Mathematical Modality
An investigation of set-theoretic contingency

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Abstract

A increasing amount of contemporary philosophy of mathematics posits, and theorizes in terms of special kinds of mathematical modality. The goal of this paper is to bring recent work on higher-order metaphysics to bear on the investigation of these modalities.

The main focus of the paper will be views that posit mathematical contingency or indeterminacy about statements that concern the 'width' of the set theoretic universe, such as Cantor's continuum hypothesis. In the higher-order framework I show that contingency about the width of the set-theoretic universe refutes two orthodoxies concerning the structure of modal reality: the view that the broadest necessity has a logic of $\mathsf{S5}$, and the 'Leibniz biconditionals' stating that what is possible, in the broadest sense of possible, is what is true in some possible world. Nonetheless, I argue that the underlying picture of modal set-theory is coherent and has natural models.

A increasing amount of contemporary philosophy of mathematics posits, and theorizes in terms of special kinds of mathematical modality.[1] These mathematical modalities are neither identical to, nor restrictions of the more familiar notion of metaphysical necessity as it is normally understood.

The goal of this paper is to bring recent work on higher-order metaphysics to bear on the investigation of these modalities. The main focus of the paper

[1]Examples include, but are not limited to: Shapiro (1985), Hellman (1989), Parsons (1983), Fine (2006), Linnebo (2013), Studd (2013), Hamkins and Linnebo (2022), Scambler (2021), Builes and Wilson (2022), Brauer (2020). This trend is certainly not limited to contemporary philosophy of mathematics: there are, for instance, many connections between intuitionistic mathematics and modal logic.

will be views that posit mathematical contingency or indeterminacy about statements that concern the 'width' of the set theoretic universe—a prime example being Cantor's continuum hypothesis. To this end, I will present a recent higher-order theory of modalities—mathematical or otherwise—in which we can frame questions about the structure of modality generally.[2] With this account I will draw some implications about the structure modal reality, and of properties and propositions, from the width contingency hypothesis. I will show (in sections 5 and 6 respectively) that contingency about the width of the set-theoretic universe refutes two orthodoxies concerning the structure of modal reality: the view that the broadest necessity has a logic of $\mathsf{S5}$, and the 'Leibniz biconditionals' stating that what is possible, in the broadest sense of possible, is what is true in some possible world. Nonetheless, I argue that the underlying picture of modal set-theory is coherent: one can formulate, in higher-order logic, the principle that every 'forcing notion' corresponds to a possible way for the set-theoretic universe to be, and can provide it with some natural models.[3] While my focus will be contingency about the width of the set-theoretic universe the framework itself can be fruitfully applied to other forms of mathematical contingency, including contingency about the 'height' of universe too.

1 Higher-Order Logic and Zermelo's Theorem

We begin by setting out the framework of higher-order logic. We will then state an important theorem of higher-order logic, due to Zermelo, that purports to show that, in a certain non-modal sense, the width of the set-theoretic hierarchy is fixed. This will be a useful place to begin our discussion of set-theoretic contingency and indeterminacy.

Second-order logic contains devices that let us express generality in predicate position. To explain what this means it is instructive to draw an analogy with the first-order quantifiers, which express generality in name position. The first-order claim '$\forall x$, x talks' expresses universal generality with respect to the *instances* of this universal—the sentences obtained by replacing the variable x in 'x talks' with particular names: 'Socrates talks', 'Plato talks', 'Aristotle talks', and so on. We can pin down the logical role of the classical first-order quantifiers uniquely by its logical relationship to these instances—a universal claim logically entails each of its instances, and a proof that establishes an arbitrary instance suffices to prove the universal. Harris (1982) has shown that

[2]The theory is essentially that of Bacon (forthcoming) chapter 10.

[3]These models will be the subject of future work.

any two quantifiers satisfying these two conditions are logically equivalent.[4]

Using the exactly same idea we can say what it means to express generality in predicate position. The second-order claim '$\forall X$, Socrates Xs' expresses universal generality with respect to the *instances* of this universal—the sentences obtained by replacing the predicate variable X in 'Socrates Xs' with particular predicates: 'Socrates talks', 'Socrates walks', 'Socrates eats', and so on. The logical role of the classical second-order quantifiers also uniquely determined by its logical relationship with these instances—a universal claim logically entails each of its instances, and a proof that establishes an arbitrary instance suffices to prove the universal. These rules too pin down the second-order quantifiers uniquely up to logical equivalence. By this same sort of logical analogizing we can introduce devices that allow us to generalize into any grammatical position, including sentence position, operator position, the positions occupied by the first-order and second-order quantifiers themselves, and so on. A language with these devices is called a *higher-order* language. I am not here going to defend the view that this process of logical analogizing is successful; my remarks should be taken as merely explanations of the meanings of the higher-order quantification on the assumption that it was.

Some authors introduce second and higher-order languages by providing them with a set-theoretic semantics in which the domains of the second-order quantifiers consists of sets of elements taken from the first-order domain. Others take them to simply be a notation for quantifying over sets or properties. These interpretations of higher-order languages should be sharply distinguished from our own. On the set-theoretic interpretation, the claim that 'Socrates talks' does not logically secure the existential '$\exists X$ Socrates Xs'. Logic doesn't take sides on the existence of sets. By contrast, on my interpretation a nominalist who believes that 'Socrates talks' should also accept the existential '$\exists X$ Socrates Xs'. For predicate generalizations were introduced by their logical relationship to their instances. If you believe there is a claim bearing the right logical relationship to these instances—e.g. that is logically secured by any one of the sentences of the form 'Socrates talks', 'Socrates walks', 'Socrates eats'—then the second-order existential '$\exists X$ Socrates Xs' is simply our notation for expressing it. Similarly the second-order existential '$\exists X$, every set Xs if and only if it is non-self-membered' is secured by the truth of any sentence of the form 'every set is ... if and only if it is non-self-membered', and so is secured by the logical truth 'every set is non-self-membered if and only if it is non-self-membered'. On a set-theoretic interpretation of the second-order

[4] If $\forall_1$ and $\forall_2$ are two first or second-order quantifiers, these logical rules let us prove the biconditional $\forall_1 X.A \leftrightarrow \forall_2 X.A$.

quantifier, by contrast, the existential sentence is false on pain of paradox. (The property theoretic interpretation fairs no better, for we want by similar reasoning '$\exists X$, every property Xs if and only if it is non-self-instantiating'.) On the present interpretation of higher-order languages it is, of course, hopeless to try and provide the semantics of a higher-order language in a first-order metalanguage, even if one that avails itself of high-powered set-theoretic machinery. But this is hardly surprising. We would never dream of attempting to state the semantic clause for the first-order quantifier without using first-order quantification in the metalanguage—why should we expect second-order and higher-order quantification to be any different?

Having distinguished higher-order quantification from first-order quantification over properties and sets, we face a more practical problem of expressing claims like '$\exists X$ Socrates Xs' in ordinary English, which is not a higher-order language.[5] It is convenient to simply use sentences like 'Socrates has some property' or 'Socrates bears some relation to Plato' to *indicate* (without being synonymous with) a higher-order generalization belonging to a proper higher-order language. Having clarified the proper way to interpret higher-order sentences no confusion should arise from this practice.

Let us begin by describing the sort of higher-order language we will work in. We assume that every expression of our language has a grammatical type: e is the type of names, t the type of sentences, and $(\sigma \to \tau)$ the type of expressions that combine with expressions of type σ to form expressions of type τ (in accordance with standard conventions we will often omit outermost brackets). Thus $t \to t$ is the type of operator expressions, $e \to t$ the type of predicates, $(e \to t) \to t$ the type of of a first-order quantifier, and so on. We will also make use of logical expressions of different types. We will assume the usual truth-functional connectives—$\to$, $\wedge$ and $\vee$ of type $t \to (t \to t)$ and $\neg$ of type $t \to t$—and for each type σ a higher-order quantifier $\forall_\sigma$ of type $(\sigma \to t) \to t$ that bears the same logical relationship to expressions of type σ as the first-order universal quantifier (i.e. $\forall_e$) bears to names, and a relation $=_\sigma$ of type $\sigma \to (\sigma \to t)$ that bears the logical relationship to expressions of type σ as first-order identity (i.e. $=_e$) bears to names.[6] Complex expressions may be created in accordance with the gloss we provided above. We can

[5]English is higher-order in the sense that it has devices that let us generalize in to *some* non-nominal positions — see for instance Prior (1971), Rayo and Yablo (2001). But it does not contain all the higher-order quantifiers: it appears to have no device for generalizing into, say, sentence position.

[6]It is possible to define the identity symbol from the quantifiers as $\lambda xy.\forall_{\sigma\to t}X(Xx \to Xy)$. However, it will be useful to have a primitive notion of identity in the language when we consider weakenings of the classical quantifier laws (i.e. free logic) in section 7.

apply expressions of type $(\sigma \to \tau)$ to expressions of type σ, so for instance given a predicate F (type $(e \to t)$) and a name a (type e), we can form the sentence (Fa) (with type t). We also will make use of Church's device of λ-abstraction, which roughly lets us create complex predicates from open formulas, and performs similar jobs at other types. Thus, for each type σ, we assume there are infinitely many variables of type σ, and when A is a term of type τ and x a variable of type σ, $(\lambda x.A)$ is a term of type $(\sigma \to \tau)$. So, for instance, given a variable x of type e and $\neg$ and F as above, we can form the predicate of being *not* F as $\lambda x.(\neg(Fx))$.

The most basic higher-order logic we call H. It is characterized by four sorts of principles: (i) the axioms and rules of the propositional calculus (PC and MP), (ii) the axioms and rules of quantificational logic (for first and higher-order quantifiers alike—see UI and Gen), (iii) the axioms of identity (for first and higher-order identity alike—see Ref and LL), and (iv) an axiom governing the λ-device[7] (see $\beta\eta$). These are listed in figure 1. By a *theory* we will mean a set of formulas (i.e. expressions of type t) containing all instances of the five axiom schemas and closed under the rules MP and Gen, and H itself refers to the smallest theory. A noteworthy feature of H is that it is quite neutral about the granularity of propositions, properties and so on. In particular it does not imply that propositions, properties and relations are individuated by their extensions. This assumption originates with Frege and has seeped its way into many applications of higher-order logic in mathematics, but is also very unfriendly not only to mathematical contingency, but to any contingency whatsoever. If there are only two propositions, a true and a false one, all operations must be truth-functional.

Higher-order logic is very useful for reasoning about sets. One can express generalizations in the position of predicates — like the membership relation — that are not coextensive with any set of ordered pairs, allowing us to talk about structural features of the entire universe of sets that cannot be said, or even approximated, using first-order quantification over sets. One can also replace first-order schemas with infinitely many instances with single universally quantified axioms, allowing us to axiomatize large fragments of accepted mathematics with finitely many axioms. In 1908 Zermelo gave us a higher-order theory axiomatizing what we now know as the *iterative* conception of sets (Zermelo (2010a)).[8] According to the iterative conception of sets, sets are

[7] Two sentences, A and B, are immediately $\beta\eta$ equivalent if one can be obtained from the other by substituting $(\lambda x.M)N$ for $M[N/x]$, provided N is free for x in M, or by substituting $\lambda x.Mx$ for M, provided x is not free in M.

[8] I do not know whether Zermelo's interpretation of the higher-order formalism matches my own, but he certainly didn't identify second-order quantification with quantification over

PC A whenever A is a tautology.

UI $\forall_\sigma F \to Fa$, where $F : \sigma \to t$, $a : \sigma$

REF $a =_\sigma a$

LL $(a =_\sigma b) \to (Fa \to Fb)$

$\beta\eta$ $A \to B$ where A and B are immediately $\beta\eta$ equivalent.

MP If $\vdash A$ and $\vdash A \to B$, then $\vdash B$.

Gen If $\vdash A \to B$, and $x : \sigma$ does not occur free in A, $\vdash A \to \forall_\sigma x B$.

Figure 1: The axioms and rules H

built up in an increasing sequence of stages: $V_0, V_1, V_2, \ldots$. One starts with the set containing no members at all, the empty set, and one constructs new stages in one of two ways. Whenever you have some sets that were all constructed at previous stages, one can form a set of them, and we can collect all sets formed this way into a single set—the *powerset* of the previous stage. Alternatively, one can collect all the sets appearing in a given sequence of stages of sets into a single set provided that sequence can be indexed by a set that has already been constructed. Zermelo provided (with an addition by Fraenkle) an axiomatization of this iterative conception of set in a higher-order language. A simplified version of these axioms are presented in figure 2.[9] We employ the abbreviation $\exists_e x \in y.A$ for $\exists_e x(x \in y \wedge A)$, $x \subseteq y$ for $\forall_e z(z \in x \to z \in y)$, $\emptyset$ is short for a definition description for the set with no elements, and $x \cup \{x\}$ for the set whose elements are x and the elements of x.

The language in which this theory is formulated has, apart from the higher-order logical constants, one non-logical constant, $\in$, meaning *is a member of*. We will make one important adjustment: the standard axiomatization, presented in figure 2 assumes a restricted interpretation of the quantifiers as ranging over pure sets, whereas we will adopt the logical unrestricted interpretation of the quantifiers. This means that all occurrences of $\forall_e x \ldots$ and $\exists_e x \ldots$ in figure 2 are replaced by explicitly restricted quantifiers $\forall_e x(\mathrm{Set}\, x \to \ldots)$ and $\exists_e x(\mathrm{Set}\, x \wedge \ldots)$, where being a pure set can be defined in terms of membership

sets. There is more evidence that Russell's interpretation of higher-order logic matches that given above, and Zermelo follows Russell in terminology for quantification into predicate position as quantification over 'propositional functions', which indicates they are talking about the same thing.

[9] For concision I have omitted some redundant axioms that are sometimes included.

Extensionality $\forall_e xy(\forall_e z(z \in x \leftrightarrow z \in y) \to x =_e y)$

Union $\forall_e x \exists_e y \forall_e z(z \in y \leftrightarrow \exists w \in x.z \in w))$

Powerset $\forall_e x \exists_e y \forall_e z(z \in y \leftrightarrow z \subseteq x)$

Foundation $\forall_e x(\exists_e y.y \in x \to \exists_e y \in x \neg \exists z \in x.z \in y)$

Replacement $\forall_{e\to e\to t} R \forall_e x(\forall_e yzz'(Ryz \wedge Ryz' \to z = z') \to \exists_e z \forall_e y(y \in z \leftrightarrow \exists w \in x.Rwy))$

Infinity $\exists_e x(\exists y \in x(\forall z.z \notin y) \wedge \forall y \in x(y \cup \{y\} \in x)$.

Figure 2: The axioms of ZF

by setting Set $:= \lambda x \exists_e y(x \in y)$.[10] The conjunction of these five axioms is a single second-order sentence $\mathrm{ZF}^{\in}$. By replacing the constant $\in$ with a binary relation variable, R, and abstracting out, we get a definition belonging to the language of *pure* higher-order logic:

$$\mathrm{ZF} := \lambda R.\, \mathrm{ZF}^{\in}[R/\in]$$

Now because higher-order languages have the ability to generalize into the position that binary predicates, like $\in$, occupy we may talk more generally about 'ZF-relations', just as we might talk about other relations that have other mathematically interesting properties, like being symmetric, a partial order, and so on. Because this is all defined in purely logical terms, the study of ZF relations is a purely logical one, and we can consequently use our theory H to draw some substantive conclusions.

Any two relations R and S that have the higher-order property of being ordered like the rational numbers, are isomorphic, whereas two relations having the property of being ordered like a tree need not be. How different, then, can two ZF relations be? Zermelo (2010b) gives a precise answer to this question: ZF relations can differ about their 'height'—how long the iteration process continues—but cannot differ about the 'width'—what the sets are like at a particular stage. If you take the series of stages $V_0^R, V_1^R, \ldots$ of a ZF relation R—constructed as above, except using R wherever I used membership—they will be isomorphic to the stages $V_0^S, V_1^S, \ldots$ of another ZF relation S, provided

[10] For this definition to be sensible we must interpret $x \in y$ as meaning 'x and y are pure sets and x belongs to y'. So as we interpret it, urelements and impure sets do not bear the $\in$ relation to any other impure sets. A proper treatment of impure sets may require a further primitive, but since the focus of this paper is the structure of the pure sets this is a complication I want to background.

those stages both are reached in the respective constructions. It follows that the structure of the pure sets is pinned down uniquely up to any given stage: the only freedom one has concerns how long the sequence of stages extends. Using $R \preceq S$ to formalize the second-order statement that the stages of R are isomorphic to an initial segment of the stages of S, we have:

Zermelo's Quasi-Categoricity Theorem $\forall RS(\mathrm{ZF}(R)\wedge\mathrm{ZF}(S) \to R \preceq S \vee S \preceq R)$

Note, again, that Zermelo's theorem is a purely logical statement—it is formulated entirely in the language higher-order logic without non-logical constants—and is a theorem of our very minimal axiomatic higher-order logic H.It has as good a claim as any higher-order sentence to being a 'logical truth'.[11]

In some presentations Zermelo's theorem is reformulated in terms of isomorphisms between *set-theoretic models* of $\mathrm{ZF}^{\in}$—specifically 'full' models—and so formulated, Zermelo's theorem is a sentence of set-theory itself and not a sentence of higher-order logic.[12] This version of the result not only fails to be true to Zermelo, but underplays its significance. For one, it invites an irrelevant set of concerns about the adequacy of full models in modeling higher-order logic. More importantly, this version speaks only to ZF-relations R that can be represented by a *set* of ordered pairs (in the sense of the particular ZF relation $\in$), thus excluding from its range restrictions of $\in$ like Gödel's constructible universe. While our version of Zermelo's theorem closes off the possibility of other binary predicates satisfying ZF describing an alternative and very different hierarchy of stages, the model-theoretic version leaves this possibility open.

A word of caution: there are various binary predicates definable in the language of set-theory—'inner models'—that are useful in the study of set-theory (especially first-order set-theory), which will quite often turn out not to be ZF-relations.[13] One example would be the binary predicate $\in_L$ obtained by restricting $\in$ to Gödel's constructible sets. Assuming $V \neq L$, we have

[11] The fact that the theorem is a second-order generalization is not as important as it might at first seem: for most applications the schema one gets by deleting those quantifiers and letting R and S be schematic variables is enough. For instance, there is an instance of the theorem where R is $\in$ and R' is $\in_L$, where this is $\in$ restricted to the constructable sets. Of course, assuming $V \neq L$, it's not the case that ZF $\in_L$: indeed $\in_L$ doesn't even satisfy all instances of the axioms of first-order ZFC, provided you include among the instances of separation and replacement instances involving $\in$. (Of course, L satisfies all instances of separation and replacement that only involve $\in_L$.)

[12] See, for instance, Shapiro (1991) pp.188-189.

[13] Schepherdson 1952 part I-III gives a general theory of inner models. Bizarrely Shepherdson dismisses the significance of Zermelo's theorem (p227).

$\neg\,\mathrm{ZF}(\in_L)$—the Separation axiom fails for $\in_L$ because, if you can find a set x in $V_\alpha \setminus L_\alpha$, then we cannot apply separation to the set L_α using the predicate of belonging to x, $\lambda y\, y \in x$. Note that L doesn't even satisfy the unrestricted *first*-order separation schema: $\lambda y\, y \in x$ is a first-order definable property — what makes L an inner model is that it satisfies all the first-order sentences of Separation *that only involve* the $\in_L$ predicate; it will fail many first-order instances of Separation that involve $\in$.

2 Set-Theoretic Contingency: Height and Width

I will begin our investigation of mathematical contingency by delineating some different motivations for positing it. We will look at three different motivations for positing contingency about the sets in the literature, and distinguish two distinctive sorts of contingency which I'll gloss as *height* and *width* contingency.

Motivations for positing height contingency can be traced back to Cantor himself. Cantor's view was that the transfinite ordinals—mathematical objects representing the order-types of well-orders—continued indefinitely through the operations of taking successors and limits. As we saw in the previous section, up to isomorphism, ZF-relations differ from one another only regarding how far the iterative process is continued along the ordinals. Zermelo's picture, like Cantor's picture of the transfinite ordinals, was that this iterative process, continued forever—the iterative hierarchy is 'indefinitely extensible'.

Some have taken Zermelo's remarks to suggest a kind of mathematical contingency.[14] Not any collection of sets form a set, on pain of Russell's paradox. But they nonetheless *could* have formed a set—a stage V_κ of a possible larger set-theoretic universe. Charles Parsons (Parsons (1983)), and several subsequent authors, have been more explicit about the modal in this formulation.[15] For now we'll give this idea the following gloss:[16]

Height Extensibility Necessarily, the sets (whatever they may be) are possibly a proper initial segment of all the sets.

[14] [ANON].

[15] See Fine (2006), Linnebo (2013), Studd (2013).

[16] Actually Zermelo's idea seems to be importantly different from that of Parsons', and the one given below. Zermelo above is concerned with the structure of ZF-relations generally, without selecting any particular one for attention, so his form of indefinite extensibility is one formulable in the language of pure higher-order logic alone (see [ANONYMIZED][**?**]). By contrast set-theory is often taken to be the study of one particular ZF-relation, membership, and Parsons, Linnebo and Studd each formulate their versions of indefinite extensibility in terms of it.

Here the sense of possibility in play is, presumably, not metaphysical possibility but a primitive kind of mathematical possibility in need of further explication (I will offer some suggestions in section 4).

More recently there has been significant interest in a different kind of indefinite extensibility inspired by Paul Cohen's method of forcing. Joel Hamkins, in a number of papers, has suggested that, even when we restrict ourselves to a particular infinite stage V_α—'the sets of rank α'—one can always consider a larger set theoretic universe that contains more sets of that rank.[17] For instance, the method of forcing lets one describe, within any given set-theoretic universe, a larger one that contains more sets of natural numbers.[18] We have an explicitly modal articulation of related ideas in Scambler (2021), Pruss (2020), Builes and Wilson (2022).[19] Let's give this idea the following gloss:

Width Extensibility Necessarily, the sets of rank α (whatever they may be) are possibly properly contained in the sets of rank α.

Again, the notion of possibility here is a primitive mathematical one, which may be identical to or orthogonal to the one appealed to above. These latter authors are typically interested in Width Extensibility because they want to make sense of the idea that all sets of countable in a strictly modal sense:

Countabilism Every set is 'countable' in the sense that for any set x, it is possible that there is an injection from the natural numbers to x.

For some motivations for Countabilism see Meadows (2015) and Builes and Wilson (2022).[20] I myself am interested in this principle because it is equivalent to the following principle:[21]

[17]See, for instance, Hamkins (2012).

[18]One can even describe what these new sets of natural numbers will have to look like, although they will be in some sense ω-inconsistent from the perspective of the present universe.

[19]Hamkin's also uses modal logic in his work to spell out the multiverse view–Hamkins (2003), Hamkins et al. (2015)–but it seems clear that his uses of the modal operator $\Box A$ are really abbreviations for something quantificational: $\Box A$ means A holds in all forcing extensions of the universe, where this is a statement that can be articulated in the extensional language of first-order set theory.

[20]Builes and Wilson (2022) argue that while height extensibilism can be motivated by a certain sort of attitude to Russell's paradox—the non-self-membered sets do not in fact form a set, they could have done—Countabilism follows from taking a parallel attitude toward Cantor's theorem.

[21]This principle is closely related to the principle **HE** from Scambler (2021). The proof that Forcing Possibilism and Countabilism are equivalent are essentially theorems 3 and 4 of Scambler (2021) (see p.1092). One difference is that Scambler's principle is formulated using plural quantification.

Forcing Possibilism For any partial order, $\mathbb{P}$, and set D of dense subsets of $\mathbb{P}$, it is possible that there exists a filter on $\mathbb{P}$ that intersects every element of D.

Forcing Possibilism legitimizes a certain practice that seems commonplace among set-theorists. The set-theorist I have in mind sets out theorizing in the language of set theory. They may then consider various partial orders $\mathbb{P}$ belonging to the cumulative hierarchy, and its associated collection of dense subsets D, and ask seemingly modal questions of the form 'what *would* the set-theoretic universe have looked like if there had been a filter that had intersected every element of D?'. For instance, $\mathbb{P}$ might consist of finite bits of information about a potential function from ω to $\{0, 1\}$ ordered by informativeness, and the postulated filter then consists of a collection of these bits of information that approximate a total function $f : \omega \to \{0, 1\}$ which differs from every actually existing function over some finite bit of information (since, for any actual function, the set finite partial functions not contained in that function is dense).[22] Consequently, by positing the possibility of such a filter we describe a possible set-theoretic universe containing a new function from the naturals to $\{0, 1\}$. One might attempt to make sense of this practice by interpreting the set-theorist's quantifiers as initially ranging over a restricted portion of the 'real' sets, and the possibility containing the new filter as simply arising from enlarging the range of those quantifiers. But this approach assumes there is a background universe of 'real' sets—consisting of *all* the sets there are—yet the procedure for describing new sets can be applied just as readily to this background universe of sets as it can to any of its restrictions. Granted, it is possible still to reduce this seemingly modal talk to extensional quantification over possibilities, even when it is applied to the entire universe of sets. Each element of $\mathbb{P}$ may be thought of as a 'possibility'—in our running example, the possibility specifies the behaviour of a new function on ω on finitely many of its arguments. In the language of set theory, one can define a relation of a sentence being 'true at' a possibility, from which we may paraphrase any claims of possibility and necessity extensionally, in terms of existential or universal quantification over possibilities. Nonetheless, I find the idea that the set-theorist is describing *genuine* possibilities for the set-theoretic universe to be deeply attractive.

[22] More precisely, $\mathbb{P}$ consists of the finite partial functions $\omega \rightharpoonup \{0, 1\}$ ordered by inculsion. Every actually existing function $f : \omega \to \{0, 1\}$ determines a filter of finite partial functions F (its finite subsets), but will also be disjoint from one of the actual dense subsets of $\mathbb{P}$, namely $\mathbb{P} \setminus F$. Thus if there had been a filter of partial functions that intersected every element of D, it's union would have to be a totally defined function on ω that doesn't actually exist.

Apart from Countabilism and Forcing Possibilism, contingency about the width of the set-theoretic hierarchy can also be motivated by considerations of set-theoretic indeterminacy. Cantor's continuum hypothesis, which we will abbreviate CH, is the claim that every infinite set of real numbers (identified with a certain set in our iterative hierarchy) can either be put in one-to-one correspondence with the set of all real numbers or can be put in one-to-one correspondence with the natural numbers. This claim is, surprisingly, left unsettled by presently accepted mathematics: no currently accepted axiomatic theory (whether first-order or higher-order) implies CH or implies its negation.[23] Perhaps this is a symptom of a deeper kind of indeterminacy about the truth value of this statement. According to this picture, our state of ignorance about the continuum hypothesis is akin to our ignorance about whether a borderline heap is a heap or not: there is simply no fact of the matter, and so additional investigation would yield no headway. Indeterminacy seems to be a kind of contingency, and in order for the continuum hypothesis to contingent in this sense, it must be contingent what real numbers there are.

Do the motivations we have discussed above really require one to posit mathematical contingency? Some authors have attempted to make sense of indefinite extensibility and indeterminacy in extensional terms. I will set aside attempts to make sense of height extensibility in extensional language, since that is not our primary concern.[24]

On the face of it, Zermelo's theorem makes it hard to see how one could articulate the above ideas concerning the width in extensional terms. Let's begin with the motivation from mathematical indeterminacy. Most have taken this kind of indeterminacy, like vagueness more generally, to involve the failure of language to single out a unique meaning for the membership symbol '$\in$'. In the language of classical supervaluationism:

> The symbol '$\in$', as used by mathematicians, has multiple admissible interpretations. Under some such interpretations 'CH' is true, and under others it is false.

'Admissibility' should be understood here as a higher-order predicate of type

[23] It is sometimes suggested that second-order set theory settles the continuum hypothesis (see Kreisel (1967) p[REF]), but these authors have a semantically defined theory in mind when they talk about by 'second-order set theory'.

[24] One common approach is to deny possibility of unrestricted quantification, so that whenever one attempts to say something about 'all' sets one only ends up talking about a restricted universe (see, for instance, the papers collected in Rayo and Uzquiano (2006)). I present a different extensional articulation of the indefinite extensibility idea, inspired by some remarks of Zermelo, in [**?**][ANON].

$(e \to e \to t) \to t$—a property of binary relations that are candidates to be the meaning of '$\in$'—so that the quantification in question is higher-order.

This formulation of indeterminacy is spelled out extensionally, for we are just quantifying over admissible interpretations of the membership symbol. But what relations are admissible interpretations of the membership predicate? Presumably they should at least be ZF-relations, for these are overwhelmingly more natural than any of the alternatives that accord with out mathematical practices.[25] But then the continuum hypothesis is not linguistically indeterminate: Zermelo's theorem tells us that any two ZF-relations are isomorphic up to the first inaccessible, and thus agree about the truth value of the continuum hypothesis:

$$\forall RS.(\mathrm{ZF}(R) \wedge \mathrm{ZF}(S) \to (\mathrm{CH}(R) \leftrightarrow \mathrm{CH}(S)).$$

In order to explore a different diagnosis let us suppose, for the sake of argument, that there is no linguistic indeterminacy in the symbol '$\in$': there are special objects, *sets*, and a special relation, *membership*, about which set-theory is unambiguously concerned. Nonetheless, there might still be indeterminacy in *the sets themselves*, concerning how they are *related* to one another by the membership relation.[26] This indeterminacy would not be metalinguistic, but a kind of contingency concerning the pattern of the membership relation among the individuals. Here we must thus assume a notion of propositional indeterminacy that is not reduced or explained in terms of linguistic indeterminacy, but is simply another propositional operator alongside the other more familiar modal operators.[27]

Could it be that the sets are indeterminately arranged to the extent required for certain claims, like the continuum hypothesis, to come out indeter-

[25]A lot has been made of the fact one can find relations that accord with mathematical practice—in particular that make all the sentences of ZF true—that are not ZF-relations in our defined sense. It is pretty clear that in order for the predicate 'is a ZF relation' to apply to something that isn't a ZF relation, at least one of the logical constants in our purely logical definition of 'is ZF relation' has to have a non-disquotational interpretation. Skolem, and several subsequent authors, have maintained that the words for universal quantification—first and higher-order alike—can sometimes refer to restrictions of the universal quantifiers. However, even if we concede that quantificational words can sometimes refer to these restrictions, the resulting relations are far less natural than ZF relations, and make unlikely candidates for what we are talking about when we are doing set theory.

[26]Compare Goodsell (2022) on the possibility of non-linguistic arithmetical indeterminacy"On this conception, for arithmetic to be indeterminate is for the numbers themselves to have an indeterminate structure, independently of how we speak about them".

[27]Bacon (2018b) defends a propositional account of vagueness more generally. See also Goodsell (2022) for a discussion of a propositinal account of mathematical indeterminacy specifically.

minate? One might have thought not. After all, we have just argued that CH cannot be linguistically indeterminate, and how could the case of propositional and linguistic determinacy be *so* different? Matters are in fact quite different, however. Zermelo's theorem applies to simultaneously existing relations ('from the same possibility'), which is to say, necessarily, any two ZF relations are isomorphic up to any rank they have in common. Let us imagine trying to formulate a "trans-world" version of Zermelo's theorem. Let us suppose we could rigidly pick out to the things which are in fact sets and rigidly pick out the membership relation. Call this rigid relation $\in^*$. One might hope to argue, as above, that necessarily, $\in^*$ must be isomorphic to an initial segment of $\in$—i.e. that the *actual sets* are isomorphic to the sets under the membership relation, whatever that might be at the relevant possibility—or, conversely, that $\in$ be isomorphic to an initial segment of $\in^*$. In which case $\in$ and $\in^*$ would agree about CH, and of course the value of CH according to $\in^*$ is not contingent given $\in^*$ is by stipulation rigid. However, in order to apply Zermelo's theorem, we need that $\in^*$ is not only a ZF relation in actual fact, but necessarily a ZF relation. But $\in^*$ could fail to satisfy the separation axiom, especially if we consider possibilities at which there are new properties for the second-order quantifier to range over. For instance, if it is possible, in the new determinacy theoretic sense, that there be a set, $x \subseteq \mathbb{N}$, of natural numbers that don't in fact exist (as one would expect for the contingency of CH) then there is a new property, $\lambda y.y \in x$, which does not define a subset* of $\mathbb{N}$ according to $\in^*$.

Similar morals may be drawn for the Width Extensibilist. What we observe, firstly, is that the possibility of a ZF relation containing more sets of rank α cannot be actually witnessed, for by Zermelo's theorem any two ZF relations are isomorphic up to a given rank (provided they both extend that far). Nonetheless, Width Extensibility is on first-looks consistent with Zermelo's theorem because the actual sets of rank α, whatever they might be, could possibly fail to contain *all* the sets of rank α, assuming there could have been more properties and thus more conditions with which to define subsets of sets with rank below α. We see then that both sorts of width contingency require the possibility not merely of 'new' type e entities, but also of 'new' type $e \to t$ properties.

3 The Structure of Modal Reality

The sorts of mathematical contingency posited will have implications for the structure of modal reality. In this section and the next three we articulate some of these connections.

Several things require untangling before we can draw these implications. For a start, what do we mean by the structure of modal reality? Often metaphysicians mean by this various theses formulated in terms of a particular kind of modality, Kripke's notion of 'metaphysical necessity'. But as this modality is used by Kripke and subsequent philosophers, mathematics is metaphysically necessary (see Kripke (1980) p36). It follows that the mathematical contingency and indeterminacy appealed to above cannot be explained in terms of metaphysical contingency, and the possibilities on which mathematical indeterminacy and extensibility theses are predicated are not metaphysical possibilities. These theses concern the structure of modality in general, but not the structure of metaphysical necessity.

To theorize about the structure of modal reality in its entirety, we must be able to talk about all the modal notions there are, metaphysical modality and otherwise, and talk about the logical relationships between these modal notions. Crucial to this enterprise is the ability to specify what it means for an operator to be a *modality*, and to specify the logical relationships between modalities—when one modality is *as broad as* another. (For instance, we have seen above that metaphysical necessity is not as broad as mathematical necessity or determinacy.[28]) Indeed, higher-order logic provides us with the perfect framework to carry this out, for in the language of higher-order logic one can quantify directly into the positions occupied by sentences and by sentential operators allowing one to formulate definitions of these notions. Once this is done it is possible to then introduce a notion of *broad necessity*, an operator defined as possessing every necessity. We will argue that the study of the structure of broad necessity has a good claim to being the study of the 'structure of modal reality' simpliciter.[29] The possibilities posited by this notion can be seen, by definition, to subsume the determinacy-theoretic possibilities and mathematical possibilities. It follows that any possibilities in which the continuum hypothesis has a different truth value, or in which there are more ordinals than there in fact are, will automatically be broad possibilities. So principles about the structure of broad necessity can have a direct bearing on the question of mathematical contingency and *vice versa*.

In order to start theorizing about modalities we face a choice. If we assume a certain thesis about the granularity of reality—roughly, that propositions, properties and relations are individuated relatively coarsely, by provable equiv-

[28]The case that metaphysical necessity is not as broad as determinacy can be made even with respect to non-mathematical claims of vagueness, given the supervenience of the vague propositions on the precise; see Bacon (2018a), Bacon (2018b).

[29]Certainly it has a better claim to this than the study of metaphysical necessity, given the remarks above.

H The axioms and rules of H (see figure 1).

RE If $\vdash P\overline{x} \leftrightarrow Q\overline{x}$, then $\vdash P =_{\overline{\sigma}\to t} Q$.

Figure 3: Axiomatization of C

alence in H—it is possible to give completely reductive definitions of *being a modality*, *being as broad as*, and *broad necessity*. If we wish to be neutral on the matter of propositional granularity, we appear to need another primitive. A higher-order predicate, Nec, *being a necessity*, of type $(t \to t) \to t$, is a natural primitive for this purpose.[30] We will take the former route of pursuing a logicist account of modality at the expense of neutrality of grain, but if you do not accept this theory of granularity everything I say can in a precise sense be translated into the latter framework by disregarding our definition of 'Nec', and replacing subsequent uses of it with the primitive.[31]

The system we will work in, *Classicism*, or simply C, adds to H the Rule of Equivalence, which ensures that the theory proves the claim that two propositions, properties or relations R and S are identical whenever it can prove that R and S are coextensive. C it is thus the smallest theory closed under the Rule of Equivalence: See figure 3.[32] It is the last rule that that distinguishes Classicism from more structured theories of granularity: it implies, for instance, that *being old and wise* and *being wise and old* are the very same property ($\lambda x.(Fx \wedge Gx) =_{e\to t} \lambda x.(Gx \wedge Fx)$) on account of their being provably coextensive from the laws of classical logic.

Using the purely logical language of higher-order logic it is possible to say that a given operator, X of type $t \to t$, has a 'normal modal logic'. Roughly, it is normal if the smallest collection of propositions containing (i) the tautologies, (ii) closed under modus ponens, (iii) containing the claim that X satisfies the normality axiom, and (iv) closed under X-necessitation

[30] One could instead take broad necessity as the primitive, an approach taken in Dorr et al. (2021). However, by taking *being a necessity* as primitive we can provide a *justification* for the posit of a broadest necessity, rather than imposing that assumption by fiat.

[31] The logicist account is spelled out in more detail in Bacon (2018a) and Bacon (forthcoming) chapter [REF], and the theory with a primitive necessity predicate, Nec, in Bacon and Zeng (2022). The latter shows that the theory Classicism used in Bacon (2018a) and Bacon (forthcoming) is interpretable in their theory, and that their theory is indeed neutral about the granularity of reality.

[32] Other axiomatizations of this system can be found in Bacon (2018a) and Bacon and Dorr (forthcoming).

are all true. Because we can quantify into sentence position we can state what it means for an operator to be closed under modus ponens with a single generalization:

$$\text{MP-Closed} := \lambda X.\forall p(X(p \to q) \to Xp \to Xq)$$

Given a modal operator $\Box$, we can similarly say what it means for a 'collection' of propositions, represented by an operator Y of type $t \to t$, to be closed under necessitation for $\Box$: $\forall p(Yp \to Y(\Box p))$.

$$\text{Nec-Closed} := \lambda XY.\forall p(Yp \to Y(Xp))$$

We can then state that p is in the normal modal logic for $\Box$ by saying that p belongs to any collection of propositions that contains the tautology, is closed under modus ponens and necessitation for X, and contains the claim that X is closed under modus ponens (i.e. the K axiom):

$$\begin{aligned}\text{InNormalModalLogicOf} := \lambda Xp.\forall Y(Y\top \wedge \text{MP-Closed}\, Y \wedge \\ Y(\text{MP-Closed}\, X) \wedge \text{Nec-Closed}(X, Y) \to Yp)\end{aligned}$$

Definition 3.1 (Weak Necessity)**.** *An operator, X, is a weak necessity iff every proposition in its 'normal modal logic' is true.*

$$\text{WNec} := \lambda X.\forall p((\text{InNormalModalLogicOf}\, X)p \to p)$$

The notion of a weak necessity is sufficient for applications of normal modal logic: if one considers an interpreted propositional modal language in which '$\Box$' is interpreted by a weak necessity, then every theorem of the smallest normal modal logic, K, will be true. For instance, in epistemic logic it is common to assume we are studying the knowledge of a logically perfect agent who also knows they are logically perfect, knows they know this, and so on. This agents knowledge will satisfy the conditions for being a weak necessity, but even this highly idealized kind of knowledge is not a necessity in the metaphysically relevant sense. It is physically possible, say, that the agent sustain a head injury and fail to be logically omniscient. A true necessity, by the metaphysicians lights, is necessarily, in all the relevant senses of 'necessarily', a weak necessity.

Definition 3.2 (Strong necessity)**.** *An operator X is a strong necessity iff, for every weak necessity Y, it is Y-necessarily a weak necessity.*

$$\text{Nec} := \lambda X.\forall Y(\text{WNec}\, Y \to Y(\text{WNec}\, X))$$

We can now spell out what it means for one necessity to be as broad as another: there must be a strict implication from one necessity to the other. It would be arbitrary to single out any particular necessity to articulate this strict implication, so we require the implication to be strict in every possible sense. In fact, broadness is a special case of the more general notion of *entailment*. In the below we write $\overline{x}$ for a sequence of varibles $x_1...x_n$, $\overline{\sigma}$ for a sequence of type $\sigma_1...\sigma_n$, and $\overline{\sigma} \to \tau$ for the type $\sigma_1 \to ... \to \sigma_n \to \tau$.

Definition 3.3 (Entailment)**.** *Given two relations, R and S, of type $\overline{\sigma} \to t$, R entails S iff for every necessity Z, it's Z-necessary that any things standing in R stand in S.*

$$\leq_{\overline{\sigma}} := \lambda RS.\forall_{t\to t} Z(\mathrm{Nec}\, Z \to Z\forall\overline{x}(R\overline{x} \to S\overline{x}))$$

We can also introduce 'multi-premise' entailment. If X of type $(\overline{\sigma} \to t) \to t$ represents a collection of propositions, properties or relations we say it entails another proposition, property or relation R iff anything entailing everything in X entails R, and we write this $X \leq R$:

$$\leq := \lambda XR\forall S(\forall T(XT \to S \leq T) \to S \leq R)$$

Given two necessity operators, X and Y, we say that X is as broad as Y iff, X entails Y, i.e. $X \leq_{t\to t} Y$.

We modeled our notion of a necessity on the idea of a normal modal operator. In a normal modal logic one can prove that if some finite list of propositions, $p_1, ..., p_n$, are each necessary, so is anything that they jointly entail. The analogous infinitary principle, that anything entailed by an arbitrary collection of necessary propositions is also necessary by contrast, cannot be proven from the principles of normal modal logic.[33] Arguably there are necessities, such as *having an objective chance of 1*, that do not satisfy this further principle, so we do not build it in to our definition. At any rate, we can specify this further property:[34]

Definition 3.4 (Infinitely closed necessity)**.** *A necessity, X, is infinitely closed iff, whenever a proposition is entailed by the collection of all necessary propositions, that proposition is also necessary: $\forall q(\mathrm{X} \leq q \to Xq)$.*

$$\mathrm{Nec}_\infty := \lambda X(\mathrm{Nec}\, X \wedge \forall q(\mathrm{X} \leq q \to Xq))$$

[33] If we add infinite conjunctions to propositional modal logic, this strengthening is valid in the usual Kripke semantics, but not in variant semantics such as the topological semantics for $\mathsf{S4}$, and so is not derivable from K augmented with the logical laws governing infinitary conjunction.

[34] See Bacon and Zeng (2022) p[**?**].

Finally, we define broad necessity as being necessary in every sense of necessity

Definition 3.5 (Broad Necessity)**.** *p is broadly necessary iff it's X-necessary for every necessity X*

$$\Box := \lambda p.\forall X(\mathrm{Nec}\, X \to Xp)$$

In order to justify the title 'broad necessity' one must show that $\Box$ does indeed meet our criteria for being a necessity, and that it is as broad as any other necessity. These are verified by the following theorem.[35]

Theorem 1. *The following are theorems of Classicism*

1. $\mathrm{Nec}\,\Box$
2. $\mathrm{Nec}_\infty\,\Box$
3. $\Box(\forall X(\mathrm{Nec}\,X \to \Box \leq X)$

Next we list some theorems of Classicism that concern the logic of broad necessity.

Theorem 2. *The following are theorems of Classicism or rules under which it is closed:*

K $\forall_t p \forall_t q(\Box(p \to q) \to \Box p \to \Box q)$

T $\forall_t p(\Box p \to p)$

4 $\forall_t p(\Box p \to \Box\Box p)$

CBF$^\sigma$ $\forall_{\sigma\to t} F(\Box\forall_\sigma x\, Fx \to \forall_\sigma x \Box Fx)$

NE$^\sigma$ $\forall_\sigma x \Box \exists_\sigma y.x =_\sigma y$

Necessitation *If A is a theorem of Classicism, so is $\Box A$*

Note that the first three axioms and Necessitation ensures the theorems of S4 for $\Box$ belong to Classicism. The first three axioms straightforwardly fall out of the fact that $\Box$ is the broadest necessity. K is guaranteed by the fact that $\Box$ is a necessity. T follows from the fact that the truth operator $(\lambda p.p)$ is a

[35] Proofs of all the theorems to follow may be found in Bacon (forthcoming) chapter [REF]).

necessity, and $\Box$ is as broad as it; $\mathsf{4}$ follows from the idea that the composition of two necessities is a necessity, so that $\Box$ must be as broad as $\lambda p.\Box\Box p$.

The converse Barcan formula and the necessity of existence correspond to instances of CBF^σ and NE^σ where σ is e. They are both consequences of the idea that the theorems of classical logic are necessary. They are consequently theorems of modal logics based on classical logic, Classicism included.[36] The latter principles apparently tells us that whatever exists does so necessarily. When applied to concrete individuals this is, of course, a contentious thesis.[37] It seems that I could have failed to exist—for instance if my parents had never met. This is true in many senses of 'could', and so must be true for the broadest kind of possibility. Some authors—'contingentists'—have attempted to avoid this consequence by weakening the principles of classical quantificational theory those of free logic.[38] I am myself sympathetic to the idea that quantificational expressions in English rarely express the sort of quantifier for which the classical laws are necessary (and consequently for which the necessity of existence and CBF are true). But the classical quantifiers are nonetheless incredibly useful for the sort of general theorizing distinctive to disciplines such as logic, mathematics and and metaphysics, and can often be introduced by definition. For instance, if we have an actuality operator and I want to say that every possible individual is F I can say 'necessarily, everything is F in actuality'. The formula $\Box\forall_e x@Fx$ thus simulates "possibilist quantification" over all possible individuals provided at the actual world. While this paraphrase is materially adequate, this fact is, of course, highly contingent: had different things been F, that paraphrase would still evaluate with respect what is actually F and deliver incorrect results.[39] Kit Fine (Prior and Fine (1979) p144) thus paraphrases quantification over all possible Fs by saying 'the true world proposition w (whatever it might be) is such that necessarily everything is entailed by w to be F'. So a candidate classical quantifier Π might be:

$$\Pi := \lambda X.\exists w(\text{World}\, w \wedge w \wedge \Box\forall x\Box(w \rightarrow Xx))$$

We will refine this idea in section 7, and in appendix B show that in the dialectical context in which it will be employed, this quantifier is indeed classical.[40]

[36] NE^e, for instance, can be proven from these principles as follows. $\exists_e y.x =_e y$ is a theorem of Classical logic, and so is necessary: $\Box\exists_e y.x =_e y$. Applying the principle of Gen we can infer $\forall_e x\Box\exists_e y.x =_e y$.

[37] For a thorough defense of this thesis, see Williamson (2013).

[38] Specifically by restricting the principle of universal instantiation. See Kripke (1963), Lambert (1963).

[39] See the discussion in Williamson (2010) p685-686.

[40] [ANONYMIZED].

But even if, for whatever reason, we cannot define the classical quantifiers, they can at least be introduced as primitives, pinned down by their introduction and elimination rules. Even skeptics of CBF and NE concede all this, and will often avail themselves of 'outer quantifiers' which range over 'merely possible individuals'. According to Kit Fine, for instance, outer quantification is perfectly 'legitimate but not basic'.[41] Since CBF and NE are valid for these quantifiers, we may proceed to use these quantifiers to interpret the theorems of Classicism. At any rate, we will give the free logicians and contingentists a fair shake in section 7 so, for the time being, let us set them aside.

4 Mathematical Necessity

This concludes our general theory of modality in higher-order logic. In order to apply it to the present topic of mathematical modality and indeterminacy we must introduce new non-logical operator constants to the logical language to stand for these operations. I will use the symbol $\blacksquare$, and we will read it as the relevant sort of mathematical necessity or as determinacy depending on the application, although for convenience we will use the terms 'mathematically necessary' and 'mathematically possible' in a way that is neutral between these interpretations. Call the language of pure higher-order logic $\mathcal{L}$, and the result of adding $\blacksquare$ to it $\mathcal{L}^{\blacksquare}$.

We must, of course, assume $\blacksquare$ is a necessity. However, it seems plausible that it is also closed under arbitrary logical consequences so we will make the stronger assumption:[42]

Mathematical Necessity $\mathrm{Nec}_{\infty}\,\blacksquare$

Let $\mathsf{C}^{\blacksquare}$ be the theory obtained by adding this principle to Classicism.

There is a long standing question for the modal extensibilists about the interpretation of mathematical modality (see §2.3 of Studd (2013)). Øystein Linnebo simply writes:

[41] See Prior and Fine (1979) pp118-119). See also Forbes (1985) and Pollock (1985).

[42] Hartry Field (Field (2003)) has suggested that rejecting the closure of determinacy under infinitary consequence is key to making sense of the paradoxes of higher-order vagueness, but I am not convinced. The assumption that determinacy is closed under infinitary consequence is explicitly argued for in Bacon (2020b) section III and another route to avoiding those paradoxes is given there. Roughly put, since the operation of conjunction (finitary or infinitary) is precise, then a conjunction of precise truths must also be precise. This can be used to show that a conjunction of determinate truths is determinate, and thus that any consequence of those determinate truths is determinate.

> *This is not metaphysical modality in the usual post-Kripkean sense.* Rather, the modality [...] is related to that involved in the ancient distinction between a potential and an actual infinity. (Linnebo (2013)p207)

But this tells us very little, and different authors have posited all sorts of modalities to fill this role. Fine (2006), for instance, posits, an 'interpretational' modality, whereas Scambler a dynamic one relating to the abilities of an ideal reasoner (Scambler (2021) p1100). Studd (2013), rejects these proposals, and likens the mathematical modalities more to tense operators, although does not find an interpretation he fully happy with. To my mind, these replacements offer no more clarity.

The present framework, however, has an alternative to offer, namely that the relevant sort of necessity is just broad necessity. Any charge of unclarity here is easily met, for the notion of broad necessity is as clear as the logical operations from which it is defined — quantification and the truth-functional operations.

The Broad Necessity of Mathematics $\blacksquare =_{t\to t} \Box$

Under this hypothesis, the subsequent discussion would be greatly simplified. Nonetheless, there are some philosophical views we wish to remain neutral about that require us to keep them separate. Clearly any mathematical possibility is possible in the broadest sense, so it is the converse entailment that is at stake: is every broad possibility mathematically possible? One might worry that broad possibility is *too* broad. For instance, some authors have entertained the hypothesis that there is a notion of *logical* necessity in which even mathematical theories, such as $\mathrm{ZF}^{\in}$, could be contingent.[43] One might also want to accommodate views in which distinct individuals can be broadly possibly identical, which would cause trouble for the attractive idea that it's mathematically necessary which elements a set has.[44] But the failure of our hypothesis above doesn't rule out precisely defined notions filling the roles that we care about. For instance consider:

$$\Box_{\mathrm{ZF}} := \lambda p.\, \mathrm{ZF}^{\in} \leq p$$

$$\Box_{\neq} := \lambda p.\exists_t q(q \wedge \Diamond q \leq p)$$

[43] These ideas can be formulated precisely in the present higher-order framework of Classicism — see, for instance, Bacon (2020a), Bacon and Dorr (forthcoming) section [REF], Bacon (forthcoming) chapter [REF].

[44] Using Set Rigidity, stated below, one can prove by transfinite induction that sets are mathematically necessarily distinct. See Lemma 21 in appendix A.

The former builds in the necessity of $\mathrm{ZF}^{\in}$, whereas the latter the necessity of distinctness.[45] The latter also has the virtue that it can be reductively defined in purely logical terms, and even someone who believed in logical possibility could maintain the necessity of $\mathrm{ZF}^{\in}$ with the force of $\Box_{\neq}$.

Once we have singled out a suitable closed necessity $\blacksquare$, we can formulate various theses about the interaction of mathematical necessity with mathematical primitives. Let us add to the language of pure higher-order logic, $\mathcal{L}$, a binary predicate $\in$ of type $e \to e \to t$ and a modality $\blacksquare$. Call the resulting language $\mathcal{L}^{\in\blacksquare}$. We will firstly assume that it is mathematically necessary that $\in$ satisfies the axioms of higher-order ZF.

The Necessity of Set Theory $\blacksquare\,\mathrm{ZF}^{\in}$

As we noted before, this is compatible with the view that $\mathrm{ZF}^{\in}$ is logically contingent, and so contingent in the broadest sense.

We will assume, in addition to this, that sets are 'rigid' in the sense that they cannot gain or lose members. We can require rigidity with respect to many different modalities. Rigidity with respect to the broadest modality implies rigidity with respect to any weaker modality. Since we wish to remain neutral about *logical* contingency, including logical contingency about the make up of a set, we will require sets only to be rigid with respect the mathematical modality/determinacy operator $\blacksquare$. Here is how we say that a set, x, cannot gain members: if any property, F, possibly applies to some member of x then there is in fact a member of x to which F possibly applies (for otherwise x could have members that are not among its actual members). Here is how we say that it cannot lose members: if, for any property F, there is some member of x that is possibly F, then it's possible that some member of x is F (for otherwise there is some actual member of x that is possibly not a member of x). This means we want $\forall_{e\to t}F(\exists y \in x \wedge \blacklozenge Fx \leftrightarrow \blacklozenge\exists_e y(y \in x \wedge Fx))$. This is essentially the dualized form of the Barcan formula for the quantifiers restricted to $\in x$. In general we will define what it means for a relation $R : \sigma_1 \to ... \to \sigma_n \to t$ to be rigid as follows, writing $\overline{x}$ for a sequence of varibles $x_1, ..., x_n$, $R\overline{x}$ for $Rx_1...x_n$, $\forall\overline{x}$ for $\forall_{\sigma_1}x_1...\forall_{\sigma_n}x_n$, and $\overline{\sigma} \to t$ for $\sigma_1 \to ... \to \sigma_n \to t$.

$$\mathrm{Rigid}_{\blacksquare} = \lambda R\blacksquare\forall_{\overline{\sigma}\to t}S(\blacksquare\forall\overline{x}(R\overline{x} \to S\overline{x}) \leftrightarrow (\forall\overline{x}R\overline{x} \to \blacksquare S\overline{x}))$$

So we can now state our principle that sets are rigid:

Sets are Rigid $\forall_e x(\mathrm{Set}\, x \to \mathrm{Rigid}_{\blacksquare}\, \lambda y.y \in x)$

[45] See Dorr et al. (2021) and Bacon and Dorr (forthcoming).

Mathematical Necessity $\mathrm{Nec}_\infty \blacksquare$
The Necessity of Set Theory $\blacksquare \mathrm{ZF}^{\in}$
Set Rigidity $\forall_e x(\mathrm{Set}\, x \to \mathrm{Rigid}_\blacksquare\, \lambda y.y \in x)$

Figure 4: $\mathsf{C}^{\blacksquare\in}$ adds these axioms to Classicism.

Rigidity here is stated with respect to $\blacksquare$, although there is a stronger notion of rigidity stated in terms of broad necessity.

Observe that this principle is *not* the claim that the *binary* membership relation is rigid: it is the claim that, for each set x, the unary property of belonging to x is rigid. If membership were rigid there could be no contingency in the pattern of membership claims. Let me head off one potential confusion. If x is the set of all sets of rank α, the claim Sets are Rigid implies that x cannot gain or lose elements. However, this does not mean that there couldn't have been more sets of rank α, it rather implies that if there had been more sets of rank α x wouldn't have contained them all. This confusion becomes particularly tempting when we start using putative singular terms like V_α or $P(\mathbb{N})$ to refer to sets. V_α is not itself a term in the language of set theory, it is really a definite description and so the property of belonging to V_α, $\lambda x.x \in V_\alpha$, can fail to be rigid consistently with the principle Sets are Rigid.[46].

We will call the system we get by adding these principles to Classicism $\mathsf{C}^{\blacksquare\in}$, $\mathsf{C5}^{\blacksquare\in}$ is the result of also include $\Box\mathsf{B}$. They are summarized in figure 4

Of course, theses we have considered earlier can now be formulated precisely:

Indeterminacy of CH $\blacklozenge \mathrm{CH} \wedge \blacklozenge \neg \mathrm{CH}$

Countabilism $\forall_e x(\mathrm{Set}\, x \to \blacklozenge \exists_e f : \mathbb{N} \to x(\forall_e zw(fz =_e fw \to z =_e w)))$

Forcing Possibilism $\forall_e xy(\mathrm{PO}\, x \wedge \mathrm{Dense}(y, x) \to \blacklozenge \exists_e z(\mathrm{Filter}\, z \wedge \forall_e u \in y(u \cap z \neq \emptyset)))$

here $f : \mathbb{N} \to x$ means that f is a function from $\mathbb{N}$ to x, $\mathrm{PO}\, x$ states that x is a partial order, $\mathrm{Dense}\, yx$ the claim that y is the set of all dense subsets of x, and $\mathrm{Filter}\, z$ the claim that z is a filter on x. These further principles will not be part of our neutral theory of mathematical necessity and sets.

[46]Observe too that our principle entails that individuals with no members — the empty set and urelements–necessarily have no members. It doesn't quite imply that urelements are necessarily urelements: for all we've said an urelement might become identical to the emptyset because the system we are in does not rule out the necessity of distinctness

5 Brouwer's principle and the Barcan formula

We noted in the previous section that Classicism proves that, in any sense of 'necessary', every individual is necessarily identical to something—$\forall_e x \blacksquare \exists_e y.x =_e y$. This is true in at least the stipulated sense of $\exists$ as the classical existential quantifier, if not the ordinary quantifiers of English. Whatever qualms you might have about this general consequence, there are a number of independent arguments for the necessary existence of pure sets. One route to this conclusion flatly appeals to the necessary existence of abstract objects in general. But even if we allow, in general, for contingently existent abstract objects we have special reasons to think that *sets* exist necessarily. Sets are commonly thought to be constituted by their elements: some say they are nothing over and above their elements, others that they are entirely grounded by their elements—their existence is entirely determined by the existence of their members. As a special case, if the members of a set exist necessarily, then so does the set. We might formalize this as follows:

$$\forall_e x(\mathrm{Set}\, x \to \forall_e y \in x \blacksquare(\exists_e z.z = y) \to \blacksquare(\exists_e z.z = x))$$

Within the sort of free logic that is friendly to contingent existent in general, a straightforward transfinite induction establishes, from this principle about set existence, that every pure set exists necessarily. We defer the argument for our discussion of free logic in section 7.

Now according to the standard view of *metaphysical* necessity, not only do mathematical objects exist necessarily the truths of mathematics are are necessary. Do these two theses—concerning mathematical existence and mathematical truth—stand or fall together? Since we are at present concerned with the view that there are wider kinds of possibility in which the truths of set-theory are contingent, we must reconcile this with the necessary existence of sets.

The two kinds of set-theoretic contingency we discussed in section 2 both involve the idea that there could have been new sets that don't in fact exist, either by there being more stages of sets or by a given stage containing more sets than it in fact contains. Both of these ideas are consistent with the necessary existence of sets. Sets that didn't previously exist can come into existence, but once they exist they do so necessarily. However, in order for this package to be consistent we must reject the necessity of the Brouwerian principle—that what is true is necessarily possible. Indeed, we must reject the Brouwerian principle not just for the sense of 'could' in which 'there could have been new sets' is true, but also for the broadest sense of 'could'.

B $\forall_t p(p \to \Box\Diamond p)$

Brouwer's principle for broad necessity is a part of the 'orthodox' package of views about modal reality, exemplified in, for instance, Lewis (1986) and Stalnaker (1976), and in the explicitly higher-order context Williamson (2013), Fritz and Goodsell and Yli-Vakkuri (MS). This package usually comes along with the view that metaphysical necessity is the broadest necessity, and its logic is $\mathsf{S5}$. Of course, this picture holds that B is not only true, but broadly necessary. This is equivalent to saying that truth entails being possibly necessary:

$\mathsf{B}^{\leq}$ $\lambda p.p \leq_{t \to t} \lambda p.\Box\Diamond p$

Let us write $\mathsf{C5}$ for the result of adding $\Box\mathsf{B}$, or $\mathsf{B}^{\leq}$, to Classicism. $\mathsf{C5}$ contains all the theorems of the modal logic $\mathsf{S5}$ for broad necessity.

Why does the possibility of new individuals require the failure of Brouwer's principle? The usual model theoretic explanation of this rests on a certain possible worlds model theory in which worlds are possible *relative* to other worlds, and the Brouwerian principle corresponds to the symmetry of this relation of relative possibility.[47] If one world, w, considers another world, v, to be possible and to contain individuals in its domain that do not belong to ws domain, then every world possible relative to v must contain those individuals (since individuals exist necessarily). This means the original world w cannot be possible according to v, so the relation of relative possibility is not symmetric.

This explanation is unsatisfactory due its reliance on a particular model theory, as well as possible worlds assumptions that we will have reason to question shortly. But we can provide another argument that is entirely independent of the existence of possible worlds, due in essence to Arthur Prior.[48] We will represent our talk of 'new individuals that don't in fact exist' by some property, F, that is possibly instantiated by some individual but such that no actual individual possibly instantiates it—whatever possibly falls under F must be 'new' or 'non-actual' in the relevant sense. But if it's possible that something is F, then it's possible that something is necessarily possibly F (appealing to the necessity of the principle that whatever is true is necessarily possible). If there is some individual x that is necessarily [possibly F], x must necessarily exist by the necessity of existence: so it's necessary that something, namely x, is [possibly F]. Thus, if it's possible that something is necessarily possibly F, it's possible that necessarily, something is possibly F. But then it follows that something is possibly F, for whatever is possibly necessary is

[47]See Cresswell and Hughes (1996) pp17-21.

[48]I offer an informal argument below, the formal version of this proof is found in Prior (1967) p146 and attributed to E.J. Lemmon. It is based on an earlier argument due to Prior (1956).

true, by the contrapositive form of Brouwer's principle. So there cannot be new individuals in the relevant sense.

We have effectively established $\forall_{e\to t}F(\Diamond\exists_e xFx \to \exists_e x\Diamond Fx)$. By taking the contrapositive and applying the duality of $\Box$ and $\Diamond$ we get what is known as the Barcan formula, which we take to be our official way to articulate the idea that there can't be new individuals. Like with the converse Barcan formula there is a version of it at each type, but I state here the version for type e:

$\mathsf{BF}^e_\Box$ $\forall_{e\to t}F(\forall_e x\Box Fx \to \Box\forall_e x\, Fx)$

This style of argument can be run at any type whatsoever. In fact, we appealed no nothing special about broad necessity in this argument. For any necessity, X, let us write BF^σ_X and B_X for the Barcan formula and Brouwerian principle concerning X (i.e. $\forall_{\sigma\to t}F(\forall_\sigma x.X(Fx) \to X(\forall_\sigma x\, Fx))$ and $\forall_t p(p \to X\neg X\neg p)$). Prior's argument establishes that, for any necessity whatsoever, the X-necessity of Brouwer's axiom for X, i.e. $X\mathsf{B}_X$, implies the Barcan formula for X, BF^σ_X. So to summarize:

Theorem 3 (Prior)**.**

1. *C proves* $\forall X(\mathrm{Nec}\, X \to X\mathsf{B}_X \to \mathsf{BF}^\sigma_X)$
2. *$\mathsf{C5}$ contains $\Box\mathsf{B}$ and thus proves $\mathsf{BF}^\sigma_\Box$.*

If mathematical contingency requires failures of $\mathsf{BF}^e_\Box$, as we have been suggesting, it means we must reject the orthodox logic of $\mathsf{C5}$. Luckily Classicism on its own includes neither the Barcan formula or Brouwer's principle, and there very are natural models in which they fail.[49]

Theorem 4. *The following are **not** theorems of Classicism*

B $\forall_t p(p \to \Box\Diamond p)$

5 $\forall_t p(\Diamond p \to \Box\Diamond p)$

$\mathsf{BF}^\sigma_\Box$ $\forall_{\sigma\to t}F(\forall_\sigma x\Box Fx \to \Box\forall_\sigma x\, Fx)$

For the free logicians and contingentists who are following all of the above in terms of stipulatively defined or primitive outer quantifiers, the crucial point is this. While the stipulations guarantee that CBF^σ (and consequently NE^σ)

[49] Several sorts of models are described in the appendices to Bacon (2018a) and Bacon and Dorr (forthcoming) and in chapters [REF] of Bacon (forthcoming).

are true when the quantifiers are interpreted stipulatively as outer-quantifiers, they *do not* ensure that $\mathsf{BF}^\sigma_\Box$ holds. For these contingentists that admit failures even of the outer Barcan formula, there is an important distinction between properties of mere possibilia (properties that are possibly instantiated but for which there is actual thing which possibly instantiates it). Sometimes a possibly instantiated property will be such that there is nothing 'within the inner domain' that possibly instantiates it, but will there will be something in the outer sense that possibly instantiates the property. In other cases—when the property characterizes 'true' mere possibilia—the property will be possibly instantiated when nothing, even in the outer sense, possibly instantiates it.[50]

How do these principles about broad necessity relate to other modalities, like mathematical modalities and determinacy? Intuitively, if there couldn't be new things in the broadest sense of 'could', then there couldn't be new things any more restrictive sense. One might naively take this to mean that the broad Barcan formula implies the Barcan formula for any modality. But this is not quite true. Some counterexamples to the Barcan formula have nothing to do with the possibility of new individuals, but to do with the failure of the necessity to be closed under infinite conjunctions. For every individual there's a chance of 1 that if it's a point on the dartboard the dart won't land on it, but it doesn't follow that there's a chance of 1 that the dart won't land on any point on the dartboard; so if *having chance 1* is a necessity, it doesn't respect the Barcan formula irrespective of the status of the broad Barcan formula. However, the Barcan formula for broad necessity implies the Barcan formula for any necessity that is infinitely closed. It follows, too, that Brouwer's axiom for broad necessity implies the Barcan formula for every necessity that is infinitely closed. We summarize this with the following theorem of the orthodox system $\mathsf{C5}$

Theorem 5.

1. $\mathsf{BF}^\sigma_\Box \to \forall X(\mathrm{Nec}_\infty X \to \mathsf{BF}^\sigma_X)$
2. *In* $\mathsf{C5}$, $\forall X(\mathrm{Nec}_\infty X \to \mathsf{BF}^\sigma_X)$

[50] The question naturally arises whether it is possible to stipulatively introduce an even wider quantifier that ensures that even $\mathsf{BF}^\sigma_\Box$ is valid. A full treatment of this question is beyond the scope of this paper. One can introduce 'free quantifiers' governed by a free logic, but they will be proper restrictions of the outer quantifier (restricted by a 'rigid' property). On the other hand, in certain free higher-order logics one can prove there is at most one classical quantifier. These questions will be treated in more detail in future work with [ANONS].

The proof is included in appendix A. As a straightforward corollary, we see that the behaviour of mathematical modalities and determinacy are tightly constrained by the behaviour of broad necessity:

Corollary 6 ($\mathsf{C}^{\blacksquare}$)**.** *$\Box\mathsf{B}$ and $\mathsf{BF}^{\sigma}_{\Box}$ both imply $\mathsf{BF}^{\sigma}_{\blacksquare}$.*

So we are now in a position to state our first implication.

> Width contingency requires failures of the 'orthodox' theory of modality according to which the broadest kind of modality is governed by a logic of $\mathsf{S5}$.

The first-order of business is to define the property V_α: being a set whose rank is no greater that α. This is done by transfinite recursion:[51]

$$V_0 := \lambda x.\bot$$

$$V_\alpha := \lambda x \forall y (y \in x \to \exists \beta \in \alpha V_\beta y).$$

Where α is an ordinal (i.e. a transitive set that is totally ordered by membership: $\forall \beta\beta' \in \alpha(\beta \neq \beta' \to \beta \in \beta' \vee \beta' \in \beta)$). It is usual in set-theory texts to use V_α as a name for a set, whereas here it is a predicate. Our choice discourages the temptation to think of V_α as automatically rigid in virtue of being a set, as we earlier cautioned against.

The claim that there is no contingency about the width of the universe, then, is the claim that for every ordinal α, V_α is rigid. Note that everything we say here is entirely consistent with height contingency: for all we say, there could be new ordinals γ and as a result new sets belonging to V_γ.

Theorem 7 ($\mathsf{C}^{\blacksquare\in}$)**.** *Given $\mathsf{BF}^{e}_{\Box}$ (for broad necessity), being of stage α (i.e. V_α) is rigid for every ordinal α.*

It is shown by transfinite induction. The crux of the proof involves showing that, for any F, if it's possible that some V_α set is F then some V_α set is possibly F. If it's possible that some V_α set is F, then $\mathsf{BF}^{e}_{\blacksquare}$ (a consequence $\mathsf{BF}^{e}_{\Box}$ ond $\mathrm{Nec}_\infty\blacksquare$) ensures that some actual individual, x, is possibly V_α and F. It suffices to show that x is in fact V_α. We can use the fact that x has its members rigidly (if it is a set), the inductive hypothesis that V_β is rigid, and the rigidity of being an ordinal less than α, to show that xs members are all V_β for some $\beta < \alpha$.

[51] This can be made into an explicit definition in the usual way $V = \lambda\gamma x(\forall Y(\forall y \neg Y0y \wedge \forall\alpha(\mathrm{Ord}\,\alpha \wedge \forall y \forall \beta \in \alpha(y \subseteq Y\beta \to Y\alpha y)) \to Y\gamma x)$.

The significance of this result is that, for any ordinal α, there cannot be new sets of rank α. Among other things, this implies the non-contingency of the continuum hypothesis, and thus its determinacy on one way of reading $\blacksquare$. For in order for it to be indeterminate whether the continuum hypothesis is true one has to introduce new sets with small ranks ($\omega + n$ for finite n): new sets of natural numbers, or new bijections between sets of reals and reals. One can similarly refute countablism: if there is no injection from $\mathbb{N}$ to x then, this fact is necessary, for there cannot be any new injections given $\mathsf{BF}^e_\square$. To make these remarks precise we introduce some useful concepts and propositions, which are proven in appendix A.

A formula of first-order set-theory is absolute with respect to a model iff (i) when it is satisfied by the same objects of the initial models it is satisfied in all extensions of that model and (ii) if it is not satisfied by those objects it is not satisfied by them in any extension of the model. This has an obvious modal analogue:

Definition 5.1 (Modal Absoluteness)**.** *A formula $A(\overline{x}, \overline{y})$ is* modally absolute *iff the formulas*

- $\forall_e \overline{x}(\mathrm{Set}\,\overline{x} \wedge A(\overline{x}) \to \blacksquare A\overline{x})$
- $\forall_e \overline{x}(\mathrm{Set}\,\overline{x} \wedge \neg A(\overline{x}) \to \blacksquare \neg A\overline{x})$

are both true, where $\overline{x}$ is short for a sequence of variables $x_1 ... x_n$, and $\mathrm{Set}\,\overline{x}$ is short for the conjunction $\mathrm{Set}\,x_1 \wedge ... \wedge \mathrm{Set}\,x_n$.

A sufficient condition for a formula of first-order set-theory to be absolute is if all of the quantifiers in the formula are restricted by formulas that are not only absolute, but do not change their extensions across models. The modal analogue of this stronger property is rigidity. In the present higher-order setting, we can similarly define a class of first-order sentences that are provably modally absolute: the smallest set of sentences containing $x \in y$ and containing $\neg A$, $A \wedge B$, $\forall_e x(C \to A)$ whenever A and B are in the set, and C is a rigid property of sets ($\mathrm{Rigid}_\blacksquare(\lambda x.C)$ and $\lambda x.C \leq_{e \to t} \mathrm{Set}$ are true).

Theorem 8 ($\mathsf{C}^{\blacksquare\in}$)**.** *Suppose $A(\overline{x})$ is a first-order set-theoretic formula with free variables $\overline{x}$. If all the quantifiers in $A(\overline{x})$ are restricted to rigid properties of sets, then A is modally absolute.*

The proof of this theorem is provided in the appendix. In practice we could dispense with the metalinguistic notion of modal absoluteness: cases where we apply theorem 8 to a particular formula $A(\overline{x})$ can be replaced by proving the rigidity of a finite number of particular properties in the object language (those restricting the quantifiers in A). But we need the concept to state theorem 8, and the theorem provides useful perspective on what we are actually doing when carry out an argument that a particular formula defines a rigid property because it is general, whereas these particular arguments are not. There too the metalinguistic ascension is often dispensible, and harmless.

Given theorem 8 and Set Rigidity, any formula that's provably equivalent to one whose quantifiers are all restricted by set membership will be modally absolute. This lets us derive the following useful facts:

Theorem 9 ($\mathsf{C}^{\blacksquare\in}$)**.** *For any ordinal α, the following conditions are modally absolute.*

1. *being an ordinal less than α.*

2. *being a limit ordinal less than α.*

3. *being the smallest limit ordinal, the successor of the smallest limit ordinal, the successor of the succcessor of the smallest limit ordinal...*

moreover, the properties in 3. are rigid.

Corollary 10 ($\mathsf{C}^{\blacksquare\in}$)**.**

1. $\mathsf{BF}^e_\Box \to \blacksquare CH \lor \blacksquare\neg CH$.

2. $\mathsf{BF}^e_\Box \to \forall_e x(\text{Uncountable}\, x \to \blacksquare\, \text{Uncountable}\, x)$

where $\text{Uncountable}\, x := \forall_e x(\forall_e f : \mathbb{N} \to x \neg\, \text{Injection}\, f)$

The consequents of these conditionals are thus outright theorems of $\mathsf{C5}$:

Corollary 11 ($\mathsf{C5}^{\blacksquare\in}$)**.**

1. $\blacksquare CH \lor \blacksquare\neg CH$.

2. $\forall_e x(\text{Uncountable}\, x \to \blacksquare\, \text{Uncountable}\, x)$

The reason this is true, roughly, is that CH is about sets of rank $V_{\omega+2}$: it's equivalent to a formula whose quantifiers are restricted to $V_{\omega+2}$. But given the modal absoluteness of V_α and of $\omega + 2$ (proven above) it follows by theorem 8 that CH is modally absolute. Note that the modal absoluteness of CH implies $\text{CH} \to \blacksquare\, \text{CH}$ and $\neg\, \text{CH} \to \blacksquare\neg\, \text{CH}$, so $\blacksquare CH \lor \blacksquare\neg CH$ follows from an instance of excluded middle.

More generally, by 7, any set theoretic statement that is equivalent to a sentence that can be formulated using quantifiers restricted to V_α for some α will be determinately true or false, and non-contingent in other senses of contingency. Thus these arguments extend straightforwardly to other contentious axioms of set theory such as the generalized continuum hypothesis up to some cardinal κ, Martin's axiom for partial orders up to cardinality κ, and so on. They do not extend to claims about the 'height' of the universe, such as large cardinal hypotheses.

It's also worth taking a moment to look at the limitations of our theorem. Our result does not rule out contingency about the height of the universe. While, for a given ordinal α, there could not be any new sets belonging to V_α, we have said nothing to rule out the possibility of new ordinals, and thus new sets belonging to new stages indexed by these sets. For while the property of being an ordinal is modally absolute, we have not shown that it has the stronger property of being rigid. If a set is an ordinal it is necessarily so, and if a set is not an ordinal it is necessarily not an ordinal—but this leaves room for something that is not a set possibly becoming an ordinal, or becoming a set of some other kind. However, while this leaves room for height contingency even with the Barcan formula, but it is a fairly weak kind. There could have been more stages of sets, but the new sets must already exist in actuality as non-sets. It is only by denying the Barcan formula can we get the stronger kind of height contingency. (On the other hand, contingentists and free logicians who are not interpreting our symbol '$\exists$' in terms of the ordinary quantificational idioms, but in terms of the stipulated outer quantifiers, can take some solace in this, for these possible sets do not actually exist in the philosophically interesting sense.)

Might one take this to be an argument against width contingency? After all, isn't $\mathsf{S5}$ in some sense the standard logic of necessity? I am not persuaded. If there ever was an implicit decision within the philosophical community about which logic of necessity is 'standard' it happened before mathematical modalities and determinacy operators were being discussed widely, and most likely was made with Kripke's notion of metaphysical necessity in mind. The failures of Brouwer's principle posited here are entirely compatible with its holding for the more restricted notion of metaphysical necessity. It follows too that the Barcan formula may be valid for metaphysical necessity, and that the continuum hypothesis is either metaphysically necessarily true or necessarily false. And this too is entirely compatible with our diagnosis of the continuum hypothesis as indeterminate and thus contingent in the broadest sense. When it comes to positive arguments for the $\mathsf{S5}$ principle, they are thin on the ground. Some considerations are abductive, and come from the relative simplicity and power of S5—the only schemas of propositional modal logic it doesn't imply are clearly invalid, whereas S4 leaves the validity of many modal principles open.[52] To this I say the theoretical virtues of simplicity and power must be weighed against the countervailing virtue of truth—after all $\perp$ is simple and very powerful. The theorist already convinced of the indeterminacy of the continuum hypothesis will find much less mileage in these abductive considerations. Other arguments for the S5 principle are far less compelling, for they often appeal to the model theory of modal logic in a patently illegitimate way—e.g. appealing to the idea that to be the broadest necessity it must quantify over 'all' possible worlds in some set-theoretic model, without taking into account that in the intended model (if there is one!) what worlds in the model represent genuine possibilities could well be contingent.[53] Finally, we should also emphasize that broad necessity, as it has been introduced here, is not necessarily a notion we had pretheoretically—intuitions about how it should behave should be taken with a generous pinch of salt, and it is generally better to simply work with its formal definition, *being necessary for every necessity*, and see where our philosophical theorizing takes us.

6 The Leibniz Biconditionals

Let us now turn to another pervasive idea in modal metaphysics, the notion of a *possible world*. Possible worlds can be wielded as a purely model theoretic tool for establishing metalogical properties like consistency and invalidity. In a model of a modal language sentences might be interpreted by arbitrary sets of possible worlds, and these might serve as the domain for quantifiers binding sentence variables if the language has them. In the present higher-order setting, this ensures various theorems of Classicism are valid—Boolean identities, like $\forall_t pq((p \wedge q) =_t (q \wedge p))$—but also ensures validities beyond Classicism. Because there are propositions modeled by the singleton of a possible world, $\{w\}$, every consistent proposition is entailed by one of these special world propositions, leading to distinctive validities. World propositions

[52]Several broadly abductive arguments are made in Williamson (2013). Scroggs (1951) shows that the only modal logics extending S5 contain schemas to the effect that there are only n possibilities, for some finite n. Fine (1974) shows there are continuum many modal logics extending S4.

[53]See Bacon (2018a) §5.4 for a critical discussion of these arguments. The point here is that the mathematical objects of the relevant model that in fact are representing genuine possibilities may not represent genuine possibilities had things been sufficiently different. We should also keep track of the fact that if there is mathematical contingency, the model itself might change its mathematical structure.

are special because they are either fully contained or disjoint from any other set of possible worlds.

However, metaphysicians often take possible world talk to be more than a mere model theoretic tool. Someone taking the possible world model of propositions metaphysically seriously should believe that these special world propositions exist.[54] Given our previous observation that singletons are consistent, and contained or disjoint from (i.e. contained in the complement of) any other proposition, we will adopt the following definition of a world proposition:

$$\text{World} = \lambda w.(\Diamond w \wedge \forall_t p(w \leq_t p \vee w \leq_t \neg p))$$

World propositions are broadly possible propositions such that any other proposition is either entailed by it or inconsistent with it. The latter condition ensures that worlds settle all questions. The possible worlds metaphysics ought, then, to subscribe to the *Leibniz Biconditionals*: that something is possible if and only if it is entailed by a world proposition.

$\mathbf{LB}^t$ $\forall_t p(\Diamond p \leftrightarrow \exists w(\text{World}\, w \wedge w \leq_t p))$

As with Brouwer's principle, we might also consider the necessitation of the Leibniz biconditionals, $\Box\mathsf{LB}^t$. The necessitation is stronger and equivalent to the claim that *being possible* is the same as *being true at a possible world.*

$\mathbf{LB}^{t=}$ $\Diamond =_{t\to t} \lambda p \exists w(\text{World}\, w \wedge w \leq_t p)$

It is worth noting that the possible worlds metaphysics encoded in LB^t is a substantive further commitment—it is not already a theorem of Classicism. Indeed, it doesn't follow from the Barcan formula, or even the Brouwerian axiom.[55]

Theorem 12. LB^t *is not a theorem of* $\mathsf{C5}$*.*

I have here brushed over an important choicepoint that arises in contexts where the propositional Barcan formula, $\mathsf{BF}^t_\Box$, fails. In this setting there could be 'new' questions concerning the truth of propositions that do not in fact exist: in that case, we might want to consider a strengthening of our definition

[54] Whether world propositions simply *are* possible worlds, as Prior and Fine maintained (Prior and Fine (1979)), or simply guaranteed by the existence of possible worlds will not be important in what follows. Once you have taken enough possible world machinery seriously, including notions like *possible world* and *true at*, then for any world w, there is the proposition that *every proposition true at w is true simpliciter.* Propositions of this form can play the role of world propositions in what follows.

[55] See the first model described in appendix D of Bacon and Dorr (forthcoming).

of World ensuring that worlds necessarily settle all the questions, even new ones. This strengthening can be obtained by prefixing a $\Box$ to the second conjunct in our definition: a strong world is possible and *necessarily* settles every question.[56]

$$\text{SWorld} := \lambda w.(\Diamond w \wedge \Box\forall_t p(w \leq_t p \vee w \leq_t \neg p))$$

Anything that's a strong world is a world, and the result of replacing world with strong world in LB^t yields a strengthening we will call the Strong Leibniz Biconditionals:[57]

SLB^t $\forall_t p(\Diamond p \leftrightarrow \exists w(\text{SWorld}\, w \wedge w \leq_t p))$

I myself am of the view that stronger notion of world better fits the notion at issue in possible world metaphysics. But since the results I prove here do not need the full strength of the strong Leibniz biconditionals, I'll work with the weaker notion in this section. Theorems we prove later from the Leibniz biconditionals thus can also be proven with the strong Leibniz biconditionals so that nothing turns on our choice about how to define world.

Like other principles we have encountered, such as the Barcan formula, there are generalizations of the Leibniz biconditionals to other types. For instance, a property theoretic version states that a property is possible (i.e. possibly instantiated) iff it is entailed by a world property. In general:

$\mathsf{LB}^{\overline{\sigma}}$ $\forall_{\overline{\sigma}\to t} R(\Diamond_{\overline{\sigma}} R \leftrightarrow \exists_{\overline{\sigma}\to t} W(\text{World}_{\overline{\sigma}}\, W \wedge W \leq_{\overline{\sigma}\to t} R)$

where these notions are defined as follows.

Definition 6.1. *Let $\overline{x}$ be a sequence of variables $x_1...x_n$ of types $\overline{\sigma} = \sigma_1, ..., \sigma_n$.*

$$\Diamond_{\overline{\sigma}} := \lambda R \Diamond \exists \overline{x} R \overline{x}$$

$$\neg_{\overline{\sigma}} := \lambda R \lambda \overline{x} \neg (R\overline{x})$$

$$\text{World}_{\overline{\sigma}} := \lambda W(\Diamond_{\overline{\sigma}} W \wedge \forall_{\overline{\sigma}\to t} S(W \leq_{\overline{\sigma}\to t} S \vee W \leq_{\overline{\sigma}\to t} \neg_{\overline{\sigma}\to t} S))$$

For those used to thinking in the possible worlds framework, an intension of type $e \to t$ (i.e. a function from worlds to extensions) is a world property at a given world w if it has a non-empty extension at exactly one world that's

[56] See Bacon (forthcoming) chapter [REF].

[57] The right-to-left direction of LB^t is in fact a theorem of Classicism, since anything entailed by a possible proposition (such as a world proposition) must be possible. The left-to-right direction of LB^t follows from SLB^t, for if p is possible it is entailed by a strong world it is entailed by a world, since every strong world is a world.

possible relative to w, and at that world its extension contains exactly one individual. Thus $\mathsf{LB}^{e\to t}$ is valid in model theories where the second-order quantifiers range over arbitrary functions from worlds to extensions.

I take it that the Leibniz biconditionals are also part of the 'orthodox' view about modal reality, found in, for instance, Lewis and Stalnaker.[58] We are now in a position to state our second connection between width contingency and the structure of modal reality:

> Width contingency requires possible failures of the Leibniz biconditionals.

What implications do the Leibniz biconditionals have for mathematical modalities? Firstly we can show that if something is mathematically possible then it is true at a mathematically possible world.[59]

Theorem 13 ($\mathsf{C}^{\blacksquare}$)**.** *Given* LB^t, $\blacklozenge p \leftrightarrow \exists w(\text{World}\, w \wedge w \leq p \wedge \blacklozenge w)$

Since $\mathsf{C}^{\blacksquare}$ only adds to Classicism the assumption that $\blacksquare$ is an infinitely closed necessity, it is a quite general theorem of Classicism with the Leibniz biconditionals that for any infinitely closed modality, X, a proposition is X-possible iff it is true at an X-possible world. (It does not hold for necessities that are not infinitely closed. Supposing, again, that having chance 1 is a necessity, then one can have chance-possible propositions that are not true at any chance-possible world propositions. For instance, its chance-possible that our dart hits the dartboard, because it has non-zero chance. But each broadly possible world where it hits the dartboard has chance 0, since a broadly possible world will settle the exact point that that the dart lands.)

We can now prove that the Leibniz biconditionals imply the rigidity of each stage of sets.

Theorem 14 ($\mathsf{C}^{\blacksquare\in}$)**.** $\mathsf{LB}^{t\to t}$ *and* LB^t *imply that* V_α *is rigid for every ordinal* α.

There is a way of glossing this argument with quantification over 'possible sets', which is strictly speaking inaccurate but which nonetheless gives an intuition for what is going on. The idea is to find, for any possible set, x, a world property W that applies to just that set. From W we can define an actual set, y, containing just those actual things that would have belonged

[58]They are explicitly postulated, or derived, in the theories of Williamson (2013), Fritz, Goodsell and Yli-Vakkuri (MS).

[59]Unless otherwise stated, proofs of all numbered theorems and propositions to follow may be found in appendix A.

to the W set, if W had been instantiated. Now the members of y are all of lower rank, so we may assume for induction that the actual sets of that rank are in fact the only possible sets of that rank, so x and y have the same members, are identical, and thus that x actually exists. The proof in appendix A is essentially an attempt to make this informal idea precise without any illegitimate quantification.

As before, we can obtain as two straightforward corollaries from the rigidity of V_α the determinacy of the continuum hypothesis, and the necessity of uncountability (and so a refutation of Countabilism).

Corollary 15 ($\mathsf{C}^{\blacksquare\in}$)**.**

1. $\mathsf{LB}^t \wedge \mathsf{LB}^{e\to t} \to \blacksquare CH \vee \blacksquare\neg CH.$
2. $\mathsf{LB}^t \wedge \mathsf{LB}^{e\to t} \to \forall_e x(\text{Uncountable}\, x \to \blacksquare\, \text{Uncountable}\, x)$

Corollary 16 ($\mathsf{C}^{\blacksquare\in}\mathsf{LB}^\sigma$)**.**

1. $\blacksquare CH \vee \blacksquare\neg CH.$
2. $\forall_e x(\text{Uncountable}\, x \to \blacksquare\, \text{Uncountable}\, x)$

Observe that our proof rested not only on the existence of world propositions, but also on a slightly less familiar consequence of naïve use of the possible worlds framework — the existence of world properties. The propositional Leibniz biconditionals do not appear to entail the property Leibniz biconditionals. In light of this, I offer another route to property Leibniz biconditionals using a strengthening of the axiom of choice. An ordinary second-order choice principle can be formulated by saying that the universe of individuals can be well-ordered. By necessitating this principle we ensure that there is a well-order of the universe at every possible world, although it might witnessed by 'new' well-orders—that is to say, a world w might entail that there is a global well-order, while there is no relation such that w entails that *it* is a global well-order. The strengthening of necessitated choice we will investigate is the idea that for each world, there is a relation which that world entails to be a well-order

Strong Modal Choice $\forall_t w(\text{World}\, w \to \exists_{e\to e\to t} R\, w \leq \text{WO}\, R)$

With this principle we can close the gap between the propositional and property Leibniz biconditionals.

Theorem 17 (Classicism)**.** *Strong Modal Choice and* LB^t *entail* $\mathsf{LB}^{\overline{\sigma}\to t}$*.*

It should be noted that there could be width contingentists who reject the necessity of the axiom of choice on the grounds that it, like the continuum hypothesis, is indeterminate or mathematically unsettled. This would, of course, be grounds to reject the stronger principle of De Re Modal Choice. However this is a minority view, and most mathematicians take the axiom of choice to be settled and in as good a standing as other principles of set theory. The necessity of choice is validated, for instance, in the framework of Hamkins, where $\Box$ is interpreted as meaning truth in all generic forcing extensions, since the truth value of the axiom of choice (unlike CH) is preserved in generic extensions.

7 Free logic

Our theory $\mathsf{C}^{\blacksquare}$—Classicism plus the claim that $\blacksquare$ is a necessity that is closed under infinitary consequence—has lead us to some striking results. First, Classicism, in virtue of being closed under classical quantificational logic and necessitation for broad necessity, proves the broad necessity of existence, NNE^e, and a closely related principle, CBF^e. Second, supplementing Classicism with the principles of $\mathsf{S5}$ for the broadest necessity lets us derive the non-contingency of the set theoretic universe up to a given stage given some modest assumptions of modal set-theory. Third, supplementing Classicism with the Leibniz biconditionals lets us do the same.

Could the lover of width contingency restore orthodoxy in the second and third respects, by rejecting it in the first respect? That is, could they retain $\mathsf{S5}$ and the Leibniz biconditionals by weakening quantificational logic and adopting instead a free logic for the quantifiers? Unlike in classical logic, it is not possible to derive the necessity of existence or the converse Barcan formula in free logic. Moreover, the Prior-Lemmon proof of the Barcan formula within $\mathsf{S5}$ is not sound in free logic. In short, things can come and go into existence freely once classical quantificational logic is weakened, giving us more options for making sense of mathematical contingency about which sets exist.

Classicism individuates propositions, properties and relation by provable equivalence in classical higher-order logic. So in order to explore this idea, we should look into the parallel theory that individuates entities instead by provable equivalence in free logic. That is we weaken the quantificational axioms of H along the lines of a free logic and close under the rule of equivalence. Call this system *Free Classicism*, or FC—it is defined in appendix B. Within this framework one can provide definitions of broad necessity and other notions of section 3. Strengthening this system with the principles of $\mathsf{S5}$ and the Strong

Leibniz Biconditionals yields the system being proposed, which we can call FC5(SLB). The reader can find the details in appendix B.

There is a vast literature on the topic of contingent existence in the framework of higher-order logic that I will not attempt to contribute to.[60] I will limit myself instead to a couple of local points about its application to mathematical contingency.

First, the view under consideration must not only reject the necessity of existence, but must do so for mathematical objects like sets in particular. But we can now show that this is inconsistent with the pervasive—and I think independently attractive—idea that a set is determined by its members (which we alluded to in section 5). This idea is articulated in various ways in contemporary philosophy — sometimes it is the idea that the existence of a set is completely grounded in the existence of its members, or that a set is 'nothing over and above' its members.[61] According this idea, while a set could fail to exist at a world if one of its members fails to exist, if all of its members at that world exist, the set itself must exist. More generally, if a proposition (a world proposition or otherwise) entails the members of x exist it must also entail x exists:

$$\forall_e x(\mathrm{Set}\, x \to \forall_t p(\forall_e y \in x(p \leq \exists_e z.z = y) \to p \leq \exists_e z.z = x))$$

If the proposition is tautologous, we can infer that if the members of a set necessarily exist, then so does the set

$$\forall_e x(\mathrm{Set}\, x \to \forall_e y \in x\Box(\exists_e z.z = y) \to \Box(\exists_e z.z = x))$$

We can derive the inconsistency as follows. Suppose that there is a set, x, that doesn't necessarily exist. By the well-foundedness of membership, we may assume without loss of generality that x is a possible non-existent of minimal rank, so that all of its members necessarily exist. But then we have a contingently existing set whose members necessarily exist. Assuming the necessity of our principles about set existence, and the well-foundedness of membership, this reasoning can be necessitated so necessarily every set necessarily exists. Given the principles of S5 this lets us derive the Barcan and converse Barcan formulas for quantification restrict to sets, allowing us to reconstruct our arguments from section 5 and prove the non-contingency of set-theoretic claims.

[60]See Fine (1977), Williamson (2013), Stalnaker (2012), Fritz and Goodman (2016), Fritz (2018a), Fritz (2018b).

[61]See Fine (1994). Roberts (2022) also articulates precisely the related idea that pluralities are nothing over and above their members.

My second point relates to the previously mentioned fact that we can define 'outer-quantifiers' in Free Classicism along the lines Fine's definition (see theorem 24 in appendix B). The assumptions built into $\mathsf{FC5(SLB)}$—specifically the assumption that every possible proposition is entailed by a strong world—ensures that they are classical quantifiers. This means, among other things, that they satisfy the converse Barcan formula and prove the necessity of existence. As have observed already, we need additional modal assumptions about the modal logic of $\Box$ to show that classical quantifiers satisfy the Barcan formula—the $\mathsf{S5}$ principles—but these are built into $\mathsf{FC5(SLB)}$ as well. Indeed, not only is every theorem of Classicism derivable with respect to the outer quantifiers in $\mathsf{FC5(SLB)}$, but also the theorems we get by adding $\mathsf{S5}$ and the Strong Leibniz Biconditionals to Classicism.

Theorem 18. $\mathsf{FC5(SLB)}$ *interprets* $\mathsf{C5(SLB)}$.

Thus, for every theorem of $\mathsf{C5(SLB)}$ there is a corresponding a theorem (under translation) of $\mathsf{FC5(SLB)}$.

Now, as we have pointed out in section 3, the free logician can read this paper by interpreting our uses of the quantifiers $\forall$ and $\exists$ stipulatively, so that they satisfy classical laws, severing any connection between these symbols and ordinary quantificational idioms of English, like 'all' and 'some'. But in doing so, they may find the *non-logical* assumptions we appealed to—like Set Rigidity and The Necessity of Set theory—to be no longer motivated or appealing. Free logicians who accept the 'being constraint' maintain that an individual ceases to have any properties or stand in any relations when it doesn't exist. They may, then, object to Set Rigidity on the grounds that a possible set that does not actually exist does not actually have any members (even if those members actually exist), although had it existed it would have had members.

Note this is a problem for Set Rigidity even interpreted in terms of the inner quantifiers. For as we pointed out earlier if there are sets that exist contingently (in the free logicians sense of 'exist') and one cannot have members unless you exist, then such sets contain their members only contingently. I am myself inclined to apply modus tollens to this argument and conclude that, if we are to hold fixed the being constraint, we should simply accept the necessary existence of sets for the idea that sets are rigid is non-negotiable. This is not the place to offer a full discussion of the being constraint. Here I will just observe that it's generally acknowledged that it must be restricted when it comes to logical words if we wish to preserve other aspects of classical logic whilst accommodating contingent existence in various logical types.[62] Already

[62]This point is made in Fritz and Goodman (2016), footnote 14.

in the first-order case we see some proponents of the being constraint relaxing it for the identity relation, in order to preserve the necessity of the reflexivity of identity, $\forall_e x \Box x =_e x$. In the higher-order setting, these sorts of considerations carry over to the truth functional connectives and quantifiers: if we want, for instance, the conditional axiom $p \to p$ to be necessarily true, even if p had not existed, i.e. $\forall_t p \Box(p \to p)$, standing in the $\to$ relation should not require existence. Systems that relax classical propositional logic are unwieldy and not within the scope of this discussion.[63] Exceptions to the being constraint are infectious. It would be arbitrary to allow exceptions for the logical words we happened to take as primitive, like $\to$ and $\forall_\sigma$, but not the logical words we didn't like existential quantification and conjunction ($\lambda F(\neg\forall_e x \neg Fx)$ and $\lambda pq.\neg(p \to \neg q)$).So the being constraint should really be relaxed for any closed term that is defined in purely logical terminology—including familiar logical operations, like existential quantification, and less familiar ones that can nonetheless be defined in purely logical terms, like ternary conjunction, 'there are finitely many Fs', and so on.

Exceptions to the being constraint are therefore pervasive in the language of pure higher-order logic. I will take this as an excuse to set it aside, at least when we are reasoning in a purely logical language. It is striking how much can be stated in purely logical terms. In this paper we have concerned ourselves with the set-theoretic continuum hypothesis, which is stated in terms of the non-logical predicate $\in$. However, there is another purely logical claim that is closely related to the set-theoretic continuum hypothesis. Let's call it the higher-order continuum hypothesis. It is possible in higher-order logic to say that a property's extension is (i) countably infinite, (ii) that is has the size of the first uncountable infinity (there is a bijection between it and the well-orders-up-to-isomorphism on a countably infinite property) and (iii) has the size of the continuum (there is a bijection between it and subproperties-up-to-extension of a countably infinite property). We call these properties $\aleph_0$, $\aleph_1$ and Continuum. See Shapiro (1991) p105. Then we may formulate the continuum hypothesis as follows:

Higher-Order CH $\forall_{e\to t} X(\text{Continuum}\, X \leftrightarrow \aleph_1 X)$

Higher-order CH entails the set-theoretic continuum hypothesis, since if x is an uncountable set of real numbers, the property of belonging to x, $\lambda y.y \in x$, must be at least $\aleph_1$ sized, and at most continuum sized, and so Higher-Order CH implies it is continuum sized.

[63] The only developed system I am aware of that takes this route is Prior's system Q (see Prior (1955)); some fairly decisive problems are discussed in Menzel (1991).

What does our free logician say about higher-order CH? Of course in Free Classicism, properties and relations, like sets, can exist contingently: what subproperties a countably infinite property has may exist contingently, and the relevant bijective relations could also fail to exist making it very plausible that one could construct models in which Higher-order CH is contingent. But suppose we consider yet another variant of the continuum hypothesis, now formulated using the classical outer quantifiers. Let us write A^* for the result of replacing each occurrence of the free quantifiers in A with the corresponding outer quantifier. We are now concerned with (Higher-Order CH)*.

One might reasonably ask what relation this sentence bears to the mathematical question of the continuum hypothesis. For that is formulated in familiar quantificational terms, whereas we have granted that the outer quantifiers may bear no relation to ordinary quantificational words, as they appear in ordinary English or in mathematics. I won't insist that we refer to (Higher-Order CH)* as a 'version of the continuum hypothesis'. However, the question of whether it is true or not is nonetheless something that can be raised and investigated in the pure language of higher-order logic. And, like the set-theoretic continuum hypothesis and its vanilla higher-order variant, it does not seem to be something we can settle using any mathematical or logical methods presently available to us. The reasons we have to think that Higher-order CH is indeterminate extend to (Higher-Order CH)*.

The problem we are presented with is this. If we add to Classicism the principles of $\mathsf{S5}$ and the Leibniz biconditionals (or the Strong Leibniz Biconditionals) one can prove the following schema, stating that there is no broad contingency in things stated in purely logical terms:

No Pure Contingency $P \to \Box P$, where P is closed and contains no non-logical vocabulary.

If purely logical statements cannot be broadly contingent, they cannot be mathematically contingent either. The argument for this is due to Zach Goodsell.[64]

Theorem 19 (Goodsell)**.** $\mathsf{C5(LB)}$ *proves No Pure Contingency.*

But given theorem 19, for every theorem of $\mathsf{C5(LB)}$ translates to a theorem of $\mathsf{FC5(SLB)}$ implying that (Higher-Order CH)* is not broadly contingent (and consequently is not indeterminate or mathematically contingent).

[64] A proof is presented in Bacon and Dorr (forthcoming)[REF].

8 Conclusion

I have argued that certain kinds of set-theoretic contingency require surrendering two pieces of modal orthodoxy: that the broadest necessity has a logic of $\mathsf{S5}$, and the Leibniz biconditionals connecting what is possible with what holds at some maximally specific possibility. Both of these modal doctrines deserve some scrutiny. The simplest kind of model of modal logic employs possible worlds, and treats the broadest necessity as quantifying over unrestrictedly over all worlds in the model, so it is easy to see where the orthodoxy may have originated. But model theory alone does not make for a positive argument. We now know how to model modal logic without building in either of assumptions.[65] One of these generalizations, possibility semantics—which replaces the complete worlds of possible world semantics with incomplete possibilities—was in fact implicit in Cohen's original papers introducing the forcing method of the independence.[66] Indeed, there are several positions in higher-order metaphysics that require rejecting $\mathsf{S5}$ for the broadest necessity—the philosophical terrain here is still largely unexplored.[67] But before we can sign off on width contingency, we need some guarantee that there aren't any unforeseen inconsistencies in the view. A strong version of width contingency maintains, putting it informally, that all forcing extensions of the set-theoretic universe are mathematically possible—the principle I earlier called Forcing Possibilism. I conclude, then, with the following theorem ensuring that no such inconsistency can be found.[68]

Theorem 20. *Forcing Possibilism is consistent with* $\mathsf{C}^{\blacksquare\in}$.

[65] Sometimes it is argued that the broadest necessity must be modeled by a universal accessibility relation (see for instance Lewis (1986)). A similar argument can be made in the possibility framework. But this appeal to model theory is questionable, and ignores the possibility that which 'worlds' of the model represent genuine possibilities might itself be contingent, and so depend on what world you are evaluating at. For further discussion of these sorts arguments, see Bacon (2018a) §5.4.

[66] Work on possibility semantics for modal logic was initiated in Humberstone (1981), and has been continued more recently by Holliday and coauthors (see, for instance, Holliday (forthcoming)). Prior even to possible world semantics, we had the algebraic approach to modal logic, found in Bjarni Jonsson (1953), that makes no assumptions akin to possible worlds.

[67] Bacon (forthcoming) chapter [REF], and Bacon and Dorr (forthcoming) section 2.4-2.6 overview some of the options here.

[68] The proof of this theorem will have to wait until a future occasion.

A Appendix: Proofs of Theorems

Theorem 5. ($\mathsf{C5}$)
$\forall X(\mathrm{Nec}_\infty X \to \mathsf{BF}^\sigma_X)$

Proof. $\mathsf{C5}$ contains the broad Barcan formula, $\mathsf{BF}^\sigma_\Box$.

Suppose that X is infinitely closed and that $\forall_\sigma x\, X(Fx)$. We want to show that $X(\forall_\sigma x Fx)$. Since X is infinitely closed, it suffices to show that anything entailing every X-necessary proposition also entails $\forall_\sigma x Fx$. Suppore r entails every X-necessary proposition. Since Fx is X-necessary for every x, $\forall_\sigma x.\Box(r \to Fx)$. By the broad Barcan formula, $\Box\forall_\sigma x(r \to Fx)$ and so $\Box(r \to \forall_\sigma x Fx)$. Thus r entails $\forall_\sigma x Fx$ as required. Since X is closed under entailment, $X(\forall_\sigma x Fx)$. $\Box$

Theorem 7. ($\mathsf{C}^{\blacksquare\in}$)
Given $\mathsf{BF}^e_\Box$ (for broad necessity), being of stage α (i.e. V_α) is rigid for every ordinal α.

Proof. As we have noted (theorem 5), $\mathsf{BF}^e_\Box$ for broad necessity implies the Barcan formula for $\blacksquare$, $\mathsf{BF}^e_\blacksquare$. Subsequent uses of the word 'possibly' and 'necessarily' in the proof refer to $\blacklozenge$ and $\blacksquare$.

The proof is by transfinite induction. V_0 is necessarily empty, and so vacuously rigid.

Suppose that α is an ordinal, and for each $\beta \in \alpha$, V_β is rigid. We want to show that V_α is rigid. Suppose there could have been an F set of stage α ($\blacklozenge\exists x(V_\alpha x \wedge Fx)$). We must show there is in fact a set of stage α that could have been F ($\exists x(V_\alpha x \wedge \blacklozenge Fx)$). By the Barcan formula there is some x that is possibly of stage α and F, so it suffices show that this set is in fact of stage α. That is, we must prove that if $y \in x$, ys rank is less than α, so suppose $y \in x$. By Set Rigidity, y is necessarily in x, and since x is possibly of stage α, y is possibly of stage β for some $\beta \in \alpha$: $\blacklozenge\exists\beta \in \alpha.V_\beta y$. Since, by Set Rigidity, belonging to α is a rigid property, for some $\beta < \alpha$ y possibly is of stage β. But by the inductive hypothesis being of stage β is rigid, so y is in fact of stage β.

For the converse direction we must show that if something, x say, of stage α is possibly F, then it's possible that something of stage α is F. It would suffice to show that x is necessarily of stage α. Every $y \in x$ is of a stage β for $\beta < \alpha$, and so by the inductive hypothesis, y necessarily of stage β. Since β is necessarily less than α, every $y \in x$ is necessarily of stage less than α. By Set Rigidity this implies that, necessarily, every $y \in x$ is of stage less than α, i.e. necessarily x is of stage α.

$\Box$

Lemma 21 ($\mathsf{C}^{\blacksquare\in}$)**.** *Sets are mathematically necessarily distinct:* $\forall_e xy(\mathrm{Set}\,x \wedge \mathrm{Set}\,y \rightarrow x \neq y \rightarrow \blacksquare x \neq y)$

Proof. Suppose the claim is false for contradiction. Choose x to be $\in$-minimal such that x possibly identical to some set it is distinct from. Choose y to be $\in$-minimal such that it is distinct from, but possibly identical to x.

Since x and y are distinct we may suppose, without loss of generality, that there is some set z belonging to x but not belonging to y. By Set Rigidity, $\blacksquare z \in x$. So $\blacklozenge z \in y$, since $\blacklozenge x = y$. Since $\blacklozenge \exists z' \in y.z' = z$ it follows by Set Rigidity that $\exists z' \in y \blacklozenge z' = z$. Since x is an $\in$-minimal failure of the necessity of distinctness, z cannot be possibly identical to anything distinct from it. It follows that whatever member of y that is possibly identical to z is in fact identical to z, so that z is a member of y after all, a contradiction. □

Theorem 8. ($\mathsf{C}^{\blacksquare\in}$)
Suppose $A(\overline{x})$ is a first-order formula with free variables $\overline{x}$. If all the quantifiers in $A(\overline{x})$ are restricted to rigid properties of sets, then A is modally absolute with respect to the parameters $\overline{y}$.

Proof. By Set Rigidity, $x \in y$ is modally absolute, since if y is a set and $x \in y$ then by Set Rigidity x is necessarily in y. And if $x \notin y$ and y is a set, then by the necessity of distinctness of sets x could not be identical to a member of y. The necessity of identity and distinctness for sets ensures the modal absoluteness of $x = y$. Suppose A and B are modally absolute. If for any sequence of sets $\overline{x}$, $A(\overline{x})$ and $B(\overline{x}$, then the modal absoluteness of A and B ensures that $\blacksquare A(\overline{x})$ and $\blacksquare B(\overline{x})$ and so $\blacksquare(A(\overline{x}) \wedge B(\overline{x}))$. Similarly if $\neg(A(\overline{x}) \wedge B(\overline{x}))$ either $\neg A(\overline{x})$ or $\neg B(\overline{x})$ and so given the modal absoluteness of A and B we have either $\blacksquare\neg A(\overline{x})$ or $\blacksquare\neg B(\overline{x})$ and in either case $\blacksquare\neg(A \wedge B)$ as required. The disjunction case is a dualization of the above, and the negation case is trivial.

Now suppose $B(y\overline{x})$ is modally absolute, and $\lambda y.A(y\overline{x})$ is a rigid property of sets ($\lambda y.A(y\overline{x})$ entails Set). We will show the modal absoluteness of $\forall_e y(A(y\overline{x}) \rightarrow B(y\overline{x})$. Let $\overline{x}$ be a sequence of sets, and suppose $\forall_e y(A(y\overline{x}) \rightarrow B(y\overline{x})$. By the modal absoluteness of B we can conclude $\forall_e y(A(y\overline{x}) \rightarrow \blacksquare B(y\overline{x}))$, and by the rigidity of A we can get $\blacksquare\forall_e y(A(y\overline{x}) \rightarrow B(y\overline{x})$. On the other hand, if $\neg\forall_e y(A(y\overline{x}) \rightarrow B(y\overline{x})$ then for some set y, $(A(y\overline{x}) \wedge \neg B(y\overline{x})$. By the modal absoluteness of B, $\blacksquare\neg B(y\overline{x})$ and by the rigidity of A, $\blacksquare A(y\overline{x})$ so $\blacksquare\exists_e y(A(y\overline{x}) \wedge \neg B(y\overline{x}))$, as required. The existential case involves dualizing this argument. □

Theorem 9. ($\mathsf{C}^{\blacksquare\in}$)

Given the truth of the theorems of $\mathsf{C}^{\blacksquare\in}$, the following formulas are modally absolute.

1. *being an ordinal.*

2. *being a limit ordinal.*

3. *being the smallest limit ordinal, the successor of the smallest limit ordinal, the successor of the succcessor of the smallest limit ordinal...*

moreover, the properties in 3. are rigid.

Proof. α is an ordinal if and only if α is (i) transitive $\forall x \in \alpha \forall y \in x.y \in \alpha)$ and (ii) linearly ordered by membership $(\forall x \in \alpha \forall y \in \alpha(x \neq y \rightarrow x \in y \vee y \in x)$. All the quantifiers in these definitions are restricted by conditions of the form $\in z$, which is rigid by Set Rigidity, and entails sethood (by the definition of Set as $\lambda y \exists x.y \in x)$. Thus they are all modally absolute.

α is a limit ordinal if it is an ordinal and additionally $\forall x \in \alpha \exists y \in \alpha(x \in y)$ and $\exists x \in \alpha$. These have the same property. α is the smallest limit ordinal iff it is a limit ordinal, and for every $x \in \alpha$ x is not a limit ordinal. α is the successor of the smallest limit ordinal iff every member of α is either belongs to the smallest limit ordinal or is identical to it. Again, all quantifiers are restricted by membership to some set.

Finally we can show that the properties in 3 are rigid. Let ω be the set that is actually the smallest limit ordinal. By the modal absoluteness, ω is necessarily the smallest limit ordinal, and uniquely so, since is a theorem of ZF that if two sets are the smallest limit ordinal they are identical. Suppose it is possible that something is the smallest limit ordinal is also F. Then it is possible that ω is F, and thus there is an actual smallest limit ordinal, ω, which is possibly F. Similar strategies apply to the other properties listed in 3. $\square$

Theorem 11. ($\mathsf{C5}^{\blacksquare\in}$)

1. $\blacksquare CH \vee \blacksquare \neg CH$.

2. $\forall_e x(\text{Uncountable}\, x \rightarrow \blacksquare\, \text{Uncountable}\, x)$

Proof. Let $V_{\omega+2}y$ be the property 'λy.for some set α, α is the successor of the successor of the smallest limit ordinal, and y is V_α'. Using the results above, it is easily seen that this property is rigid.

The continuum hypothesis can be formulated in such a way that all quantifiers are restricted by the predicate $V_{\omega+2}$. Since this predicate is rigid, CH is modally absolute: $CH \to \blacksquare CH$ and $\neg CH \to \blacksquare CH$. This establishes 1.

Let x be an uncountable set, and suppose that α is an ordinal such that $x \in V_\alpha$. Then the claim that $x \in V_\alpha$ and is uncountable is equivalent to the claim that $x \in V_\alpha$ and there is no set of ordered pairs belonging to $V_{\alpha+3}$ that is an injective function from the smallest limit ordinal to x. All of the quantifiers in this claim are similarly restricted to rigid properties.

□

Theorem 13. ($\mathsf{C}^{\blacksquare}$)
Given LB^t, $\blacklozenge p \leftrightarrow \exists w(\text{World}\, w \wedge w \leq p \wedge \blacklozenge w)$

Proof. Mathematical Necessity states that anything entailed by the $\blacksquare$-necessities must be itself $\blacksquare$-necessary. So any $\blacklozenge$-possibility is such that its negation is not entailed by the $\blacksquare$-necessities.

Thus if $\blacklozenge p$, $\blacksquare \not\leq \neg p$. That is, for some r such that $\forall q(\blacksquare q \to r \leq q)$, $r \not\leq \neg p$. This means $\Diamond(r \wedge p)$, so by LB^t, there is a world proposition w that entail $r \wedge p$. We finally can see that w must be $\blacklozenge$-possible. For if not, then $\blacksquare\neg w$, and since r entails every $\blacksquare$-necessity, $r \leq \neg w$. But since $w \leq r$, $w \leq \neg w$, contradicting the assumption that w is a world.

The right-to-left direction is obvious. □

Theorem 14. ($\mathsf{C}^{\blacksquare\in}$)
$\mathsf{LB}^{t\to t}$ *and* LB^t *imply that* V_α *is rigid for every ordinal* α.

Proof. The proof is by transfinite induction. V_0 is necessarily empty, and so vacuously rigid.

Suppose that α is an ordinal, and for each $\beta \in \alpha$, V_β is rigid. We want to show that V_α is rigid. Suppose $\blacklozenge\exists x(V_\alpha x \wedge Fx)$. We must show $\exists x(V_\alpha x \wedge \blacklozenge Fx)$.

Since $\lambda x(V_\alpha x \wedge Fx)$ is broadly possibly instantiated, it follows by the Leibniz Biconditionals, $\mathsf{LB}^{e\to t}$, that there is a world property W that that entails it, and by proposition 13 it will be a world property that is mathematically possibly instantiated.[69] We can use this world property to define the actual member of V_α that's possibly F explicitly:

$$x' := \{y \in \bigcup_{\beta\in\alpha} V_\beta \mid \blacksquare\forall x(Wx \to y \in x)\}$$

[69] $\blacklozenge\exists x(V_\alpha x \wedge Fx)$ implies by theorem 13 that there is a mathematically possible world proposition $w \leq \exists x(V_\alpha x \wedge Fx)$, and since $\Diamond\exists x(w \wedge V_\alpha x \wedge Fx)$ there is a world property W entailing $\lambda x(w \wedge V_\alpha x \wedge Fx)$.

Roughly W singles out a merely possible set. x' is the set of ys in V_α that would have belonged to the merely possible object picked out by W if it had existed. We can now show that x' is identical to the merely possible W: i.e. we show $\blacksquare\forall x(Wx \to x = x')$. Given the mathematical necessity of Set Extensionality and the mathematical possibility of W it suffices to show that necessarily whatever is W is coextensive with x': $\blacksquare\forall x(Wx \to \forall y.(y \in x \leftrightarrow y \in x')$. We break this up into two claims:

1. $\blacksquare\forall x(Wx \to \forall y(y \in x' \to y \in x)$
2. $\blacksquare\forall x(Wx \to \forall y(y \in x \to y \in x')$

We establish 1 first. From the definition of membership in x', we immediately have $\forall y \in x'\blacksquare\forall x(Wx \to y \in x)$. Since Sets are Rigid, it follows that x'-restricted quantification satisfies BF, so we can infer $\blacksquare\forall y \in x'\forall x(Wx \to y \in x)$. By applying first-order logic under the scope of $\blacksquare$, this is equivalent to 1.

To establish 2, we first show $\forall\beta \in \alpha\forall y(V_\beta y \to \blacksquare\forall x(Wx \to (y \in x \to y \in x'))$. Let $\beta \in \alpha$ and let y be an arbitrary set of rank β. Now either $y \in x'$ or $y \notin x'$. Suppose the former. Then by the rigidity of set membership y is necessarily in x' and so $\blacksquare\forall x(Wx \to (y \in x \to y \in x'))$ follows. Suppose then that $y \notin x'$. By the condition for belonging to x', this means that W doesn't entail the property of containing y. Since W is a world property, it must entail the property of not belonging to y, and thus must also mathematically necessitate it: $\blacksquare\forall x(Wx \to y \notin x)$. So this means $\blacksquare\forall x(Wx \to (y \in x \to y \in x'))$, by applying some straightforward logic under the $\blacksquare$ (namely that $y \notin x$ entails $y \in x \to y \in x'$).

This completes the argument that $\forall\beta \in \alpha\forall y(V_\beta y \to \blacksquare\forall x(Wx \to (y \in x \to y \in x'))$. By the inductive hypothesis, V_β is rigid, and so we can infer $\forall\beta \in \alpha\blacksquare\forall y(V_\beta y \to \forall x(Wx \to (y \in x \to y \in x'))$. Since α is a set and sets are rigid, we can also infer $\blacksquare(\forall\beta \in \alpha\forall y(V_\beta y \to \forall x(Wx \to (y \in x \to y \in x'))$. Thus $\blacksquare\forall x(Wx \to \forall y(y \in x \to \exists\beta \in \alpha.V_\beta y \to y \in x')))$ applying first-order logic under $\blacksquare$. Recall that necessarily whatever the W set is, it's V_α: thus, necessarily, whatever the W set is, if y belongs to it, y is in V_β for some $\beta \in \alpha$ (by the definition of V_α). That is we have (a) $\blacksquare\forall x(Wx \to V_\alpha x)$, (b) $\blacksquare\forall x(V_\alpha x \wedge y \in x \to \exists\beta \in \alpha.V_\beta y)$ (by definition of the V relation and the mathematical necessity of ZF). So putting this together $\blacksquare\forall x(Wx \to \forall y(y \in x \to y \in x')))$ as required.

Since W mathematically necessitates being identical to x' ($\blacksquare\forall x(Wx \to x = x')$, and W is mathematically possible, it follows that $\blacklozenge Wx'$. Finally, since W entails F it follows that $\blacklozenge Fx'$. By construction $V_\alpha x'$ so $\exists x(V_\alpha x \wedge \blacklozenge Fx)$ as required. $\square$

Theorem 16. ($\mathsf{C}^{\blacksquare\in}\mathsf{LB}^{\sigma}$)

1. $\blacksquare CH \vee \blacksquare\neg CH$.

2. $\forall_e x(\text{Uncountable}\, x \to \blacksquare\, \text{Uncountable}\, x)$

Theorem 17. (C)
De Re Modal Choice and LB^{t} *entail* $\mathsf{LB}^{\overline{\sigma}\to t}$.

Proof. We show $\mathsf{LB}^{e\to t}$, since that is the instance required for theorem 14, however the proof generalizes trivially.

Suppose that $\Diamond\exists x Fx$. By LB^{t}, there is a world proposition w such that $w \leq_t \exists x Fx$. Let R be a relation which is necessarily a well-order, and consider the property of being the R minimal F while w is true: $W := \lambda x(w \wedge \text{Min}\, RFx)$ where $\text{Min} = \lambda RFx(Fx \wedge \forall y(Fy \to Rxy \vee x = y))$. Clearly W entails F. Let G be another property. Since there is at most one minimal F of a well-order, we know that $\Box(\text{WO}\, R \to \forall x(\text{Min}\, RFx \to Gx) \vee \forall x(\text{Min}\, RFx \to \neg Gx))$, and since $\Box\, \text{WO}\, R$, $\Box(\forall x(\text{Min}\, RFx \to Gx) \vee \forall x(\text{Min}\, RFx \to \neg Gx))$. Since w settles every question it either entails every R-minimal F is G, or that it's not, $\Box(w \to \forall x(\text{Min}\, RFx \to Gx)) \vee \Box(w \to \forall x(\text{Min}\, RFx \to \neg Gx))$. Rearranging a little and appealing to the definition of W this is $\Box\forall x(Wx \to Gx) \vee \Box\forall x(Wx \to \neg Gx)$

$\square$

B Appendix: Free Logic

In this appendix we provide the necessary background for the results discussed in section 7.

Free logic replaces the law of universal instantiation with its universal closure, $\forall_\sigma y(\forall_\sigma x\, Fx \to Fy)$. We must then also add the principle that universal quantification distributes over conditionals. We of course, may apply the analogous substitutions at other types.

Free Instantiation $\forall_\sigma y(\forall_\sigma x\, Fx \to Fy)$ provided y is not free in F.

Quantifier Normality $\forall_\sigma x(A \to B) \to (\forall_\sigma x A \to \forall_\sigma x B)$

The remaining principles of H—Gen, and the laws governing the truth-functional connectives and λ—remain the same. Let FH, 'free higher-order logic', be the result of making these substitutions to H, ad Free Classicism, FC, the result of closing FH under the rule of equivalence.

Because the logic of the quantifiers in Free Classicism is weaker than Classicism, notions we defined using the quantification over all necessities—entailment, broad necessity, world, etc—may behave in undesirable ways. For example, a natural quantificational definition of property entailment in Free Classicism, $\Box\forall_e x\Box(Fx \to Gx)$, is consistent with pathological situations where F entails G, a is F but a is not G.[70] However, in Classicism many of the notions that we defined in terms of the classical quantifiers can be given equivalent definitions in terms of identity, and because the logic of identity in Free Classicism is classical, we can recover the desired behaviour by using the identity-theoretic definitions instead. For instance, there is a long tradition in logic, tracing back to George Boole, of defining entailment in terms of identity. For properties F and G, F entails G when the property conjunction of F with G (i.e. $\lambda x(Fx \wedge Gx)$) just is F.[71] The pathological situation mentioned above cannot arise, for if F entails G and then $F = (\lambda x.Fx \wedge Gx)$. So by Leibniz's law $Fa \to (\lambda x.Fx \wedge Gx)a$, and thus $Fa \to Ga$ by β and propositional logic.

Proposition 22. *In Classicism, the following identities are derivable:*

1. $\Box = \lambda p.p =_t \top$

2. $\leq_{\overline{\sigma}} = \lambda RS(R \wedge_{\overline{\sigma}} S =_{\overline{\sigma}\to t} R)$

3. $\text{SWorld} = \lambda w((w \neq_t \bot) \wedge (\lambda p(w \leq p \vee w \leq \neg p) =_{t\to t} \lambda p.\top))$

Proofs of 1 and 2 may be found in Bacon (forthcoming)p[**?**]. The first conjunct of the RHS of 3, $w \neq_t \bot$, is equivalent to $\Diamond w$ by 1, and the second conjunct is equivalent to $\Box\forall_t p(\top \to (w \leq p \vee w \leq \neg p))$ by 2, and thus to $\Box\forall_t p(w \leq p \vee w \leq \neg p)$.

Perhaps it is possible to augment Free Classicism with further principles that would rule out these pathological situations, but we will avoid the need for any further assumptions by adopting the identity theoretic definitions of these three notions listed in proposition 22 as our official ones when working in Free Classicism.

We can now define a possibilist quantifier along the lines of Fine's definition discussed in section 3:

$$\Pi_\sigma := \lambda F\exists_t w(\text{SWorld}\, w \wedge w \wedge (\lambda x.w \leq_\sigma F)$$

[70] Models of this will interpret a with an individual that does not belong to the domain of quantification at any world. It is quite easy to generate an extensional model of Free Classicism in which $\forall_e x(Fx \to Gx)$ (and thus $\Box\forall_e x\Box(Fx \to Gx)$), Fa and $\neg Ga$ are all true.

[71] Boole (1847), p20 project Gutenberg.

where F has type $\sigma \to t$ and x has type σ. $\Pi_e F$ means that, when w is the true strong world proposition, the vacuous property of being such that w entails F. We have replaced Fine's $\Box\forall_\sigma x\Box(w \to Fx)$—the potentially ill-behaved notion of entailment mentioned above—with the corresponding identity theoretic entailment for reasons detailed above.

We are now in a position to formulate the orthodox possible worlds metaphysics within Free Classicism. We can do this by adding to FC the Strong Leibniz Conditionals and the B schema and closing under the rule of equivalence as well as the background logical rules, remembering, of course, that SWorld, $\leq$, $\Diamond$, etc are now given in identity theoretic terms.

SLB^t $\Diamond A \leftrightarrow \exists_t w(\mathrm{SWorld}\, w \wedge w \leq A)$

B $A \to \Box\Diamond A$

We will call the result $\mathsf{FC5(SLB)}$. Note that because necessitated quantificational claims are weak in Free Classicism, merely adding the necessitations of the universal closures of these principles to Free Classicism would fail to deliver identities that one could obtain from the result of closing under the rule of equivalence. We could acheive the same effect as closing under the rule of equivalence by adding a pair of identities to Free Classicism. The claim that *to be possible is to be true at some possible world*, and the claim that *to be true* entails *to be necessarily possible*.

SLB^t_λ $\Diamond =_{t\to t} \lambda p(\exists_t w(\mathrm{SWorld}\, w \wedge w \leq p)$

B_λ $\lambda p.p \leq_{t\to t} \lambda p.\Box\Diamond p$

$\mathsf{FC5}$ and $\mathsf{FC(SLB)}$ stand for the result adding, in the same way, only one of these principles.

Lemma 23. $\mathsf{FC5(SLB)}$ *contains* $A \to \exists w(\mathrm{SWorld}\, w \wedge w \wedge w \leq_t A)$.

Proof. First we show $\mathrm{SWorld}\, w \to \Box\, \mathrm{SWorld}\, w \wedge \Box(\exists_t p.w = p)$. Since $\mathrm{SWorld}\, w$ is the conjunction of a distinctness claim and an identity claim, the necessity of the first conjunct follows from the necessity of distinctness and the necessity of identity both of which are well-known theorems of $\mathsf{S5}$ with the classical axioms of identity.[72] Using SLB, and the fact that w is necessarily possible, $\Box\exists_t v(\mathrm{SWorld}\, v \wedge v \leq w)$. It's also necessary that for any strong world $v \leq w$, $w \leq v$. For w is necessarily a strong world, and so must entail v or

[72]For the necessity of identity see Kripke (1971), for the necessity of distinctness see Prior and Prior (1955), pp. 206–7.

$\neg v$ for any strong world $v \leq w$, and it couldn't entail $\neg v$ since otherwise $v \leq \neg v$ by the transitivity of entailment, contradicting the fact that v is possible. So necessarily, any strong world entailing w is identical to w, thus $\Box\exists_t v(\mathrm{SWorld}\, v \wedge v =_t w)$.

Now we argue that every strong world, w, entails (i) w, (ii) that w is an existent strong world, and (iii) $A \to (w \leq A)$. (i) is trivial, (ii) is established above. For (iii), $\lambda p.w \leq_{t\to t} \lambda p.(w \wedge (p \to (\lambda p\top)p)$ since $p \to \top$ is a tautology. And since $\lambda p.\top =_{t\to t} \lambda p(w \leq p \vee w \leq \neg p)$ (since w is a strong world) we have $\lambda p.w \leq \lambda p(w \wedge (p \to (w \leq p \vee w \leq \neg p)))$. We also have $\lambda p.w \leq \lambda p(w \wedge p \to w \not\leq \neg p)$ since $w \wedge p \to w \not\leq \neg p$ is a theorem of Free Classicism. Since operator entailment is closed under propositional logic, $\lambda p.w \leq \lambda p(p \to w \leq p)$. Apply both these operators to A and using β we get w and $A \to w \leq A$, and since the former operator entails the latter, $w \leq (A \to w \leq A)$.

Putting (i),(ii) and (iii) together, we have that for every strong world, w, $w \leq (w \wedge \mathrm{SWorld}\, w \exists_t p(p = w) \wedge (A \to w \leq A))$. Using the fact that entailment is closed under free logic we get $w \leq (A \to \exists_t w(w \wedge \mathrm{SWorld}\, w \wedge w \leq A)))$. Since every strong world entails $A \to \exists_t w(w \wedge \mathrm{SWorld}\, w \wedge w \leq A))$ we can infer $\Box(A \to \exists_t w(w \wedge \mathrm{SWorld}\, w \wedge w \leq A)))$ by SLB.

$\Box$

Theorem 24. $\mathsf{FC5(SLB)}$ *interprets* $\mathsf{C5(SLB)}$

Proof. We map each term M of $\mathcal{L}$ to M^*, the result of substituting each free quantifier $\forall_\sigma$ with Π_σ. We wish to show that whenever A is a theorem of Classicism, A^* is a theorem of Free Classicism$^+$.

Each tautology, instance of B, and instance of $\beta\eta$ are mapped to tautologies instances of B or instances of $\beta\eta$. Uses of modus ponens and the rule of equivalence are similarly mapped to themselves. It remains to show that UI* and SLB^* are theorems of $\mathsf{FC5(SLB)}$, and, for Gen, that if $(A \to B)^*$ is a theorem of Free Classicism$^+$, so is $(A \to \forall x B)^*$.

Let's begin with UI. We will show generally that $\Pi_\sigma F \to Fa$. Suppose $\Pi_\sigma F$, so that there is some truth, p, such that $\lambda x(p \wedge Fx) =_{\sigma\to t} \lambda x.p$. Want to show Fa. $(\lambda x.p)a =_t p$ by β, and since p is true, we can conclude $(\lambda x.p)a$. By the above identity, $\lambda x(p \wedge Fx)a$, so $p \wedge Fa$, and finally, Fa as required.

For the right-to-left direction of SLB^* we show the dualized contrapositive version. We will suppose that $\Pi_t w(\mathrm{SWorld}\, w \to w \leq A)$ and show $\Box A$. Expanding the definition of Π, the true strong world, v, is such that $\lambda w.v \leq \lambda w.(\mathrm{SWorld}\, w \to w \leq A)$. Applying $\forall_t$ to both sides we see that the claim that everything is such that v (i.e. $\forall_t p.v)$) entails that every strong world is entails A (i.e $\forall_t w(\mathrm{SWorld}\, w \to w \leq A)$). Since v is true, everything is such that v, and so every strong world entails A. By SLB, $\Box A$.

For the converse of SLB^* suppose $\Sigma_t w(\mathrm{SWorld}\, w \wedge w \leq A)$—i.e. $\lambda w.v \nleq \lambda w(Sw \to w \nleq A)$ where v is a true strong world. We want to show $\Diamond A$. It suffices to show $\exists_t u(\mathrm{SWorld}\, u \wedge u \leq A)$. Suppose for contradiction that $\forall_t u(\mathrm{SWorld}\, u \to u \nleq A)$. By lemma there is a strong world v that is true and entails $\forall_t u(\mathrm{SWorld}\, u \to u \nleq A)$, delivering also the corresponding entailment between vacuous operators: $\lambda w.v \leq \lambda w \forall u(\mathrm{SWorld}\, u \to u \nleq A)$. Since being a strong world entails existence, we have $\lambda w.v \leq \lambda w(\mathrm{SWorld}\, w \to \exists_t r.r = w)$. Since the right-hand-sides of entailments are closed under free logical consequences, we have $\lambda w.v \leq \lambda w(\mathrm{SWorld}\, w \wedge \exists_t r.r = w \to w \nleq A)$ and so $\lambda w.v \leq \lambda w(\mathrm{SWorld}\, w \to w \nleq A)$. This contradicts our assumption.

For Gen it suffices to show that whenever we have a proof of $A \to B$ where x is not free in B there is also a proof of $A \to \Pi_\sigma x B$. Since we can prove $A \to B$, we can prove $(\lambda x(A \to B))y \leftrightarrow (\lambda x.\top)y$ using β and so by the rule of equivalence we then have $\lambda x(A \to B) = \lambda x.\top$.

Now we will show that $A \to \exists w(w \wedge \mathrm{SWorld}\, w \wedge \lambda x.w \leq \lambda x.B$. Suppose A, and let w be the true strong world entailing A (appealing to lemma 23). So $w \wedge \neg A =_t \bot$. Clearly $\lambda x.w \leq \lambda x(A \to B)$ since $\lambda x.w \leq \lambda x.\top$.

$\lambda x(w \wedge (A \to B)) =_{\sigma \to t} \lambda x.w$. The left-hand-side is $\lambda x.((w \wedge \neg A) \vee (w \wedge B))$ using Boolean equivalences that can be obtained from the Rule of Equivalence. Since x isn't free in A and $w \wedge \neg A = \bot$ we can infer the the left-hand-side is $\lambda x.(w \wedge B)$ by Leibniz's law and Boolean equivalences. So $\lambda x(w \wedge B) = \lambda x.w$ as required. $\square$

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