guaranteeing the existence of infinitely many individuals. This too seems to be consistent with all consistent combinations of our principles; indeed, we can have an axiom of infinity for all types including t, which is obviously inconsistent with Extensionalism. The *denial* of infinity for type t obviously implies both Atomicity and Actual World.

Apart from being interesting in its own right, Plenitude also serves as a useful bridge between several of the other principles we have discussed. Here is one crucial fact:

Proposition 2.13. Plenitude implies ND.

For if $x \neq_\sigma y$, there is a functional relation that maps x to $\top$ and everything else to $\bot$ (namely, $\lambda zw.(x = z \wedge w = \top) \vee (x \neq z \wedge w = \bot)$), so Plenitude implies that there is a Z (of type $\sigma \to t$) such that $Zx = \top$ and $Zy = \bot$. Since $\Box(\bot \neq \top)$, it follows that $\Box(Zx \neq Zy)$, and hence that $\Box(x \neq y)$.

We can also use Plenitude to prove the leftover left-to-right direction of Proposition 2.5, as a consequence of the following facts:

Proposition 2.14. Boolean Completeness implies Plenitude in C5.

Proposition 2.15. Plenitude implies Actuality.

For Proposition 2.14, suppose that R is a functional relation between propositions (the general case is proved similarly). The trick to obtaining the required Y of type $t \to t$ is to construct it as a limit from below: take the least upper bound of the properties Z that map the proposition p to something entailing the proposition to which p bears R.[48] For Proposition 2.15, consider the functional relation that maps every truth to itself and every falsehood to the tautology: $\lambda pq.(p \wedge p = q) \vee (\neg p \wedge q = \top)$. By Plenitude is exists a Z of type $t \to t$ such that $\forall p\, Rp(Zp)$: i.e. $Zp = p$ whenever p is true and $Zp = \top$ whenever p is false. Thus the proposition $\forall p\, Zp$, that everything is Z, is true. Moreover this proposition entails Zp for every proposition p, and thus entails p whenever p is true; so it is a true atom.

We can also prove a variant of Proposition 2.14 in which Booolean Completeness is strengthened to Rigid Comprehension, while $\Box$ND is weakened to ND:

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[48] Suppose R of type $\sigma \to t \to t$ is functional. Let $F_R : (\sigma \to t) \to t$ be $\lambda X^{\sigma \to t}.\forall y \forall p(Ryp \to Xy \leq p)$. By Boolean Completeness, F_R has a least upper bound: an operation $G_R : \sigma \to t$ such that (i) whenever $F_R X$, $X \leq_{\sigma \to t} G_R$, and (ii) whenever $\forall X(F_R X \to X \leq Y)$, $G_R \leq Y$. We will show that for any given a, $Ra(G_R a)$, so that G_R witnesses the truth of Plenitude. Fix a, and let p_a be the proposition such that Rap_a, so we want to show that $p_a = G_R a$.

We first show that $p_a \leq G_R a$. Let $H_a : \sigma \to t$ be $\lambda x.x = a \wedge p_a$. By ND, $\forall x(x \neq a \to (H_a x = \bot))$, and hence $\forall xp((x \neq a \wedge Rxp) \to (H_a x \leq p))$; meanwhile $H_a a = p_a$, and so $\forall xp((x = a \wedge Rxp) \to (H_a x \leq p))$. Putting these facts together, we have that $\forall xp(Rxp \to (H_a x \leq p))$, i.e. $F_R H_a$. Since G_R is the GLB of all the F_R operations, we can conclude that $H_a \leq G_R$, and hence $p_a = H_a a \leq G_R a$.

It remains to show that $G_R a \leq p_a$. Let $J_a : \sigma \to t$ be $\lambda x.x \neq a \vee p_a$. By ND, $\forall x(x \neq a \to (x \neq a = \top))$, hence $\forall x(x \neq a \to (J_a x = \top))$, and hence $\forall X(F_R X \to \forall x(x \neq a \to (Xx \leq J_a x)))$. We also have $\forall X(F_R X \to \forall x(x = a \to (Xx \leq J_a x)))$, since $J_a a = p_a$ and if $F_R X$ and Xa, $X \leq p_a$. Putting these facts together we have $\forall X(F_R X \to \forall x(Xx \leq J_a x))$. By BF (which follows from $\Box$ND), this implies $\forall X(F_R X \to X \leq J_a)$: i.e. J_a is an upper bound of the F_R operations. Hence $G_R \leq J_a$, and so $G_R a \leq J_a a = p_a$.

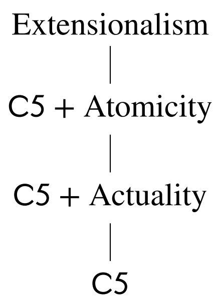

Figure 6. Between C5 and Extensionalism

Proposition 2.16. Rigid Comprehension and ND jointly imply Plenitude.[49]

The results we have proven imply that as regards views about the five principles that took centre stage in this and the previous section, there are just two combinations strictly between C5 and Extensionalism, namely C5 + Actuality (= C5 + Boolean Completeness = C5 + Rigid Comprehension = C5 + Plenitude), and the stronger C5+Atomicity (= C5 + $\Box$Actuality = C5 + $\Box$Boolean Completeness = C5 + $\Box$Rigid Comprehension = C + $\Box$Plenitude). Appendix D shows that all these inclusions are strict, so the map is as in Figure 6.

For systems not including C5, by contrast, the map of possible combinations of principles is far more complicated. We have not been able to identify any interesting logical relationships between the principles discussed above (and their negations and necessitations) other than those just established. Appendix D establishes the consistency of some of these packages, but falls short of a truly systematic exploration of the rather large set of possible distributions of the statuses of necessary truth, contingent truth, contingent falsity, and necessary falsity over the above principles whose consistency is not ruled out by the results of this section. We are hopeful that the model-theoretic techniques introduced in this appendix will also be useful for establishing the consistency of some other combinations.

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[49] Suppose that R is a functional relation of type $\sigma \to t \to t$. (The general case for type $\sigma \to \tau \to t$ is analogous.) Let R^* be a rigid relation coextensive with R, and let Z be $\lambda y^\sigma.\forall p.R^*yp \to p$. Suppose Rxq. Then R^*xq, so $\Box R^*xq$ by the persistence of R^*, hence $q = R^*xq \to q$. But then by Instantiation, $Zx \leq R^*xq \to q$; so $Zx \leq q$. Also, since R^* is coextensive with the functional R, $\forall pp'(R^*pp' \to (p \neq x \vee p' = q))$. By ND (and the necessity of identity), $\forall pp'(R^*pp' \to \Box(p \neq x \vee p' = q))$; by the inextensibility of R^*, this implies $\Box\forall pp'(R^*pp' \to (p \neq x \vee p' = q))$, hence $\Box(q \to \forall p(R^*xp \to p))$, i.e. $q \leq fx$. Hence $q = Zx$, so $Rx(Zx)$.

2.4 Strengthenings in the direction of fineness: Maximalist Classicism

Extensionalism is intuitively an extremely coarse-grained view in higher-order logic. Indeed, if we say that a theory is coarser-grained than another just in case the former contains every identity claim (i.e. every closed sentence of the form $A = B$) the latter contains, and the latter contains every distinctness claim (i.e. sentence of the form $A \neq B$) the former contains, then Extensionalism can correctly be characterized as a maximally coarse-grained theory. In Extensionalism $A = B$ is equivalent to $(A = B) \neq \bot$ and $A \neq B$ is equivalent to $(A \neq B) = \top$, so every identity is equivalent to a distinctness claim, and every distinctness claim is equivalent to an identity. You cannot consistently add a new identity without also adding a distinctness claim, and you cannot subtract a distinctness claim without also subtracting an identity claim (since that identity claim entails the distinctness claim in H). However, Extensionalism is not in this sense the *unique* maximally coarse-grained extension of Classicism, since the above reasoning applies to any strengthening of Extensionalism.[50]

It's also worth exploring extensions of Classicism that are more fine-grained than it. Indeed, one might wonder if there are *maximally* fine-grained extensions of C. The answer turns out to be: yes. In fact, by contrast with the case of coarse-grainedness, there's an extension of C at least as fine-grained as each and every consistent extension of C. The weakest such theory we call *Maximalist Classicism*. It is the result of extending C with every closed distinctness claim that is consistent with C. Clearly, so long as Maximalist Classicism is consistent, it's at least as fine-grained as any consistent extension of C. What's not obvious is that Maximalist Classicism is consistent, i.e. that the set of closed distinctness claims that are individually consistent with C are jointly consistent with C. After all, this isn't true with identity claims: C is consistent both with the proposition that there is exactly one object being identical to $\top$ and with it being identical to $\bot$. The proof of the consistency of Maximalist Classicism is a central application of the model theoretic techniques discussed in the next section.

Maximalist Classicism isn't a recursively axiomatizable theory.[51] But if we don't mind talking about axiom schemas whose instances aren't recursively enumerable, we can obviously axiomatize Maximalist Classicism by the schema:

Distinctness $A \neq B$, where $A = B$ is closed and not a theorem of Classicism.

Or, equivalently, we could use the schema:

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[50] E.g. by adding the claim that there are exactly three things of type e.

[51] If it were, C would have to be decidable, since we could enumerate the non-theorems of C by enumerating the theorems of Maximalist Classicism of the form $\Diamond\neg A$, and stripping off the $\Diamond\neg$. But C is not decidable, for the same reason first-order logic is not.

Possibility $\Diamond A$, where A is closed and consistent with Classicism.

The failure of Maximalist Classicism to be recursively axiomatizable makes it quite hard to apply theoretical considerations such as simplicity to it. On the one hand, in one sense of ‘simplicity’, such theories might be considered very unsimple. On the other hand, we have given an extremely simple *characterization* of it, as the maximally fine-grained extension of Classicism. At any rate, it occupies an important position in the space of extensions of Classicism that makes it eminently worthy of serious engagement.[52]

While these two schemas are obviously equivalent given Classicism, thinking of the view in terms of Distinctness is perhaps more effective for bringing out its appeal: the thought is that when it comes to question of identity—at least questions that can be formulated as closed sentences of higher-order logic—there are no surprises. It would be suprising if, for instance, the proposition that there are three individuals were the same as the proposition that there are four individuals; Distinctness, by contrast, ensures the only true identities are those forced on us by logic (i.e. Classicism). Any such further identities would reflect an aspect of the natures of the logical constants, and the way that they fit together, that standard classical logic tells us nothing about. By contrast, Distinctness can be thought of as saying that all there is to the natures of the logical constants are their logical roles, as captured in the usual logical rules (i.e. those of H).

Maximalist Classicism feels more tendentious when stated in terms of Possibility. Given our earlier results about the non-theorems of Classicism, instances of Possibility include the broad possibility of Functionality, Boolean Completeness, the Fregean Axiom, and so on, as well as their negations. They also include the broad possibility of many claims we have not discussed, such as higher-order versions of contentious set theoretic principles, like the axiom of choice and the continuum hypothesis. This will feel alien to many metaphysicians, who are used to thinking that when it comes to claims that are sufficiently general, whatever is true

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[52] Those who like the impulse behind Maximalist Classicism should be interested in the project of finding strong axiomatizable fragments of Maximalist Classicism. One strategy is to pick some way of encoding “‘$\boldsymbol{P}$” is consistent in C’ as a sentence of higher-orderese, $\mathrm{Con}(\ulcorner \boldsymbol{P} \urcorner)$, in which case one can formula a decidable axiom-schema $\mathrm{Con}(\ulcorner \boldsymbol{P} \urcorner) \rightarrow \Diamond \boldsymbol{P}$ (where $\boldsymbol{P}$ is closed). One can then derive any instance of Possibility from the corresponding consistency assumption. By Gödel’s second incompleteness theorem, this will never be a consequence of C, but it will often be derivable from further well-motivated claims. One such well-motivated claim is $\Diamond \exists R^{e \rightarrow e \rightarrow t}.\, \mathrm{ZFC}(R)$, where $\mathrm{ZFC}(\in)$ is the conjunction of the nine axioms of second-order ZFC. Notice that this claim is also an instance of Possibility, assuming $\mathrm{ZFC}(\in)$ is consistent. The theory comprising this claim together with the aforementioned schema can prove everything that can be proved in ZFC to be in Maximalist Classicism. So in practice there isn’t much difference between being “committed to” Maximalist Classicism and being “committed to” this particular fragment.

is necessarily true and whatever is false is necessarily false. But of course, proponents of Maximalist Classicism could accept this impulse as far as metaphysical possibility is concerned, in which case they should take this to be another reason to deny that metaphysical necessity is identical to broad necessity. There's another way of talking that metaphysicians often slip into that fits fairly naturally with Maximalist Classicism, namely working with an operator 'it is logically necessary that', or 'it is logically possible that', in a way that takes for granted that you can go back and forth between the metalinguistic status of logical truth for sentences, and object language formulations in terms of operators. Although there is a lot about this practice to be suspicious of (see Bacon 2018b, ch.4.), that fact that Maximalist Classicism is consistent means that there is a real vision in the vicinity that isn't merely a use-mention fallacy.

Maximalist Classicism belongs to a broader family of theories in a similar spirit. For any theory T in higher order logic, we can define the *maximization* of T to be the result of adding to it all closed sentences $A \neq B$ which are consistent with T. Many theories other than C have consistent maximalizations, which offer interesting alternatives to Maximalist Classicism.[53] For example, we could consider the maximalist versions of the results of adding various combinations of the principles considered in sections 2.1 and 2.2 to C, or the maximalizations of theories weaker than C such as H. So long as T can prove that $\Box$ has a reasonable modal logic including the necessity of identity, its maximalization will be equivalent to the result of adding $\Diamond P$ for every closed P consistent with T, and will thus support the naïve practice of talking about "logical necessity" as an operator in the same way as Maximalist Classicism. However, not all theories have consistent maximalizations. For example the maximalization of $\mathsf{C5}$ is inconsistent, since for many choices of A—for example 'there are exactly three individuals'—$\mathsf{C5}$ entails $A = \top \vee (\neg A) = \top$ but does not entail either of its disjuncts, so that its maximalization entails both their negations.[54] Much of what we say about Maximalist Classicism below will also apply to these alternative maximalized theories.

2.5 Maximalist Classicism and non-logical constants

So far we have been talking about Maximalist Classicism as a theory in the purely logical signature. There is an analogue of this theory for any way of extending

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[53] Fritz, Lederman and Uzquiano (forthcoming) prove the consistency of, the maximalizations of H_0 (see note 7) and H.

[54] The property of having a consistent maximilization is related to the property of coherence in modal logics (see Meyer 1971), and is studied more generally in the context of higher-order theories in Bacon Bacon n.d.(a). In E we'll discuss a construction that can be used to show the consistency of maximizations of several extensions of C.

the signature by adding non-logical constants. But for many choices of non-logical constants, such a view seems deeply implausible. For example, if the signature includes predicates 'bachelor' and 'married', then the version of Possibility for that signature will include the claim that it's broadly possible that some bachelor is married, i.e. $\Diamond \exists x.\,\mathrm{Bachelor}\,x \wedge \mathrm{Married}\,x)$. And if it also has a constant meaning 'man', then the version of Distinctness for that signature will also have the claim that it is not the case that to be a bachelor is not to be an unmarried man—i.e. $\mathrm{Bachelor} \neq (\lambda x.\,\mathrm{Man}(x) \wedge \neg\,\mathrm{Married}(x))$. This seems misguided to us—surely natural languages very often provide us with simple expressions that refer to entities that can also be referred to with more complex expressions.[55]

However, there is considerable attraction to the idea that if all of the nonlogical constants in some signature denoted distinct *fundamental* entities, then the version of Maximalist Classicism for that signature would be true. When formulated in a fundamental language like this, the resulting principle imposes substantive constraints on the fundamental entities denoted by constants of that language: that any 'logically consistent' thing we can say about them in this language correspond to a way for them to broadly possibly be (see also the principle LC from Bacon 2020). The thought that fundamental entities are in some demanding sense "independent" of one another has been a guiding idea for a broad range of theorists, especially in the "Humean" tradition. Some ways of cashing out this vision take us extremely close to the Logical Maximalist point of view: for example, Dorr and Hawthorne 2013 discuss a view they call 'combinatorialism', according to which "in an appropriate language in which all predicates express perfectly natural properties, the only sentences that express metaphysically necessary propositions are the logical truths". While the meaning of 'logical truth' here is up for grabs, *being a theorem of Classicism* certainly looks like a principled way of filling in the idea, and one that answers to the Humean impulse that generates it.[56] Of course, even those who

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[55] There is a radical view worth engaging with that denies all higher-order identity claims where the terms flanking the identity symbol are closed and structurally non-isomorphic. But this view also denies many of the theorems of Classicism, and so is not relevant in the present context.

[56] An alternative interpretation of the slogan would cash out "logical truth" in the manner of Williamson (2013) (derived from Tarski 1959 and Bolzano 2004), such that logical truth coincides with plain truth when it comes to closed sentences involving only logical vocabulary. That interpretation suggests a schema weaker than Possibility:

$$(*) \qquad \exists \vec{x}.A \rightarrow \Diamond A[\vec{a}/\vec{x}]$$

where A is a formula with no nonlogical constants and only the variables $\vec{x}$ free, $\vec{a}$ is a list of distinct constants, and $A[\vec{a}/\vec{x}]$ is the result of substituting them for the variables in A. This schema, unlike Possibility, is consistent with No Pure Contingency (see §2.2). The combination of these two schemas is equivalent to the result of strengthening the → in (*) to a ↔: this is the schema Bacon (2020) calls 'Logical Necessity'. Note that much of the exploration in that paper concerns consequences of vehemently reject this sort of combinatorialist thinking so far as metaphysical necessity is concerned might still accept it for *broad* necessity. Indeed, expressions of anti-Humeanism often slip into assuming something like Maximalist Classicism, by treating failures of the metaphysical-possibility version of Possibility as establishing that metaphysical possibility is a more demanding status than "logical possibility" (see, e.g., Wilson 2010).

Maximalist Classicism in a given signature is equivalent to the combination of "Pure Maximalist Classicism"—i.e. the result of adding all instances of Distinctness involving only logical constants to Classicism in the signature—with the following schema (from Bacon 2020):

Separated Structure $Fa = Ga \to F = G$, where a is a non-logical constant and F and G are closed terms not including a.

Even by itself, Separated Structure is completely implausible for languages with arbitrary non-logical constants. But with the assumption that the non-logical constants denote distinct fundamental entities, it is an appealing principle in its own right, even to those that reject not only Maximalist Classicism, but Classicism. It can be thought of as offering an important grain of truth in the 'structured' picture of propositions shown to be inconsistent by the Russell-Myhill paradox (see Dorr 2016, §6, Goodman 2017).

To see that Separated Structure follows from Maximalist Classicism in a given signature, note that since C is closed under uniform substitution, if $Fa = Ga$ is a theorem of C so is $Fx = Gx$, and hence also $F = G$ by ζ.[57] If $Fa = Ga$ is not a theorem of C, then its negation, and hence also the conditional, is a theorem of Maximalist Classicism. To see that Maximalist Classicism (for a given signature) follows from Pure Maximalist Classicism and Separated Structure, we can begin by showing Separated Structure to be equivalent to the schema

$$Fa_1...a_n = Ga_1...a_n \to F = G$$

where F and G are closed and contain *no* non-logical constants, and $a_1...a_n$ are distinct non-logical constants. This can be shown by an obvious induction on the number of nonlogical constants in F and G. So, suppose that $A \neq B$ is consistent with C. Enumerating the non-logical constants in A and B as $a_1...a_n$, we can, using β show that $A = Fa_1...a_n$ and $B = Ga_1...a_n$ for some closed pure terms F and

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the direction of Logical Combinatorialism equivalent to (*) above, which follows from Maximalist Classicism; indeed most of the paper concerns the more abstract feature of "stablility" which is common to Maximalist Classicism and Logical Combinatorialism.

[57] The fact that a doesn't appear in F or G is crucial here, since only in that case is $Fa[x/a]$ the same as Fx, and similarly for G.

G. $F \neq G$ must also be consistent with C, so Pure Maximalist Classicism implies $F \neq G$. Hence by the above equivalent of Separated Structure, $A = Fa_1...a_n \neq Ga_1...a_n = B$.

Note that the above reasoning applies equally well to maximizations of many other theories. So long as T is closed under uniform substitution and ζ, the maximization of T will be equivalent to the conjunction of the purely logical instances of Distinctness (or Possibility) and Separated Structure.[58]

The conviction that certain specific properties, relations, and objects are fundamental might motivate someone to accept Maximalist Classicism for a signature that includes constants for those entities. Disagreements about which entities are fundamental will lead to disagreements about which instances of Possibility to accept, but those sympathetic to the Humean idea can at least agree that, whatever language turns out to be fundamental, all instances of Possibility will be true in that language.

Rather than formulate the idea in this metalinguistic way, we could introduce predicates into object language for talking about the status of fundamentality itself, in the form of a predicate ‘Fun_σ’ (of type $\sigma \to t$) for each type σ. For a sequence of variables $\vec{v} = v_1, \dots, v_n$ of types $\sigma_1, \dots, \sigma_n$, let $\mathrm{Fun}(\vec{v})$ abbreviate the claim that all of $\vec{v}$ are fundamental and (when of the same type) distinct:

$$\bigwedge_{i \leq n} \mathrm{Fun}_{\sigma_i}(v_i) \wedge \bigwedge_{i<j\leq n:\sigma_i=\sigma_j} v_i \neq v_j$$

In this language, we can capture the combinatorialist thought with the following schema:

Fundamental Possibility $\mathrm{Fun}\,\vec{x} \to \Diamond P$, where P is any formula containing only logical constants consistent with Classicism all of whose free variables are among $\vec{x}$.[59]

One noteworthy consequence of Fundamental Possibility is that the denotations of

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[58] The Logical Necessity schema from Bacon 2020, for a given signature, is by this general sort of argument, equivalent to the conjunction of Separated Structure and No Pure Contingency. Thus while the two views both offer precisifications of the Humean combinatorialist vision concerning the fundamental entities, they are as different as could be concerning the purely logical claims, for according to the former view there is no contingency in that domain.

[59] One worrisome consequence of Fundamental Possibility is that if a binary relation is fundamental, its converse is not fundamental. Since the formula $r \neq (\lambda xy.ryx) \wedge s \neq (\lambda xy.ryx)$ is consistent in C,

$$\mathrm{Fun}\,r \wedge \mathrm{Fun}\,s \wedge (r \neq s) \to \Diamond(r \neq (\lambda xy.ryx) \wedge s \neq (\lambda xy.ryx))$$

is an instance of Fundamental Possibility. By the necessity of identity, the $\Diamond$ is redundant in the consequent, and the entire formula is in fact equivalent to

$$\mathrm{Fun}\,r \to \neg\,\mathrm{Fun}(\lambda xy.ryx)$$

the logical constants are not themselves fundamental: for example, since the formula $\exists p.p \wedge Xp$ is consistent in C, an instance of Fundamental Possibility is $\mathrm{Fun}\,X \to \Diamond\exists p.p \wedge Xp$. Instantiating X with $\neg$ gives $\mathrm{Fun}\,\neg \to \Diamond\exists p.p \wedge \neg p$, which implies $\neg\,\mathrm{Fun}(\neg)$ given C. The vision thus stands in contrast with the kind of account we find in Sider (2011), where certain logical constants are supposed to have exactly the same fundamentality-theoretic status as, e.g., certain predicates needed for physics. The maximalist picture goes more naturally with the kind of vision we find in Bacon 2020 and Dorr 2016, on which there is a different but also metaphysically important status of ‘Purity’ (or ‘Logicality’), which the denotations of closed terms containing only logical constants all have, but nothing fundamental has.

2.6 Extensions of Maximalist Classicism

Although Maximalist Classicism is maximally fine-grained in our technical sense, it is far from being maximally strong. In this section we will mention various extensions of it that strike us as interesting and attractive, although we have almost no proofs of consistency.

First: we could consider adding some of the further principles discussed in sections 2.1 and 2.2 to Maximalist Classicism. (This is different from adding the principles and *then* maximizing, which also yields interesting theories.) Clearly we cannot consistently add ND, since for any closed A and B such that $A \neq B$ and $A = B$ are both consistent in Classicism, Maximalist Classicism implies $A \neq B \wedge \Diamond(A = B)$. Nor can we add the *necessitations* of any of the other principles. But for all this tells us, we might be able to consistently add the non-necessitated BF, Boolean Completeness, Actuality, Atomicity, or Rigid Comprehension principles. In fact Zach Goodsell (p.c.) has shown that Rigid Comprehension is inconsistent with Maximalist Classicism; the consistency of the other combinations remains to be explored.

Second: consider the following scenario, which is compatible with Maximalist Classicism. There are two magic individuals a and b such that $a \neq b$ entails every true proposition. All kinds of divergences from actuality are broadly possible, but in order to diverge in any other way the first thing we have to do is to identify a

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This consequence is rather alarming: it conflicts with the plausible idea that a relation and its converse are ‘metaphysically on a par’. Bacon (n.d.[b]) takes this to suggest we should eschew the ideology of ‘fundamentality’, and instead theorize in terms of a polyadic notion of ‘cofundamentality’. Alternatively, we can weaken Fundamental Possibility in such a way as to avoid it, by strengthening the definition of $\mathrm{Fun}(\bar{v})$ to include, alongside the conjuncts expressing the *distinctness* of distinct v_i (of the same type), further conjuncts requiring other kinds of “logical independence” among distinct v_i, including $v_i \neq (\lambda xy.v_j yx)$ when v_i and v_j are of type $\sigma \to \sigma \to \tau$. (Dorr (2016, §9) suggests, in a non-Classicist setting, a picture of fundamentality on which fundamental entities come in clusters related by certain kinds of “interdefinability” operations.)

and b. This doesn't feel very much in keeping with the "combinatorialist" spirit that motivates Maximalist Classicism, so it's natural to look for strengthenings that rule it out.

One kind of strengthening: make a special provision for type e by strengthening Possibility to

Possibility+ $$(\bigwedge_{1\leq i<j\leq n} x_i \neq x_j) \rightarrow \Diamond P$$

where P has only free variables $\vec{x}$, all of type e.[60] This seems attractive, at least on some ways of thinking about what's special about type e. The proof we give of the consistency of Possibility extends easily to Possibility+— indeed it shows that Possibility is consistent with there being only one individual.

But it doesn't go as far as we might like since it is consistent with the same $a \neq b$ phenomenon arising in, e.g., type $e \rightarrow t$.

For a different kind of strengthening, let's say that a proposition is $\neq$-necessary just in case it is entailed by the possibility of each truth:

$$\Box_{\neq} := \lambda p.\exists q.q \land \Box(\Diamond q \rightarrow p)$$

Dorr, Hawthorne and Yli-Vakkuri (n.d., appendix D) show that ND holds (in every type) for $\Box_{\neq}$, and moreover that on a natural definition of "ND-respecting modal operation", $\Box_{\neq}$ is equivalent to *having every ND-respecting modal operation*: intuitively, $\Box_{\neq}$ is the minimal restriction of $\Box$ compatible with imposing ND. So we can capture the idea that lots of things can happen without any distinct entities of any type having to become identical with a schema with instances of the form $\Diamond_{\neq} P$ for some wide range of values of P. The most obvious idea would be to strenthen Possibility by replacing $\Diamond$ with $\Diamond_{\neq}$. But this is clearly inconsistent: the Fregean Axiom, according to which there are only two propositions, is consistent with Classicism, but $\Diamond_{\neq}$(Fregean Axiom) implies the Fregean Axiom, since if there were three distinct propositions p, q, r, $p \neq q \land p \neq r \land q \neq r$ would be a $\Box_{\neq}$ truth incompatible with the Fregean Axiom. More generally, whenever some closed P is consistent with Classicism but not with Maximalist Classicism, $\Diamond_{\neq} P$ will also be inconsistent with Maximalist Classicism. So the furthest we could hope to go in this direction consistent with Maximalist Classicism is the following schema:

Strong Possibility $\Diamond_{\neq} P$, where P is closed and consistent with Maximalist Classicism.

Note that adding Strong Possibility to Classicism makes Possibility redundant. For

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[60] This is tantamount to saying 'Everything of type e is fundamental'.

if P is consistent with Maximalist Classicism, $\Diamond P$ is too, so $\Diamond_{\neq}\Diamond P$ is an instance of Strong Possibility; but $\Diamond_{\neq}\Diamond P$ implies $\Diamond\Diamond P$ and hence $\Diamond P$.

We do not know whether Strong Possibility is consistent.

3 Model theory for Classicism

In studying systems of higher-order logic, including Classicism, model theory is a crucial tool. A model for a certain higher-order language is a mathematical construct according to which we can assign "denotations" to the terms of that language, and ultimately truth values to its formulas. A class of models, C, is sound for a logic, T (understood as a set of formulae), just in case every member of T is true in every model in C, and complete for T just in case every formula true in every model in C is in T. Soundness theorems are particularly useful for proving consistency results, but the enterprise also has considerable heuristic value in generating intuitions about the metaphysical worldviews these theories are describing.

Our guiding idea in the search for a useful notion of model for Classicism will be a deep parallel between Extensionalism and Classicism. We began to notice this already in §1.4 where we saw that, where Extensionalism could be axiomatized by certain material conditionals (The Fregean Axiom + Functionality, Extensionality, etc), turning those material conditionals into rules of proof delivered parallel characterizations of Classicism. Our model theory will be based on similar parallels. In model theoretic terms, a material conditional corresponds to an inference rule preserving truth over a single model, and a rule of proof to the preservation of truth-in-all-models from a certain class, or 'category' of models. First we present a general model theory for H and see that models of this sort satisfying a certain natural "extensionality" condition characterize Extensionalism. Models that satisfy this condition can be simplified into a familiar form due to Henkin. Then we look at categories of models of H and formulate a condition on such categories which we call "intensionality", which is in a natural sense a generalization of to the extensionality condition on single models, and corresponds to rule form of the Extensionality axiom. With this condition in hand, we can similarly define a simpler class of models that stands to Classicism as Henkin models stand to Extensionalism. In §3.5 and §3.6 we will construct some simple examples of these "action models", use them to verify some of the consistency claims we made in part 3.5, and explain how they relate to and generalize existing notions of model for Classicism.

3.1 BBK-models

Benzmüller, Brown and Kohlhase (2004) provide a concept of model that they show to be sound and complete for H. We will need a few preliminary definitions. A ***typed***

collection C is a function that maps each type σ to a nonempty set C^σ. When C and D are typed collections, a ***mapping*** from C to D is a function h that maps each type σ to a function h^σ from C^σ to D^σ. A ***variable assignment*** g for a typed collection C is a mapping to C from some typed collection of variables; a variable assignment is *adequate for* a term if it is defined on all variables free in that term. A model consists of (i) a domain, $\mathbf{M}^\sigma$, for each type σ, from which the interpretations of terms of that type are drawn and the quantifiers of that type range, (ii) an interpretation function $[\![\cdot]\!]^\cdot$ mapping terms to their interpretations relative to variable assignments, and (iii) a device for determining which elements of $\mathbf{M}^t$ (the propositions) are true or false:

Definition. A ***BBK-model*** for $\mathcal{L}$ is a triple $\mathbf{M} = \langle \mathbf{M}^\cdot, [\![\cdot]\!]^\cdot_\mathbf{M}, \mathrm{val}_\mathbf{M} \rangle$, where:

(i) $\mathbf{M}^\cdot$ is a typed collection.

(ii) $[\![\cdot]\!]^\cdot_\mathbf{M}$ is a function that maps each type-σ term A of $\mathcal{L}$ and variable assignment g for $\mathbf{M}^\cdot$ that is adequate for A to an element $[\![A]\!]^g$ of $\mathbf{M}^\sigma$, subject to the following constraints:

  a. $[\![v]\!]^g = g(v)$

  b. If $[\![A]\!]^g = [\![C]\!]^h$ and $[\![B]\!]^g = [\![D]\!]^h$ then $[\![AB]\!]^g = [\![CD]\!]^h$.

  c. $[\![A]\!]^g = [\![A]\!]^h$ when g and h agree on all variables free in A.

  d. $[\![A]\!]^g = [\![B]\!]^g$ when A and B are $\beta\eta$-equivalent.[61]

(iii) $\mathrm{val}_\mathbf{M}$ (the 'valuation') is a function from $\mathbf{M}^t$ to $\{0, 1\}$, subject to the following constraints, where '$\mathbf{M}, g \Vdash P$' means '$\mathrm{val}_\mathbf{M} [\![P]\!]^g_\mathbf{M} = 1$':

  a. $\mathbf{M}, g \Vdash \neg P$ iff $\mathbf{M}, g \nVdash P$.

  b. $\mathbf{M}, g \Vdash P \wedge Q$ iff $\mathbf{M}, g \Vdash P$ and $\mathbf{M}, g \Vdash Q$.

  c. $\mathbf{M}, g \Vdash P \vee Q$ iff $\mathbf{M}, g \Vdash P$ or $\mathbf{M}, g \Vdash Q$.

  d. $\mathbf{M}, g \Vdash \forall_\sigma F$ iff $\mathbf{M}, g[v \mapsto \mathbf{a}] \Vdash Fv$ for every $\mathbf{a} \in A^\sigma$ (v not free in F).

  e. $\mathbf{M}, g \Vdash \exists_\sigma F$ iff $\mathbf{M}, g[y \mapsto \mathbf{a}] \Vdash Fv$ for some $\mathbf{a} \in A^\sigma$ (v not free in F).

  f. $\mathbf{M}, g \Vdash A = B$ iff $[\![A]\!]^g_\mathbf{M} = [\![B]\!]^g_\mathbf{M}$.[62]

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[61] In the most general notion of model explored in Benzmüller, Brown and Kohlhase 2004, η isn't baked in. And indeed, one could even be more general by dropping the requirement that of β if one wanted models of logics not including β, as Muskens (2007) does.

[62] Given clause (ii.b), we only need the special cases of these conditions where P, Q, F, A, and B are variables.

Note that by clause (ii-c), $\llbracket A \rrbracket^g$ is independent of g when A is closed; in this case we just write $\llbracket A \rrbracket$. ***P* holds** in $\mathbf{M}$ iff $\mathbf{M}, g \Vdash P$ for all g adequate for P; the ***theory of*** a class of models is the set of all formulae that hold in all of them.

The point of these definitions arises from the following key theorem:

Theorem 3.1. The class of BBK-models is sound and complete for H: for any signature $\mathcal{L}$, every theorem of H holds in every BBK-model, and every set of sentences consistent in H holds in some BBK-model. Moreover, if $\mathcal{L}$ is countable, all of the above is true for models in which each domain $\mathbf{M}^\sigma$ is a subset of some given countable set, say $\mathbb{N}$.

We sketch the proof of this in a footnote.[63]

Theorem 3.1 automatically yields soundness and completeness theorems for all manner of theories extending H: whenever T extends H, the class of all BBK-models of T is sound and complete for T. This is not the most *useful* sort of soundness and completeness result, since the definition of BBK-model does not suggest any methods for constructing BBK-models. However, these automatic soundness and completeness theorems can provide the basis for more useful soundness and completeness theorems where the models are characterized in a more intrinsic, concrete, compositional way. Our goal is to do this for Classicism. But we will begin by seeing

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[63] Soundness is routine. To show that it's complete, we must first show that any H-consistent set of formulae T in a given $\mathcal{L}$ can be extended to a consistent set of formulae T^+ in an expanded language $\mathcal{L}^+$ that adds new constants to $\mathcal{L}$, where T^+ is both negation complete ($\neg A \in T^+$ whenever $A \notin T^+$) and witness-complete (whenever $\exists F \in T^+$, $Fc \in T^+$ for some constant c). The proof of this fact, "Henkin's Lemma", is exactly the same as the corresponding proof for first-order logic — indeed the original version of this result by Henkin (1950) was in a higher order setting. Given a consistent, negation-complete, witness-complete T^+, we form a BBK-model $\mathbf{M}_{T^+}$ for $\mathcal{L}$ as follows. Each domain $\mathbf{M}^\sigma_{T^+}$ is the set of equivalence classes of closed type-σ terms of $\mathcal{L}^+$ under the equivalence relation $\approx_{T^+}$, where $A \approx_{T^+} B$ iff $T^+ \vdash A = B$. $\llbracket A \rrbracket^g_{\mathbf{M}_{T^+}}$ is the equivalence class of all the closed terms that can be derived from A by replacing every free occurrence in A of any variable v with any member of gv (an equivalence class of closed terms). And for $\mathbf{p} \in \mathbf{M}^t_{T^+}$, $\mathrm{val}_{\mathbf{M}_{T^+}}(\mathbf{p}) = 1$ iff $P \in T$ for any (or equivalently, all) $P \in \mathbf{p}$. It is then a straightforward matter to show that $\mathbf{M}_{T^+}$ is indeed a BBK-model in which every member of T holds. The properties of substitution secure that $\llbracket\cdot\rrbracket^{\cdot}_{\mathbf{M}_{T^+}}$ meets constraints (ii-a–c), while the fact that T^+ is closed under $\beta\eta$-equivalences secures (ii-d). The consistency and negation-completeness of T^+ and the PC-rules for $\wedge$ and $\vee$ guarantee that $\mathrm{val}_{\mathbf{M}_{T^+}}$ is well/behaved with respect to $\neg$, $\wedge$, and $\vee$; the witness/completeness of T^+ takes care of one direction of the biconditionals for $\forall$ and $\exists$; the fact that T^+ contains every instance of UI and EG takes care of the other directions; finally the fact that T^+ contains Ref and LL yield the biconditional for $=$. Note that if $\mathcal{L}$ is countable, we can set things up so that $\mathcal{L}^+$ is also countable, in which case the domain of $\mathbf{M}_{T^+}$ in each type is countable as well.

The proof in Benzmüller, Brown and Kohlhase 2004 establishes the completeness of BBK-models for a certain *cut free sequent calculus*, which requires a proof substantially more complicated than the proof we have sketched here, which is essentially due to Henkin (1950).

how it can be done for the much stronger theory Extensionalism (see §1.4), which will provide a helpful starting point for the generalization to Classicism. To get a better sense of the distinctive properties of BBK-models of Extensionalism, we can introduce some salient structural operations on, and properties of, BBK-models.

Certain elements of a BBK model can be associated with functions determined by how those elements apply to their arguments, and with extensions determined by the truth values of the results of those applications. These notions let us distinguish some special classes of BBK models: the functional and functionally full models, and the extensional and extensionally full models.

Definition. Where $\mathbf{M}$ is a BBK-model:

- The ***applicative behaviour***, $\text{app}_{\mathbf{M}}\,\mathbf{d}$, of an element $\mathbf{d} \in \mathbf{M}^{\sigma\to\tau}$, is the function $\mathbf{a} \mapsto [\![X y]\!]^{[X\mapsto\mathbf{d}, y\mapsto\mathbf{a}]}$.[64]
- The ***extension***, $\text{ext}_{\mathbf{M}}\,\mathbf{d}$, of an element $\mathbf{d} \in \mathbf{M}^{\sigma_1\to\cdots\to\sigma_n\to t}$ is the set

$$\{\langle \mathbf{a}_1, \ldots, \mathbf{a}_n\rangle \in \mathbf{M}^{\sigma_1} \times \cdots \times \mathbf{M}^{\sigma_n} \mid \mathbf{M}, [X \mapsto \mathbf{d}, y_i \mapsto \mathbf{a}_i] \Vdash X\vec{y}\}$$

Using these concepts, we can pick out certain special classes of BBK-models:

Definition. A BBK-model $\mathbf{M}$ is:

- ***functional*** iff its applicative behaviour functions are injective for all σ, τ: that is, whenever $\mathbf{d} \neq \mathbf{d}'$, there is some $\mathbf{a} \in \mathbf{M}^{\sigma}$ such that $\text{app}\,\mathbf{d}(\mathbf{a}) \neq \text{app}\,\mathbf{d}'(\mathbf{a})$.
- ***extensional*** iff its extension functions are injective: whenever $\text{ext}\,\mathbf{a} = \text{ext}\,\mathbf{b}$, $\mathbf{a} = \mathbf{b}$.
- ***Fregean*** iff val is injective. In other words, the domain of type t has exactly two elements.[65]

These three principles bear a special relationship to the Functionality schema ($\forall z(Xz = Yz) \to X = Y$), Extensionality schema ($\forall\vec{z}(X\vec{z} \leftrightarrow Y\vec{z}) \to X = Y$), and the Fregean Axiom ($(p \leftrightarrow q) \to p = q$): a model is functional (extensional, Fregean) iff all instances of the Functionality schema (Extensionality schema, The Fregean Axiom) hold in it. See §1.4 for discussion of these principles. Extensionality is thus equivalent to the combination of functionality and Fregeanness. As a consequence of Theorem 3.1, we have:

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[64] By clauses (ii-b) and (ii-c), it doesn't matter which variables we pick to be X and y. Benzmüller, Brown and Kohlhase (2004) treat the application map to be a separate ingredient in the definition of "model", but since it can be recovered from $[\![\cdot]\!]$ we omit it.

[65] It can't have less than two elements since no element of type t can have the same truth value as its negation.

Theorem 3.2. The class of extensional BBK-models is sound and complete for Extensionalism.

It is also interesting to think about BBK-models in which the applicative or extension maps are *surjective*. A model $\mathbf{M}$ is:

- ***functionally full*** iff each $\mathrm{app}_{\mathbf{M}}$ is surjective on each domain: every function from $\mathbf{M}^\sigma$ to $\mathbf{M}^\tau$ is the applicative behaviour of some element of $\mathbf{M}^{\sigma\to\tau}$.
- ***extensionally full*** iff $\mathrm{ext}_{\mathbf{M}}$ is surjective on each domain: every subset X of $\mathbf{M}^{\sigma_1} \times \cdots \times \mathbf{M}^{\sigma_n}$is the extension of some $\mathbf{d} \in \mathbf{M}^{\sigma_1\to\cdots\to\sigma_n\to t}$.

Functional fullness implies, though it is not implied by, extensional fullness. Analogous to the ways in which the Extensionality and Functionality schemas characterise the eponymous properties of BBK-models, one might hope to find some axioms which characterise functional or extensional fullness. But this is not so: Gödel's first incompleteness theorem implies that neither of the properties is captured by any recursively enumerable axiom-scheme. (For a bit more detail see Dorr 2016, n. 106.)

3.2 Henkin models

Every functional and Fregean (i.e., extensional) BBK-model is equivalent to a model where the elements of a given functional type are simply *identical* to their applicative behaviours, and the elements of propositional type are simply identical to their truth values. The operation of turning certain kinds of models into more "concrete" ones will be important later in analogous settings, so we shall present it in some detail. The relevant concrete models are known as *Henkin* models, (after Henkin 1950). In a Henkin model $\mathbf{H}$, $\mathbf{H}^e$ can still be any set, but $\mathbf{H}^t$ must be $\{0, 1\}$, and $\mathbf{H}^{\sigma\to\tau}$ must be some subset of $(\mathbf{H}^\tau)^{\mathbf{H}^\sigma}$.

Working with this more concrete kind of model has one very significant advantage. BBK-models that are not concrete have few practical uses because they are not constructed compositionally from the interpretations of the non-logical constants; $[\![\cdot]\!]$ is a function defined on all terms of the language and have to satisfy some highly non-trivial constraints. By contrast, to specify a Henkin model one only need to specify the interpretation of the non-logical constants, and provided they exist, this interpretation extends uniquely to an interpretation of all the terms. However, we do need to ensure that the domains are sufficiently full that we will be able to provide an appropriate interpretation for every term relative to every variable assignment. To capture this, we first define a notion of *premodel* whose domains need not be sufficiently full; then recursively define a *partial* interpretation function for any premodel; and finally define a *model* to be a premodel whose interpretation function is full. Spelling this out, we get the following.

Definition. (i) A ***Henkin premodel*** for a signature $\mathcal{L}$ is an ordered pair $\mathbf{H} = \langle \mathbf{H}^{\cdot}, \mathcal{I} \rangle$, where $\mathbf{H}^{t} = \{0, 1\}$ and $\mathbf{H}^{\sigma \to \tau} \subseteq (\mathbf{H}^{\tau})^{\mathbf{H}^{\sigma}}$, and $\mathcal{I}$ is a mapping that takes each nonlogical constant c of type σ to an element of $\mathbf{H}^{\sigma}$.

(ii) When $\mathbf{H}$ is a Henkin premodel for $\mathcal{L}$, $\llbracket \cdot \rrbracket^{\cdot}_{\mathbf{H}}$ is the partial function that takes a type-σ term A and a A-adequate assignment function g for $\mathbf{H}^{\cdot}$ to something of the right sort to be in $\mathbf{H}^{\sigma}$, in accordance with the following clauses:

$$\begin{array}{rlrl}
\llbracket AB \rrbracket^{g} &= \llbracket A \rrbracket^{g}(\llbracket B \rrbracket^{g}) & \llbracket \lambda v.A \rrbracket^{g} &= \mathbf{a} \mapsto \llbracket A \rrbracket^{g[v \mapsto \mathbf{a}]} \\
\llbracket c \rrbracket^{g} &= \mathcal{I}(c) & \llbracket v \rrbracket^{g} &= g(v) \\
\llbracket \neg \rrbracket^{g} &= n \mapsto 1 - n & \llbracket =_{\sigma} \rrbracket^{g} &= \mathbf{a} \mapsto \left(\mathbf{b} \mapsto \begin{cases} 1 \text{ if } \mathbf{a} = \mathbf{b} \\ 0 \text{ otherwise} \end{cases} \right) \\
\llbracket \wedge \rrbracket^{g} &= n \mapsto (m \mapsto \min\{n, m\}) & \llbracket \vee \rrbracket^{g} &= n \mapsto (m \mapsto \max\{n, m\}) \\
\llbracket \forall_{\sigma} \rrbracket^{g} &= \mathbf{d} \mapsto \min\{\mathbf{d}(\mathbf{a}) \mid \mathbf{a} \in \mathbf{H}^{\sigma}\} & \llbracket \exists_{\sigma} \rrbracket^{g} &= \mathbf{d} \mapsto \max\{\mathbf{d}(\mathbf{a}) \mid \mathbf{a} \in \mathbf{H}^{\sigma}\}
\end{array}$$

(The first clause means that $\llbracket AB \rrbracket^{g}$ exists so long as $\llbracket A \rrbracket^{g}$ and $\llbracket B \rrbracket^{g}$ exist, and is in that case equal to $\llbracket A \rrbracket^{g}(\llbracket B \rrbracket^{g})$.)

(iii) A ***Henkin model*** for $\mathcal{L}$ is a Henkin premodel $\mathbf{H}$ for $\mathcal{L}$ such that $\llbracket A \rrbracket^{g}_{\mathbf{H}}$ exists and is in $\mathbf{H}^{\sigma}$ for every type-σ term A and assignment function g adequate for A.[66]

(iv) A formula P ***holds in*** $\mathbf{H}$ on g—in symbols, $\mathbf{H}, g \Vdash P$—iff $\llbracket P \rrbracket^{g} = 1$. P holds in $\mathbf{H}$ iff $\mathbf{H}, g \Vdash P$ for every g adequate for P.

Henkin (1950) (as corrected by Andrews 1972) establishes that the class of Henkin models is sound and complete for Extensionalism. To set the stage for our discussion of Classicism below, we observe that this fact can be derived as a corollary of Theorem 3.1. The derivation uses the following two results.

Proposition 3.3. Every Henkin model is an extensional BBK-model, with its defined interpretation function and the identity on $\{0, 1\}$ as valuation.

Proposition 3.4. For every extensional BBK-model $\mathbf{M}$, there is a Henkin-model $\mathbf{H}_{\mathbf{M}}$ in which the same formulae hold.

Verifying Proposition 3.3 boils down to checking that when A and B are $\beta\eta$-equivalent terms, $\llbracket A \rrbracket^{g}_{\mathbf{H}} = \llbracket B \rrbracket^{g}_{\mathbf{H}}$ whenever defined. For Proposition 3.4, the idea is to construct

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[66] More "intrinsic" ways of expressing this condition are known. For example, it can be shown (see, e.g. Bacon n.d.[a]) that Henkin premodel $\mathbf{H}$ is a Henkin model so long as (i) the domains are closed under application; (ii) the denotations of the logical constants (as given above) all belong to the domain of the appropriate type, and (iii) for any type σ and relational types ρ, τ, $\mathbf{H}^{(\sigma \to \rho \to \tau) \to (\sigma \to \rho) \to \sigma \to \tau}$ contains the function $S_{\sigma,\rho,\tau} \coloneqq \mathbf{d} \mapsto \mathbf{b} \mapsto \mathbf{a} \mapsto \mathbf{d}(\mathbf{a})(\mathbf{b}(\mathbf{a}))$ and $\mathbf{H}^{\tau \to \sigma \to \tau}$ contains the function $K_{\sigma,\tau} \coloneqq \mathbf{a} \mapsto \mathbf{b} \mapsto \mathbf{a}$.

each domain $\mathbf{H}^\sigma_\mathbf{M}$ as the range of an injective function f^σ on $\mathbf{M}^\sigma$, where f^e is just the identity; f^t is $\mathrm{val}_\mathbf{M}$; and $f^{\sigma\to\tau}$ is defined by recursively replacing each element $\mathbf{d}$ with $\mathrm{app}_\mathbf{M}\,\mathbf{d}$.

By Proposition 3.3 and the soundness part of Theorem 3.2, every theorem of Extensionalism holds in every Henkin model. And by Proposition 3.4 and the completeness part of Theorem 3.2, every formula consistent with Extensionalism holds on some assignment in some Henkin model. Thus:

Theorem 3.5. The class of Henkin models is sound and complete for Extensionalism.

3.3 Categories of BBK-models

To do for Classicism what Henkin did for Extensionalism, we will consider properties of *collections* of BBK-models analogous to the properties of individual BBK-models that make for the truth of Extensionalism. It will turn out that the properties of interest are properties not of mere collections of BBK-models but of *categories* of BBK-models, i.e. collections of models with a specified collection of homomorphisms between them. So, the first thing we will need is an appropriate notion of homomorphism for BBK-models (for a given signature). As usual in model theory, a homomorphism is a mapping that preserves interpretations. More carefully, a homomorphism h from $\mathbf{M}$ to $\mathbf{N}$ is a typed family of functions h where $h^\sigma : \mathbf{M}^\sigma \to \mathbf{N}^\sigma$ and for any term A and assignment function g for $\mathbf{M}$ that is adequate for A,

$$h^\sigma [\![A]\!]^g_\mathbf{M} = [\![A]\!]^{h\circ g}_\mathbf{N}$$

Here, $h\circ g$ is the assignment function for $\mathbf{N}$ that maps each type-σ variable v to $h^\sigma(gv)$.[67]

Evidently, the composition of any two homomorphisms is a homomorphism, and the identity mapping that maps every element of every domain of a model to itself is a homomorphism from that model to itself. This means that if we take any class of BBK-models and any class of homomorphisms between those models, so long as the latter class is closed under composition and contains all the identity homomorphisms on the models, they will form a ***category*** according to the standard definition: a class of "objects" (here, the models) and a class of "arrows" (here, the homomorphisms) together with a pair of mappings src and trg that assigns each arrow a

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[67] Note that the valuation functions $\mathrm{val}_\mathbf{M}$ and $\mathrm{val}_\mathbf{N}$ play no role in this definition: models that are isomorphic (in the sense that there are mutually inverse homorphisms between them) can thus make different sentences true. Given this there is a natural sense in which homomorphisms relate what BBK call "structures" rather than models. Nevertheless, we will think of the source and target of homomorphisms as models, and indeed take these to be "built in", so that for each homomorphism, there is a unique model that is its source and another unique model that is its target.

unique "source" and "target" object; a mapping $\circ$ (here, function-composition) that takes any arrows f and g where the source of g is the target of f to an arrow $g \circ f$ which shares a source with f and a target with g; and a mapping Id that takes every object A to an arrow 1_A with source and target A, such that $h \circ (g \circ f) = (h \circ g) \circ f$, $f \circ 1_A = f$ and $1_A \circ f = f$.

Given a BBK-model $\mathbf{M}$ and a category C to which it belongs, we can think of the set of homomorphisms in C with source $\mathbf{M}$, which we will call $\mathbf{M}^{\mathcal{P}}$, as playing a role similar to that of possible worlds in the standard semantics for modal logic. Any element of $\mathbf{M}^t$ can be assigned not just a truth-*value* but a "truth value profile"—the subset of $\mathbf{M}^{\mathcal{P}}$ comprising those homormorphisms that map it to a truth. Likewise, an element of a relational type can be assigned not just an *extension* but an *intension* which we get by looking at the extensions of the results of transporting it using the homomorphisms, and each element of a functional type can be assigned an *applicative behaviour profile* by looking at the applicative behaviours of the results of transporting it using the homomorphisms.

Definition. Where C is a category of BBK-models and $\mathbf{M}$ is a member of C,

- The ***applicative behaviour profile*** $\mathrm{app}^C_{\mathbf{M}}\,\mathbf{d}$ of any $\mathbf{d} \in \mathbf{M}^{\sigma \to \tau}$ is the function such that for any pair $\langle h, \mathbf{a} \rangle$ where for some $\mathbf{N}$, $h : \mathbf{M} \to \mathbf{N}$ and $\mathbf{a} \in \mathbf{N}^{\sigma}$,

$$\mathrm{app}^C_{\mathbf{M}}\,\mathbf{d}\langle h, \mathbf{a} \rangle = \mathrm{app}_{\mathbf{N}}(h^{\sigma \to \tau}\mathbf{d})\mathbf{a}$$

- The ***intension*** $\mathrm{int}^C_{\mathbf{M}}\,\mathbf{d}$ of any $\mathbf{d} \in \mathbf{M}^{\sigma_1 \to \cdots \to \sigma_n \to t}$ is the set of all tuples $(h, \mathbf{a}_1, \ldots, \mathbf{a}_n)$ such that for some $\mathbf{N}$, $h : \mathbf{M} \to \mathbf{N}$ and $\langle \mathbf{a}_1, \ldots, \mathbf{a}_n \rangle \in \mathrm{ext}_{\mathbf{N}}(h^{\sigma_1 \to \cdots \to \sigma_n \to t}\mathbf{d})$

- The ***truth value profile*** $\mathrm{val}^C_{\mathbf{M}}\,\mathbf{p}$ of any $\mathbf{p} \in \mathbf{M}^t$ is the set of all arrows with source $\mathbf{M}$ that map $\mathbf{p}$ to a truth:

$$\bigcup_{\mathbf{N}} \{ h : \mathbf{M} \to \mathbf{N} \mid \mathrm{val}_{\mathbf{N}}(h^t \mathbf{p}) = 1 \}$$

These notions generalize the analogous operations, app, ext and val, for single BBK-models in the sense that they end up being equivalent on a category consisting of a single BBK-model with the identity homomorphism.

We can then consider some special categories of BBK-models in which these functions are injective.

Definition. A category C of BBK-models is

- ***quasi-Fregean*** iff $\mathrm{val}^C_{\mathbf{M}}$ is injective for each $\mathbf{M}$ in C.

- ***quasi-functional*** iff $\mathrm{app}^C_{\mathbf{M}}$ is injective for each $\mathbf{M}$ in C.

- ***intensional*** iff $\text{int}^{C}_{\mathbf{M}}$ is injective for each $\mathbf{M}$ in C.

Informally, the first condition corresponds to the idea that propositions are individuated by their truth-values across modal space, the second to the idea that relations are individuated by their applicative behaviour across modal space, and the last condition to the idea that relations are individuated by their extensions across modal space. These are similarly generalizations of the notions of Fregeanness, functionality, and extensionality: they coincide in a one-object category with just the identity homomorphism. Moreover, in analogy with the discussion of extensional models, we can see that intensionality is equivalent to the combination of quasi-functionality and quasi-Fregeanness.[68]

And as with our discussion of Extensionalism we will see that there is a close correspondence between these conditions and the rule corresponding to the Fregean Axiom, Functionality and Extensionality, namely Equiv, ζ, and Equiv+. This correspondence lets us prove the following key result:

Theorem 3.6. The class of BBK-models that belong to an intensional category of BBK-models is sound and complete for Classicism.

For the soundness direction, the central observation is that the theory of any quasi-Fregean category of BBK-models (the set of formulae that hold in all of them) is closed under Equiv. Intuitively: if $P \leftrightarrow Q$ is true on all assignments in every model in C, there's no way for a homomorphism to pull apart the truth values of the interpretations of P and Q in a model, which given quasi-Fregeanness means that denotations must be identical. Likewise, the theory of any quasi-functional category of BBK models is closed under ζ. For if $Fx = Gx$ is true on all assignments in every model in C, then there is no way for a homomorphism to pull apart the applicative behaviours of the interpretations of F and G in any model on any assignment, which given quasi-functionality forces those denotations to be identical.

For the completeness direction, the key fact is that for any theory T closed under both Equiv and ζ—and hence in particular Classicism—the category of *all* models of T and all homomorphisms between these models is quasi-Fregean and quasi-functional.[69]

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[68] Quasi-Fregeanness is just the $n = 0$ special case of intensionality; given quasi-functionality, we can extend this by induction on n.

[69] For the Equiv part, suppose we have a model $\mathbf{M}$ of T with $\mathbf{p}, \mathbf{q} \in \mathbf{M}^t$ such that $\mathbf{p} \neq \mathbf{q}$. Consider the expanded language $\mathcal{L}_{\mathbf{M}}$ in which every element of $\mathbf{M}^\sigma$ is a constant of type σ. Let $\mathbf{M}^+$ be the model derived from $\mathbf{M}$ by extending its interpretation function to terms of $\mathcal{L}_{\mathbf{M}}$ with the new constants interpreted as denoting themselves.[70] Let T^+ be the result of adding to T all closed identities in $\mathcal{L}_{\mathbf{M}}$ that are true in $\mathbf{M}^+$, along with $\neg(\mathbf{p} \leftrightarrow \mathbf{q})$. T^+ must be consistent. For if it were inconsistent there would be a finite collection of identities, $A_1, \dots, A_n$ in T^+ inconsistent with $\neg(\mathbf{p} \leftrightarrow \mathbf{q})$: i.e. $A_1 \wedge \cdots \wedge A_n \to (\mathbf{p} \leftrightarrow \mathbf{q})$ would be a theorem of H, and thus a theorem of T. But since T is closed

Indeed, we can sharpen the proof of Theorem 3.6 to show that for any infinite set—e.g. $\mathbb{N}$—the category of BBK-models of Classicism whose domains are all subsets of this set is also quasi-Fregean and quasi-functional, so we also have a soundness and completeness theorem for BBK models that belong to categories that are constrained in this way. Note that, unlike the category of all BBK models of Classicism, such categories are small — i.e. there is a set of objects and arrows. This strengthening of Theorem 6 will be useful in the next section.

In §3.1 we also discussed "functional fullness" and "extensional fullness" conditions on models, defined by the surjectiveness of the applicative behaviour mapping and extension mapping. These also have analogues at the level of categories, but we need to think a little about what sets it would make sense to require the relevant functions to be surjective to. For any category of BBK-models $\mathcal{C}$ and model $\mathbf{M}$ in $\mathcal{C}$, define

- $\mathbf{M}^{[\sigma_1,\dots,\sigma_n]}$ to be the powerset of the set of all $n+1$-tuples $\langle h, \mathbf{a}_1, \dots, \mathbf{a}_n\rangle$, where for some $\mathbf{N}$, $h : \mathbf{M} \to \mathbf{N}$ and each $\mathbf{a}_i \in \mathbf{N}^{\sigma_i}$.
- $\mathbf{M}^{\sigma\Rightarrow\tau}$ to be the set of all functions α which take an ordered pair $\langle h, \mathbf{a}\rangle$ such that for some $\mathbf{N}$, $h : \mathbf{M} \to \mathbf{N}$ and $\mathbf{a} \in \mathbf{N}^{\sigma}$, and yield an element of $\mathbf{N}^{\tau}$, and are "well-behaved" in the following sense: for any $h : \mathbf{M} \to \mathbf{N}$, $i : \mathbf{N} \to \mathbf{O}$, and $\mathbf{a} \in \mathbf{N}^{\sigma}$, $\alpha\langle i \circ h, i^{\sigma}\mathbf{a}\rangle = i^{\tau}(\alpha\langle h, \mathbf{a}\rangle)$.

We can then define a category of $\mathcal{C}$ of BBK-models to be:

- ***quasi-functionally full*** iff $\mathrm{app}^{\mathcal{C}}_{\mathbf{M}}$ is a surjection from each $\mathbf{M}^{\sigma\to\tau}$ to $\mathbf{M}^{\sigma\Rightarrow\tau}$.
- ***intensionally full*** iff $\mathrm{int}^{\mathcal{C}}_{\mathbf{M}}$ is a surjection from each $\mathbf{M}^{\sigma_1\to\cdots\to\sigma_n\to t}$ to $\mathbf{M}^{[\sigma_1,\dots,\sigma_n]}$.

To motivate the first of these definitions, note that by the definition of $\mathrm{app}^{\mathcal{C}}_{\mathbf{M}}$, $\mathrm{app}^{\mathcal{C}}_{\mathbf{M}}\,\mathbf{d}$ must always obey the well-behavedness condition, since homomorphisms must commute with application.

By contrast with the case of individual BBK-models where the $n = 0$ case of extensional fullness is automatically satisfied (since each truth-value is guaranteed to

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under the rule of Necessitation, we can derive $\Box A_1 \wedge \cdots \wedge \Box A_n \to \Box(\mathbf{p} \leftrightarrow \mathbf{q})$. But since $A_1, \dots, A_n$ are identities, they imply their own necessitations, so $\Box(\mathbf{p} \leftrightarrow \mathbf{q})$ must be a theorem of T^+, and thus also $\mathbf{p} = \mathbf{q}$, which is impossible since $\mathbf{M}^+$ is a model of T^+ in which $\mathbf{p} = \mathbf{q}$ is false. So there must be a model $\mathbf{N}^+$ of T^+, by the completeness theorem for BBK-models. Let N be the model of $\mathcal{L}$ obtained by restricting $\mathbf{N}^+$'s interpretation function to $\mathcal{L}$, and let h be the typed family of functions obtained by mapping each element $\mathbf{a}$ of $\mathbf{M}^+$ to $[\![\mathbf{a}]\!]_{\mathbf{N}^+}$, i.e. the interpretation of $\mathbf{a}$ (considered as a constant of $\mathcal{L}_{\mathbf{M}}$) in $\mathbf{N}^+$. It's easy to see that h is a homomorphism from $\mathbf{M}$ to $\mathbf{N}$. So, as desired, we get a member of our category and a homomorphism that maps $\mathbf{p}$ and $\mathbf{q}$ to elements with different truth values.

To finish the proof, we can note that if Modalized Functionality—which can be derived from Equiv and ζ—holds in a quasi-Fregean category, the category must be quasi-functional.

be had by some type-t element), the $n = 0$ case of intensional fullness is non-trivial. We'll call a category of BBK-models that satisfies this condition ***propositionally full***: every set of homomorphisms from a given model $\mathbf{M}$ is the truth-value profile of some element of $\mathbf{M}^t$.[71]

3.4 Action models

We saw earlier that every extensional BBK-model is isomorphic to a "concrete" Henkin model, with a compositionally specified interpretation function, in which propositions are identical to their truth values, and elements of functional type are identical to their applicative behaviour. Our strategy in this section will be to find similarly "concrete" and compositional models for each intensional category of BBK-models, in which propositions are identical to their truth-value *profiles* (which specify their truth value under each homomorphism) and elements of functional type are identical to their applicative behaviour *profiles* (which specify their applicative behaviour under each homomorphism). The main upshot of this, apart from introducing a workable notion of model for proving consistency results, is that we will be able to transfer our soundness and completeness theorems for intensional categories of BBK-models to our more concrete models, just as we did for extensional BBK-models and Henkin models.

Here the key concept we will need in order to carry out this strategy is that of an *action* on some category (a.k.a. a functor from that category to Set), which will play roughly the role of a "domain" that was being played by mere sets in Henkin models.

Definition. An ***action*** on a category C is a function $-^*$ that associates each object A of C with a set A^*, and each arrow $h : A \to B$ of C with a function $h^* : A^* \to B^*$, in such a way that

- $i^* \circ h^* = (i \circ h)^*$ for any composable arrows i and h
- when 1_A is the identity arrow on A, 1_{A^*} is the identity function on A^*.

Example. Where C is a category of BBK-models, each type σ determines an action $-^\sigma$ of applying the superscript to a model or homomorphism respectively:

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[71] The connection between these properties is a little more intricate that in the extensional case. Whereas functional fullness implied extensional fullness, quasi-functional fullness does not imply intensional fullness (since the latter does whereas the former does not imply propositional fullness). However, the conjunction of functional and propositional fullness does imply intensional fullness. Meanwhile, in intensional categories, intensional fullness coincides with the combination of quasi-functional and propositional fullness.

- $-^\sigma$ takes a model M in the category to the set M^σ
- $-^\sigma$ takes a homomorphism $h : \mathbf{M} \to \mathbf{N}$ in the category to the function $h^\sigma : \mathbf{M}^\sigma \to \mathbf{N}^\sigma$.

As another example, we can treat all possible truth-value profiles as an action:

Example. For any category C, the ***powerset action*** on C is the action $-^{\mathcal{P}}$ where for any object A, $A^{\mathcal{P}}$ is the powerset of the set of all arrows with source A, i.e.

$$\mathcal{P}(\bigcup_B \mathrm{Hom}(A, B))$$

and for any arrow $h : A \to B$ and set of arrows $X \in A^{\mathcal{P}}$, $h^{\mathcal{P}} X$ is the set of all arrows with source B which yield a member of X when composed with h, i.e.

$$\bigcup_C \{i : B \to C \mid i \circ h \in X\}.$$

(This is sometimes called the result of 'dividing' X by h.)

The truth-value profile of any element of $\mathbf{M}^t$ is an element of $\mathbf{M}^{\mathcal{P}}$. Moreover, the action of any homomorphism of BBK-models on the propositional elements induces an action on their truth-value profiles: the truth-value profile of $h^t p$ is the set of homomorphisms i such that $i^t(h^t p)$ is true; i.e. the set of i such that $i \circ h$ belongs to the truth-value profile of p. So the induced action of h on a truth-value profile X can be stated intrinsically in terms of X and is just $h^{\mathcal{P}} X$ as defined above.

We can likewise treat all possible applicative behaviour profiles as an action:

Example. Suppose $-^*$ and $-^\dagger$ are actions on C. Then the ***exponential action*** is the action $-^{*\Rightarrow\dagger}$ on C such that:

(i) For every object A, $A^{*\Rightarrow\dagger}$ is the set of all functions α whose domain is the set of all pairs $\langle h, x\rangle$, where for some object B, $h : A \to B$ and $x \in B^*$; which map each such $\langle h, x\rangle$ to a member of $B^\dagger$; and which are well-behaved in the sense that $i^\dagger(\alpha\langle h, x\rangle) = \alpha\langle i \circ h, i^* x\rangle$ for any $h : A \to B$, $i : B \to C$, and $x \in B^*$.

(ii) For every arrow $h : A \to B$, $h^{*\Rightarrow\dagger}$ is the function such that for any $\alpha \in A^{*\Rightarrow\dagger}$, $i : B \to C$, and $x \in C^*$, $(h^{*\Rightarrow\dagger}\alpha)\langle i, x\rangle = \alpha\langle i \circ h, x\rangle$.

The applicative behaviour profile of any element of $\mathbf{M}^{\sigma\to\tau}$ is an element of $\mathbf{M}^{\sigma\Rightarrow\tau}$. Moreover, the action of any homomorphism of BBK-models on elements of functional type induces an action on their applicative behaviour profiles: the applicative behaviour profile of $h^{\sigma\to\tau}\mathbf{d}$ maps $\langle i, \mathbf{a}\rangle$ to $i^{\sigma\to\tau}(h^{\sigma\to\tau}\mathbf{d})(\mathbf{a})$. So the induced action of h on an applicative behaviour profile α can be stated intrinsically in terms of α and is just $h^{\sigma\Rightarrow\tau}\alpha$ as defined above.

In general the truth-value profiles of $\mathbf{M}^t$ will be a subset of $\mathbf{M}^{\mathcal{P}}$ and the applicative behaviour profiles of elements of $\mathbf{M}^{\sigma\to\tau}$ a subset of $\mathbf{M}^{\sigma\Rightarrow\tau}$. However, in a category of BBK-models $-^t$ and $-^{\sigma\to\tau}$ will determine *subactions* of $-^{\mathcal{P}}$ and $-^{\sigma\Rightarrow\tau}$:

Definition. One action $-^*$ is a ***subaction*** of another action $-^\dagger$ iff $A^* \subseteq A^\dagger$ for every object A and $h^*(x) = h^\dagger(x)$ for every $h : A \to B$ and $x \in A^*$.

With these concepts under our belt, we can finally introduce the promised analogue of Henkin models for Classicism. As before, we can start with a notion of premodel; define a *partial* notion of interpretation for premodels; and then define a model to be a premodel which is "sufficiently full" in the sense that its interpretation function is total. Our starting point is a ***rooted category***: an ordered pair $\langle \mathcal{C}, W_0\rangle$, where W_0 is an object of $\mathcal{C}$ with an arrow to every other object of $\mathcal{C}$.[72]

Definition. (i) An ***action premodel*** $\mathbf{A}$ for a signature $\mathcal{L}$ is a tuple $\langle \mathcal{C}, W_0, -^{\cdot}, \mathcal{I}\rangle$, where $\langle \mathcal{C}, W_0\rangle$ is any rooted category, and for each type σ, $-^\sigma$ is an action of $\mathcal{C}$, such that:

1. $-^e$ is any action of $\mathcal{C}$ such that W^e is nonempty for every object W;
2. $-^t$ is a subaction of $-^{\mathcal{P}}$ (the powerset action on $\mathcal{C}$);
3. $-^{\sigma\to\tau}$ is any subaction of $-^{\sigma\Rightarrow\tau}$ (the exponential action on $\mathcal{C}$ for $-^\sigma$ and $-^\tau$); and
4. For each type-σ nonlogical constant c of $\mathcal{L}$, $\mathcal{I}(c) \in W_0^\sigma$.

The notion of interpretation function for an action premodels will be a bit different from the interpretation functions we have been working with up to now. These interpretation functions require not just a term A and an assignment function g, but an arrow h with source W_0: g is an assignment function for h's target.

Definition. When $\mathbf{A}$ is an action premodel for $\mathcal{L}$, the ***interpretation function*** of $\mathbf{A}$ is a partial function $[\![\cdot]\!]_{\mathbf{A}}$ that takes an arrow $h : W_0 \to W$, a term A of some type σ, an assignment function g for W adequate for A, and returns something of the right sort to belong to W^σ, such that:

$$[\![c]\!]^g_h = h(\mathcal{I}c)$$

---

[72] Disallowing objects without an arrow from W_0 is just a convenience, since they would make no difference if they were present.

$$
\begin{aligned}
&\llbracket v\rrbracket_h^g = g(v)\\
&\llbracket\neg\rrbracket_h^g = \langle i,\mathbf{p}\rangle \mapsto \bigcup_V\{j : \mathsf{trg}(i)\to V\}\setminus\mathbf{p}\\
&\llbracket\wedge\rrbracket_h^g = \langle i,\mathbf{p}\rangle \mapsto (\langle j,\mathbf{q}\rangle \mapsto j^t\mathbf{p}\cap\mathbf{q})\\
&\llbracket\vee\rrbracket_h^g = \langle i,\mathbf{p}\rangle \mapsto (\langle j,\mathbf{q}\rangle \mapsto j^t\mathbf{p}\cup\mathbf{q})\\
&\llbracket\forall_\sigma\rrbracket_h^g = \langle i,\alpha\rangle \mapsto \bigcup_V\{j : \mathsf{trg}(i)\to V \mid 1_V\in\alpha(j,\mathbf{a}) \text{ for every } \mathbf{a}\in V^\sigma\}\\
&\llbracket\exists_\sigma\rrbracket_h^g = \langle i,\alpha\rangle \mapsto \bigcup_V\{j : \mathsf{trg}(i)\to V \mid 1_V\in\alpha(j,\mathbf{a}) \text{ for some } \mathbf{a}\in V^\sigma\}\\
&\llbracket=_\sigma\rrbracket_h^g = \langle i,\mathbf{a}\rangle \mapsto (\langle j,\mathbf{b}\rangle \mapsto \{k \mid k^\sigma(j^\sigma\mathbf{a}) = k^\sigma\mathbf{b})\}\\
&\llbracket AB\rrbracket_h^g = \llbracket A\rrbracket_h^g\langle 1_{\mathsf{trg}(h)}, \llbracket B\rrbracket_h^g\rangle\\
&\llbracket\lambda v.A\rrbracket_h^g = \langle i,\mathbf{a}\rangle \mapsto \llbracket A\rrbracket_{i\circ h}^{(i\circ g)[v\mapsto\mathbf{a}]}
\end{aligned}
$$

For now, we won't try to justify these clauses; we will soon provide a way of restating their essential effect in a more familiar-looking format.[73]

Definition. An action premodel $\mathbf{A}$ is an ***action model*** iff for every type-σ $\mathcal{L}$-term A, arrow $h : W_0 \to V$, and assignment g for V adequate for A: $\llbracket A\rrbracket_h^g$ exists and belongs to V^σ.[74]

When $\mathbf{A}$ is an action model, $h : W_0 \to V$ and g is an assignment function for V, formula ***P holds in* A *on*** h, g ($\mathbf{A}, h, g \Vdash P$) iff $1_V \in \llbracket P\rrbracket_h^g$. ***P holds in* A *on*** g ($\mathbf{A}, g \Vdash P$) iff $\mathbf{A}, 1_{W_0}, g \Vdash P$. ***P holds in* A** ($\mathbf{A} \Vdash P$) iff $\mathbf{A}, g \Vdash P$ whenever g is an assignment for W_0 adequate for P. P is ***valid in*** a class X of action models ($X \Vdash P$) iff P holds in every model in X.

Using this notation, we can parlay the interpretations of the logical constants

---

[73] One thing to note is that the logical constants all denote functions from ordered pairs that are indifferent to the identity of its first element (a homomorphism). This is to be expected: in a category of BBK-models, any homomrphisms $h, i : \mathbf{M} \to \mathbf{N}$ must agree on $\llbracket c\rrbracket_{\mathbf{M}}$ for any logical constant c, so $\mathrm{app}_{\mathbf{M}}^{\mathcal{C}}\llbracket c\rrbracket_{\mathbf{M}}\langle h,\mathbf{a}\rangle = \mathrm{app}_{\mathbf{M}}^{\mathcal{C}}\llbracket c\rrbracket_{\mathbf{M}}\langle i,\mathbf{a}\rangle$ for any $\mathbf{a}$ in the appropriate domain of $\mathbf{N}$.

[74] Analogous to the fact about Henkin models reported in note 66, we can also give a more "intrinsic" version of the sufficient fullness condition: an action premodel $\mathbf{A}$ is an action model iff (i) the domains are closed under application: $\alpha\langle 1_W, \mathbf{a}\rangle \in W^\tau$ when $\alpha \in W^{\sigma\to\tau}$ and $\mathbf{a} \in W^\sigma$; (ii) the denotations of all the logical constants (as specified above) all belong to the appropriate domains; and (iii) for any type σ, relational types ρ, τ, and object W, $W^{(\sigma\to\rho\to\tau)\to(\sigma\to\rho)\to\sigma\to\tau}$ and $W^{\tau\to\sigma\to\tau}$ respectively contain the following functions:

$$
\begin{aligned}
S_{\sigma,\rho,\tau} &\coloneqq \langle i,\alpha\rangle \mapsto \langle j,\beta\rangle \mapsto \langle k,\mathbf{a}\rangle \mapsto \alpha\langle k\circ j,\mathbf{a}\rangle\langle 1_{\mathrm{trg}\,k}, \beta\langle k,\mathbf{a}\rangle\rangle\\
K_{\sigma,\tau} &\coloneqq \langle i,\mathbf{b}\rangle \mapsto \langle j,\mathbf{a}\rangle \mapsto j^\tau\mathbf{b}.
\end{aligned}
$$

into the following more helpful form. We can show that when $\mathbf{A}$ is an action model, $h : W_0 \to V$, g is an assignment for V adequate for the relevant formula, and v is a variable of type σ:

$$\begin{array}{ll}
\mathbf{A}, h, g \Vdash \neg P & \text{iff } \mathbf{A}, h, g \nVdash P \\
\mathbf{A}, h, g \Vdash P \wedge Q & \text{iff } \mathbf{A}, h, g \Vdash P \text{ and } \mathbf{A}, h, g \Vdash Q \\
\mathbf{A}, h, g \Vdash P \vee Q & \text{iff } \mathbf{A}, h, g \Vdash P \text{ or } \mathbf{A}, h, g \Vdash Q \\
\mathbf{A}, h, g \Vdash \forall v.P & \text{iff } \mathbf{A}, h, g[v \mapsto \mathbf{a}] \Vdash P \text{ for all } \mathbf{a} \in V^\sigma \\
\mathbf{A}, h, g \Vdash \exists v.P & \text{iff } \mathbf{A}, h, g[v \mapsto \mathbf{a}] \Vdash P \text{ for some } \mathbf{a} \in V^\sigma \\
\mathbf{A}, h, g \Vdash A = B & \text{iff } [\![A]\!]^g_h = [\![B]\!]^g_h
\end{array}$$

and in consequence,

$$\mathbf{A}, h, g \Vdash \Box P \quad \text{iff } \mathbf{A}, i \circ h, i \circ g \Vdash P \text{ for all } i \text{ with source } V.$$

The theorem that action models are sound and complete for Classicism is proved in Appendix C; here we just sketch the main ideas. For the soundness part, the important fact is

Proposition 3.7. Any action model can be turned into a BBK-model which makes the same formulae true, and in which every instance of Logical Equivalence is true.

The informal idea is to let the domains at each type σ be given by W_0^σ, the interpretation function given by $[\![\cdot]\!]^{\cdot}_{1_{W_0}}$, and val $\mathbf{p} = 1$ iff $1_{W_0} \in \mathbf{p}$. The BBK clauses for the connectives and quantifiers follow immediately from the above biconditionals involving $\Vdash$; the only non-trivial aspect is showing the $\beta\eta$ holds.

For the completeness part, the important fact is

Proposition 3.8. For every small, intensional category of BBK-models $\mathcal{C}$, and model $\mathbf{M}_0$ in $\mathcal{C}$ there is an action model $\mathbf{A}^{\mathcal{C}}_{\mathbf{M}}$ in which the same formulae hold.

The idea here is that the rooted category of the action model will be $\langle \mathcal{C}, \mathbf{M}_0 \rangle$ (if necessary throwing away any models in $\mathcal{C}$ without homormorphisms from $\mathbf{M}_0$), constructing the domain of each by (roughly) leaving individuals alone, replacing propositional elements with their truth-value profiles, and iteratively replacing elements of functional type with their applicative behaviour profiles. Note that we could derive propositions 3.3 and 3.4, relating extensional BBK-models to Henkin models, from special cases of propositions 3.7 and 3.8 applied to categories with one object with its identity arrow.

By Proposition 3.7 and the soundness of BBK models for H, every theorem of Classicism holds in every action model. And by Proposition 3.8 and the completeness part of Theorem 3.6, every formula consistent with Classicism holds on some assignment in some action model. Thus:

Theorem 3.9. The class of action models is sound and complete for Classicism.

3.5 Exploring action models

One special case in which we can already do a lot is the case where the base category has only one object. A category with only one object is called a monoid, and an action of a monoid is called an M-set. Bacon (n.d.[b]) discusses the special one-object case of action models under the label 'M-set models'. Every instance of the schema No Pure Contingency from §2.2 — i.e. $P \to \Box P$ for closed P with no nonlogical constants — is true in any M-set model, since as noted above when P contains no nonlogical constants $[\![P]\!]^g_h = [\![P]\!]^g_{h'}$ for any arrows h and h' with the same target, and when P contains no free variables $[\![P]\!]^g_h = [\![P]\!]^{g'}_h$ for any assignment functions g and g'.

We can already use full M-set models to show that Classicism is consistent with failures of ND and BF in arbitrary types, using the following facts:

1. ND_σ holds in an action model **A** iff h^σ is injective for every arrow h from **A**'s base object.[75]

2. If h^σ is is surjective for every arrow h with source W_0 then BF_σ holds in **A**.[76]

3. If **A** is functionally full and BF_σ holds in **A**, then every h^σ with source W_0 is surjective.[77]

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[75] Proof: $\mathbf{A}, 1_{W_0} \Vdash \forall xy.x \neq y \to \Box x \neq y$ iff $\mathbf{A}, 1_{W_0}, g \Vdash \Box x \neq y$ for all g such that $gx \neq gy$, iff $\mathbf{A}, h, h \circ g \Vdash x \neq y$ for all such g, all objects V, and all $h : W_0 \to V$, iff $h(gx) \neq h(gy)$ for all such h and g, iff $h\mathbf{a} \neq h\mathbf{b}$.

[76] Proof: Suppose every h^σ with source W_0 is surjective, and $1_{W_0}, g \Vdash \forall y.\Box P$. Let V be any object, $\mathbf{a} \in V^\sigma$, and $h : W_0 \to V$. Then there is some $\mathbf{b}$ such that $\mathbf{a} = h\mathbf{b}$, and hence $(h \circ g)[y \mapsto \mathbf{a}] = h \circ (g[y \mapsto \mathbf{b})$. But since $1_{W_0}, g[y \mapsto \mathbf{b}] \Vdash \Box.P$, $h, h \circ (g[y \mapsto \mathbf{b}]) \Vdash P$; thus $h, (h \circ g)[y \mapsto \mathbf{a}] \Vdash P$. Since **a** was arbitrary, it follows that $h, h \circ g \Vdash \forall y.P$. And since h was arbitrary, this implies $1_{W_0}, g \Vdash \Box \forall y.P$

[77] Suppose **A** is functionally full. Then $W_0^{\sigma \to t}$ contains the function α such that for any $i : W_0 \to U$ and $\mathbf{b} \in U^\sigma$, $\alpha(i, \mathbf{b}) = \{k \mid k\mathbf{b} = k(i\mathbf{a})$ for some $\mathbf{a} \in W_0^\sigma\}$. Note that for every $\mathbf{a} \in W_0^\sigma$, $\alpha(1_{W_0}, \mathbf{a})$ is the set of all arrows from W_0, thus $\mathbf{A}, id_{W_0}, [X \mapsto \alpha] \Vdash \forall y.\Box Xy$. So if BF_σ is true in **A**, we have $\mathbf{A}, id_{W_0}, [X \mapsto \alpha] \Vdash \Box \forall y.Xy$, hence $\mathbf{A}, h, [X \mapsto h\alpha] \Vdash \forall y.Xy$ for every arrow $h : W_0 \to V$, and hence $1_V \in h\alpha(1_V, \mathbf{b})$ for every $h : W_0 \to V$ and $\mathbf{b} \in V^\sigma$, which means that for all such **b** there is some $\mathbf{a} \in W_0^\sigma$ such that $\mathbf{b} = h\mathbf{a}$, i.e. h^σ is surjective.

Consider an action model whose base category has a single object W_0 and two arrows, $1 = 1_{W_0}$ and k, with $k \circ k = k$. If we choose an action for type e—e.g. just have W_0^e be a singleton—this, together with an interpretation of any nonlogical constants, uniquely determines a full model. Its propositional domain W^t contains four propositions, $\emptyset, \{1\}, \{k\}$ and $\{1, k\}$. 1^t is of course the identity function; $k^t(\emptyset) = k\{1\} = \emptyset$, $k^t(\{k\}) = k^t(\{1, k\}) = \{1, k\}$. Thus ND_t and BF_t both fail: the former since $k^t(\emptyset) = k\{1\}$; the latter, since the model is functionally full and $\{1\}$ is not in the range of k^t.

By contrast, if we choose the base category to have one object and two arrows with $k \circ k = i$, we get a model in which $\Box$ND is true. Still there are four propositions — $\emptyset$, $\{i\}$, $\{k\}$, and $\{i, k\}$ — so the Fregean Axiom is false, establishing the (already well-known) fact that C5 is weaker than Extensionalism.

For an M-set model with BF but not ND, we can consider a full model where the base category is the monoid of all *surjective* functions on some infinite set X, and choose $-^e$ in such a way that each h^e is also surjective. We can show [*cite TBN*] that in that case h^σ must also be surjective for every σ, which is sufficient for the truth of BF_σ for every type σ; but ND_t still fails, since when h is not injective $h^t\{1\} = \{i \mid i \circ h = 1\} = \emptyset = h^t\emptyset$, so h^t is not injective.

To model failures of No Pure Contingency, we can turn to categories with multiple objects. For example, consider a full model based on a category with two objects W_0 and W_1 and three arrows 1_{W_0}, 1_{W_1}, and $k : W_0 \to W_1$. Then the Fregean Axiom is false since there are four propositions $\emptyset, \{1_{W_0}\}, \{k\}, \{1_{W_0}, k\}$, but it is not necessarily false since it holds relative to k. For another example, consider a full model based on the category with two objects W_0 and W_1 and three non-identity arrows $h : W_0 \to W_1$, $j : W_1 \to W_0$, and $k : W_1 \to W_1$ (as well as the identity arrows 1_{W_0} and 1_{W_1}), with $k \circ k = h \circ j = k$. W_0^t is the four-membered powerset of $\{1_{W_0}, h\}$; W_1^t is the eight-membered powerset of $\{1_{W_1}, j, k\}$. ND_t is false at W_1, since $j^t\{1_{W_1}\} = j^t\{k\} = \{h\}$. To show that ND_t is true at W_0 it suffices to show that h–the only non-identity arrow with source W_0–acts injectively on W_0^t. This is true, since $h^t\emptyset = \emptyset$, $h^t\{1_{W_0}\} = \{j\}$, $h^t\{h\} = \{j, k\}$, and $h^t\{1_{W_0}, h\} = \{1_{W_1}, j, k\}$. So ND_t is true in the model whereas $\Box\mathrm{ND}_t$ is false. Note that while No Pure Contingency fails here, we do (unlike in the previou example) have the weaker schema $P \to \Box\Diamond P$ for all closed P. This will hold in any action model where every object has an arrow from every other object.

Another case of special interest is that of action models where the base category is a *preorder category*–one with at most one arrow having any given source and target. (Any set with a transitive and reflexive relation R can be turned into such a category by counting each ordered pair in R as an arrow from its first element to its second element.) In this case, the analogy between objects in the category and worlds in an S4 Kripke model becomes much closer; the objects with an arrow from a given object work like the worlds accessible from a world in a Kripke model with a reflexive and transitive accessibility relation. In contrast with the most familiar way of developing Kripke models for quantified modal logics including CBF, there is no requirement that the domain of an accessible world contain the domain of the accessing world. The role of identity across domains is played instead by the "transition functions", h^σ, which provide elements in the domain of one world corresponding to elements in the domain of another. In the case of type e we could thus recover the standard treatment of expanding domains by identifying the transition maps with the inclusion mappings from a set to a superset. However this forces the truth of ND_e, and more generally, failures of ND at any type require non-injective transition maps. What one can do, if one wants to have the domains of all worlds be subsets of one big domain is to associate each world W with a partial equivalence relation $\sim_W$ on that domain (i.e. a reflexive and symmetric relation): failures of injectivity in the transition function from W to W' correspond to the case where two things are related to themselves but not to each other at W, and are related to each other at W', and failures of surjectivity correspond to the case where something is related to itself at W' but not at W. For the details of these alternative "Expanding Modalized Domain Models" see Bacon 2018a. There is a natural recipe for transforming one sort of model into the other.[78]

In one way we would lose nothing by confining our attention to the class of action models based on preorders: this class is also sound and complete for Classicism. This follows from the fact that there is a procedure that "unravels" any action model $\mathbf{A}$ based on an arbitrary category into a new model $\mathbf{A}^*$ based on a preorder category, in which exactly the same formulae are true. The objects (worlds) of the preorder are composable finite sequences of arrows of the old category, starting from the base world: one such sequence is accessible from another (i.e. has an arrow to it) iff it is an initial segment of it. We can think of each such sequence as a copy of the old object that is the target of its final arrow, and there is a natural way of reading off domains in each type for every sequence from the domains of that object. The interpretations of terms in the old model map straightforwardly into the new model and the mapping preserves truth. However, the models output by this unravelling procedure are neither propositionally nor functionally full (except in degenerate cases). And indeed, the logic of propositionally and functionally full action models based on preorders is a strict strengthening of the logic of all propositionally and functionally full action models. For instance, the following sentence belongs to the logic of

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[78] The models in Bacon 2018a correspond to full preorder action models. The basic idea behind the correspondence is this: any action $-^*$ of a preorder $\mathcal{P}$ corresponds to a "modalized domain" $\langle D^*, \sim^* \rangle$ where the elements of D^* are what we might call "modal worms"—maximal partial functions f that map each world W to an elements of W^* in such a way that whenever $h : W \to W'$, $h^*(f(W)) = f(W')$—and $f \sim^*_W g$ iff $f(W) = g(W)$.

propositionally full action models based on preorders:

$$\Box\exists x.\exists y(x \neq y \wedge \Diamond x = y) \to \exists p_1 ... p_n(\bigwedge_{i \neq j} p_i \neq p_j)$$

For to make the antecedent true, every world must see a world that does not see it back, and so there must be infinitely many worlds, which in a propositionally full model means that the porpositional domain at the base world. By contrast, we already saw a one object action model that makes the antecedent true, that has only two arrows, and four propositions. This affords us extra flexibility in constructing models: full models are easy to construct, whereas checking that non-full action models meet "sufficient fullness" condition is tricky.

Nevertheless, even the logic of all full action models is still rather strong. It may be shown, by appeal to Gödel's incompleteness theorems, that it is undecidable.[79] But more importantly for our purposes, this logic also includes several of the more controversial principles surveyed in §2.2. Atomicity and Actuality are both true, and indeed necessarily true, in every propositionally full action model, for the obvious reason: they contain all the singletons. And Rigid Comprehension and its necessitation are true in every intensionally full action model, since for every subset X of W_0^σ, the element of α_X of $W_0^{\sigma\to t}$ defined by $\alpha_X\langle h, \mathbf{a}\rangle = \{i \mid i^\sigma(a) = (i \circ h)^\sigma(b)$ for some $b \in X\}$ is coextensive with X, persistent, and inextensible.[80] So to explore the consistency of packages in which some of these principles are false, or at least possibly false, we will need ways of constructing non-full models. Appendix D develops one method of constructing such models which can be used to verify the consistency of many combinations of the controversial principles and their necessitations.

3.6 The consistency of Maximalist Classicism

Recall that for any theory T extending C (in a given signature $\mathcal{L}$), Max T, the maximalization of T, is the result of adding $\Diamond P$ to T for every closed $\mathcal{L}$-formula P consistent with T. In this section we will see how action models can be used to

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[79] There is a computable mapping from the language of arithmetic to that of pure higher-order logic that maps all the arithmetical truths the validities in this class of models, and all the arithemtical false to invalidities.

[80] One further limitation of full models is worth noting: if BF is true (in a given type), then so is $\Box$BF. For in a full model, $W_0^{\sigma\to t}$ contains the function α defined by $\alpha(h, x) = \{i \mid i^\sigma x = i^\sigma(h^\sigma y)$ for some $y \in W_0^\sigma\}$. $\forall y.\Box X y$ is true on the assignment that maps X to this α, so by the truth of BF so is $\Box\forall y.X y$. This means that for every $h : W_0 \to W$, $1_W \in \alpha(h, y)$ for every $y \in W_0^\sigma$; i.e. h^σ is surjective. But if h^σ is surjective for every h with source W_0, i^σ must be surjective for every i whose source is reachable from W_0, and hence BF$_\sigma$ must be true at every object reachable from W_0, and hence $\Box$BF$_\sigma$ must be true at W_0.

prove the consistency of the maximalizations of Classicism and many other theories extending Classicism.

A crucial concept will be that of a *truncation* of an action model by an arrow $h : W_0 \to V$. Informally a truncation is what you get by treating V as your new base world, and throwing away objects with no arrow from V.[81]

Definition. When $\mathbf{A} = \langle C, W_0, -^{\cdot}, \mathcal{I} \rangle$ is an action premodel and $h : W_0 \to V$, the ***truncation*** of $\mathbf{A}$ by h is the action premodel $\mathbf{A}_h = \langle C', V, -^{\cdot'}, \mathcal{I}' \rangle$ whose base category C' contains all the arrows of C with an arrow from V and all arrows between them; whose action $-^{\sigma'}$ in each type is just the restriction of $-^{\sigma}$ to C', and where for each nonlogical constant c, $\mathcal{I}'c = h(\mathcal{I}c)$.

It is easy to show (by induction on the complexity of terms) that for any term A, $[\![A]\!]^g_{\mathbf{A}_h,i} = [\![A]\!]^g_{\mathbf{A},i \circ h}$: thus $\mathbf{A}_h$ is in fact an action model, not just a premodel. Moreover, since the truth of $\Diamond P$ for a closed sentence P amounts to its being true under some arrow (i.e. $1_{\operatorname{trg} h} \in [\![P]\!]_h$), we can use this fact to show that for a closed sentence P:

> $\Diamond P$ holds in an action model iff P holds in one of its truncations.

To show the consistency of Max T, it is thus sufficient to find an action model $\mathbf{A}$ of T such that T is complete with respect to the set of all truncations of $\mathbf{A}$. Indeed, the converse is also true: any action model of Max T must be such that T is complete with respect to its truncations, since if P is consistent with T, $\Diamond P$ holds in the model, and thus P must hold in one of its truncations.[82]

We can thus establish the consistency of Maximalist Classicism by finding an action model such that Classicism is complete with respect to its truncations. In Appendix E, we will establish a stronger result which implies this:

Theorem 3.10. Every set X of action models whose base categories are disjoint has a *coalesced sum*—an action model such that every member of X is among its pruned truncations.

The informal idea behind this construction is this: lay out the rooted categories corresponding to the action models in X side by side and add a new object, W_0, at the bottom with a new arrow from W_0 to the roots of the old categories and an identity arrow for W_0. (At this point you will have to add new arrows obtained by composing the new arrows with the old.) Now you have a big rooted category containing each of the rooted categories from X as truncations. This is turned into a big action model by using the domains from the models in X to determine the

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[81] It is related to the notion of a generated submodel from a world from modal logic.

[82] If T is closed under necessitation, then truncations of models of T will also be models of T, so T will be sound as well as complete with respect to the the truncations of a model of Max T.

domains of the old objects, and at W_0 we take the domains to be "as full as possible". The intensions of the interpretations of relational nonlogical constants are chosen so that their extensions relative to non-identity arrows are given by the old models, and their actual extensions may be chosen freely.

This gives us what we need, since as we pointed out in §3.4, C is not only sound and complete with respect to the proper class of *all* action models, but also with respect to various *sets* of action models. And given any set of action models for which C is complete, we can easily turn it into a set whose base categories are disjoint just by replacing each base category with an isomorphic copy.

Theorem 3.10 has other interesting consequences. In any category, a *weakly initial* object is any object that has at least one arrow to every other object of that category. If a category C of BBK-models of a certain signature $\mathcal{L}$ contains a weakly initial object $\mathbf{M}$, then $\mathbf{M}$ must be a model of $\mathrm{Max}\,\mathrm{Th}\,C$ (where $\mathrm{Th}\,C$ is the theory of C), since the existence of a homomorphism $h : \mathbf{M} \to \mathbf{N}$ for BBK-models $\mathbf{M}$ and $\mathbf{N}$ means that $\Diamond P$ holds in $\mathbf{M}$ whenever P is closed and P holds in $\mathbf{N}$. Thus in particular, Max C would have to hold in any weakly initial object in the category of all BBK-models of C. But models of Max C don't *have* to be weakly initial in this category, so the consistency of Max C doesn't immediately imply the existence of such a weakly initial object. But using Theorem 3.10, we can show that there is such an object (at least when $\mathcal{L}$ is countable).

The key for this result is a version of the downward Löwenheim-Skolem theorem for BBK-models: for every BBK-model $\mathbf{M}$ for a countable signature $\mathcal{L}$, there is a BBK-model $\mathbf{M}^{\downarrow}$ for $\mathcal{L}$ in which all the domains are countable such that there is a truth-preserving homomorphism $h : \mathbf{M}^{\downarrow} \to \mathbf{M}$. ("Truth-preserving" in the sense that whenever $\mathbf{M}^{\downarrow}, g \Vdash P$, $\mathbf{M}, h \circ g \Vdash P$.) The proof of this uses a similar technique to Theorem 3.1 (the completeness theorem for BBK-models). Starting with our BBK-model $\mathbf{M}$ for $\mathcal{L}$, we can extend $\mathcal{L}$ to a larger (but still countable) language $\mathcal{L}^+$, and simultaneously extend $\mathbf{M}$ to a model $\mathbf{M}^+$ for $\mathcal{L}^+$ in such a way that whenever a sentence $\exists F$ of $\mathcal{L}^+$ is true in $\mathbf{M}^+$, FA is also true for some closed term A of $\mathcal{L}^+$. We can then make a new model $\mathbf{M}^-$ for $\mathcal{L}$ by throwing away all the elements of the domains of $\mathbf{M}^+$ that are not denoted by any closed term of $\mathcal{L}^+$. The identity function on each domain is a homomorphism from $\mathbf{M}^-$ to $\mathbf{M}$. And since $\mathcal{L}^+$ still only has countably many terms in each type, $\mathbf{M}^-$ is countable in every type.

We can replace all the elements of the domains of $\mathbf{M}^-$ with natural numbers in some arbitrary way to get a homomorphism to $\mathbf{M}$ from a BBKℕ-model (a BBK-model where all domains are subsets of ℕ). So for each BBK-model $\mathbf{M}$ of some theory T (in a countable signature), there is a BBKℕ model of T, $\mathbf{M}^{\downarrow}$, and an injective homomorphism $h : \mathbf{N} \to \mathbf{M}$. We also know that when T includes Classicism, each BBK-model (and thus each BBKℕ model) of T is (BBK-)isomorphic to an action model. So pick a set K big enough to index the BBKℕ models, and choose for each BBK$\mathbb{N}$ model $\mathbf{N}_k$ a corresponding action model $\mathbf{A}_k$, in such a way that the base categories of any two of these action models are disjoint. We can thus apply Theorem 3.10 to show that there exists a coalesced sum $\mathbf{A}$ of all these action models. We can consider this $\mathbf{A}$ as a BBK-model: it has a homomorphism to every BBK$\mathbb{N}$ model, since it has a homomorphism to each of its truncations and each BBK$\mathbb{N}$-model is isomorphic to one of its truncations. But every BBK-model of C has a homomorphism from some BBK$\mathbb{N}$-model; composing this with the homomorphism to that model from $\mathbf{A}$, we can deduce that $\mathbf{A}$ (considered as a BBK-model) is weakly initial in the category of BBK-models of T. In particualr, there is a weakly initial object in category of BBK-models of Classicism for any countable signature.

The existence of a weakly initial object in the category of BBK-models of T (e.g. Classicism for a given signature) leaves several questions open:

(iii) Is there an *initial* object in the category of all BBK-models of T—i.e. an object with *exactly one* homomorphism into each BBK-model of T?

(iv) Is there an object in the category of all BBK-models of T with an *injective* homomorphism into every weakly initial object in the category?

If the answer to the first question is ‘yes’, the answer to the second question must also be ‘yes’, since if $\mathbf{M}$ is initial and $\mathbf{N}$ is weakly initial, there must be a homomorphism $h : \mathbf{M} \to \mathbf{N}$ and a homomorphism $i : \mathbf{N} \to \mathbf{N}$, and moreover since $1_{\mathbf{M}}$ is the only homomorphism from $\mathbf{M} \to \mathbf{M}$, we must have $i \circ h = 1_{\mathbf{M}}$ which implies that h is injective. If the answer to the second question is ‘yes’, then then the “Strong Possibility” schema for T—whose instances are $\Diamond_{\neq} P$ for all closed P consistent with T—is consistent, since the existence of an injective homomorphism from $\mathbf{M}$ to $\mathbf{N}$ means that whenever P is closed and true in $\mathbf{N}$, $\Diamond_{\neq} P$ is true in $\mathbf{M}$. Unfortunately, the models we construct in our proof of Theorem 3.10 are generally very large; their homomorphisms to their truncations are very far from being injective. So establishing a positive answer to either of the above questions would, at least, require a fairly extensive modification of the model-construction technique used in the proof of Theorem 3.10.

We hope that action models will be a useful tool for the investigation of these and many other open questions concerning the space of consistent extensions of Classicism.

A Closure of Classicism under Equiv+

This appendix will show that the theory that results from adding the Boolean and Classicist Identities to H is closed under Equiv+. Since any H-theory closed under Equiv+ must contain every instance of Logical Equivalence, and the Boolean and Classicist Identities are all $\beta\eta$-equivalent to instances of Logical Equivalence, it follows that Classicism can be characterised either as the smallest H-theory closed under Equiv+, the smallest H-theory containing every instance of Logical Equivalence, or the smallest H-theory containing the Boolean and Classicist Identities.

In what follows, $\vdash$ denotes provability from H + the Boolean and Classicist Identities.

Lemma A.1. $\vdash \Box\forall x(\top)$

Proof.

(1) $\vdash (\lambda X.\forall X \lor p) = (\lambda X.\forall y(Xy \lor p))$ (Absorption-∀∨)
(2) $\vdash (\lambda X.\forall X \lor p)X\top = (\lambda X.\forall y(Xy \lor p))X\top$ 1,LL
(3) $\vdash \forall X \lor \top = \forall y(Xy \lor \top)$ 2, $\beta\eta$
$\vdash \top = \forall y(\top)$ (3, Booleanism) □

Lemma A.2. For any formula P and sequence of variables $\vec{v}$, if $\vdash P$ then $\vdash (\lambda\vec{v}.P) = (\lambda\vec{v}.\top)$.

Proof. By induction ("on the length of proofs"). *Base cases:*

(i) P is an instance of PC. Then $(\lambda\vec{v}.P) = (\lambda\vec{v}.\top)$ follows from the Boolean identities.

(ii) P is an instance $\forall F \to Fa$ of UI. Then:

(1) $\vdash (\lambda Xy.Xy) = (\lambda Xy.Xy \lor \forall X)$ (Absorption-∨∀)
(2) $\vdash (\lambda\vec{v}.\forall F \to (\lambda Xy.Xy)Fa) = (\lambda\vec{v}.\forall F \to (\lambda Xy.Xy \lor \forall X)Fa)$ 1, LL
(3) $\vdash (\lambda\vec{v}.\forall F \to FA) = (\lambda\vec{v}.\forall F \to (FA \lor \forall F))$ 2, β
(4) $\vdash (\lambda\vec{v}.\forall F \to FA = \lambda\vec{v}.\top)$ (3, Booleanism)

(iii) P is an instance $Fa \to \exists F$ of EG. Similar to (ii), using Absorption-∧∃

(iv) P is an instance $a = a$ of Ref. Then:

(1) $\vdash (\lambda yz.y = z) = (\lambda yz.\forall X.Xy \leftrightarrow Xz)$ Identity Identity
(2) $\vdash (\lambda\vec{v}.a = a) = (\lambda\vec{v}.(\lambda yz.y = z)a)$ β
(3) $\vdash (\lambda\vec{v}.a = a) = (\lambda\vec{v}.(\lambda yz.\forall X.Xy \leftrightarrow Xz)a)$ 1, 2, LL
(4) $\vdash (\lambda\vec{v}.a = a) = (\lambda\vec{v}.\forall X.Xa \leftrightarrow Xa)$ 3, β
(5) $\vdash (\lambda\vec{v}.a = a) = (\lambda\vec{v}.\forall X.\top)$ 4, Booleanism
(6) $\vdash (\lambda\vec{v}.a = a) = (\lambda\vec{v}.\top)$ 5, Lemma A.1

(vi) P is an instance $a = b \to Fa \to Fb$ of LL. Similar to (iv).

(vii) P is an instance $P' \leftrightarrow Q$ of β or η. Then $(\lambda\vec{v}.P' \leftrightarrow Q) = (\lambda\vec{v}.Q \leftrightarrow Q)$ is also an instance of β or η (respectively). But by Booleanism $(\lambda\vec{v}.Q \leftrightarrow Q) = (\lambda\vec{v}.\top)$; so by LL, $\vdash (\lambda\vec{v}.P' \leftrightarrow Q) = (\lambda\vec{v}.\top)$.

(viii) P is a closed identity $A = B$ that is one of the Boolean or Classicist Identities. Then $\vdash (\lambda\vec{v}.A = B) = (\lambda\vec{v}.A = A)$ by the relevant identity, Ref, and LL, so $\vdash (\lambda\vec{v}.A = B) = (\lambda\vec{v}.\top)$ by Ref and LL.

Inductive steps:

(i) P follows by MP from some Q and $Q \to P$. By IH, $\vdash (\lambda\vec{v}.Q) = (\lambda\vec{v}.\top)$ and $\vdash (\lambda\vec{v}.Q \to P) = (\lambda\vec{v}.\top)$. Then we can appeal to the Boolean identities to derive that $(\lambda\vec{v}.P) = (\lambda\vec{v}.\top)$:

$$\begin{aligned} \lambda\vec{v}.P &= \lambda\vec{v}.P \vee (Q \wedge (Q \to P)) \\ &= \lambda\vec{v}.P \vee ((\lambda\vec{v}.Q)\vec{v} \wedge (\lambda\vec{v}.Q \to P)\vec{v}) \\ &= \lambda\vec{v}.P \vee ((\lambda\vec{v}.\top)\vec{v} \wedge (\lambda\vec{v}.\top)\vec{v}) \\ &= \lambda\vec{v}.P \vee (\top \wedge \top) \\ &= \lambda\vec{v}.\top \end{aligned}$$

(ii) P is of the form $P' \to \forall u.Q$ and follows by Gen from some previously proved $P' \to Q$. By the induction hypothesis, $\vdash (\lambda\vec{v}u.P' \to Q) = (\lambda\vec{v}u.\top)$. So we have

(1)
$(\lambda Xp.\forall X \vee p) = (\lambda Xp.\forall u.Xu \vee p)$ (Dist-$\vee\forall$)

(2)
$(\lambda\vec{v}.(\lambda Xp.\forall X \vee p)(\lambda u.Q)(\neg P')) = (\lambda\vec{v}.(\lambda Xp.\forall u.Xu \vee p)(\lambda u.Q)(\neg P'))$ 1, Ref, LL

(3)
$(\lambda\vec{v}.\lambda Xp.(\forall u.Q) \vee \neg P') = (\lambda\vec{v}.\lambda Xp.\forall u.(\lambda\vec{v}u.Q \vee \neg P')\vec{v}u$ 2, β

(4)
$(\lambda\vec{v}.P' \to \forall u.Q) = (\lambda\vec{v}.\forall u.(\lambda\vec{v}u.P' \to Q)\vec{v}u)$ (3, def. $\to$)

(5)
$(\lambda\vec{v}.P' \to \forall u.Q) = (\lambda\vec{v}.\forall u.(\lambda\vec{v}u.\top)\vec{v}u)$ 4, IH

(6)
$(\lambda\vec{v}.P' \to \forall u.Q) = (\lambda\vec{v}.\forall u.\top)$ 5, β

(7)
$(\lambda\vec{v}.P' \to \forall u.Q) = (\lambda\vec{v}.\top)$ (6, Lemma A.1)

(iii) P is of the form $(\exists v.P') \to Q$ and follows from some previously proved $P' \to Q$ by Inst. Similar to (ii) using Dist-$\wedge\exists$. □

Monotonicity-∀	$\lambda X^{\sigma\to t}Y^{\sigma\to t}.\forall_\sigma X \leq \lambda XY.\forall_\sigma(X \vee_{\sigma\to t} Y)$
Instantiation	$\lambda X^{\sigma\to t}y^{\sigma}.\forall_\sigma X \leq \lambda X.X$
Vac-∀	$\lambda p.p \leq \lambda p.\forall x^{\sigma}.p$
Monotonicity-∃	$\lambda X^{\sigma\to t}Y^{\sigma\to t}.\exists_\sigma(X \wedge_{\sigma\to t} Y) \leq \lambda XY.\exists_\sigma X$
Generalization	$\lambda X^{\sigma\to t}.X \leq \lambda Xy^{\sigma}.\exists_\sigma X$
Vac-∃	$\lambda p.\exists x^{\sigma}.p \leq \lambda p.p$

Figure 7. The Adjunctive Entailments

Proposition A.3. Classicism is closed under Equiv+.

Proof. Suppose $\vdash A \leftrightarrow B$; then $\vdash (\lambda\vec{v}.A \leftrightarrow B) = (\lambda\vec{v}.\top)$ by the previous lemma. Then:

(1)
$\vdash (\lambda\vec{v}.A) = (\lambda\vec{v}.(B \wedge (A \leftrightarrow B)) \vee (\neg B \wedge \neg(A \leftrightarrow B)))$ Booleanism

(2)
$\vdash (\lambda\vec{v}.A) = (\lambda\vec{v}.B \wedge (\lambda\vec{v}.A \leftrightarrow B)\vec{v} \vee (\neg B \wedge \neg(\lambda\vec{v}.A \leftrightarrow B)\vec{v}))$ $\beta\eta$

(3)
$\vdash (\lambda\vec{v}.A) = (\lambda\vec{v}.B \wedge (\lambda\vec{v}.\top)\vec{v} \vee (\neg B \wedge \neg(\lambda\vec{v}.\top)\vec{v}))$ Lemma A.2

(4)
$(\lambda\vec{v}.A) = (\lambda\vec{v}.B \wedge \top \vee (\neg B \wedge \neg\top))$ $\beta\eta$
$(\lambda\vec{v}.A) = (\lambda\vec{v}.B)$ Booleanism $\square$

B An axiomatization in terms of entailment

This appendix will discuss a couple of other axiomatizations of Classicism which give a central role to the entailment relations $\leq_\tau$. Recall (from Figure 1) that $\leq_\tau$ is short for $\lambda XY.Y = Y \vee_\tau X$. Booleanism proves that $\leq$ is reflexive, transitive, and antisymmetric in each type, and that it is identical to $\lambda XY.X = X \wedge_\tau Y$. The first axiomatization we'll discuss is given by adding the schemas in Figure 7, along with the Identity Identity, to an axiomatization of Booleanism. Deriving these from the Classicist Identities (Figure 4) is straightforward. Note that ∀-Instantiation and ∃-Generalization just rewrite Absorption-∨∀ and Absorption-∧∃ using $\leq$. Vac-∀

and Vac-$\exists$, meanwhile, can be derived from Dist-$\vee\forall$ and Dist-$\wedge\exists$ by instantiating X with $(\lambda x.\bot)$ and $(\lambda x.\top)$, respectively.

To preserve the duality of the axioms, we stated Monotonicity axioms using $\vee$ for $\forall$ and $\wedge$ for $\exists$, but we could just as well have used the same connective in both cases. The axioms imply that if $X \leq_{\sigma\to t} Y$ (i.e. $Y = X \vee_{\sigma\to t} Y$), then $\forall_\sigma X \leq \forall_\sigma Y$ and $\exists_\sigma X \leq \exists_\sigma Y$. To see what's going on with the remaining Adjunctive Entailments, we can suggestively rewrite (β-equivalents of) them using the following abbreviations:

$$\begin{aligned} I_\tau &\coloneqq \lambda x^\tau.x \\ K_{\sigma,\tau} &\coloneqq \lambda x^\sigma y^\tau.x \\ A\circ B &\coloneqq \lambda x.A(Bx) \end{aligned}$$

The relevant four Adjunctive Entailments can now be rewritten as follows:

Instantiation	$K_{t,\sigma}\circ\forall_\sigma \leq I_{\sigma\to t}$
Vac-$\forall$	$I_t \leq \forall_\sigma\circ K_{t,\sigma}$
Generalization	$I_{\sigma\to t} \leq K_{t,\sigma}\circ\exists_\sigma$
Vac-$\exists$	$\exists_\sigma\circ K_{t,\sigma} \leq I_t$

In the theory of partial orders, when we have two partially ordered sets, $(X, \leq_1)$ and $(Y, \leq_2)$, a function $f : X \to Y$ is *monotonic* just in case whenever $x \leq_1 x'$, $f(x) \leq_2 f(x')$. When $f : X \to Y$ and $g : Y \to X$, we say that f is a *right adjoint* of g, and g a *left adjoint* of f, just in case both are monotonic and:

(i) $\quad x \leq_1 g(f(x)) \quad$ for every $x \in X$

(ii) $\quad f(g(y)) \leq_2 y \quad$ for every $y \in Y$

Using common notational shorthands, (i) and (ii) can be rewritten respectively as $1_X \leq_1 g\circ f$ and $f\circ g \leq_2 1_Y$, mirroring the pair of Vac-$\forall$ and Instantiation, or Generalization and Vac-$\exists$..[83] These axioms can thus be summed up by saying that universal and existential quantifiers in type $(\sigma\to t)\to t$ are respectively a right-adjoint and a left-adjoint of the K combinator in type $t \to (\sigma \to t)$. H already implies that the K combinator is monotonic: if $p \leq q$, then $q = p \vee q$, so $\lambda x.q = \lambda x.p \vee q = (\lambda x.p) \vee_{\sigma\to t} (\lambda x.q)$, i.e. $K_{\sigma,t}p \leq K_{\sigma,t}q$.[84]

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[83] Here 1_Z stands for the identity function on the set Z, and where h and h' are functions from some set Z to a partial order, $h \leq h'$ means that $h(z) \leq h'(z)$ for all $z \in Z$.

[84] More generally, for any type τ (ending in t), the above axioms entail that the "lifted" quantifiers $\forall_{\sigma,\tau}$ and $\exists_{\sigma,\tau}$ (see Figure 1) are respectively right and left adjoints of $K_{\tau,\sigma}$. By contrast, if we only had the quantified versions of the axioms—e.g. $\forall p.(p \leq \forall x.p)$ instead of $(\lambda p.p) \leq (\lambda p.\forall x.p)$—we would not be able to derive this generalization (unless we rely on some new axioms or rules

Our definition of 'f is a right adjoint of g' is easily seen to be equivalent to the following: for every $x \in X$ and $y \in Y$, $x \leq_1 g(y)$ iff $f(x) \leq_2 y$.[85] This biconditional definition of adjointness suggests yet another axiomatization of Classicism, which adds the following biconditionals to Booleanism (together with the Identity Identity):

Adjunction-∀ $((\lambda v_0\vec{v}.Q) \leq (\lambda v_0\vec{v}.P)) \leftrightarrow ((\lambda\vec{v}.Q) \leq (\lambda\vec{v}.\forall v_0.P))$

Adjunction-∃ $((\lambda v_0\vec{v}.P) \leq (\lambda v_0\vec{v}.Q)) \leftrightarrow ((\lambda\vec{v}.\forall v_0.P) \leq (\lambda\vec{v}.Q))$

where in each case v_0 is not free in Q.[86] Dorr (2014) shows how these principles can be regarded as capturing the "validity" of the standard natural deduction quantifier rules for ∀ and ∃, in a certain natural sense of "validity" on which linguistic facts about validity boil down to nonlinguistic facts about entailment.

To derive the left-to-right direction of **Adjunction-∀** from the Adjunctive Entailments, note that we have $(\lambda\vec{v}.Q) \leq (\lambda\vec{v}.\forall v_0.Q)$ by Vac-∀, which given the left-hand side, Monotonicity-∀, and the transitivity of entailment gives $(\lambda\vec{v}.Q) \leq (\lambda\vec{v}.\forall v_0.P)$. To derive the right-to-left direction of **Adjunction-∀**, note that we have $((\lambda v_0\vec{v}.\forall v_0.P) \leq (\lambda v_0.\vec{v}.P))$ by Instantiation, which given the right-hand-side, Monotonicity-∀, and the transitivity of entailment gives $((\lambda v_0\vec{v}.Q) \leq (\lambda v_0\vec{v}.P))$. In the other direction, the Adjunctive Entailments for ∀ follow from the following three instances of **Adjunction-∀**:

(i) $((\lambda yX.\forall y.Xy) \leq (\lambda yX.Xy)) \leftrightarrow ((\lambda X.\forall y.Xy) \leq (\lambda X.\forall y.Xy))$

(ii) $((\lambda xp.p) \leq (\lambda xp.p)) \leftrightarrow ((\lambda p.p) \leq (\lambda p.\forall x.p))$

(iii)
$((\lambda zXY.\forall z.Xz) \leq (\lambda zXY.Xz \vee Yz)) \leftrightarrow ((\lambda XY.\forall z.Xz) \leq (\lambda XY.\forall z.Xz \vee Yz))$

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to be discussed below). This situation, where failures of functionality motivate replacing quantified identities involving functions with identities between functions, instantiates a pattern that is pervasive in category theory. In this context, standard set theoretic definitions involving functions understood set theoretically can be turned into definitions that make sense in some more general category, by first formulating them in a way that doesn't directly involve quantifying over members of the set, and consequently doesn't make any strong functionality assumptions.

[85] If $x \leq_1 g(y)$, then $f(x) \leq_2 f(g(y))$ by the monotonicity of f, so $f(x) \leq_2 y$ by (ii) and the transitivity of $\leq_2$; if $f(x) \leq_2 y$, then $g(f(x)) \leq_1 g(y)$ by the monotonicity of g, so $x \leq_1 g(y)$ by (i) and the transitivity of $\leq_1$. In the other direction, (i) and (ii) follow immediately from the reflexivity of $\leq_1$ and $\leq_2$ respectively, while the monotonicity of f and g follows from their transitivity.

[86] Dorr (2016, note 59) states these bicondtionals (using the definition of entailment in terms of identity), describing them rather inscrutably, as the 'natural analogues of Booleanism for the quantifiers'. The principle 'Adjunction' in Goodman 2016 is strictly weaker: it is equivalent to the special case of Adjunction-∀ where $\vec{v}$ is empty.

The right side of (i) and the left side of (ii) are both consequences of the reflexivity of $\leq$, and their other sides are $\beta\eta$-equivalent to Instantiation and Vac-$\forall$ respectively. Meanwhile, the left side of (iii) follows from Instantiation given Booleanism, and its right side is $\beta\eta$-equivalent to Monotonicity-$\forall$.

C Soundness and completeness of action models for Classicism

This appendix will prove the following two results stated in §3.4, which together with Theorem 3.6 imply the soundness and completeness of action models for Classicism.

Proposition 3.7. Any action model can be turned into a BBK-model which makes the same formulae true, and in which every instance of Logical Equivalence is true.

Proposition 3.8. For every small, intensional category of BBK-models $\mathcal{C}$, and model $\mathbf{M}_0$ in $\mathcal{C}$ there is an action model $\mathbf{A}^{\mathcal{C}}_{\mathbf{M}}$ in which the same formulae hold.

For Proposition 3.7, suppose $\mathbf{A}$ is an action model with base object W_0 and $h : W_0 \to V$. Then we will construct BBK-model $\mathbf{M}^h_{\mathbf{A}}$ by setting each type-σ domain to be V^σ for each type σ, $\llbracket A \rrbracket^g = \llbracket A \rrbracket^g_{\mathbf{A},h}$ for every term, and $\operatorname{val}\mathbf{p} = 1$ iff $1_V \in \mathbf{p}$. Looking at the definitions of BBK-model and action model, it is obvious that $\mathbf{M}^h_{\mathbf{A}}$ obeys all the conditions to be a BBK-model apart from condition (ii.d) (that $\llbracket A \rrbracket^g = \llbracket B \rrbracket^g$ when A and B are $\beta\eta$-equivalent). To establish this, we must first show that our interpretation functions are well-behaved in a few other ways.

To begin with, we need the following fundamental fact about the interpretation functions:

Proposition C.1. In any action premodel, when $h : W_0 \to W_1$ and $i : W_1 \to W_2$, g is an assignment for W_1, and A is a type-σ term such that $\llbracket A \rrbracket^g_h$ is defined,

$$\llbracket A \rrbracket^{i \circ g}_{i \circ h} = i^\sigma \llbracket A \rrbracket^g_h$$

Proof. By induction on the complexity of terms. It is immediate for variables and nonlogical constants. For the logical constants, the claim follows from the fact that they do not care about the first co-ordinate of their argument (the arrow). For example, $i^{t\to t\to t}\llbracket \wedge \rrbracket^g_h = i^{t\to t\to t}((l, \mathbf{p}) \mapsto (j, \mathbf{q}) \mapsto j^t(\mathbf{p}) \cup \mathbf{q}) = (m, \mathbf{p}) \mapsto (j, \mathbf{q}) \mapsto j^t(\mathbf{p}) \cup \mathbf{q} = \llbracket \wedge \rrbracket^{i \circ g}_{i \circ h}$. For an abstraction $\lambda v.A$ where v is of type σ and A of type τ, $\llbracket \lambda v.A \rrbracket^{i \circ g}_{i \circ h}\langle j, \mathbf{a} \rangle = \llbracket A \rrbracket^{(j \circ i \circ g)[v \mapsto \mathbf{a}]}_{j \circ i \circ h} = \llbracket \lambda v.A \rrbracket^g_h \langle j \circ i, \mathbf{a} \rangle = (i^{\sigma\to\tau}\llbracket \lambda v.A \rrbracket^g_h)\langle j, \mathbf{a} \rangle$ (since $i^{\sigma\to\tau}$ acts by division). For an application AB where A is of type σ and B of type τ, we have $\llbracket AB \rrbracket^{i \circ g}_{i \circ h} = \llbracket A \rrbracket^{i \circ g}_{i \circ h}\langle 1_{W_2}, \llbracket B \rrbracket^{i \circ g}_{i \circ h} \rangle = (i^{\sigma\to\tau}\llbracket A \rrbracket^g_h)\langle 1_{W_2}, i^\sigma \llbracket B \rrbracket^g_h \rangle$ (by the induction hypothesis) $= \llbracket A\rrbracket^g_h\langle i, i^\sigma\llbracket B\rrbracket^g_h\rangle$ (since $i^{\sigma\to\tau}$ acts by division) $= i^\tau(\llbracket A\rrbracket^g_h\langle 1_{W_1}, \llbracket B\rrbracket^g_h\rangle)$ (since $\llbracket A\rrbracket^g_h \in W_1^{\sigma\Rightarrow\tau}$) $= i^\tau\llbracket AB\rrbracket^g_h$. ☐

Proposition C.2. Suppose that in an action model $\mathbf{A}$, $\llbracket A\rrbracket^g_h = \llbracket B\rrbracket^g_h$ whenever either side is defined. Then $\llbracket \Phi[A]\rrbracket^g_h = \llbracket \Phi[B]\rrbracket^g_h$ whenever $\Phi[A]$ and $\Phi[B]$ are terms that differ only by the replacement of an occurrence of A for one of B.

Proof. By induction on the construction of $\Phi[A]$ from A. ☐

Proposition C.3. $\llbracket A[B/v]\rrbracket^g_h = \llbracket A\rrbracket_h^{g[v\mapsto\llbracket B\rrbracket^g_h]}$ whenever both sides are defined.

Proof. By induction on the complexity of terms. ☐

Proposition C.4. $\llbracket(\lambda v.A)B\rrbracket^g_h = \llbracket A[B/v]\rrbracket^g_h$.

Proof. When $h : W_0 \to V$, $\llbracket(\lambda v.A)B\rrbracket^g_h = \llbracket\lambda v.A\rrbracket^g_h(1_V, \llbracket B\rrbracket^g_h) = \llbracket A\rrbracket_h^{g[v\mapsto\llbracket B\rrbracket^g_h]} = \llbracket A[B/v]\rrbracket^g_h$ by Proposition C.3. ☐

Proposition C.5. $\llbracket\lambda v.Av\rrbracket^g_h = \llbracket A\rrbracket^g_h$ when v is not free in A.

Proof.

$$\begin{aligned}
\llbracket\lambda v.Av\rrbracket^g_h &= \langle i,\mathbf{a}\rangle \mapsto \llbracket Av\rrbracket_{i\circ h}^{(i\circ g)[v\mapsto\mathbf{a}]} && \text{by the clause for abstraction}\\
&= \langle i,\mathbf{a}\rangle \mapsto \llbracket A\rrbracket_{i\circ h}^{(i\circ g)[v\mapsto\mathbf{a}]}\langle 1_{\operatorname{trg} i}, \llbracket v\rrbracket_{i\circ h}^{(i\circ g)[v\mapsto\mathbf{a}]}\rangle && \text{by the clause for application}\\
&= \langle i,\mathbf{a}\rangle \mapsto \llbracket A\rrbracket_{i\circ h}^{i\circ g}\langle 1_{\operatorname{trg} i}, \mathbf{a}\rangle && \text{since } v \text{ isn't free in } A\\
&= \langle i,\mathbf{a}\rangle \mapsto (i^{\sigma\to\tau}\llbracket A\rrbracket^g_h)\langle 1_{\operatorname{trg} i}, \mathbf{a}\rangle && \text{by Proposition C.1}\\
&= \langle i,\mathbf{a}\rangle \mapsto \llbracket A\rrbracket^g_h\langle i,\mathbf{a}\rangle && \text{by the definition of } i^{\sigma\to\tau}
\end{aligned}$$

☐

Proposition C.6. When A and B are $\beta\eta$-equivalent, $\llbracket A\rrbracket^g_h = \llbracket B\rrbracket^g_h$ whenever g is adequate for both.

Proof. By Proposition C.4, Proposition C.5, and Proposition C.2. ☐

This implies that $\mathbf{M}^h_{\mathbf{A}}$ is always a BBK-model, in which a formula P holds on an assignment g iff $\mathbf{A}, h, g \Vdash P$. We can also show that every instance of Logical Equivalence holds in every action model (relative to every h, g):

Proposition C.7. If $\mathsf{H} \vdash P \leftrightarrow Q$ then $\mathbf{A}, h, g \Vdash (\lambda\vec{v}.P) = (\lambda\vec{v}.Q)$ for all h, g, $\vec{v}$.

Proof. Suppose $\mathsf{H} \vdash P \leftrightarrow Q$. Then by what we have just proved, for any h from $\mathbf{A}$'s base object W_0 to some V, $\mathbf{A}, h, g \Vdash P$ iff $\mathbf{A}, h, g \Vdash Q$. Hence $[\![P]\!]^g_h = [\![Q]\!]^g_h$ for all h and g, and so by Proposition C.2, $[\![\lambda\vec{v}.P]\!]^g_h = [\![\lambda\vec{v}.Q]\!]^g_h$ for all h and g, so $\mathbf{A}, h, g \Vdash (\lambda\vec{v}.P) = (\lambda\vec{v}.Q)$ for all h and g. □

It follows that class of all action models is sound for Classicism.

Turning to Proposition 3.8, suppose C is a small, intensional category of BBK-models, and $\mathbf{M}_0$ is an object in it. We'll use these construct an action model $\mathbf{A}^C_{\mathbf{M}_0}$ and show that the same formulae hold in it. First we truncate C by throwing away any models with no homomorphism from $\mathbf{M}_0$. Then we choose $\mathbf{A}^C_{\mathbf{M}_0}$'s base category to be C and its base object to be $\mathbf{M}_0$. For each $\mathbf{M}$ in C and each type σ, we let $\mathbf{M}$'s type-σ domain (considered now as an object in $\mathbf{A}^C_{\mathbf{M}_0}$) be the range of the function $f^\sigma_{\mathbf{M}}$ whose domain is $\mathbf{M}^\sigma$ ($\mathbf{M}$'s built-in type-σ domain), defined recursively as follows.

$$\begin{aligned} f^e_{\mathbf{M}}\mathbf{a} &= \mathbf{a} \\ f^t_{\mathbf{M}}\mathbf{p} &= \mathrm{val}^C_{\mathbf{M}}\,\mathbf{p} \\ f^{\sigma\to\tau}_{\mathbf{M}}\mathbf{d} &= \langle h, \mathbf{a}\rangle \mapsto f^\tau_{\mathrm{trg}\,h}(@^C_{\mathbf{M}}\mathbf{d}\langle h, {f^\sigma_{\mathrm{trg}\,h}}^{-1}\mathbf{a}\rangle) \end{aligned}$$

To establish the legitimacy of this definition, we must simultaneously prove that each $f^\sigma_{\mathbf{M}}$ is injective. The first clause automatically secure this for type e; the quasi-Fregeanness of C secures it for type t, and the quasi-functionality of C guarantees it for types $\sigma \to \tau$.

$\mathbf{A}^C_{\mathbf{M_0}}$ so defined is automatically an action premodel. Moreover, we can show by induction on the complexity of formulae that for any term A of type σ, any $h : \mathbf{M}_0 \to \mathbf{M}$, and any assignment function g for $\mathbf{M}$ adequate for A,

$$[\![A]\!]^{f_{\mathbf{M}}\circ g}_h = f^\sigma_{\mathbf{M}}[\![A]\!]^g_{\mathbf{M}}$$

Since each $f^\sigma_{\mathbf{M}}$ is a bijection, every assignment function g for $\mathbf{M}$ considered as an object of $\mathbf{A}^C_{\mathbf{M}_0}$ is identical to $f_{\mathbf{M}}\circ f^{-1}_{\mathbf{M}}\circ g$. So as a consequence of the previous equation, we have

$$[\![A]\!]^g_h = f^\sigma_{\mathbf{M}}[\![A]\!]^{f^{-1}_{\mathbf{M}}\circ g}_{\mathbf{M}}$$

It follows that $[\![A]\!]^g_h$ is always well defined, i.e. that $\mathbf{A}^C_{\mathbf{M}_0}$ is an action model. Moreover, the mapping preserves truth value: for any formula P and assignment g for $\mathbf{M}_0$ adequate for P, $\mathrm{val}_{\mathbf{M}_0}\,\mathbf{p} = 1$ iff $1_{\mathbf{M}_0} \in f^t_{\mathbf{M}_0}\mathbf{p}$, hence

$$\mathrm{val}_{\mathbf{M}_0}[\![P]\!]^g_{\mathbf{M_0}} = 1 1_{\mathbf{M}_0} \in [\![P]\!]^{f_{\mathbf{M}_0}\circ g}_{1_{\mathbf{M}_0}}$$

or in other words,

$$\mathbf{M}_0, g \Vdash P \text{ iff } \mathbf{A}^{\mathcal{C}}_{\mathbf{M}_0}, 1_{\mathbf{M}_0}, f_{\mathbf{M}_0} \circ g \Vdash P.$$

Since $f_{\mathbf{M}_0}$ is bijective, it follows that P holds in $\mathbf{M}$ iff it holds in $\mathbf{A}^{\mathcal{C}}_{\mathbf{M}_0}$.

One noteworthy feature of this construction is that since any two homomorphisms from one BBK-model to another must agree on the interpretations of all nonlogical constants, the generated action models $\mathbf{A}^{\mathcal{C}}_{\mathbf{M}_0}$ will always obey the following condition of ***non-logical harmony***: $h(\mathcal{I}c) = i(\mathcal{I}c)$ whenever $h, i : W_0 \to V$ and c is a nonlogical constant. In a non-logically harmonious model, all that matters about an arrow as far the interpretation function is concerned is its target: i.e. when $h, i : W_0 \to V$, $[\![A]\!]^g_h = [\![A]\!]^g_i$ when defined.[87] Our proof of Proposition 3.8 shows that Classicism is also complete for non-logically harmonious action models. But we find the more general notion more intuitive, in that any model for in which $\exists x.Fx$ is true can be extended into a model for a larger signature in which Fc is true for some new constant, and also more useful, in that it allows one to construct smaller models of certain theories involving nonlogical constants.

D Consistency results using non-full action models

This appendix will introduce a technique for defining certain non-full action models, and use it to verify the consistency of certain packages of principles from §2.

We start with the following definition.

Definition. Suppose $-^*$ and $-^\dagger$ are two actions on a category $\mathcal{C}$, A is an object of $\mathcal{C}$, $X \subseteq A^\dagger$ and $y \in A^*$. Then y is ***pinned down*** by X iff for any object B and arrows $h, i : A \to B$, if $h^\dagger x = i^\dagger x$ for all $x \in X$, then $h^* y = i^* y$.

Definition. Suppose $-^*$ and $-^\dagger$ are actions on $\mathcal{C}$ and $-^\ddagger$ is a subaction of the "power action" on $-^\dagger$ (i.e., $A^\ddagger \subseteq A^\dagger$ for each object and $h^\ddagger X = \{h^\dagger y \mid y \in X\}$). Then $y \in A^*$ is ***pinned down*** by $-^\ddagger$ iff y is pinned down by some $X \in A^\ddagger$. And $-^*$ is pinned down by $-^\ddagger$ iff for every A, every $x \in A^*$ is pinned down by $-^\ddagger$.

Definition. When $\mathbf{A} = \langle \mathcal{C}, W_0, -^{\cdot}, \mathcal{I} \rangle$ is an action premodel, $-^\dagger$ is an action on $\mathcal{C}$ and $-^\ddagger$ is a subaction of its power action, $\mathbf{A}$ is $-^\ddagger$***-full*** iff

(i) $-^e$ is any action pinned down by $-^\ddagger$

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[87] This is shown by a straightforward induction on the complexity of terms: the logical constants all have this property, since they denote functions whose value on a given pair $\langle h, \mathbf{a} \rangle$ does not depend on h.

(ii) For each W, $W^t = \{\mathbf{p} \in W^{\mathcal{P}} \mid \mathbf{p}$ is pinned down by $-^{\ddagger}\}$

(iii) For each W, $W^{\sigma\to\tau} = \{\alpha \in W^{\sigma\Rightarrow\tau} \mid \alpha$ is pinned down by $-^{\ddagger}\}$.

We have the following sufficient condition for $-^{\ddagger}$-full action premodel to be an action *model*:

Proposition D.1. If $\mathbf{A}$ is a $-^{*}$-full action premodel and $-^{\ddagger}$ is closed under finite unions—i.e., $X \cup Y \in A^{\ddagger}$ whenever $X \in A^{\ddagger}$ and $Y^{\ddagger}$—then $\mathbf{A}$ is an action model.

Proof. We use the alternative version of the sufficient fullness condition given by note 74. Since the denotation of each logical constant relative to any $h : W_0 \to V$ is a function α in some $V^{\sigma\to\tau}$ with the property that $\alpha\langle i, \mathbf{a}\rangle = \alpha\langle j, \mathbf{a}\rangle$ whenever $i, j : V \to U$ and $\mathbf{a} \in U^{\sigma}$, $i^{\sigma\to\tau}\alpha = j^{\sigma\to\tau}\alpha$ for any two such parallel i, j, so α is pinned down by $\emptyset \in V^{\ddagger}$. Similarly for the S and K combinators. So all we need to show is that when $\alpha \in W^{\sigma\Rightarrow\tau}$ and $\mathbf{b} \in V^{\sigma}$ are both pinned down by $-^{\ddagger}$ and $h : W \to V$, $\alpha\langle h, \mathbf{b}\rangle$ is also pinned down by $-^{\ddagger}$. Let $X \in W^{\ddagger}$ pin down α and $Y \in V^{\ddagger}$ pin down $\mathbf{b}$. Then we can show that $h^{\ddagger}X \cup Y$, which belongs to $V^{\ddagger}$ since $-^{\ddagger}$ is an ideal action, pins down $\alpha\langle h, \mathbf{b}\rangle$. For suppose $i, j : U \to V$ agree on $h^{\ddagger}X \cup Y$. Then they agree on Y, so $i^{\sigma}\mathbf{b} = j^{\sigma}\mathbf{b}$. And they agree on $h^{\ddagger}X$, which implies that $i \circ h$ and $j \circ h$ agree on X, which implies that $(i \circ h)^{\sigma\to\tau}\alpha = (j \circ h)^{\sigma\to\tau}\alpha$. Hence, $i^{\tau}(\alpha\langle h, \mathbf{b}\rangle) = \alpha\langle i \circ h, i^{\tau}\mathbf{b}\rangle) = ((i \circ h)^{\sigma\to\tau}\alpha)\langle 1_U, i^{\tau}\mathbf{b}\rangle) = ((j \circ h)^{\sigma\to\tau}\alpha)\langle 1_U, j^{\tau}\mathbf{b}\rangle) = \alpha\langle j \circ h, j^{\tau}\mathbf{b}\rangle) = j^{\tau}(\alpha\langle h, \mathbf{b}\rangle)$. □

The payoff of this is that we have a way of building non-full models to verify the consistency of various packages of principles. All we have to specify is the underlying category C, an action $-^{\dagger}$ on C, an subaction $-^{\ddagger}$ of its power action that is closed under finite unions, and an action at type e (making sure that this is pinned down by $-^{\ddagger}$). In what follows, we will often (for the sake of concreteness) choose C to be a subcategory of Set, and $-^{\dagger} = -^{e}$ to be the identity action on C (i.e. $W^{\dagger} = W^{e} = W$ and $h^{\dagger} = h^{e} = h$). Often, we will take $W^{\ddagger}$ to be set of all *finite* subsets of $W^{\dagger}(= W)$.

Let's consider what the action model constructed in this way will look like where C is the permutation group on $\mathbb{N}$: i.e., the category with just one object W_0, namely $\mathbb{N}$, and whose arrows are all the bijections $\mathbb{N} \to \mathbb{N}$.

- $W_0^e = \mathbb{N}$; $h^e = h$ for all $h : \mathbb{N} \to \mathbb{N}$.
- W_0^t is the set of all $\mathbf{p} \subseteq \mathbb{N}^{\mathbb{N}}$ which are pinned down by some finite $X \subseteq \mathbb{N}$. That is: whenever $hx = ix$ for all $x \in X$, $h \in \mathbf{p}$ iff $i \in \mathbf{p}$.

- $W_0^{\sigma\to\tau}$ is the set of all $\alpha \in W_0^{\sigma\Rightarrow\tau}$ which are pinned down by some finite $X \subseteq \mathbb{N}$. That is: whenever $hx = ix$ for all $x \in X$, $\alpha(h, \mathbf{b}) = \alpha(i, \mathbf{b})$ for all $\mathbf{b} \in W_0^\sigma$.

The intuition for this model is that there are infinitely many individuals, all playing distinct qualitative roles, and over which the roles can be redistributed in any way. Each arrow h represents the possibility where each individual n plays the role actually played by $h(n)$. But the only propositions and properties are ones that are "about" some finite collection of objects, and thus indifferent to the question how the qualitative roles are distributed over objects not in that collection.

What sentences hold in this model?

- ND and BF are true: every arrow has an inverse, which implies its action is bijective in each type. $\Box$ND and $\Box$BF are also true—since this is a one-object model, No Pure Contingency holds in it. (See 3.5.)

- Actuality is false. Take any $\mathbf{p} \in W_0^t$ that is true (i.e. contains $1_{\mathbb{N}}$). $\mathbf{p}$ is pinned down by some finite $X \subseteq \mathbb{N}$. Choose $n \notin X$, and let $\mathbf{q} = \{h \in \mathbf{p} : hn = n\}$. $\mathbf{q}$ clearly is pinned down by $X \cup \{n\}$, and thus also belongs to W_0^t. It's also true since it contains 1. $\langle \mathbf{q}, \mathbf{p} \rangle$ is in the extension of $[\![\leq]\!]$, since $\mathbf{q}$ is a subset of $\mathbf{p}$. But also $\mathbf{q} \neq \mathbf{p}$. For suppose $h \in \mathbf{q}$, and let h' be the function that agrees with h except that $h'n = n + 1$. Then $h' \notin \mathbf{q}$ since $h'n \neq n$, but $h' \in \mathbf{p}$ since $h' \sim_X h$. Indeed by similar reasoning, we can show that *Atomlessness*—$\forall p(\Diamond p \to \exists q(q \leq p \wedge q \neq p \wedge \Diamond q)$—is true in the model.

We thus verify that the bottom inclusion in Figure 6 is strict: C5 does not imply Actuality.

To get ND to fail while keeping BF, we can just change the underlying monoid to include all the surjective functions on $\mathbb{N}$. ND will now fail, since some arrows fail to be injective (in type e and hence in every type). BF still holds: since every h has a right inverse i such that $h \circ i = 1_{\mathbb{N}}$, h^σ must be surjective in each type (since anything not in its range couldn't be in the range of $(h \circ i)^\sigma$). Atomlessness is still true for the same reason as before, which implies that Actuality and Atomicity are both false. Moreover, Rigid Comprehension is false. Consider the function $\alpha : \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathcal{P}(\mathbb{N})$ where for any h and n, $\alpha(h, n) = \{i : in \text{ is even}\}$. This belongs to $W_0^{e\Rightarrow t}$. Moreover, it is pinned down by $\emptyset$ (since it doesn't care about its h argument at all), and thus is in $W_0^{e\to t}$. Its extension is the set of even numbers: we can think of it as expressing the disjunction of all the qualitative roles corresponding to even numbers. By contrast, for $\beta \in W_0^{e\to t}$, β is in the extension of $[\![\text{Rigid}]\!]$ iff for some $X \in N$, $\beta(h, n) = \{i \mid in = i(h(m)) \text{ for some } m \in X\}$, in which case X is the extension of β. But when X is infinite, such a β is not pinned down by any finite

X, and thus is not in W_0^t. Thus, no rigid property in the model has the set of even numbers as its extension, making α a counterexample to Rigid Comprehension.

What if we want BF to fail too? We might expect that we could do this by choosing C to be the monoid of *all* functions $\mathbb{N} \to \mathbb{N}$. But BF turns out to hold in this model as well—surprisingly, since intuitively the function that, e.g., maps every number to 0 represents a possibility in which there are lots of new individuals. We first observe that if $h : \mathbb{N} \to \mathbb{N}$ is surjective, it has a right inverse i such that $h \circ i = 1_\mathbb{N}$ so that h^σ must be surjective for every σ. Now note that for any finite $X \subset \mathbb{N}$ and any $h : \mathbb{N} \to \mathbb{N}$, there is a surjective function that agrees with h on X. Suppose for contradiction that for some $\alpha \in W_0^{\sigma \to t}$, $\forall y.\Box X y$ is true on $[X \mapsto \alpha]$ even though $\Box \forall y.X y$ is false on the same assignment. The first assumption implies that $\alpha\langle 1, \mathbf{a}\rangle = \mathbb{N}^\mathbb{N}$ for all $\mathbf{a} \in W_0^\sigma$. The second implies that there are h, i and $\mathbf{a}$ such that $i \notin \alpha(h, \mathbf{a})$. But since $\alpha \in W_0^{\sigma \to t}$, α is pinned down by some finite $X \subseteq \mathbb{N}$. Let j be surjective and agree with h on X. Since α is pinned down by X, $\alpha(j, \mathbf{a}) = \alpha\langle h, \mathbf{a}\rangle$, so $i \notin \alpha(j, \mathbf{a})$. But since j^σ is surjective, there is some $\mathbf{b}$ such that $j^\sigma \mathbf{b} = \mathbf{a}$, and hence $\alpha(j, \mathbf{a}) = j^t(\alpha(1, \mathbf{b})) = j^t(\mathbb{N}^\mathbb{N}) = \mathbb{N}^\mathbb{N}$: contradiction.

So what *can* we do to make BF come out false? One way is to choose the monoid to contain just the functions $h : \mathbb{N} \to \mathbb{N}$ such that for some n: $h(m) = m$ for all $m < n$, and for all $m \leq n$, $h(m)$ is even and $\leq n$. (This family is closed under composition: if h and h' both meet the condition for n and n' respectively, where $n \leq n'$, then $h \circ h'$ and $h' \circ h$ both meet it for n.) Let $\alpha = \langle h, n\rangle \mapsto \{i : in \text{ is even }\}$. This is pinned down by $\emptyset$ since it is indifferent to its h argument: intuitively, it is the qualitative property of being even. Consider the interpretations of $\forall y.\Box(Xz \to Xy)$ and $\Box \forall y.Xz \to Xy$ on the assignment $[X \mapsto \alpha, z \mapsto 1]$. The latter is false, since $\forall y.Xz$ denotes $\emptyset$ on this assignment. But the former is true, since every arrow that sends 1 to an even number sends every number to an even number.

Atomlessness still holds in that model for the same reason as before—we can make any consistent proposition more informative by making it care about numbers that weren't in the set that pinned down the original proposition. To get a model where Actuality holds but BF still fails, we can consider the monoid of functions h where either $h(m)$ is even for all m, or h is $1_\mathbb{N}$. Now Actuality holds, since $\{1\}$ is pinned down by $\{1\}$: it is $\{h \mid h1 \text{ is even}\}$. But ND, BF, Atomicity, and Rigid Comprehension are all false, for the same reasons as before. (Indeed, the actual world is the *only* atomic proposition: the restriction of Atomlessness to false propositions holds in the model.)

For a model where Actuality and BF both hold while Atomicity and Rigid Comprehension fail, let C be the monoid of all *surjections* $h : \mathbb{N} \to \mathbb{N}$ such that either $h = 1_\mathbb{N}$ or $h0 = h1$.

For a model where Atomicity holds while BF, Actuality, and Rigid Comprehension fail, let the monoid contain just $1_\mathbb{N}$ together with the functions g_n, where

$g_n(m) = m$ when $m \leq n$ and $g_n(m) = 0$ when $m > n$. (This is closed under composition: when $n \leq n'$, $g_n \circ g_{n'} = g_{n'} \circ g_n = g_n$.) Actuality fails for the same reason as before. But Atomicity now holds. For $n > 0$, $\{g_n\}$ is pinned down by $\{n-1, n, n\}$, since it is $\{h \mid h(n-1) \neq h(n) \text{ and } h(n) = h(n+1)\}$; $\{g_0\}$ is pinned down by $\{0, 1\}$, since it is $\{h \mid h(0) = h(1)\}$. Thus every $\{g_n\}$ is pinned down by some finite set; moreover, every nonempty proposition in the domain contains some g_n.

For a model where Atomicity, Actuality, and BF hold while Rigid Comprehension and ND fail, let the monoid contain just the functions f_n for even n, where $f_n(m) = 0$ when $m \leq n$ and $f(n) = m - n$ otherwise. These are all surjective, so BF holds. The atom $\{f_n\}$ is $\{h \mid h(n-1) = h(n) \text{ and } h(n) \neq h(n+1)\}$, and $\{f_0\}$ is $\{h \mid h(0) \neq h(1)\}$, so all they are all finitely pinned down; but $\{f_0\}$ is $\{1_{\mathbb{N}}\}$ and thus witnesses Actuality.

All of the models considered so far in this appendix have been based on monoids, so that No Pure Contingency holds in them. To model cases where some of the principles are only contingently true or false, we can turn to multi-object models. For example, for any of the above models, we can adjoin a second object W_1 which is a second copy of $\mathbb{N}$. The arrows from W_0 to itself are as before; every permutation of $\mathbb{N}$ corresponds to an arrow from W_1 to itself; every function $\mathbb{N} \to \mathbb{N}$ corresponds to an arrow from W_0 to W_1; there are no arrows from W_1 to W_0. W_0^* is the set of finite subsets of $\mathbb{N}$, but W_1^* is the set of all subsets of $\mathbb{N}$. Then all the same principles (from among ND, BF, Atomicity, Actuality, Rigid Comprehension) are true, but we also have $\Diamond(\Box\text{ND} \wedge \text{Atomicity})$: it is possible that C5 true.

A wide range of other distributions of necessity, contingent truth, contingent falsehood, and impossibility over the principles can be modelled in a similar way. One particularly interesting result is that we can have BF without $\Box$BF. For this, we can use a two-object model $W_0 = \mathbb{N}$ and $W_1 = \{0\}$, with all functions as arrows. As before, $-^*$ is the ideal of finite subsets; thus $W_1^* = \{\emptyset, \{0\}\}$. BF is false at W_1, since $\forall y.\Box x = y$ is true and $\Box\forall y.x = y$ on the assignment $[x \mapsto 0]$. But BF is true at W_0, for the same reason that we found it to be true before when we looked at the monoid of all functions on $\mathbb{N}$: for any function and finite set, there is a surjective function that agrees with that function on that set.

One thing we haven't yet confirmed is that Actuality doesn't imply Atomicity in C5. We can show this using a mild generalization of the techniques of this appendix. Consider a category with two objects W_0 and W_1 which are both copies of $\mathbb{N}$, where the arrows between any pair of objects correspond to the permutations of $\mathbb{N}$ and composition is composition. In constructing $-^t$ we impose a new constraint: for a set of arrows to belong to W_0^t or W_1^t, it must not only be pinned down by some subset of W_0 or W_1 (respectively), but must also be such that whenever it contains an arrow $h : W_i \to W_1$, it also contains $g \circ h$ for every $g : W_0 \to W_0$. In higher types we proceed as before: $W_i^{\sigma\to\tau}$ contains just those $\alpha \in W_i^{\sigma\to\tau}$ that are pinned down by some finite set. In this model, the smallest set in W_0^t that contains 1_{W_0} is the set of all arrows $W_0 \to W_0$, so this set witnesses the truth of Actuality at W_0. But Actuality is false at W_1 for the usual reason, and so $\Box$Actuality is false at W_0.

There is a lot more that could be done using the methods of this appendix. For example, we have not investigated whether Boolean Completeness holds in the models we have considered, or whether we can modify them to change its status. We have seen how to secure Rigid Comprehension in type e without ND, but have not investigated how this works for other types. We hope that others will be able to make a more systematic exploration.

E Coalesced sums and Maximalist Classicism

This appendix will prove the following theorem, whose significance is explained in §3.6:

Theorem 3.10. For any set of action models with disjoint base categories, there is an action model such that every member of that set is among its truncations.

We first introduce an operation for combining an arbitrary set of non-overlapping rooted categories into a new category, their "coalesced sum". Intuitively, the coalesced sum is the smallest category containing all the objects and arrows of the input categories, one new object W_0, and one new arrow from W_0 to the base world of each input category, without making any unnecessary identifications.

Definition. Given a set of rooted disjoint categories $\langle \mathcal{C}_k, W_k \rangle$ for $k \in K$, their ***coalesced sum***, $\nabla_{k \in K} \langle \mathcal{C}_k, W_k \rangle$, is the minimal category that includes all the objects and arrows of each of the $\mathcal{C}_k$; one new object W_0; and a new distinguished arrow k from W_0 to each W_k such that whenever $h \circ k = h' \circ k$, $h = h'$.[88]

To prove Theorem 3.10, we define a corresponding operation on sets of action models.

Definition. Given a set of action models $\mathbf{A}_k = \langle \mathcal{C}_k, W_k, -^{\cdot}_k, \mathcal{I}_k \rangle$ for $k \in K$ (for a given signature) whose underlying categories are disjoint, their ***coalesced sum*** $\nabla_{k \in K} \mathbf{A}_k$ is an action premodel $\langle \mathcal{C}, W_0, -^{\cdot}, \mathcal{I} \rangle$ defined as follows.

- The underlying rooted category $\langle \mathcal{C}, W_0 \rangle$ is $\nabla_{k \in K} \langle \mathcal{C}_k, W_k \rangle$, the coalesced sum of the underlying categories of the models $\mathbf{A}_k$.

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[88] This definition does not actually specify the identity of the new object and new arrows, which can be anything we like. If we want to officially choose, we could require the index set K to be disjoint from each $\mathcal{C}_k$, choose $W_0 = K$, and for each object V of $\mathcal{C}_k$, choose $Hom(W_0, V)$ to be the set of ordered pairs $\{\langle W_0, h \rangle : h \in Hom(W_k, V)\}$. In this representation the distinguished arrow from W_0 to W_k is $\langle W_0, 1_{W_k} \rangle$ rather than just k.

- For any type σ, $V^\sigma = V_k^\sigma$ for every object V of $\mathcal{C}_k$, and $h^\sigma = h_k^\sigma$ for every arrow h of $\mathcal{C}_k$. (That is: the action of each type on the objects and arrows of each $\mathbf{A}_k$ is just carried over unchanged into $\nabla_{k\in K}\mathbf{A}_k$.)
- W_0^e is $\Pi_k W_k^e$ (the Cartesian product of the type-e domains of the W_k).
- When $x \in W_0^e$, $k^e(x) = \pi_k(x)$ (the projection of x onto its k'th co-ordinate).
- $W_0^t = \{X \in W^{\mathcal{P}} \mid k^{\mathcal{P}}X \in W_k^t \text{ for each } k \in K\}$.
- $W_0^{\sigma\to\tau} = \{\alpha \in W_0^{\sigma\Rightarrow\tau} \mid k^{\sigma\Rightarrow\tau}\alpha \in W_k^{\sigma\to\tau} \text{ for all } k \in K\}$.
- For a nonlogical constant c of type e, $\mathcal{I}(c) = \Pi_k\mathcal{I}_k(c)$.
- For a nonlogical constant c of type $\sigma_1 \to \cdots \to \sigma_n \to t$, we define $\mathcal{I}(c)$ by way of its *intension*:

$$\mathcal{I}c = \mathrm{Int}\{\langle h\circ k, \mathbf{a}_1, \dots, \mathbf{a}_n\rangle \mid \langle h, \mathbf{a}_1, \dots, \mathbf{a}_n\rangle \in \mathrm{App}\,\mathcal{I}_k c\}$$

Here, Int stands for the operation that turns applicative behaviour profiles in $W^{\sigma_1\to\cdots\to\sigma_n\to t}$ into their corresponding intensions, analogous to the int operation from §3.3, and App stands for the inverse operation turning intensions back into applicative behaviour profiles, analogous to app from from §3.3.[89]

To show that this is a premodel, we need to check that for a nonlogical constant c of type σ, $\mathcal{I}(c) \in W_0^\sigma$. For type e this is immediate. For type $\tau = \sigma_1 \to \cdots \to \sigma_n \to t$, we need to show that for all $k \in K$, $k^\tau\mathcal{I}c \in W_k^\tau$. But this is true: $k^\tau\mathcal{I}c = \mathrm{App}\,k^{[\sigma_1\times\cdots\times\sigma_n]}\{\langle h\circ k, \mathbf{a}_1, \dots, \mathbf{a}_n\rangle \mid \langle h, \mathbf{a}_1, \dots, \mathbf{a}_n\rangle \in \mathrm{Int}\,\mathcal{I}_k c\} = \mathrm{App}\{\langle h, \mathbf{a}_1, \dots, \mathbf{a}_n\rangle \mid \langle h, \mathbf{a}_1, \dots, \mathbf{a}_n\rangle \in \mathrm{Int}\,\mathcal{I}_k c\} = \mathrm{App}\,\mathrm{Int}\,\mathcal{I}_k c = \mathcal{I}_k c \in W_k^\tau$.

Clearly, if $\nabla_k\mathbf{A}_k$ so defined is an action model (i.e. if its domains are "sufficiently full"), then each of the $\mathbf{A}_k$ is a truncation of it. So all we need to do to prove the theorem is verify that the sufficient fullness condition is met: i.e. that for any type-σ term A and $i : W_0 \to V$, $[\![A]\!]_i^g$ exists and is in V^σ for every type-σ term A.

Any arrow from the base object W_0 is either the identity 1_{W_0} or can be written uniquely in the form $h\circ k$ for some arrow $h : W_k \to V$. In the case of a non-identity arrow $h\circ k$, a straightforward induction on the complexity of terms shows

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[89] For $X \in W^t$, $\mathrm{Int}\,X = X = \mathrm{App}\,X$. For $\alpha \in W^{\sigma_1\Rightarrow\cdots\Rightarrow\sigma_n\Rightarrow t}$

$$\mathrm{Int}\,\alpha = \{\langle h, \mathbf{a}_1, \dots, \mathbf{a}_n\rangle \mid \langle 1_{\mathrm{trg}\,h}, x_2, \dots, x_n\rangle \in \mathrm{Int}\,\alpha\langle h, x_1\rangle$$

And when X is a set of $n{+}1$-tuples $\langle h, \mathbf{a}_1, \dots, \mathbf{a}_n\rangle$ where for some V, $h : W \to V$ and each $\mathbf{a}_i \in V^{\sigma_i}$,

$$\mathrm{App}\,X = \langle h, \mathbf{a}_1\rangle \mapsto \mathrm{App}\{\langle i, \mathbf{a}_2, \dots, \mathbf{a}_n\rangle \mid \langle i\circ h, i^{\sigma_1}\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\rangle \in X\}.$$

It is readily shown that these operations are inverses and commute with arrows.

that $[\![A]\!]^{g}_{h\circ k} = [\![A]\!]^{g}_{k,h}$ for every term A (where $[\![\cdot]\!]^{\cdot}_{k,\cdot}$ is the interpretation function of $\mathbf{A}_k$), and thus in the domain of the target of h. The interesting cases here are those of the nonlogical constants, where we need to check that our chosen denotation in the coalesced sum gets mapped to the constant's denotation in $\mathbf{A}_k$. When c is of type e, $[\![c]\!]^{g}_{h\circ k} = (h\circ k)^{\sigma}(\mathcal{I}(c)) = h^{\sigma}(k^{\sigma}(\mathcal{I}(c))) = h^{\sigma}(\pi_k(\mathcal{I}(c)) = h^{\sigma}(\mathcal{I}_k(c)) = {}^{k}[\![c]\!]^{g}_{h}$. When c is of type other than e, $[\![c]\!]^{g}_{h\circ k} = (h\circ k)^{\sigma}(\mathcal{I}(c)) = h^{\sigma}(k^{\sigma}(\mathcal{I}(c))) = h^{\sigma}(\mathcal{I}_k(c)) = [\![c]\!]^{g}_{k,h}$. All the other cases of this induction are trivial.

This leaves us with one more thing to check, namely where i is 1_{W_0}, the identity arrow of the base world. But here everything goes smoothly because we chose the domains of the base world to be "as full as possible". Again we need an induction on the complexity of A.

- For a variable v of type σ $[\![v]\!]^{g}_{1_{W_0}}$ is just $g(v)$ which belongs to W_0^{σ} by definition of assignment function.

- For a nonlogical constant c, $[\![c]\!]^{g}_{1_{W_0}}$ is $\mathcal{I}(c)$ which we already showed was in the domain of W_0 as part of showing that the coalesced sum was a premodel.

- For a logical constant A of type $\sigma \to \tau$, it suffices to show that $k^{\sigma\Rightarrow\tau}[\![A]\!]^{g}_{1_{W_0}} \in W_k^{\sigma\to\tau}$ for each $k \in K$. But this is trivial: since the logical constants do the same thing relative to each arrow in any model the result of acting with $k^{\sigma\Rightarrow\tau}$ will just be the interpretation of the same logical constant in $\mathbf{A}_k$ and so will automatically belong to $W_k^{\sigma\to\tau}$. E.g., for negation we have that $k^{t\Rightarrow t}[\![\neg]\!]_{1_{W_0}}\langle i, \mathbf{p}\rangle = [\![\neg]\!]_{1_{W_0}}(i\circ k, \mathbf{p}) = \mathrm{trg}(i)^{\mathcal{P}} \setminus \mathbf{p} = [\![\neg]\!]_{k,1_{W_k}}\langle i, \mathbf{p}\rangle$, and so $k^{t\Rightarrow t}[\![\neg]\!]_{1_{W_0}} = [\![\neg]\!]_{k,1_{W_k}} \in W_k^{t\to t}$.

- For any application AB where A is of type $\sigma \to \tau$ and B is of type σ, suppose $[\![A]\!]^{g}_{1_{W_0}} \in W_0^{\sigma\to\tau}$ and $[\![B]\!]^{g}_{1_{W_0}} \in W_0^{\sigma}$. Then $k^{\tau}[\![AB]\!]^{g}_{1_{W_0}} = k^{\tau}([\![A]\!]^{g}_{1_{W_0}}\langle 1_{W_0}, [\![B]\!]^{g}_{1_{W_0}}\rangle) = [\![A]\!]^{g}_{1_{W_0}}\langle k, k^{\sigma}[\![B]\!]^{g}_{1_{W_0}}\rangle \in W_k^{\tau}$; thus $[\![AB]\!]^{g}_{1_{W_0}} \in W_0^{\tau}$.

- For an abstraction $\lambda v.A$ where v is of type σ and A is of type τ, we note that $[\![\lambda v.A]\!]^{g}_{1_{W_0}}$ definitely exists by the inductive hypothesis, so we just need to show that acting on it with $k^{\sigma\Rightarrow\tau}$ gives an element of $W_k^{\sigma\to\tau}$. But for any i and $\mathbf{a}$, $(k^{\sigma\Rightarrow\tau}[\![\lambda v.A]\!]^{g}_{1_{W_0}})\langle i, \mathbf{a}\rangle = [\![\lambda v.A]\!]^{g}_{1_{W_0}}\langle i\circ k, \mathbf{a}\rangle = [\![A]\!]^{(i\circ k\circ g)[v\mapsto\mathbf{a}]}_{i\circ k} = [\![A]\!]^{(i\circ k\circ g)[v\mapsto\mathbf{a}]}_{k,i}$ (by the induction hypothesis) $= [\![\lambda v.A]\!]^{k\circ g}_{k,1_{W_k}}$, which is in $W_k^{\sigma\to\tau}$ since $\mathbf{A}_k$ is an action model.

This concludes the proof.

We made some unforced choices when we constructed the coalesced sum. Most obviously, when choosing how to interpret the non-logical constants in the sum, any suitable element $\mathbf{a}$ such that $k^\sigma\mathbf{a} = \mathcal{I}_k(c)$ for each k would have done. This fixes the extensions of the non-logical predicates relative to every arrow other than 1_{W_0}, but leaves us free to set the extensions relative to 1_{W_0}—i.e. the "actual" extensions in the model—however we please. For concreteness, we gave every nonlogical constant of a type other than e the empty extension. We also chose a very large domain for type e at W_0, namely the Cartesian product of all of the type-e domains of the input models. However any set X equipped with functions $k^e : X \to W_k^e$ whose ranges include all the interpretations of the type-e non-logical constants in the W_k would have done. If our language doesn't have any nonlogical constants of type e, we could even have X be a singleton, while if it has at least one such constant, we could choose it to be the set of type-e constants.[90]

By taking as our input a family of action models for which Classicism is complete and building the type-e domains in this alternative way, we will get a model of the Possibility+ schema discussed in §2. However, the alternative construction still gives the base world enormous domains in every type other than W_0. Indeed, for every arrow h other than 1_{W_0}, W_0^t contains propositions $\mathbf{p} \neq \mathbf{q}$ such that $h^t\mathbf{p} = h^t\mathbf{q}$: for example, take $\mathbf{p}$ to be the set of all arrows from W_0, and $\mathbf{q}$ to be the set of all arrows other than 1_{W_0}. So with this way of constructing the models, we have the principle $\forall p(p \to \Box_{\neq} p)$: nothing can be different in any way without some distinct propositions becoming identical. They are thus as far as can be from being models of Strong Possibility, which requires that all sorts of things—anything compatible with Possibility—can happen without any propositions becoming identical. We do not know whether Strong Possibility is consistent, but any proof of its consistency using the methods of this section would have to involve a major modification of our construction that in some sense keeps the domains of W_0 as small as possible, so that there is no need to identify any elements when we follow any arrow into any model of Max C. This seems difficult to pull off.

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[90] If all of the input models are *non-logically harmonious*, then we could also build a coalesced sum on a different underlying category which has *exactly* one arrow from its base object W_0 to every object in every input category. The definition of the premodel is just as before. The only thing that needs to be redone is the proof that $\mathcal{I}c$, for a nonlogical constant c of type $\sigma_1 \to \cdots \to \sigma_n \to t$, does indeed belong to $W_0^{\sigma_1 \to \cdots \to \sigma_n \to t}$ (so that the definition is actually an action premodel).

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