A Hyperbolic Secant Welfare Function

Walter Barta

University of Houston

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Introduction

Here we introduce a formulation for describing welfare based on a hyperbolic secant function, derived from certain intuitions about the nature of material and experiential conditions, that satisfies a number of normatively critical constraints, making for an elegant and satisfactory welfarist axiology. We first introduce intuitions about experiential conditions, material conditions, and their valences; we second make a mathematical formulation of our hedonic calculus consistent with these intuitions; we third make several manipulations of our formulation in order to make calculations in population ethics; and we fourth show how the calculations are consistent with several critical normative constraints.

Intuitions

In order to get started in constructing a welfare function, we can introspectively intuit several assumptions about experiential conditions and their properties. We can come to these by merely observing the manner in which the valences of our own experiential conditions come and go with varying material conditions:

1. Experiential conditions (w) are related to material conditions (x) at some differential rate of change ($\frac{dw}{dx}$) according to some mathematical function describing the experiential/material relation ($w = f(x)$).
2. Given the limits of cognition, there must be some limits on experience, including limits on the valence of experience (good, bad, neutral), and so some maximum experiential valence (w_{max}) and a minimum experiential valence (w_{min}).
3. Given that some experiential conditions have positive valence ($w > 0$) and some negative ($w < 0$), the maximum experiential valence must be positive ($w_{max} > 0$) and minimum must be negative ($w_{min} < 0$).
4. At the optimum material condition ($x = x_{max}$), the closer the corresponding experiential condition approaches its maximum valence ($w(x_{max}) = w_{max}$).
5. Maximum experiential valence requires materiality, therefore the optimum material condition must be some material condition greater than zero ($x_{max} > 0$).
6. At the optimum material condition, the rate of change in experiential valence per change in material condition is zero ($\frac{dw}{dx}(x_{max}) = 0$).
7. The further away from the optimum material condition ($x \approx \infty$), the closer the corresponding experiential condition approaches its minimum valence ($w(\infty) = w_{min}$).
8. The furthest away from the optimum material condition, the rate of change in experiential valence per change in material condition is zero ($\frac{dw}{dx}(\infty) = 0$).

9. Some excess is negatively valenced, therefore there is some positive material condition greater than the optimum material condition ($0 < x_{max} < x_{+max}$) with a negative valence ($w(x_{+m}\quad) < 0$).
10. Some lack is negatively valenced, therefore there is some positive material condition less than the optimum material condition ($0 < x_{-max} < x_{max}$) with a negative valence ($w(x_{-max}) < 0$).

We can also identify some corollaries:

1. Since some experiential valences are positive and some negative, some positive material condition exists ($x_{neu} > 0$), such that the corresponding experiential condition has neutral valence ($w(x_{neu}) = 0$).
2. As the experiential condition departs from its maximum valence, the change in experiential valence per change in material condition accelerates towards some maximum rate of change ($\frac{dw}{dx}_{max}$).
3. After the maximum rate of change is achieved, the change in experiential valence per change in material condition decelerates towards zero as the experiential condition approaches its minimum valence.

As we shall see next, these intuitions give us enough information to narrow down and construct a simple continuous mathematical formulation of an individual welfare function.

Formulations

From these givens, we can deduce a formulation for the experiential/material relation (i.e., the welfare function).

1. The simplest continuous condition-satisfying mathematical function that is the hyperbolic secant function (Wolfram).
   This can be expressed in terms of exponentials:

$$y = \frac{2a}{e^{b(x-c)} + e^{-b(x-c)}} + d = \frac{2ae^{b(x-c)}}{e^{2b(x-c)} + 1} + d$$

   Or, in terms of the hyperbolic trigonometric function:

$$y = a(\text{sech}(b(x-c))) + d$$

2. The variables in the function can be substituted as follows.
   a. The dependent variable can be described as the valence of the experiential condition, or alternatively the welfare experienced by the individual ($y = w$).
   b. The independent variable can be described as the material condition, or alternatively the commodity consumed by the individual ($x = x$).
   c. The first coefficient can be described as the range of experiential valence, the range between the maximum and minimum experiential valence ($a = (w_{max} - w_{min})$).

d. The second coefficient can be described as the psychological sensitivity factor, an arbitrary scaling factor describing the sensitivity of changes in the experiential valence to changes in the material condition ($b = \frac{1}{x_s}$).
e. The third coefficient can be described as the material condition at experiential condition of maximum valence, peak welfare ($c = x_{max}$).
f. The fourth coefficient can be described as the experiential condition of minimum valence, peak illfare ($d = w_{min}$).
g. A further simplification can be made in which the difference between the independent variable and the third coefficient can be set to some new variable, representing the material peak distance, the distance of the material condition from the optimum material, which normalizes the material condition at maximum experiential valence to zero ($x - x_{max} = x^*$).

3. Therefore, when substituted, the function becomes (Appendix B: Figures 1 & 2). In terms of material condition:

$$w = w_{min} + (w_{max} - w_{min})\left(\operatorname{sech}\left(\frac{x - x_{max}}{x_s}\right)\right)$$

Or, when rearranged with like-to-like comparisons:

$$\frac{w - w_{min}}{w_{max} - w_{min}} = \left(\operatorname{sech}\left(\frac{x - x_{max}}{x_s}\right)\right)$$

Or, simplified, in terms of material peak distance:

$$\frac{w - w_{min}}{w_{max} - w_{min}} = \left(\operatorname{sech}\left(\frac{x^*}{x_s}\right)\right)$$

Or, even more simplified, in terms of dimensionless numbers:

$$\hat{w} = \operatorname{sech}(\hat{x})$$

Some clarificatory qualifying notes may be worthy of mention before proceeding.

First of all, other mathematical functions will suffice to satisfy our set of experiential intuitions, for example like a polynomial approximation, but the hyperbolic secant function has the virtue of being the simplest established hyperbolic function capable of doing so.

Second of all, here we have derived the correspondence of the experiential/material relation by observing its consistency with the features of the hyperbolic secant function, but the hyperbolic secant function also corresponds to some systems of partial differential equations that describes the experiential/material relation, which must relate to the underlying mechanics of the mind/body interface. A tentative derivation of the hyperbolic secant from the simplest possible system of partial differential equations in offered here (in Appendix A: Proofs: Table 1), and in terms of biochemistry this system of PDEs is closest to that of an two-component reaction with a competitive inhibitor in which the concentrations of the two components are cast as constant and the concentration of the inhibitor cast as variant (Eisenstein & Crothers, 1979, pp. 265-6), but this topic (i.e., the kinematics of hedonics) is surely an area for further study.

As we will see next, this hyperbolic secant welfare function can be manipulated in order to perform calculations in population ethics.

Manipulations

From the above formulation, we can make some further manipulations.

1. Assuming classical utilitarianism (Bentham, 1789), which sums the welfare of the population ($W_N = \sum_N w$), we can set the hedonic calculus equal to a summation of individual hyperbolic secant functions:

$$W_N = \sum_N w = \sum_N w\left(w_{min} + (w_{max} - w_{min})\left(\text{sech}\left(\frac{x - x_{max}}{x_s}\right)\right)\right)$$

2. Assuming an egalitarian population in which everyone has identical welfare levels ($W_{Ne} = wN$), we can then set the hedonic calculus equal to the product of the population and the average individual hyperbolic secant function:

$$W_{Ne} = wN = N\left(w_{min} + (w_{max} - w_{min})\left(\text{sech}\left(\frac{x - x_{max}}{x_s}\right)\right)\right)$$

3. Assuming the aforementioned definition of material peak distance ($x^* = x - x_{max}$), and assuming the material conditions are finite (x_{total}) and therefore must be distributed amongst the individual material conditions of members of the population ($x = \frac{x_{total}}{N}$), we can substitute the material peak distance for its definition, and substitute the individual material condition for the total material conditions divided by the population (Appendix B: Figure 3):

$$W_{Ne} = N\left(w_{min} + (w_{max} - w_{min})\left(\text{sech}\left(\frac{\frac{x_{total}}{N} - x_{max}}{x_s}\right)\right)\right)$$

As we will see next, this particular manipulation has properties that satisfy a handful of important constraints on population ethics.

Constraints

Using the formulations developed above, we can deduce that it satisfies several constraints that are normatively important in that they specify out unreal mathematical solutions: 1) it does not result in All-Negative Welfare, 2) it does not result in a Utility Monster, 3) it does not result in the Repugnant Conclusion, and 4) it does not result in Non-Neutral Nonexistence. We will prove that these constraints are met by showing that the domain does not incur any of these problems—contrary to some expectations (Arrhenius, 2000).

First, there is some population (N_{max}) that satisfies a Not-All-Negativity Criteria (Smart, 1958) by having positive total welfare:

$$W_{Ne}(N_{max}) = N_{max}\left(w_{min} + (w_{max} - w_{min})\left(\operatorname{sech}\left(\frac{x_{max} - x_{max}}{x_s}\right)\right)\right)$$
$$= N_{max}\big(w_{min} + (w_{max} - w_{min})\big) = N_{max}(w_{max}) > 0$$

This has the further benefit of satisfying the general normative intuition that there is some middling optimum virtue point, surrounded by sub-optimal ranges of vice (Aristotle, 2009).

Second, at population of one ($N = 1$) the formula does not fall prey to a Utility Monster (Nozick, 1974) because there is a maximum experiential condition:

$$W_{Ne}(1) = 1\left(w_{min} + (w_{max} - w_{min})\left(\operatorname{sech}\left(\frac{\frac{x_{total}}{(1)} - x_{max}}{x_s}\right)\right)\right)$$
$$= \big(w_{min} + (w_{max} - w_{min})(\ldots)\big) < w_{max}$$

Third, at population of infinity ($N = \infty$) the formula does not slip towards the Repugnant Conclusion (Parfit, 1984; Ryberg et al., 2004; Arrhenius et al., 2017) because the asymptote of an infinite population is not positive:

$$W_{Ne}(\infty) = \infty\left(w_{min} + (w_{max} - w_{min})\left(\operatorname{sech}\left(\frac{\frac{x_{total}}{(\infty)} - x_{max}}{x_s}\right)\right)\right) = \infty(w_{min})$$
$$= -\infty$$

Furthermore, because we have assumed classical utilitarianism ($W_N = \sum_N (w - \alpha)$ where $\alpha = 0$), not critical level utilitarianism ($W_N = \sum_N (w - \alpha)$ where $\alpha \neq 0$), we have avoided the Sadistic Conclusion simultaneously (Ryberg et al., 2004).

Fourth, at population of zero ($N = 0$) the formula does not obtain a valid trivial solution (Benatar, 2004) because a divide by zero error results in an indeterminate outcome:

$$W_{Ne}(0) = 0\left(w_{min} + (w_{max} - w_{min})\left(\text{sech}\left(\frac{\frac{x_{total}}{(0)} - x_{max}}{x_s}\right)\right)\right) = \frac{0}{0}$$

The avoidance of these four constraints makes for a satisfactory formulation of the welfare function, useful for calculations in population ethics, without incurring unreal solutions.

Objections and Responses

A few hypothetical objections to our procedure, and our responses to these objections, are worth short discussion.

First, any one of our intuitions could be objected to, or even the method of intuition-gathering via introspection itself could be rejected. This might prove convincing, and this author only means to honestly represent their own introspective observations to the best of their ability, which is the only method forthcoming.

Second, the formulation of the hyperbolic secant function from our intuitions could be rejected. To this we are open, as some other, perhaps more complex version of the function could be proposed, and the hyperbolic secant is merely meant as the simplest consistent formulation.

Third, some of the manipulations made could be rejected, perhaps on some mathematically or normatively sound grounds. Should this prove to be the case, we could probably accept any mathematically self-consistent adjustments, though we would argue that the results obtained here would still obtain under certain further mathematical/normative assumptions—albeit simplifications.

More specifically one could reject the distributive egalitarian utilitarianism that we have assumed, or one of many implicit utilitarian premises—measurability, commensurability, etc.—fair enough. But to this, we offer two preemptive responses. One, the constraints that our method is attempting to resolve (Monstrosity, Repugnance, etc.) are problems for utilitarianism (and other normative views that take seriously an impartial benevolence principle), and so our invocation of utilitarianism here need not be advocacy; rather, we are merely trying to some the problems within the system in which they arise. Two, even if we reject some utilitarian premise, our conclusions are not necessarily dependent upon these premises; rather, they are only dependent upon our formulation of the function itself, and we suspect they would therefore hold independently. (In other words, if our conclusions hold for distributive egalitarian utilitarian versions, then they should hold for less strict versions as well.)

Fourth, the characterization of our constraints could be objected to, or some new constraint could be proposed, perhaps inconsistent with our method. As with the manipulations, the characterizations of these constraints could be adjusted, and assumptions proposed. And some new constraint could be plausibly proposed, but because the constraints listed here have exhausted the natural number line, which plausibly mathematically restricts the domain normative considerations are made, we doubt new constraints will be realistically important.

Conclusion

Given that the consequences of the function for welfare satisfy critical normative constraints, and given that the assumptions that went into crafting the function are introspectively intuitive, it seems reasonable to accept the hyperbolic secant function as a compelling if not conclusive mathematical description of the experiential valences across a domain of material conditions. This represents a potential fruitful avenue for value theory (Hirose et al. 2015) and population ethics (Arrhenius et al., 2022).

References

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Appendix A: Proofs

Table 1: A Derivation of the Hyperbolic Secant Function from a System of Partial Differential Equations

	PDE Hyperbolic Form	PDE Simple Form	Exponential Form
A	$\frac{d(\sinh(x))}{dx} = \cosh(x)$	$\frac{d(A)}{dx} = \mathrm{B}$	$A = \frac{e^x + e^{-x}}{2}$
B	$\frac{d(\cosh(x))}{dx} = \sinh(x)$	$\frac{d(B)}{dx} = \mathrm{A}$	$B = \frac{e^x - e^{-x}}{2}$
C	$\frac{d(\mathrm{sech}(x))}{dx} = -\tanh(x)\,\mathrm{sech}(x) = -\sinh(\mathrm{x})\,(\mathrm{sech}(x))^2$	$\frac{d(C)}{dx} = -\mathrm{AC}^2$	$\mathrm{C} = \frac{2}{e^x + e^{-x}}$

Appendix B: Graphs

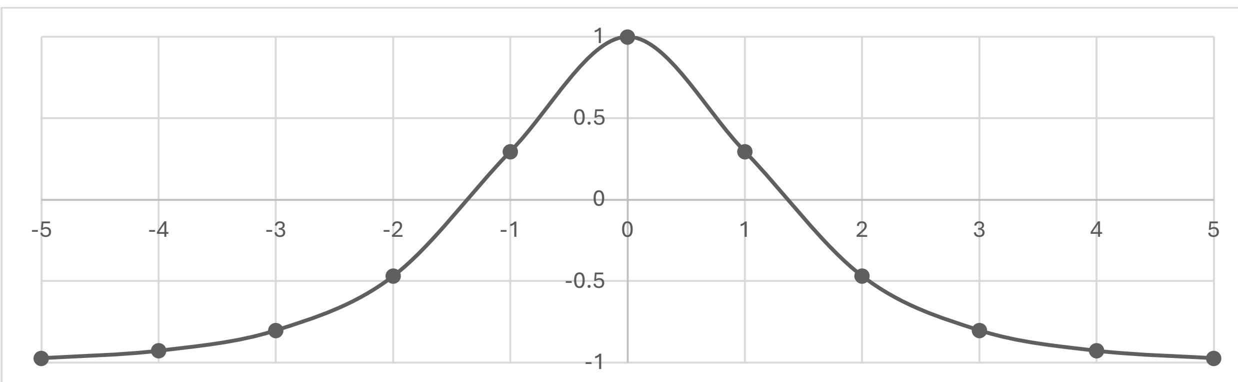

Figure 1: Individual Experiential Condition as a Function of Distance from Optimum Material Condition

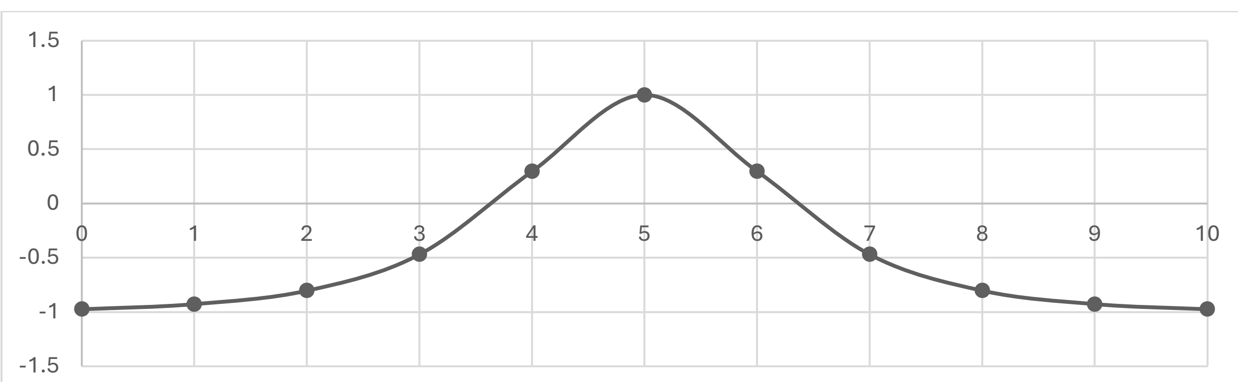

Figure 2: Individual Experiential Condition as a Function of Material Condition (x_max=5)

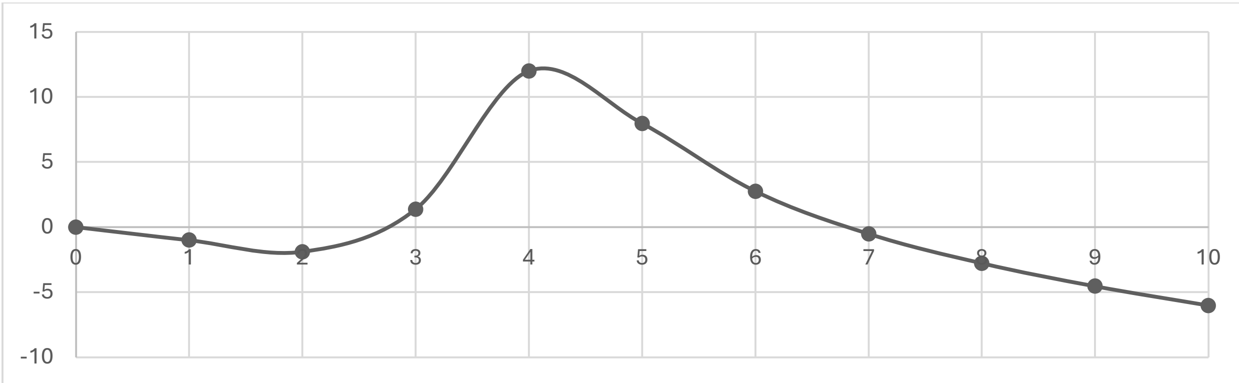

Figure 3: Total Experiential Condition as a Function of Population (x_max=5, x_total=20)