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The Problem of Extension in Natural Philosophy

Abstract

The construction of extension is a long-standing problem in natural philosophy. The primary goal of the paper is to clarify the philosophical concept of extension per se and to develop a simple working combinatorial model to illustrate the idea. Grassmann's exterior algebra is proposed as the combinatorial structure and Mach's elements are interpreted as the content.

Zusammenfassung

Die philosophische Konstruktion der Ausdehnung aus einfachem, nicht ausgedehnten Inhalt ist ein Problem mit langer Herkunft in der Philosophie der Natur. Das Ziel dieses Aufsatzes ist, den philosophischen Begriff der Aufdehnung aufzuklären und ein kombinatorisches Modell des Raumes darzustellen, in welchem Machs Elemente der Inhalt und Grassmanns Algebra der Zusammenhang des Inhalts sind.

1.

The construction of spatial extension from prior notions of quality is a long-standing problem in natural philosophy, although, strangely, the idea receives little play in contemporary thinking about space and time. Straightforwardly, one starts with unextended qualities and relations, to arrive at a solid, extended manifold suitable for grounding the world of physics, or psychology.[1] This philosophical construction of extension is not a physical investigation of an extension already assumed as given, nor is it a mathematical investigation, because mathematics is indifferent to an extended representation of numbers or manifolds, as opposed to some other. The *philosophical* investigation of extended representation must begin from scratch and cannot assume a topology, a coordinate system of numbers, a metric, not even a simple expanse or ready made drafting board on which to do constructions, if these are already considered as extended. We want to know what extended quantity is in general, not a classification of already given extensions and their properties. An initial division of labor avoids the confusion of philosophical and scientific goals, even if they later turn out to be the same.

2.

The reason why philosophers have traditionally used unextended qualities to construct extensions – unfamiliar entities in science or mathematics – is that it is the only way to avoid begging the philosophical questions. One cannot construct extension from things already extended to begin with. Some examples are Leibniz's dynamical construction of extension from within a community of unextended monads and Russell's use of unextended quality overlaps to define a topological manifold of points, defined by the intersection of qualities. Carnap and Goodman used similarity relations and patches of sensations to define perceptual manifolds of sight, touch and so on as similarity classes. To philosophers, these last two versions are probably the most familiar, as are the following objections which have cast doubt on the very idea of space constructions:

Objection 1. The construction is positivistic and can only amount to constructing the world from sense-data. But we know this fails because the world is not *made* of human sense data, but of particles and fields which existed before human experience. There is moreover no reasonable way to translate the findings of science into a purely observational or sense-data language.

Objection 2. Qualities, being mental, have no principle of composition. There is no theory of what they are or how they are connected with one another beyond a vague „what it's like" subjective criterion of identity, similarity and difference (Chalmers, 2003). Moreover, the use of a similarity relation strongly suggests a mentalistic construction instead of a real construction of physical space. In any case, the world in space and time *outside* of us does not hold together because of our subjective similarity standards, but because of its real physical connectedness.

Objection 3. The whole construction is circular if taken as a serious construction of extension, if for no other reason than that qualitative sensations of color or sound are already extended in space and time, or are „at" points somewhere in some pre-existing manifold. (Even Russell's attempts to avoid assuming extension seem to founder when he implicitly calls on a background space to constrain the topological overlaps of quality patches.)

We can meet these objections[2] by adopting the view that quality is *not* automatically mental, but an Aristotelian feature of the natural universe. We might then frame a notion of non-extended, but internally structured, quality to fit the bill along with a combinatorial relation for using them constructively. There are ample precedents. Leibniz's petites perceptions were an early example of objective qualities. Kant also claimed that sensation is the common matter of mental impressions and things in themselves, but *not* their form which is constructed around them by the imagination and understanding (Lockwood 1989, Banks 2005). And Ernst Mach proposed a theory of neutral elements neither mental nor physical (Mach 1959 Ch 1, Banks 2003). Mach's elements were qualities like sensations, but also effective dispositions, or powers, acting on one another. One construction over these elements yielded the sensory fields of psychology; another resulted in physical objects in space and time, but the elements would be the self-same items in both cases, psychology and physics being but two orderings of the same underlying natural content. William James' „pure experiences" and Russell's „unsensed sensibilia" and „event particulars" were based on Mach's elements (Banks 2003, 2004). Carnap also relates that his notion of quality was of the radical empiricist *cum* Machian kind, and he even contemplated a construction parallel to the ill-starred *Aufbau* in which, this time, the elements would be treated as realistic, energetic qualities of physics (Carnap, 1960, Richardson, 1990). Recently, objective qualities have been revived in the philosophy of mind, for example in the work of Grover Maxwell (1978), Michael Lockwood (1989) David Chalmers (1996, 2003), Galen Strawson (2006), myself and many others, but I believe Mach's elements remain the best of these because of their robust causal efficacy and full-blooded membership in the natural world (Banks 2007).

An argument for these neutral elements follows from the traditional mind-brain identity theory. If the qualitative mental processes *are* the corresponding brain states, not superadded to them, then by the symmetry of identity, the „raw look" of energies in the nervous system is actually qualitative. A mass of sensations is what an assembly of physical energies actually *looks like* in nature. In our brains we can observe this fact directly. Of course what we „see" is a resultant of an enormous agglomeration of these physical energies of our nervous systems in specific structural relations, and their relation may matter as much as the constituents do. But it is ridiculous to think our nervous systems must be unique in the universe. So, it seems reasonable to assume that physical energies outside the nervous system also have quality and intensity, but they are probably far more intense and energetic, and the complexes they form no doubt so different as to obviate any panpsychist analogies. But the idea that natural qualities occur *only* in the human nervous system seems ridiculously restrictive, and the idea that sensations are *not* self-energies in the brain seems likewise unbelievable. So the need to assume objective elements is urged upon us by two very reasonable premises.

3.

Since the seventeenth century, we have thought of the world as consisting of extended *quantities* all the way down to the smallest level of tiny, identical particles in potential fields moving against a background of space and time. But there is a countervailing view that notions of quality are metaphysically, perhaps even logically, prior to notions of quantity. We are all familiar with the idea that the frictionless planes, perfect spheres or the absolutely free particle are abstractions. Leibniz famously argued that one *only* arrives at quantitative determinations by leveling off individual qualitative differences (Leibniz 1989, 2002). The reverse is not true: one cannot abstract quality away from quantitatively identical things like sets, numbers or geometric points. Leibniz also urged that extension is ex-tended, complex and divisible, and thus cannot be an ultimate simple property. In the nineteenth century, philosophically minded mathematicians Bernhard Riemann and Herman Grassmann attempted an analysis of extension from the ground up. Both required that such an analysis should not use extended concepts at the outset, and both were led to the idea of a philosophical construction of space which they mention in the preambles of their most famous works, Grassmann's *Theory of Extension* (*Ausdehnungslehre*) and Riemann's „On the Hypotheses that lie at the Foundation of Geometry."

Grassmann's theory operates with directed line segments, which are already extended of course, but prior to that he describes constructing extension from an elementary process of „joining and separation" in which traces of previous stages in the process are reproduced and adjoined to a present stage (Grassmann 1995). He gives the example of a row of identical letters or units:

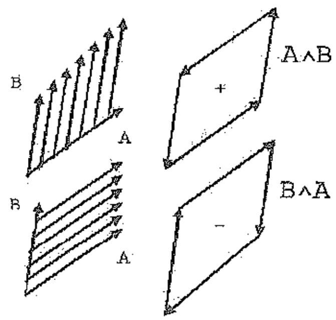
a, aa, aaa, aaaa …

which is traced out by a process that *associates* one unit with the next, finding them identical under one concept (they are both a's) and also *dissociating* them, finding them different under another concept (one ‚a' differs from its neighbors), and then collecting them all up in an extension. Moreover, the previous units generated are reproduced and added alongside the presently produced unit, so that they do not just fall away.[3]

Extension of even this primitive sort interprets the formal operation of multiplication: constantly reproducing and adding the multiplicand ‚a' to itself, separated by n units of the multiplier, in this case moments of time.[4] Although a multiplication can be defined more abstractly, this mechanism of association and dissociation is essential to extended *representations* of multiplication; this is the first main idea of the present paper. Why would such a pattern of associations and dissociations *look* like an extended quantity? For the same reason the points of a wire viewed head on do not extend unless we turn the wire through an independent, dissociating direction to set its points apart from one another, extension is a result of an associative *and* a dissociative operation.

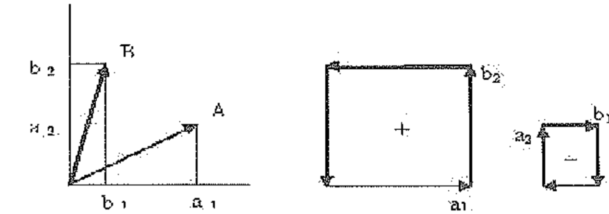
Grassmann algebra, in addition to being a universal scientific language, was also intended to answer philosophical questions about the origins of extension, as William Rowan Hamilton's quaternions were originally intended as a „science of pure time." Michael Crowe writes in his history of vector analysis that these philosophical aims strongly offended mathematicians, who preferred to think of their science as complete in itself without any superadded intuitive picture thinking (Crowe 1967, p. 95).

The operations and entities of Grassmann's algebra are ‚born' indifferent to coordinate systems of numbers and measure. Unlike vector algebra where coordinates are introduced only to abstract away from them, Grassmann algebra describes a world entirely innocent of these notions, which makes it ideally suited to a philosophical space construction. Grassmann algebra (See also Zaddach 1994) assumes a basis of mutually independent elements e_1, e_2, e_3, whose coefficients are all added and subtracted separately from one another as quantities of different kinds. Unlike the vector cross product, which yields another vector, or Hamilton's quaternion multiplication which yields another quaternion, Grassmann defined his outer „wedge product" $\wedge$ as a higher order extension, so that two directed lines will multiply to a directed area or parallelogram, by moving one directed segment along another and generating a directed extended area element:



This geometric multiplication is like arithmetic multiplication in that the parallelogram A∧B is obtained by reproducing B a total of A times along A's length and reproducing and adding the results. Geometric multiplication is however non-commutative, so that the parallelogram $e_1 \wedge e_2$ is directed in the opposite sense as $e_2 \wedge e_1$ and these are considered as opposite area-generating processes that undo each other: $e_1 \wedge e_2 = -e_2 \wedge e_1$. The need for independently directed or dissociated, segments, is responsible for the non-commutativity, for in $B \wedge A$ the segment A advances from one determinate point of B to the next while in-between taking on all values along its own length. When B takes a value at a point, A is completely dissociated from B, and takes all values. The associative-dissociative mechanism is welded into the extended product, if one takes a determinate value the other is in a superposition of values.

In coordinate terms, the magnitude of the area $A \wedge B$, where directed segment A is $(a_1e_1 + a_2e_2)$ and B is $(b_1e_1 + b_2e_2)$, is given by a determinant of the components and basis elements, where a positive or negative sign is affixed depending on whether the products are an odd or an even permutation of the basis elements, indicating underlying generating processes that cancel one another.[5]



$A \wedge B = a_1b_2 - a_2b_1 \, e_1 \wedge e_2$.

Notice that the product formed actually occupies a new, higher order space, with new area coordinates, in units of $e_1 \wedge e_2$. Other independent coordinates of the subspace of bivectors are $e_2 \wedge e_3$, and $e_1 \wedge e_3$. The next step up is a sub-space of solid parallelepiped coordinates, or trivectors obtained by triple products. For three basis elements, then, the subspaces are e_1, e_2, e_3; $e_1 \wedge e_2, e_2 \wedge e_3, e_1 \wedge e_3$; $e_1 \wedge e_2 \wedge e_3$.

Grassmann also defined a regressive product (v) which takes higher order extensions to extensions of lower order, if they share an overlap. For example $(e_1 \wedge e_2) \vee (e_2 \wedge e_3) = e_2$. In a causal qualitative sense, the regressive product behaves in a similar way to the empirical method of variations, for example when we isolate out a common element C experimentally from its concomitant occurrences in AC and BC, ‚isolating out' the common factor by holding it constant through variation with A and B. This is an experimental way of analytically reducing complexes to simpler elements, or lowering the grade of elements through a product. It is also possible in Grassmann, and related Clifford algebra, to define only one product for raising and lowering operations by employing the notion of duality (Hestenes, 1984).

4.

Riemann also began his famous geometry lecture by observing that mathematics had only considered constructions *in* space and said nothing about the construction *of* space, or the general concept of manifold extension. He then went on to define general concepts of quantity and how they arise from what he called „modes of determination" (*Bestimmungsweisen*):

> Quantitative concepts are only possible when there is a general concept that admits of various modes of determination. As these modes of determination allow a continuous transition from one to the other, or a discrete one, they form a continuous or a discrete manifold. The individual modes of determination are in the first case points in the second case elements of the manifold …In a concept whose modes of determination form a continuous manifold, if one transitions in a determinate way from one mode of determination to another, the modes of determination that are run through form a simply extended manifold, whose essential mark is this, that from a point one may progress continuously only in either of two directions, forward or back (Riemann 1876).

These are puzzling statements, to be sure, but as we now know from the excellent historical research of Erhard Scholz (1982a, b), Riemann's modes were like *qualitative* predicate concepts like color hues, or the pitches of a one-dimensional tone row, and they were drawn from those sorts of examples – which Riemann found expounded in the work of the philosopher and psychologist J.F. Herbart whose extensions of visual and tone space are explicitly generated by alternating patterns of associated and dissociated qualities. (Herbart had influenced Grassmann and Mach as well, see Banks 2003, 2005.)

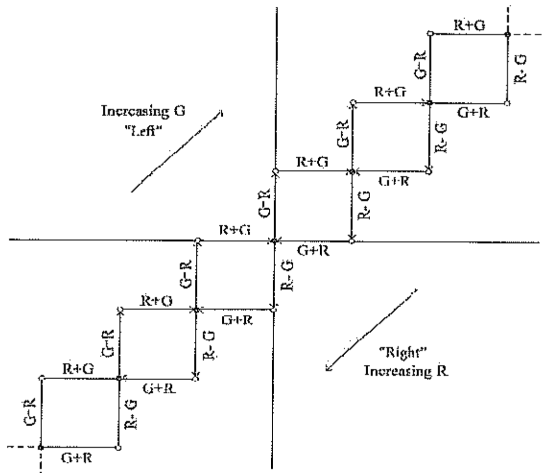
Riemann actually cites color space as an example of a triply extended manifold, where the three directions of the manifold are analogous to the three primary colors, the individual points to qualitatively distinguished color mixtures. This is a strange example because color space is anisotropic and non-homogeneous, but still counts as a space of the most fundamental kind. In a spatial extension, Riemann's modes become the directions of forward or back and describe possible orderings of points traced out in either direction to infinity. For Riemann, spatial direction is directly analogous to the qualitative orderings of hue, pitch or quality. The modes, or directions, of a manifold extension come even before the points in it, since they are the means for determining points, formed by the exchange or transition (*Übergang*) of the modes. For other readings than mine see Torretti, 1978, Dominguez 1999.

Riemann's definition of extension also demands a minimum of *two* such qualitative modes of determination and either a discrete or continuous transition between them. Why two? Why is a single quality not already an extended manifold, and why does it not suffice to order a variety of individuals under a single concept or genus, like a series of men more and less tall, or a series of shades more or less red, the traditional idea of a predicate? As Riemann says, designating individuals in an *extension*

requires various *independent* means of determining them through color, figure, size and other qualities they possess. One quality alone does not suffice to determine an individual per se, it just says that the individual falls under a qualitative genus or a class concept or intension.

Nor would two qualities determine an individual if they were too much alike in the discriminations they made. It would be like using two adjectives or predicates ‚hirsute' and ‚hairy' so dependent in meaning that they always made fused or overlapping determinations. So to really determine an individual, setting it apart from others falling under the same concept, we require at least two predicates that are dependent, in that an individual possess both of them, but also independent or dissociated, in the sense that other individuals alike in the first respect are different in another. For example, suppose we have a group of men separated by height. If we introduce another quality such as weight, men all of one constant height can be separated out by their differences in weight and vice versa, for a more complex determination of size as a matrix of the two dependent-independent orderings, a given size is either a height by weight or weight by height. Reversewise, a given size can be varied (against other sizes) to separate out the properties of constant weight and constant height from the complex size property.

We can understand Riemann's talk of determining points in a given order by a „transition" of one mode to another, I believe, as a tracing process by which two qualities determine a point by associating there, then dissociating to set the point off from its neighbors, then associating again at the next point.[6] We can imagine a continuous or a discrete transition between modes, say R and G, in which, as he himself pointed out, metric information (missing in the continuous case) is welded into the manifold directly, like a honeycomb of cells separated by gaps. These kinds of manifolds had been constructed previously by Herbart (1964, Banks 2003, 2005). Here we can think of the association (+) and dissociation (-) of modes as switching on and off discretely in-between points, to separate one individual (solid dot) from the next with a dissociating gap (open dot). Notice also the similarity of Riemann's transition between dependent-independent modes to create a richer concept and Grassmann's idea of extension as a multiplication of one element through the elements of another, „setting apart" the units of one through the units of the other, for example in the generation of an area.



Riemann's Discrete Manifold

So here is the importance of the color space example. Riemann is saying that a physical or geometric space is actually leveled off from an manifold more like a color space underneath, of qualitatively different points and individuals generated in a given order, like ordering individual men from short to tall or from light to heavy. Spaces like those of color are fundamentally nonhomogeneous and anisotropic, but by leveling off the properties of the underlying manifold, ignoring the qualitative differences by looking at the abstract *patterns* of modes and generated individuals, we interpret space as homogenenous and isotropic: a quality-less pure expanse of internally identical points, ordered in both directions indifferently to infinity. Riemann says that mathematicians can treat extensions of points, lines, areas and solids abstractly without worrying about what actual concrete things or manifolds these abstractions have been derived from, as they do with discrete manifolds of number for counting:

Notions whose modes of determination form a *discrete* manifold are so common that at least in the cultivated languages it is always possible to find a notion in which they are included. Hence mathematicians found the theory of discrete magnitudes upon the postulate that certain given things are to be regarded as equivalent (Riemann 1867).

5.

I will identify Grassmann's generating directional elements with Riemann's qualitatively directed modes of determination and give them both a further interpretation as the objectively neutral Machian elements discussed in the introduction. The elements are qualities with magnitude (push and pull) and direction (type). Just as we make higher dimensional analogies to four-dimensional or n-dimensional things, we can also make lower dimensional analogies to these more primitive elements, simpler even than points, building up extension from a more basic level. We can thus code information into an extension internally without increasing the number of extended dimensions.

Grassmann (1995) used a ‚vector' representation of color quality, where direction and qualitative type are directly analogous, and derived his empirical law of mixing by interpreting the direction of the vector as its hue on the color wheel, and the magnitude of the vector as the length from white or gray at the center to pure saturations on the rim. Like vectors, colors, ideally at least, can be expressed as linear combinations of a basis of primaries, which add together in mixtures, and the basis elements themselves can be expressed as linear combinations of any three orthogonal hues, removing the need for any absolute basis or absolute directions.

We can thus use these magnitude and directional properties to conceptualize the intensive force and type of Machian elements. Elements are dynamical entities, causally dependent and independent, and all fused together whole in a massive agglomeration or complex, such as they appear in our sensations, from which they can only be separated by the method of variations, holding a given element constant in its complex by varying the concomitant elements that occur in combination with it. As Mach often pointed out: there are no such things as isolated element „atoms" or „building blocks" of nature. An element, as he originally defined it, is something we cannot at present isolate any further or divide out from its occurrence in a complex. I must break sharply with Mach, however, in his demand that all elements be *observable* qualities as it conflicts with our intention to utilize unextended elements. Unextended, intensive elements, with their interior structure, cannot be observed, or even visualized, except as part and parcel of extended constructions, but we will still want to work with these entities in our construction.

So without visualizing elements directly and externally, we can attribute an interior structure to them: *type* and *magnitude*. The type is indifferently a property, a direction, or even a concept of classification (an intension); it plays all of these roles.[7] Type determines whether two elements are causally dependent (associated) or independent (dissociated) and thus whether they can affect each other's magnitudes or not. An element's magnitude or intensive force will be called its tension. Comparing elements to tiny springs, for example, the potential quantity of tension is like the stiffness in the coils of the spring or its overall capacity to store tension. The potential intensity is like the compression of the spring, describing how intensely it will communicate tension to other elements associable with it.[8]

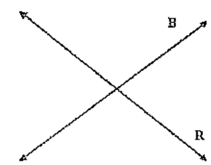
In response to the complaint that one just can't visualize these elements, I think it is better to postpone the question and imagine that we are dealing instead with some vast array of *qualitatively ordered* information, in the form of intensions or taxa. Not just any taxa or philosophers' properties,[9] either but only those that exert real influence in the causal structure of nature. The problem is how such an ordering of individuals by taxa or intentions becomes an ordering of familiarly extended and serially generated objects in space and time – just as Riemann claimed that space *qua* triple extended quantity comes from leveling off a prior manifold which orders, or classifies, its points using qualitative modes. A manifold of qualitative information *becomes* an extended manifold representation of objects in space and time when rules for that kind of representation are satisfied and the whole thing is leveled off.

6.

Nature presents us with a vast and bewildering manifold of elements, which we find combined every which way and constantly evolving toward new configurations. There are probably no such things as either purely independent or dependent elements in nature, more likely all elements have complex causal relations with one another which have to be analyzed into *grades* of dependence and independence.[10] As a first step, however, let us imagine isolating out a basis of *free* element types, {B, R, G, Y, S, W} like a set of primary colors, which will serve to express (classify) the other types as mixtures or combinations and which are a

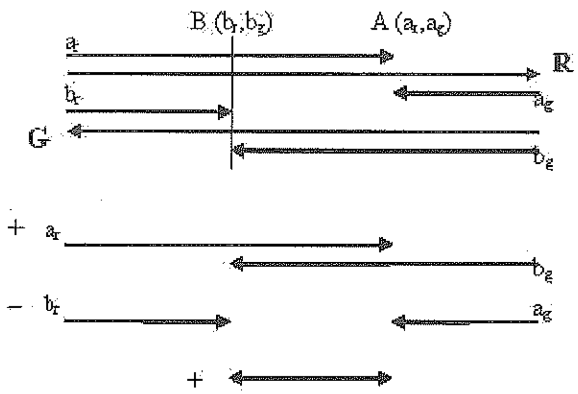
representative basis for codifying nature's qualitative variety. This need not be a privileged basis, or positivist building blocks, for, as in the case of the colors, *any* set of independent basis elements or notions might serve to express the others.

Now let's return to Riemann's construction of points, using the elements in place of his modes of determination. A point individual is determined by two elements which are dependent in that individual but occur independently elsewhere in other individuals. A bound occurs where two otherwise free elements come into dependence and bound each other from two directions or by means of two types of determinations:



We can put a causal interpretation on this classificatory notion by imagining a bound as a kind of mutual impedence, where elements meet and oppose. If we allow for these bounds to change dynamically, we can interpret the rates of change and overcoming of these bounds as a primitive intensive time determination. A multiplication of all free elements with all, will give us a mesh of elements with degrees or internal bounds as well as open zones of independence. These internally bounded elements are of a higher level of extension than free elements without internal boundaries and represent degrees ordered in one sense or direction given by the type of element. Individuals bounds in the mesh are identified by their „qualitative coordinates" so to speak. Individuals bounded by several elements to various degrees are called points.

The next step up is the multiplication of points. The points A and B are found by summing their two graded elements R, G as their „modes of determination" or qualitative coordinates in either independent direction $A = a_r + a_g$, $B = b_r + b_g$. We take a wedge product and the result is a handed spectrum of all of the point individuals that lie between A and B in two orderings (+) and (-).



$\mathbf{A} \times \mathbf{B} = a_r b_g - a_g b_r$

This spectrum extends in a new independent direction and is a more complex type. But for directions lying within the spectrum the product traces out all possible point determinations of its two component types in *both* directions between the given limits, or endpoints. While graded elements order their points in a preferred direction or class, like rays in geometry, the points within the spectrum are traced out by two classifications, so that the extended determinations of point individuals are made in at least two ways, similar to a matrix ordering of elements by row and by column. This result combines Grassmann's wedge product with Riemann's requirement that an extended manifold of points should be produced by two directions or orderings.

7.

It is now time to answer an important objection which has probably occurred to the reader. Aren't these patterned serial associations and dissociations *already* spatial and or temporal, not to mention notions of bounding or causal impedence and interaction of dynamical tension elements? Surely the serial construction of space cannot *itself* be extended in space without begging the question.

To meet this objection, I suggest we look closer at Grassmann's generating process at any given level of extension. An area element, like $a_1b_2 - a_2b_1\ \mathbf{e}_1 \wedge \mathbf{e}_2$, is generated, as we saw above, by reproducing all points along the vector A at every point of B and reproducing and collecting up the results. At its own level, it is indeed a serial process. But from the higher level of bivector space, a space with area-coordinates such as $\mathbf{e}_1 \wedge \mathbf{e}_2$, the whole area is present simultaneously and there is no serial construction except in a potential sense.

Similarly, real numbers like $\sqrt{2}$ are not treated as „evolving" limit processes, but completed infinite sets. What must be represented as a sequence of determinations at a lower order can be represented as a completed whole at its own proper level. The generating processes of finished spatial extension must appear to us as *potential* processes, beneath our own level of extension, processes completed potentially serially but for us, all at once. The unextended generating processes appear this way from within an already extended space, as simultaneous, not successive, superpositions of all the associations and dissociations that trace out the present level of extension.

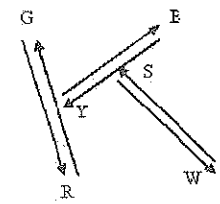
Something similar is often said of time (Barbour, 2000, Lockwood, 2007) whether time ‚flows' or not, or, as Lockwood says, where the horizon between the potential and that actual is made, can be considered an artifact of the observer's own level of extension, although it is certainly empirically real for us. In fact, all *actual* measureable quantities for us are certainly occupants of three dimensional space, or rather four dimensional spacetime as the basic arena of events described in a physical language. Possibly for this reason Heisenberg (following Bohr) claimed that three dimensional spatial language of classical physics is specially selected to be the language of actuality and measurement. It is a realistic restriction that should *not* be replaced, he claimed, even while he allowed that the abstract configuration spaces of wave functions may be real in a potential, not actual, sense because it is „not in space and time" (Heisenberg 1930 p.65). I propose to follow Heisenberg and identify the actual with measured extension in three dimensional space, or spacetime, while identifying *both* lower and higher dimensional entities and serial processes as potential in nature (Heisenberg 1958, p. 117).[11]

Hence an extended quantity like an area can be measured determinately and given a value. But if the present account is right, this area is also product of two elements of lower order associated and dissociated with one another and hence taking on no determinate, measurable values; when one takes on a value, the other is completely dissociated from it. By the same token these processes have to be real in some sense, they support the extension. But because we consider the extended quantity the fixed, actual thing, the tracing process is considered *potentially* to have run through all the associations-dissociations of elements all at once to support the actual extension. So if lower dimensional serial constructions are not actual serial tracing processes in spacetime, but simultaneous superpositions, then the question is really not begged and we are not assuming space to construct it.[12]

8.

We want to thus describe a tracing out of space in time by means of unextended spectra and their superposed associations and dissociations one level below the finished, actual extension of experience. To describe such a three dimensional tracing process in time, we have limit the construction to a certain level and not allow extensions to continue into the infinite, as Grassmann's algebra does. Suppose then that the multiplication of two spectra returns again a spectrum (*not* a higher order extension) which points in a third direction, or manifests a third type, independent of the component spectra, but still within the overall extension, not a higher order direction. (Similarly a multiplication of two square matrices with equal numbers of rows and columns ab x bc = ac, returns a matrix of the same order.)

We can understand this visually, in the above potential sense, as a rotation from two dependent-independent spectra types to a third type. A multiplication of elements can result in a change of intensive length (where they are causally dependent in a bound) or a change of directional type (where they are causally independent).[13] When two independent spectrum elements meet, they can still interact by each „pushing along its length" so to speak, resulting in a circulating, or screw-motion, transferring their degrees of potential tension into a third direction or element type. „Positive" will mean this new type is increasing, giving off tension in a new direction or new type, and „negative" will mean that it is decreasing or absorbing tension into it. To this new type will come a new partner element and a new spectrum, and so it continues, (using non-commutative multiplication as the associative-dissociative mechanism responsible for tracing the extension). A new spectrum B (below) is extended out in the third direction through the association-dissociation of G and R:



One branch of expanding combination tree starting with RG

These space generating processes spread out in a sphere of every conceivable pattern or combination of spectrum elements, in all directions. Starting with R at the base of the tree, we have five other choices. The resulting combinatorial patterns of the various spectra and their multiplications appear in the finished product, as it were, like directions at infinity, setting the ground rules for how it is possible to move within the manifold extension by combinatorially tracing out all of these possibilities as the basic directions of the manifold, just as the directions of right and left set the possibilities for a one dimensional line and represent all of the possible ways to move in it.

9.

How many of these combinatorial spectra should there be and what should the set of possible combinations look like? To set up the solution, consider what happens if we are only given a basis of four spectra types to work with {G, R, B, Y} all associable-dissociable with all. In an *extended* RG process, a line of R and G determinations, we want R and G to associate, then take other partners, then re-associate. With only four spectra to work with, if R and G are together then so are B and Y. If R is dissociated from G, it can associate with either B or Y, leaving the other for G. The best we can do is mix up the cross-directional associations randomly (R with B or Y; G with B or Y), so as not to create any lasting associations between them.

RG, BY
RY, GB
RG, BY
RB, GY
RG, BY

What these combinations produce is a discrete line of points moving in the R or G direction, set off by gaps of dissociation in the other two spectrum types. Dependent because the dissociations have also generated a B-Y line, such that wherever the R-G line has a point, the B-Y line has one too and wherever it has a gap, so does the other. We can think of these two lines as becoming extended in a dependent way, i.e., by sliding over one another in different directions.

Consider moving to at least six spectra. Now the combinatorial possibilities increase, so that it is possible to separate out three bidirectional independent lines, all free of one another and set off by the larger number of randomized cross-directional associations. For example, when R and G are associated, BY and SW may be either associated with each other, or cross-associated BS, YW, so as not to create a dependent association with the RG line.

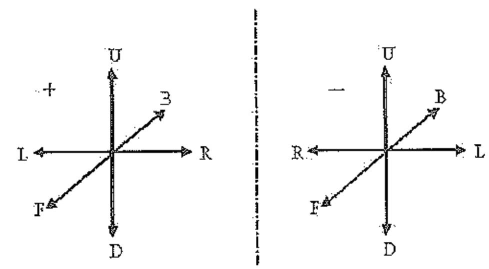
RG, BY, SW
RG, BS, YW
RG, BW, YS
RG, BY, SW

The same goes of course for the other two lines BY and SW. I am not claiming there *must be* six spectrum directions a priori; one might always have more, or less. But there is something special about the combinatorial relations that give a three dimensional space; it is the first free extension, three independent lines, completely set off against a random background.

10.

I would like to close by touching on some consequences of the above construction as it relates to a few traditional issues in philosophy of space. The comparison between this combinatorial model of space and ordinary, naïve, flat, infinite Euclidean space is interesting. Euclidean space is a completed manifold all given at once and one step above the serial construction of space. It has six directions at infinity, but its independent axes are fully opposed pairs of dependent directions (forward, back) (up, down) and (right, left). The directions of Euclidian space are in no way qualitatively different, but are completely arbitrary, or leveled off, differing not a bit among one another. Moreover the cross-directions

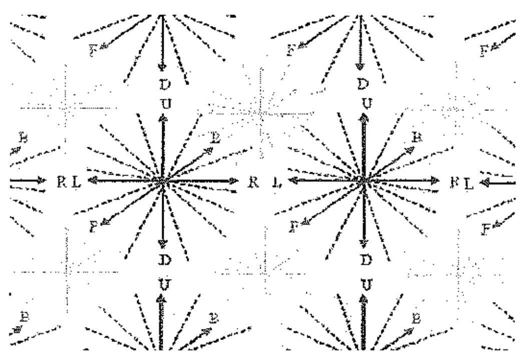
(forward, down) and (back, up) appear dissociated and orthogonal, having nothing to do with each other. Euclidean space is presented to intuition as a finished „box" with no clue how it is constructed (all geometric constructions being already embedded in space) but these features strike me as vestiges of the underlying combinatorial processes responsible for its finished form. The alternating patterns of associated-dissociated spectra, then, are what we really mean by directions in space, like right, left, up, down. Even if the spectra are specific directional properties, *patterns* of those spectra can exhibit the required isotropy of space, as we consider arbitrary rotations of axes which hide the underlying differences of the spectra making up these patterns. Likewise certain physical and geometrical considerations seem to point to a substantial built-in dependence even among the independent directions of space (vanCleve, ed. 1991; Mach 1930, Banks 2003). For example in an orientable space, exchanging left and right means exchanging up and down, or front and back, to preserve the handedness of the coordinate system, or handed volume (see also Wrede 1972, Chapter 1.6).



Because non-orientable spaces are also possible a priori, (see VanCleve, Ed. 1991), the fact that physical space is orientable suggests that spatial directions might conceal dependent properties, like the spectra, in combinations or patterns, but not as absolute directions of right and left. In the above construction, axis flips are facts about exchanges of tension in the underlying patterns of spectra that trace out the volume. Referring to the branch of the combinatorial tree pictured above (RG, BY, SW), flipping spectra, like RG to GR, has to be offset by other exchanges of tension in the overall pattern for the same handed volume element to be traced out.[14]

I have been assuming discrete generating processes and countable sets of points, but there is no reason why one might not extend these constructions to uncountable sets of continua. One could abstractly interpret the power set operations as a method for generating all the uncountable subsets of a countable set of elements by association-dissociation, interpreting the collection between brackets {a,b} to mean an association, and separating with brackets {a}, {b} to mean a dissociation of members, and the whole power set as representing the final collecting up of all the uncountable subsets of associated-dissociated members.

On the relation of geometric space to physical processes in space, it is possible to code more into the manifold internally than just the spatial directions and their extension. There is nothing wrong with treating certain of the cross-directional dependencies (like RY, BW, GS) *not* as randomized background but as „lines" in their own right, extended against the spatial background of the other directions (RG, BY, SW). Call the RY process U, the BW process V and the GS process W. In fact, we can have a *dependent* three-extension of associated spatial directions, and three extended processes *in* space, mutually embedded and encoded in one another and reciprocally differentiated, in so far as they can be differentiated by the method of variations. When the three spatial directions are associated, the three process cross-directions are either dissociated or associated, and vice versa. Any residual associations between spatial directions and physical processes are randomized and eliminated.



The original qualitative manifold of elements has been reclassed into a quantitative one, a manifold of extended process of potential tension exchange, some of which represent the tracing out of space in three independent directions (solid lines) others of which represent the propagation of physical processes of various independent kinds through space (dotted lines). The conclusion of the space construction is the beginning of geometry and physics, or rather to do the actual construction we should reason backwards from the established properties of spacetime and physical potentials to the basis elements of the construction. The combinatorial directions trace the manifold extension which can then be investigated for its topology, coordinate system, and metrics for distance and/or interval properties. The two specifications of tension by quantity and intensity serve to delineate these tracing processes in space into still more definite form, such as particulate potential sources and distances, as suggested above in footnote 6.

Is spatial extension (and its directions) separable from the processes in space (and its types)? It might be that under some circumstances (such as variably accelerated motions) we can vary the two manifolds independently and they can be separated out. In other circumstances (such as inertial motions) we cannot make the separation. So whether they really are separate might not be a metaphysical question but might depend instead on the method of variations and whether they are separable under a given circumstance.

I hope these beginnings suggest a link with a natural science, which would start from primitive notions of direction and magnitude, and a geometric algebra of neutral qualities, building up and leveling off the extension to give familiar properties of a spacetime manifold (extension, direction, dimension). These would not have to be taken for granted but could be traced back to information encoded in the manifold.

Notes

1 There are space constructions not involving qualities or the associative-dissociative mechanisms of this article (Penrose, 2006, Finkelstein, 1969, Jammer, 1993, Reichenbach, 1958, Sklar, 1987).

2 There are other objections of a technical nature, for example the uniqueness of the construction by similarity relations (quasianalysis), see Toader 2004.

3 What Grassmann says about a sequential process could well be captured by the language of set theory, in which all of the temporal stages are represented simultaneously in one coexisting set, which can exhibit a nested structure like the generating process he describes, especially in that the previous stages are reproduced and adjoined to later stages by a rule.

4 Actually Grassmann understands „a multiplication", whether commutative or not, as defined by this distributive property over addition, because multiplication is really understood as addition.

5 The operator takes the value + or – depending on whether the indices are an even or odd number of rearrangements. For example if we have 1,2,3 an odd number of exchanges results in 2,1,3 or 3,2,1 or 1,3,2 while the evens are 1,2,3 itself, 2,3,1 and 3,1,2.

6 The qualities also change in value in moving from one individual to another in a given ordering. They assume a value, change independently of one another, then assume new values at a different individual, see diagram below.

7 Not out of confusion, but because these roles have not been separated out as yet.

8 As I have suggested in Banks 2002, given a matrix or table of these intensities and the method of variations, one can differentiate the tension as due to two factors: those variations due to constant capacities that vary only in their potential differences $r_1,\ldots, r_n$ along rows (which can come to represent source particles distant from a test source) and tension due to variable capacities holding their potential levels constant (which come to represent different quantities of potential A,B,C,D ... in columns, for the source particles). This is a way of getting a particle and distance formulation out of a more abstract table of qualities and their interactions.

9 To form a philosopher's property take any word and at the suffix -ness to it.

10 It might well be that a Clifford central product like that used by Hestenes (1984) best expresses an interaction between naturally dependent-and-independent elements. But the algebera I used here separates the roles more clearly I believe.

11 As Lockwood (2007) points out, even the block universe view of time in 4-D could be said to flow when considered in a higher dimensionality, and the flowing, static views of time simply exchange places as we ascend.

12 Incidentally, on this view, beings occupying a different level of extension would not observe wave function collapse the same way. So-called collapse would then turn out to be a dimension-dependent artifact based on a certain kind of privileged extended representation that we employ as our preferred physical language. Heisenberg harshly criticized Schrödinger's and deBroglie's early attempts to interpret matter waves as actual three dimensional waves. Instead they occupy a configurational space with a potential existence. In Heisenberg 2003, pp. 70–71, he has Carl von Weizsäcker compare the distinction to that made in Kantian philosophy between empirically real space and time language and processes that are real but which may exist outside of a spacetime representation. It is then measurement that forces the natural processes into an extended representation.

13 The generating processes do not need to be unique either, when different basis elements and their associative-dissociative relations can account for the same extension.

14 Again, the geometric product captures both notions neatly.

15 If we reverse RG to GR, we get not Y, but the flipped –Y. This will lead to the tracing of a different tension element unless we flip –Y's partner element B to –B. (That's actually another exchange of the two spectra responsible for B, four total.) –B–Y however has the same sense as BY, and leads to element S as before. After these changes are made, the migration of tension proceeds in the same sense as RG, BY, SW.

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