A Polynomial Approximation Method for Welfarist Axiology

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Introduction

Several philosophers have suggested that it is impossible to formulate a theory of population ethics that simultaneously satisfies all of the necessary conditions set by our ethical intuitions (Arrhenius, 2000; Blackorby et al., 2004). However, we will attempt to demonstrate that, using the Stone-Weierstrass method, starting from scratch with a completely underspecified polynomial approximation function for commensurable cardinal utility (used here as a synonym for welfare) in a domain of commodities and populations, there are possible sets of functions that satisfy many if not all of our ethical intuitions (avoiding the Repugnant and Sadistic Conclusions, and others), thus providing a self-consistent welfarist axiology.

The Generic Proof

In what follows, we will attempt a proof of the possibility of a utility function that meets certain reasonable assumptions and also avoids certain problems. We will assume 1) von Neumann-Morgenstern cardinal utility, 2) interpersonal commensurability, 3) equality of distribution, and 4) a Stone-Weierstrass polynomial approximation and factorization. Using these assumptions, we will then establish a generic utility function. From this function, we will show how to simultaneously avoid 1) the Repugnant Conclusion (Parfit, 1984) and 2) the Sadistic Conclusion (Arrhenius, 2000) together, thus avoiding the Repugnant/Sadistic Dilemma (Blackorby et al., 1997; Carlson, 1998), as well as 3) the Monstrous Conclusion, a version of the Utility Monster in the domain of population (Nozick, 1999), and 4) Negative Utilitarianism, utilitarianism that attempts to minimize suffering forced by condition of negative utilities throughout the real domain (Smart, 1958), the Anti-Egalitarian Conclusion (Arrhenius, 2000), and satisfies a Neutral Nonexistence Intuition.

1. Assumptions

1.1 Assumption 1: Generic Cardinal Utility Function

To start, we will begin, according to the von Neumann-Morgenstern theorem, assuming that we can act *as if some* cardinal utility function can represent the decisions of persons (Neumann & Morgenstern, 1966, pp. 15-30). We will maintain some level of maximal agnosticism about the nature of this cardinal utility function, and thus define the vaguest possible cardinal utility function, completely underspecifying the constraints. We will postulate only that there must be some relationship of dependence between subjective utility experienced and objective commodities consumed by a person:

> **Generic Individual Utility Function**: the most generic possible cardinal utility function with the least possible constraints, consisting only of the individual utility experienced

(u_i) as an unspecified function of the commodities individually consumed (m_i) by some person (i).

Then, we will begin loosely constraining that function.

1.2 Assumption 2: Total Summability/Commensurability

First, we will define total commodity and total utility for the population as a summation of the individually consumed commodities and individually experienced utilities, thus assuming the summability and commensurability of commodities and utilities between persons, not unlike the open-ended utilitarian mathematics of Jeremy Bentham (Bentham, 1789):

Total Commodity Consumed: the total commodities (M_T) consumed is the sum of the commodities individually consumed (m_i) for a population (n).

Total Utility Experienced: the total utility (U_T) experienced is the sum of the utilities individually experienced (u_i) for a population (n).

We will also assume a scarcity condition that stipulates the total utilization of commodities, thus treating the total commodity value as a constant.

1.3 Assumption 3: Equal Distribution

Second, we will assume equal distribution for individual consumption:

Equal Distribution Individual Commodity Consumption: the individual commodities consumed (m_i) by every individual in the population (n) is equal.

Equal Distribution Individual Utility Experience: the individual utilities experienced (u_i) by every individual in the population (n) is equal.

Although these assumptions of equal distribution are unrealistic, in that they stipulate a perfectly egalitarian distribution of goods and a perfectly egalitarian distribution of utilities, they specify a special egalitarian case that removes the variable of inequality but still allows for population-related problems like Parfit's Repugnant Conclusion and Nozick's Utility Monster. Thus they are an acceptable simplification.

1.3* Lemma 3*: Totality/Equality Combination

From these definitions and assumptions, it follows that:

Egalitarian Total Commodity Consumption: the total commodities (M_T) consumed is the commodities individually consumed multiplied by the population (n).

Egalitarian Total Utility Experience: the total utility (U_T) experienced is the utilities individually experienced multiplied by the population (n).

1.3** Lemma 3**: Totality/Equality Utility Function

From here, substituting the generic individual utility function into the total egalitarian utility function and then substituting the Egalitarian Total Commodity function into the Generic Egalitarian Total Utility Function:

> **Generic Egalitarian Total Utility Function**: the total utility (U_T) experienced is some function (f) of individually consumed commodities (m_i), which is the ratio of the totally consumed commodities (M_T) over the population (n), all multiplied by the population.

1.4 Assumption 4: Polynomial Approximation

From here it is necessary to attempt to specify (or at least approximate) the Generic Individual Utility function. Thus, appealing to the Stone–Weierstrass theorem, we can show that any function can be approximated by some complicated higher-order polynomial (Stone, 1948). So, taking the function as the most generic possible higher-order polynomial, we can say:

> **Generic Individual Utility Polynomial Function:** the individual utility (u_i) experienced is some zth order polynomial function with coefficients ($a_0, \dots, a_z$) of some variable (x), representing the of ratio of the totally consumed commodities (M_T) over the population (n).

This polynomial could perhaps be simplified by stipulating that the zeroth order coefficient is zero, in order to exclude the possibility of a non-zero level of utility from a consumption of zero commodities, since it seems obvious that a universe with zero individually consumed mass would have zero utility. However, this move will be avoided so as to not prematurely exclude varieties of critical-level utilitarianism (Broome, 2004; Blackorby et al., 2004).

1.4 Lemma 4*: Polynomial Factorization*

We can stipulate that the polynomial must be reducible to factors in the real domain, because commodities can only be represented in the real domain (assuming that there can be no imaginary commodities). Therefore, we arrive at:

> **Generic Individual Utility Factorized Polynomial Function:** the individual utility (u_i) experienced is the product of an nth order series of first order factors with real number coefficients ($b_0, \dots, b_{z-1}; c_0, \dots, c_{z-1}$) of some variable ($x$), representing the of ratio of the totally consumed commodities (M_T) over the population (n), all multiplied by the population.

2. Generic Egalitarian Total Utility Function

Using the above assumptions, we can finally derive:

> **Generic Egalitarian Total Utility Factorized Polynomial Function:** the total utility (U_T) experienced is the product of an nth order series of first order factors with real number coefficients ($b_0, \dots, b_{z-1}; c_0, \dots, c_{z-1}$) of some variable ($x$), representing the of ratio of the totally consumed commodities (M_T) over the population (n), all multiplied by the population.

Using this to specify the set of utility functions of interest, we have enough constraints to specify the behavior of utility at asymptotes of population and thus consider whether the various problems have solutions within the possible set of utility functions we have specified.

3. Conclusions

Because we have stipulated Generic Egalitarian Total Utility Factorized Polynomial Function, we can now draw some conclusions. We can analyze the function's asymptotic behavior in order to specify the conditions under which it will approach or avoid the Monstrous Conclusion (a version of Nozick's "Utility Monster" redefined in the domain of population) and Parfit's Repugnant Conclusion. As Parfit himself puts it, we can make an attempt at "quarantining" those "cases that involve infinite quantities" (Parfit, 2004). While we are at it we can also attempt to avoid three other pressing problems, Negative Utilitarianism, the Sadistic Conclusion, and the Anti-Egalitarian Conclusion.

3.1 Conclusion 1: Population approaching zero:

First, it can be observed that one asymptote exists as the population approaches zero on the natural number line.

> **Zero Population Asymptote**: as the population (n) approaches zero, the Generic Egalitarian Total Utility Factorized Polynomial Function approaches the product (B_T) of the first-order coefficients (b_k) in the series, multiplied by infinity.

This permits two possible asymptotic behaviors: positive and negative.

> **Zero Population Asymptote Sign**: as the population (n) approaches zero, the sign of the asymptote of the Generic Egalitarian Total Utility Factorized Polynomial Function is determined by the product (B_T) of the first-order coefficients (b_k) in the series—negative infinity if negative, positive infinity if positive.

Thus, the asymptotic behavior depends upon the value of the product of the coefficients:

3.1.a Positive Zero Population Asymptote:

If the value of the product of coefficients is positive, then we arrive at:

> **The Monstrous Conclusion**: There is always some lower population with higher total utility (Nozick, 1999, p. 41).

The Positive Zero Asymptote implies the Monstrous Conclusion because there is always some lower population with some higher value for total utility.

3.1.b Negative Zero Population Asymptote:

However, if we specify the condition that the value of the of coefficients is negative, then we avoid the Monstrous Conclusion.

> **The Non-Monstrous Conclusion**: There is always some lower population with lower total utility.

The Negative Zero Asymptote thus avoids the Monstrous Conclusion.

3.2 Conclusion 2: Population approaching infinity:

Second, it can be observed that one asymptote exists as the population approaches infinity on the natural number line.

> **Infinity Population Asymptote**: as the population (n) approaches infinity, the Generic Egalitarian Total Utility Factorized Polynomial Function approaches the product (C_T) of the first-order coefficients (c_{Tk}) in the series, multiplied by infinity.

This permits two possible asymptotic behaviors: positive and negative

> **Infinity Population Asymptote Sign**: as the population (n) approaches infinity, the sign of the asymptote of the Generic Egalitarian Total Utility Factorized Polynomial Function is determined by the product (C_T) of the first-order coefficients (c_{Tk}) in the series—negative infinity if negative, positive infinity if positive.

Thus, the asymptotic behavior depends upon the value of the product of the coefficients:

3.2.a Positive Infinity Population Asymptote:
If the value of the product of coefficients is positive, then we arrive at:

> **The Repugnant Conclusion**: There is always some higher population (n) with higher total utility (Parfit, 1984, p. 388).

The Positive Infinite Asymptote implies the Repugnant Conclusion because there is always some higher population with some higher value for total utility.

3.2.b Negative Infinity Population Asymptote:
However, if we specify the condition that the value of the of coefficients is negative, then we avoid the Repugnant Conclusion.

> **The Non-Repugnant Conclusion**: There is always some higher population (n) with lower total utility.

The Negative Infinite Asymptote avoids the Repugnant Conclusion. Thus, there is no need to embrace Repugnance, as some authors do Embracing Repugnance (Tännsjö, 2008).

3.3 Conclusion 3: Not-All-Negativity Criteria

Thus, it is possible to specify a set of utility functions that avoids both the Monstrous Conclusion and the Repugnant Conclusion by specifying that the asymptotes at zero and infinity commodities approach negative numbers. This contrasts with the impossibility theorems proposed by others (Arrhenius, 2000; Arrhenius et al, 2017).

However, because a continuous function with two negative asymptotes may be entirely negative, the resulting set of utility functions includes functions that are entirely negative, thus implying negative utilitarianism.

> **Negative Utilitarianism**: There is no population (n) on the real number line that has positive total utility (U_T) (Smart, 1958).

However, we can set a further constraint on the Generic Egalitarian Total Utility Factorized Polynomial Function so as to eliminate functions that are always negative from the set of acceptable functions. Thus:

> **Not-All-Negativity Criteria**: if at least one population exists for the Generic Egalitarian Total Utility Factorized Polynomial Function where that population equals to the negative ratio of the first-order coefficient (b_k) over the zeroth-order coefficient (c_k) for some factor (k), then the total utility is equal to zero, which is non-negative.

Thus, we have specified that the set of functions that solve Monstrosity and Repugnance can also avoid Negative Utilitarianism.

3.4 Conclusion 4: The Zero Critical-Level Corollary

By assuming a function for total utility that resolves the Repugnant Conclusion, some would argue that our results must imply the so-called Sadistic Conclusion (Blackorby et al, 2004, p. 51). This is because, by deduction, critical-level utilitarian theories cannot resolve the Repugnant Conclusion and the strong version of the Sadistic Conclusion under the same critical-level specifications, a dilemma (p. 52).

However, our solution is immune to all versions of the Sadistic Conclusion because they arise if and only if we have specified a non-zero critical-level for utility, and we have not (Blackorby et al., 2004, p. 52). Rather, we have specified a zero critical-level for utility, which is to say no critical-level at all, classical utilitarianism, making us vulnerable to the Repugnant Conclusion but no Sadistic Conclusion (Mulgan, 2002).

Indeed, it follows from the above analysis that the only possible critical-level is equal to zero because, given the Generic Total Utility Factorized Polynomial Function and assuming that function is equal to some critical-level utilitarian function, by simplifying and rearranging the function, all variables must cancel out except the critical-level itself and zero. Thus:

> **The Zero Critical Level Corollary**: the Generic Total Utility Factorized Polynomial Function must have a critical-level (α) for utility of zero.

Thus, we have managed to avoid the Repugnant Conclusion because treating the utility function as an infinite polynomial series has allowed us to stipulate a polynomial curve with many roots in the domain of commodities; and, we have managed to avoid the Sadistic Conclusion because our polynomial approximation does not require a critical-level in the codomain of utilities.

3.5 Conclusion 5: The Egalitarian Corollary

Our function resolves the so-called “Anti-Egalitarian Conclusion”, which is the possibility that there may be some low utility low equality population that is greater than an equivalently sized high utility high equality population (Arrhenius, 2000, p. 258). However, we can show that our solution is adequately egalitarian, avoiding this the Anti-Egalitarian Conclusion:

> **The Egalitarian Corollary**: if an egalitarian population has individual utilities that when summed are greater than the summed individual utilities of some equally sized inegalitarian population, then the total utility of the egalitarian population is greater than that of the inegalitarian population.

Indeed, any other result would not make sense from our function, because any sum of individual utilities (equal or unequal) greater than any other given sum of individual utilities implies a corresponding total utility greater than the other given total utility, per our commensurability/summability assumption.

3.6 Conclusion 3.6: Neutral Nonexistence Corollary

Our function satisfied what we will dub a "Neutral Nonexistence Intuition", which is a considered intuition that a population of zero should not be assigned a determinate, non-zero utility. We can show that our formulation does satisfy the Neutral Nonexistence Intuition with a corollary:

> **The Neutral Nonexistence Corollary**: if a total population is zero, then the total utility of the population is undefined.

Indeed, the only reasonable result is undefined, both intuitively and mathematically, because a population with zero members cannot be intuitively said to have determinate, non-zero utility, and a polynomial function that permits multiplication and division by zero in the domain will not have determinate, non-zero output.

4. Combined Conclusion: A Possibility Theorem

Combining the conclusions above, contra to some other authors (Arrhenius, 2000) we can arrive at a possibility theorem:

> **A Special Possibility Theorem for Welfarist Axiology**: assuming a Generic Cardinal Utility Function, Commensurability, Equal Distribution, and Polynomial Approximation there is a possible set of utility functions that avoid the Repugnant Conclusion, the Monstrous Conclusion, Negative Utilitarianism, the Sadistic Conclusion, the Anti-Egalitarian Conclusion, and satisfies a Neutral Nonexistence Intuition.

This theorem holds not just for each problem solved separately but for all solved together: the Sadistic Conclusion and Anti-Egalitarian Conclusion are excluded (and Neutral Nonexistence Satisfied) by *all* forms of the Generic Total Utility Factorized Polynomial Function; the Monstrous and Repugnant Conclusions are mutually inclusive in the domain, as are their Non-Monstrous and Non-Repugnant Solutions; and Negative Utilitarianism requires the specification of only one element of the Generic Total Utility Factorized Polynomial Function, leaving all other elements open to manipulation; all of which, in combination, allows the possibility for all four problems to be solved simultaneously.

Furthermore, beyond these four problems, this possibility theorem can be generalized to a wider set of welfarist constraints:

> **A General Possibility Theorem for Welfarist Axiology**: assuming a Generic Cardinal Utility Function, Commensurability, Equal Distribution, and Polynomial Approximation there is an infinite set of real number specifications $(a_0, \ldots, a_z)$ that can satisfy an infinite number of mutually inclusive real constraints (z) within the domain.

This solution seems underexplored by previous considerations of these problems (Ryberg & Tännsjö, 2004; Arrhenius et al., 2018). Furthermore, as a promising avenue of intuitive population ethics, the possibility of this subset of utility functions is sufficient not to give up Parfit's hope for "Theory X" (Parfit, 1984, p. 390).

Objections and Reponses

Here we will make some objections and offer responses in order to support our assumptions:

Von Neumann-Morgenstern Cardinal Utility

By assuming that utilities can be analyzed in terms of measurable quantities as a continuous function, we have tacitly assumed the cardinality of utilities. For many, making such an analysis erroneously assumes that utilities are the kinds of things that can be measured, and to many economic schools of thought considerations of cardinal utility have been superseded by considerations of ordinal utility instead (Hicks & Allen, 1934).

However, it is not obvious that we can theoretically resolve the Monstrous and Repugnant problems via our method without cardinality because constructing a polynomial approximation using the Stone-Weierstrass theorem requires a continuous function of cardinal numbers on a closed interval, a move unavailable to an analysis based merely on the preference rankings of ordinal utility, at least without further assumptions. Thus, the further assumptions of cardinality (or at least some and perhaps all of them) seem to be necessary for solving these problems. Furthermore, on a reasonable reading of their original problem statements, Parfit and Nozick both seem to implicitly assume cardinality at times in, and thus resolving the problem in the cardinal form is in a sense may be a more faithful solution to their respective problems (Parfit, 1984, p. 388; Nozick, 1999, p. 41).

Fortunately, even if we favor an analysis based on ordinal utility, it has been shown by von Neumann and Morgenstern that, under a set of reasonable conditions (completeness, transitivity, continuity, and indifference of alternatives), we are "free to make use of a numerical conception of utility", which is to say cardinal utility (p. 29). Ordinal utility problems can be treated as if they are cardinal utility problems, such that solving them cardinally solves them ordinally, because cardinal numbers have an explicit amount (1, 2, 3, etc.) and an implicit position (1st, 2nd, 3rd, etc.), whereas ordinal numbers merely have an explicit position (1st, 2nd, 3rd, etc.), so ordinal numbers can be mapped onto cardinal numbers given some transform specifications (Neumann & Morgenstern, 1966, pp. 15-30). Thus, in the best case, the analysis provided by a cardinal utility function applies under those reasonable conditions specified by von Neumann and Morgenstern, and these conditions hold for the welfarist problems in question; in the worst case, one or more of these reasonable conditions (completeness, transitivity, continuity, and indifference of alternatives) breaks down for our welfarist problems, and our analysis must be reformulated explicitly in terms of ordinal utility, if that is possible.

Commensurability and Summability

By assuming that utilities and commodities can be summed, we have tacitly assumed interpersonal commensurability between individual utilities and commodities. For commodities, this seems like an acceptable assumption because commodities can be objectively measured and compared, apples to apples. For utilities, this assumption seems uncertain because utilities must be subjectively measured and compared, apples to oranges. Some are doubtful that interpersonal comparisons are possible at all, either cardinal or ordinal (Hausman, 1995). Nor is commensurability automatically assumed by our assumption of cardinal utility theory (Neumann & Morgenstern, 1953).

However, some form of summability and thus commensurability must be assumed to even formulate the problems of Monstrosity and Repugnance because both problems assume a comparison and conglomeration of total utilities between persons and populations to analyze the betterness of those different persons and populations. Indeed, Parfit and Nozick explicitly assume a form of summation/commensurability in the statements of their respectively problems; Parfit imagines comparing populations and finding one "better" than the other, and Nozick imagines individual utilities with "greater sums . . . than . . . others lose" (Parfit, 1984, p. 387; Nozick, 1999, p. 41). As an assumption implicit in the problems, we must assume commensurability when looking for solutions that meet the problems on their own terms.

Equal Distribution and Total Utilization

By assuming that all individual commodities consumed and individual utilities experienced are equally distributed for all individuals in the population, we are assuming an idealized egalitarian world. Furthermore, by assuming total utilization, a constant total commodity for the population, we are considering a population in a frozen, finite, material world. Both of these assumptions simplify the mathematics considerably, perhaps unrealistically. Indeed, glancing at the real-world numbers, perfect equality is not representative of realistic states of affairs, nor is the total commodities consumed a constant (or even close to one) ("World Inequality Report", 2022).

However, Monstrous and Repugnant Conclusions should arise for populations independent of the inequality of the distribution of commodities and the inefficiency of underutilized commodities. Inequality and inefficiency should not affect the nature of these problems, nor should it affect the nature of the solutions. Thus, solving the problems in the idealized egalitarian and perfectly efficient case also solves them in the more realistic inequality cases; and, solving them in even one case is sufficient to show that at least one set of solutions is possible in the set of possible cases.

Furthermore, Parfit and Nozick formulate their problems within the constraints of egalitarian cases and the problems thus should be resolvable in egalitarian cases. Parfit assumes this equality explicitly by stipulating in the problem statement a population in which the population consists of persons "all with a very high quality of life" and under the condition that "if other things are equal"; whereas Nozick implicitly stipulates equality by imagining "we all be sacrificed", such that the utility of every individual is of similar (and effectively equivalent) magnitude (Parfit, 1984, p. 388; Nozick, 1999, p. 41).

Polynomial Approximation and Factorization

By assuming that individual utility functions can be represented as polynomial approximations, we have foregone specifying what the actual individual utility function might look like, and therefore our answers here are only as good as our approximations. Indeed, per the Stone-

Weierstrass Theorem by which we have come to our conclusions, we are only entitled to approximation of a continuous function on a closed interval and nothing more (Stone, 1948). Furthermore, we are only entitled to factorize a polynomial in this manner if all factors are real.

However, this problem is acceptable on three counts. First, we are mathematically entitled to drawing conclusions as approximate as our assumptions. So, assuming a provisional polynomial approximation, we may make an approximate analysis of functional behaviors, and we need not extend our claims beyond that approximation, since such conclusions need not be precise anyway. Indeed, neither Parfit nor Nozick make an analytically precise problem statement, only extremely imprecise descriptions (Parfit, 1984, p. 388; Nozick, 1999, p. 41). Second though, given that the Stone–Weierstrass Theorem allows that higher-order polynomials produce more approximate results, and given that we have arbitrarily specified that our polynomial is infinitely long (to the nth order) because we have not bounded its series, we are entitled to assume that our polynomial approximation and the actual utility function are *extremely close* (Stone, 1948). And third, although the Stone-Weierstrass Theorem only applies on a closed interval and we are drawing conclusions about the behavior of the function as it approaches infinity, our conclusions do not require precision in an open interval, but only require that the polynomial approximation's asymptotic behavior has the same general asymptotic behavior—positive or negative infinity—as the actual function. If so, the analysis succeeds. Furthermore, because we have assumed real number commodities and populations, the polynomial must also have real roots, so factorizing the polynomial down into real number factors is also possible.

Indeed, assuming a polynomial approximation seems to be the critical move necessary for the avoidance of the Repugnant/Sadistic Dilemma, and previous attempts to solve the dilemma have failed because they have stopped at critical-level utilitarianism without proposing more complicated utility functions (Blackorby et al., 2004). And this must be true mathematically because each independent specification upon the welfare function requires an extra degree of freedom, and only permitting the function the flexibility of a polynomial approximation, with an infinite series of specifiable coefficients, renders such specification possible.

Thus, we may concede that perhaps the method ultimately falls short as theoretically explanatory because it requires ad hoc approximation, the method nonetheless can succeed theoretically as a possibility theorem for the simultaneous satisfaction of many of our ethical intuitions, and the method can succeed practically as a first-pass consideration in the construction of welfarist axiologies.

Conclusion

So, with minimal reasonable assumptions, we can conclude that within the set of possible functions is a set of commensurable cardinal utility functions that avoid the Repugnant Conclusion, Monstrous Conclusion, Negative Utilitarianism, the Sadistic Conclusion, and the Anti-Egalitarian Conclusion. Thus, a set of utility functions defining a self-consistent population ethics seems possible.

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World Inequality Report, 2022. World Inequality Lab, 2022. https://wir2022.wid.world/download/

Appendix: Mathematical Derivations

Where the variables are:

x is some variable

n is the population

m_i is the individual commodities consumed

M_T is the total commodities consumed by the population

M_{Te} is the total egalitarian commodities consumed by the population

u_i is the individual utility experienced

u_{ie} is the individual egalitarian utility experienced

U_T is the total utility experienced by the population

U_{Te} is the total egalitarian utility experienced by the population

a_k is a polynomial coefficient

b_k is a first-order factored polynomial coefficient

c_k is a zeroth-order factored polynomial coefficient

B_T is a product of all first-order factored polynomial coefficients

C_T is a product of all zeroth-order factored polynomial coefficients

α is the critical-level of utility

1 Assumptions

1.1 Assumption 1: Generic Cardinal Utility Function

Generic Individual Utility Function:

$$u_i = f(m_i, n)$$

1.2 Assumption 2: Total Summability/Commensurability

Defining summability:

$$M_n(m_i, n) = \sum_{i=1}^{n} m_i$$

$$U_T(u_i, n) = \sum_{i=1}^{n} u_i$$

1.3 Assumption 3: Equal Distribution

Assume equality:

$$m_1 = \cdots = m_n$$

$$u_1 = \cdots = u_n$$

1.3* Lemma 3*: Totality/Equality Combination

Combining summation and equality:

$$M_{Te}(m_i, n) = m_{ie} n$$

$$U_{Te}(u_i, n) = u_{ie} n$$

1.3** Lemma 3**: Totality/Equality Utility Function

Combining and substituting:

$$U_{Te} = nf(m_i, n) = nf\left(\frac{M_T}{n}\right)$$

1.4 Assumption 4: Polynomial Approximation

Approximating any function f as some polynomial (via Stone–Weierstrass theorem):

$$f(x) = f(x, k) = \sum_{k=0}^{z} a_k x^k = a_z x^z + a_{z-1} x^{z-1} + \cdots + a_1 x^1 + a_0 x^0$$

1.4* Lemma 4*: Polynomial Factorization

Applying factorization:

$$f(x, k) = (b_{z-1} x + c_{z-1})(b_{z-2} x + c_{z-2}) \ldots (b_1 x + c_1)(b_0 x + c_0) = \prod_{k=0}^{z-1} (b_k x + c_k)$$

2. Generic Egalitarian Total Utility Function

Assuming:

$$U_T(u_i, n) = \sum_{i=1}^{n} u_i$$

Substituting:

$$U_T(m_i, n) = \sum_{i=1}^{n} f(m_i)$$

Substituting:

$$U_T(m_i, n, k) = \sum_{i=1}^{n} \prod_{k=0}^{z-1} (b_k m_i + c_k)$$

Simplifying:

$$U_{Te}(n, k) = n \prod_{k=0}^{z-1} \left(\frac{b_k M_{Te}}{n} + c_k \right)$$

3. Conclusions

3.1. Conclusion 1: Population approaching zero:

One asymptote exists as the population approaches zero:

$$\lim_{n \to 0} U_{Te}(n, k) = (0) \prod_{k=0}^{z-1} \left(\frac{b_k M_{Te}}{0} + c_k \right)$$

$$= (0) \prod_{k=0}^{z-1} (b_k M_{Te}(\infty) + c_k)$$

$$= (0) B_T M_{Te}^z (\infty)^z$$

$$= M_{Te}^z B_T \infty$$

3.1.a Positive Zero Population Asymptote:

If:

$$B_T > 0$$

Then:

$$\lim_{n \to 0} U_{Te}(n, k) = +\infty$$

3.1.b Negative Zero Population Asymptote:

If:

$$B_T < 0$$

Then:

$$\lim_{n \to 0} U_{Te}(n, k) = -\infty$$

3.2. Conclusion 2: Population approaching infinity:

One asymptote exists as the population approaches infinity:

$$\lim_{n\to\infty} U_{Te}(n,k) = (\infty)\prod_{k=0}^{z-1}\left(\frac{b_k M_{Te}}{\infty} + c_k\right)$$
$$= (\infty)\prod_{k=0}^{z-1}(0 + c_k)$$
$$= (\infty)\prod_{k=0}^{z-1}(c_k)$$
$$= C_T\infty$$

3.2.a Positive Infinite Population Asymptote:

If:

$$C_T > 0$$

Then:

$$\lim_{n\to\infty} U_{Te}(n,k) = +\infty$$

3.2.b Negative Infinite Population Asymptote:

If:

$$C_T < 0$$

Then:

$$\lim_{n\to\infty} U_{Te}(n,k) = -\infty$$

3.3. Conclusion 3: Not-All-Negativity Criteria

If:

$$\exists n \exists k \left(n = n_0 = -\frac{b_k}{c_k}\right)$$

Then:

$$\exists n(U_T(n = n_0) = 0)$$

3.4. Conclusion 4: The Zero Critical-Level Corollary

If:

$$U_{T\alpha}(u_i, n, \alpha) = U_T(m_i, n, k)$$

Then:

$$\sum_{i=1}^{n}(u_i - \alpha) = \sum_{i=1}^{n}\prod_{k=0}^{z-1}(b_k m_i + c_k)$$

Then:

$$\sum_{i=1}^{n}\alpha = \sum_{i=1}^{n}u_i - \sum_{i=1}^{n}u_i = 0$$

Then:

$$\alpha = 0$$

3.5 Conclusion 3.5: Anti-Egalitarian Conclusion

If:

$$\frac{1}{n}\sum_{i=1}^{n}u_{ai} > \frac{1}{n}\sum_{i=1}^{n}u_{bi}$$

And:

$$u_{a1} = \cdots = u_{an}$$

Then:

$$\sum_{i=1}^{n}u_{ai} > \sum_{i=1}^{n}u_{bi}$$

Then:

$$nu_{ai} > \sum_{i=1}^{n}u_{bi}$$

Then:

$$U_{TAe}(u_{ai}, n) = nu_{ai} > \sum_{i=1}^{n}u_{bi} = U_{TB}(u_{bi}, n)$$

Therefore:

$$U_{TAe}(u_{ai}, n) > U_{TB}(u_{bi}, n)$$

3.6 Conclusion 3.6: Neutral Nonexistence Corollary

If:

$$U_T(m_i, 0, k) = \sum_{i=1}^{0}\prod_{k=0}^{z-1}(b_k m_i + c_k)$$

Then:

$$U_{Te}(0,k) = undefined$$