Copernicus and Axiomatics

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Abstract

The debate about the foundations of mathematical sciences traces back to Greek antiquity, with Euclid and the foundations of geometry. Through the flux of history, the debate has appeared in several shapes, places, and cultural contexts. Remarkably, it is a locus where logic, philosophy, and mathematics meet. In mathematical astronomy, Nicolaus Copernicus's axiomatic approach toward a heliocentric theory of the universe has prompted questions about foundations among historians who have studied Copernican axioms in their terminological and logical aspects but never examined them as a question of mathematical practice. Copernicus provides seven unproved assumptions in the introduction of the brief treatise entitled *Nicolaus Copernicus's draft on the models of celestial motions established by himself*, better known as *Commentariolus* (ca. 1515), published circa 30 years before the final composition of his heliocentric theory (*On the revolutions of the heavenly spheres*, 1543). The assumptions deal with the renowned Copernican hypothesis of considering the Earth in motion and the Sun,

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not affected by motion, near the center of the universe. Although Copernicus decides to omit the proofs for the sake of brevity, the deductions in the *Commentariolus* are supposed to be drawn from the initial seven assumptions. Questions on the nature (are they postulates or axioms?) and the logic (is there an internal rigor?) of those assumptions have yet to be fully explored. By examining Copernicus's seven assumptions as a question of mathematical practice, it is possible to hold historical, philosophical, and logical aspects of Copernican axiomatics together and understand them as part of Copernicus's intuition and creativity.

Keywords

Axioms · Axiomatics · Axiomatization · Axiomatic method · Copernicus · Commentariolus · Creativity · Discovery · Euclid · Feferman · Heliocentrism · History of axiomatics · Mathematical astronomy · Pluralism · Postulates · Proclus · Schlimm · Use of axioms

1 Introduction

The first written draft of the heliocentric theory by Nicolaus Copernicus dates back to circa 1515 and is attested in an unpublished manuscript, which was known – already among Copernicus's peers and acknowledged then among later generations of scholars and historians – as *Commentariolus* (brief commentary). The full title sounds *Draft on the hypothesis on celestial motions established by Nicolaus Copernicus* (*Nicolai Copernici de hypothesibus motuum caelestium a se constitutis commentariolus*). This work was likely composed between 1510 and 1515. It would later attract the attention of historians of astronomy, especially in the twentieth century (Swerdlow 1973; Goddu 2010, 243–274, 291; Folkert et al. 2019, 7–111). In history, among the scholars who had in their hands a copy of the *Commentariolus*, some renowned astronomers and mathematicians are to be mentioned: Joachim Rheticus, Duncan Liddel, and Tycho Brahe (Dobrzycki and Szczucki 1989; Omodeo 2016, 11; Folkert et al. 2019, 8–14). This aspect is particularly telling of the relevance that the *Commentariolus* assumed in the sixteenth century, let alone for the wide circulation and discussion of Copernican theory in Europe. Particularly noteworthy is the so-called Aberdeen edition, because it contains the pages of *Commentariolus* and *De revolutionibus orbium coelestium* (*On the revolutions of the heavenly spheres*, Copernicus's major work, see Copernicus 1978) in parallel, according to apparent affinity between the topics. This shows that the generation immediately following Copernicus considered the two works to be complementary and not mutually exclusive, though they do not exactly match: in fact, Copernicus refined, polished, and improved his theory between the composition of *Commentariolus* and the publication of *De revolutionibus* in 1543 (Dobrzycki 1973; Folkert et al. 2019: 8–9).

The *Commentariolus* has left historians puzzled on several issues. On the one hand, the aspects regarding the details of the motion of the Earth and the mathematics implicit in the text as well as the question on the derivation of the heliocentric hypothesis (i.e., whether Copernicus arrived at the models independently or was possibly inspired by geometric models developed by Arabic astronomers) are still debated to this day (Swerdlow 1973; Blåsjö 2014; Ragep 2016; Swerdlow 2017; Blåsjö 2018). On the other hand, the seven propositions that introduce the heliocentric hypothesis in the *Commentariolus* have been the object of much scholarly discussion (Swerdlow 1973; Rosen 1976; Goddu 2010; Vesel 2014; Lerner and Segonds 2015; Folkert et al. 2019, 24–28). On the latter issue, the basic questions are as follows: What kind of propositions are they? Postulates, axioms, or something else? How should they be conceived and what kind of meaning does Copernicus attach to *axioms*? Do they have an internal rigor? Was Copernicus looking for logical rigor?

Copernicus does indeed adopt an axiomatic approach to his mathematical–astronomical theory, but the nature of his principles has confounded historians of science and generated a debate which continues to this day. Questions about Copernicus's axioms have hitherto been confined to historians of science, and studies of mathematical practice have never been employed to tackle such a problem.

This chapter examines Copernicus's axiomatic approach as a question of mathematical practice. First, it illustrates the main features of pre-modern axiomatics, evaluates the interpretations on *Commentariolus*'s principles advanced so far by historians, and eventually offers a new interpretive framework in which historical, philosophical, and logical details are comprehensively considered.

2 Axiomatics Before Axiomatics

One may ask: What does Copernicus have to do with axiomatics? Copernicus and his time seem far too distant from us today to suggest that there is any link to questions concerning "axiomatics" (in the modern sense of the term). In fact, the kind of questions that recent literature on mathematical practice has been tackling regarding axioms and their use is present already in the history of the reflections, discussions, and debates on the foundations of mathematical sciences (Schlimm 2013). These debates have their roots in Greek antiquity. For instance, there are traces of debates on foundational issues of mathematics in ancient and Hellenistic times (Acerbi 2010; Acerbi 2013; De Risi 2016). For centuries, the most successful reference work for Western mathematical sciences has been Euclid's *Elements*, and this work has also been taken as emblematic of axiomatic rigor. On the one hand, the theorems taken from the *Elements* were employed to build mathematical theories; on the other hand, the hypothetico-deductive structure of *Elements* was deemed the prototype of reason and certainty. On these grounds, it soon became an important object of study not only for mathematicians, but also for philosophers and logicians. The *Elements*' deductive system is based on principles, from which all subsequent deductions are to be drawn. A set of principles opens Euclid's *Elements*, Book One:

definitions of the subject addressed, postulates, and axioms (common notions). The distinction between postulates and axioms was a topic of debates and controversies already in ancient times. However, the ancient conception of axiomatization was very different from our own. For centuries, mathematicians, logicians, and philosophers did not look upon the definition of axioms as logically necessary, but rather saw them in light of the need to justify a "weak" science, which would not need to justify its principles if it was already consistent and, roughly speaking, logically coherent. This explains why the foundations of arithmetic had been (largely though not entirely) neglected until the nineteenth century, while the foundations of geometry were discussed since Antiquity, making Euclid's *Elements* the canon for Western mathematics (De Risi 2016).

Euclid's prevalence notwithstanding, the questions raised about principles were quite heterogeneous. Notably, influential scholars of the Greek tradition, such as Archimedes, Aristarchus, Apollonius, and others, did not assume principles in the same manner as Euclid did in the *Elements* and, in applied mathematics, such as optics, Euclid himself did not employ the same axiomatics as in the *Elements* (Capecchi 2018: 6–13). However, there was no homogeneity on the nature of principles even at a more general level. Notably, Aristotle's definition of postulates does not coincide with the Euclidean use, and Proclus, reprising Geminus (first century BCE), differs from other Greek philosophical schools (Bobzien 1996; Acerbi 2010; Acerbi 2013; Bobzien 2019).

In sum, the history of axiomatics has never been monolithically Euclidean. However, the focus on axiomatization of arithmetic in the nineteenth and twentieth centuries, after the works of Dedekind, Klein, Hilbert, Peano, Frege, and so on and the reflections on axiomatics by logicians like Carnap, Gödel, and Tarski and his followers, has led historians, philosophers, and logicians to study traces of "axiomatics before axiomatics" in mathematical sciences (optics, astronomy, mechanics), logic, and philosophy of antiquity till early modernity (Henkin et al. 1959; Hintikka et al. 1981; Novaes 2020; Cantù and Luciano 2021). Some influential works are worthy of note in this regard. Arpad Szabó was pioneering in his studies of the interplay of philosophical and mathematical influences in the development of Greek axiomatics and claimed that the Eleatic school influenced the rise of the deductive method in early Greek mathematics (Szabó 1964). Contrary to Szabó, Wilbur R. Knorr argued that the deductive procedures evident in fifth-century Greek works are intrinsic to mathematics and that the influence of the Eleatic dialectic on the development of axiomatics in arithmetic and geometry could only have occurred later, in the environment of the fourth-century Academy, in which the interest in mathematics merged with a renewal of the logical and philosophical views espoused by the Eleatics (Knorr 1981). Later on, Patrick Suppes shed light on axiomatics in Greek and Hellenistic science, pointing out their similarities with contemporary philosophical and scientific endeavors:

> the admiration many of us have for the rigor and relentlessness of the axiomatic method in Greek geometry has given us a misleading view of the role of this method in the broader framework of ancient Greek mathematical sciences. By stressing the limitations of the

> axiomatic method or, more explicitly, by stressing the limitations of the role played by the axiomatic method in Greek mathematical science, I do not mean in any way to denigrate what is conceptually one of the most important and far-reaching aspects of Greek mathematical thinking. I do want to emphasize the point that the use of mathematics in the mathematical sciences and in foundational sciences, like astronomy, compare rather closely with the contemporary situation. (Suppes 1993, 25)

As the above claims suggest, the possibility of external influences (from philosophy or logic) on an axiomatic method in mathematical sciences is a widely explored topic, and questions on this issue are also raised by the use of axioms in *Commentariolus*. Whereas this text has indeed pushed historians to search for possible philosophical influences on Copernicus's axiomatic approach, the possibility of this approach as intrinsic to Copernicus's mathematical practice has yet to be considered.

3 Questions About Copernicus's Axioms

Philadelphia, PA, 1973. At a symposium marking the 500th anniversary of Copernicus's birth, the eminent historian of science Noel Swerdlow presents a paper on the *Commentariolus* along with a translation into English and a commentary thereon. He states that Copernicus incorrectly called the seven sentences in the first section of the *Commentariolus* "axioms" and that those assumptions are illogically structured – the question on axioms in the *Commentariolus* became a topic of interest immediately following Swerdlow's translation of the Latin text (Swerdlow 1973). I will therefore report Swerdlow's translation since it is the one from which the discussions on axioms proceeded.

Swerdlow's critique refers to the first passages of *Commentariolus*, where Copernicus claims that Claudius Ptolemy and his predecessors were not coherent with the axiom of uniform circular motion, for they introduced geometrical devices which ultimately contradicted that principle and made cosmology excessively complicated. Ultimately, his predecessors failed to conceive of the universe as an integral whole (Goddu 2009). In the following, the words of Copernicus, writing in first-person singular, account for this state of the art:

> Therefore, when I noticed these [difficulties], I often pondered whether perhaps a more reasonable model composed of circles could be found from which every apparent irregularity would follow while everything in itself moved uniformly, just as the principle of perfect motion requires. After I had attacked this exceedingly difficult and nearly insoluble problem, it at last occurred to me how it could be done with fewer and far more suitable devices than had formerly been put forth if some postulates, called axioms, are granted to us, which follow in this order:
>
> First Postulate There is no one center of all the celestial spheres (*orbium*) or spheres (*sphaerarum*).
>
> Second Postulate The center of the earth is not the center of the universe, but only the center towards which heavy things move and the center of the lunar sphere.

> Third Postulate All spheres surround the sun as though it were in the middle of all of them, and therefore the center of the universe is near the sun.
>
> Fourth Postulate The ratio of the distance between the sun and earth to the height of the sphere of the fixed stars is so much smaller than the ratio of the semidiameter of the earth to the distance of the sun that the distance between the sun and earth is imperceptible compared to the great height of the sphere of the fixed stars.
>
> Fifth Postulate Whatever motion appears in the sphere of the fixed stars belongs not to it but to the earth. Thus the entire earth along with the nearby elements rotates with a daily motion on its fixed poles while the sphere of the fixed stars remains immovable and the outermost heaven.
>
> Sixth Postulate Whatever motions appear to us to belong to the sun are not due to [motion] of the sun but [to the motion] of the earth and our sphere with which we revolve around the sun just as any other planet. And thus the earth is carried by more than one motion.
>
> Seventh Postulate The retrograde and direct motion that appears in the planets belongs not to them but to the [motion] of the earth. Thus, the motion of the earth by itself accounts for a considerable number of apparently irregular motions in the heavens. (Translation Swerdlow 1973, 435–436)

In response to Swerdlow's criticism, Edward Rosen attempted to do justice to Copernicus's axiomatics (Rosen 1976). The problem would also be taken up again later (Goddu 2010; Vesel 2014; Lerner and Segonds 2015; Folkert et al. 2019; Goddu 2019, 162–163; more on this below). In sum, the questions on Copernicus's axiomatic approach touch on two key areas: *terminology* and *logic*.

3.1 Terminology

Let us recall Copernicus's words in *Commentariolus*:

> [. . .] it at last occurred to me how it could be done with fewer and far more suitable devices than had formerly been put forth if *some postulates, called axioms*, are granted to us, which follow in this order. (Swerdlow 1973, 435, emphasis added)

The original Latin text reads *petitiones quas axiomata vocant*, where *petitio* conveys a variety of meanings: requirement, statement, assumption, axiom, and postulate.

According to Swerdlow (1973, 437):

> The seven postulates, incorrectly called axioms since they are hardly self-evident, take the place of the general description of the universe in the opening chapters of the *Almagest*, the *Epitome*, and later, *De revolutionibus*. [. . .] There is no reason to doubt that he also believes these postulates to be true.

Swerdlow was undoubtedly referring to a precisely connotated meaning of axioms and postulates, where axiom is exclusively connected to self-evidence. This distinction is not modern; it traces back to the debates on Euclid's *Elements*, more precisely to the Greek philosopher Geminus (first century BCE), as attested in

Proclus's *Commentary on the First Book of Euclid's Elements* (Acerbi 2010). According to Geminus-Proclus, the postulates of Euclid's *Elements* can be divided into two groups, reflecting their different nature: constructions are required in postulates 1–3, while postulates 4–5 state properties of particular geometric objects. As for the axioms (or common notions), they were generally conceived as assumptions conveying self-evident truths, hence requiring no proofs. Yet, not every author of mathematics assumed principles in the same manner as Euclid did. As anticipated above, Archimedes used axioms as true statements to describe the physical world, for example, Archimedes's *On the Sphere and the Cylinder* opens with axioms (Netz 2004, 34–36); Aristotle thought that "an axiom is a primary proposition which must be possessed by whoever is to gain any knowledge"; "the axioms are the primary propositions from which a proof proceeds" (Aristotle 2002, I, 2: 72a17; I, 10: 76b14–15); the Stoics considered axioms as kinds of assertibles but did not employ the category of self-evidence in their logic, much to Proclus's disappointment (Ierodiakonou 2006).

Edward Rosen, author of translations of several Copernican treatises, *Commentariolus* included, accused Swerdlow of anachronism (Rosen 1976): Copernicus was not taking part in any Hilbertian program. Rosen's reply was beneficial to future generations of Copernicus scholars. André Goddu (Goddu 2010) undertook detailed research on the making of Copernican cosmology and his possible sources, detected his logical and philosophical backgrounds, and surmised a plausible method of reasoning in a Socratic-dialectic process, used by Copernicus in setting out his axioms. Goddu claimed that axioms are to be intended in the sense of common notions, assumptions to be taken for granted, where no self-evidence is needed:

> Copernicus began *Commentariolus* controversially with "petitiones" that he also called "axioms." If we set aside the personal attacks by some commentators, the experts agree, even if inadvertently, that he did not mean the word "axiom" in the sense of self-evident principles but rather in the sense of assumptions or common notions. As Copernicus himself made abundantly clear, the rest follows only if the seven postulates or assumptions are granted him. It is evident that he arrived at these seven propositions by working his way back to them as the ones necessary and sufficient from which to derive the remaining propositions. (Goddu 2010, 243)

Matjaž Vesel proposed a different interpretation. The axioms are set out and Copernicus draws his conclusions directly from them, and his source should be Proclus's commentary on Timaeus:

> I believe Copernicus' manner of exposition and the nature of *petitiones quas axiomata vocant* find their explanation in Proclus' *Commentary on Plato's Timaeus* II, 3 [. . .]. Proclus explains that Plato is not an empiricist: Plato will not start with experiences and then draw conclusions from them. Plato's *methodos* is hypothetical, or, rather, Plato uses the method of the hypothesis. He sets out fundamental *axiômata* and *hypotheseis* and draws conclusions from them. Proclus presents first a list of five axiomata, and then follows another list of seven *axiômata*. Describing Plato's "hypothetical method" Proclus does not refer to Plato's own description of hypothetical method but explicitly refers to the *methodos* used by geometers.

> They first postulate, define and name their key principles before proceeding to their demonstrations based on them. And he cites an example from Euclid. On the basis of the fundamental principles or hypotheses Plato's Timaeus then proceeds, in Proclus' reading of the text, to a number of "demonstrations" (*apodeixeis*) based on them and required in order to solve the problems [. . .]. Copernicus' method in the Commentariolus is highly reminiscent of Proclus: he first establishes seven *petitiones quas axiomata vocant* and then promises to provide mathematical *demonstrationes* in a larger book. (Vesel 2014, 269–270)

Michel-Pierre Lerner and Alain Segonds agree in part with Swerdlow on the first objection, for they claim that Copernicus's axioms are not self-evident (Lerner and Segonds 2015, 233). To some extent, they agree with Goddu when he writes that "the postulates are not axiomatic in the Euclidean sense, there is not a logically deductive relationship between the postulates, and the results described later in the text derive from the postulate of a moving Earth" (Goddu 2019, 163).

All things considered, two things are noteworthy here. First, there is no occurrence of the equivalence *petitio* as a synonym of *axioma* in the Euclidean tradition (De Risi 2016). It is likely that Copernicus knew that axioms were known and evident to everybody, which could explain why he names his principles such. Second, the word *axioma* in Latin was extremely rare around 1515, when Copernicus composed the *Commentariolus*. Indeed, the only occurrence is in Giorgio Valla's *De expetendis et fugiendis rebus* (Valla 1501: Book 10, ch. 110), an encyclopedic work of sciences and arts. Valla was a humanist who was keen on transliterating Greek words into the Latin alphabet (De Risi 2016, 643; Goddu 2010, 229–236). In the passage in question, Valla is rephrasing the distinction between postulates and axioms, drawing from Proclus:

> But a postulate prescribes that we construct or provide some simple or easily grasped object for the exhibition of a character, while an axiom asserts some inherent attribute that is known at once to one's auditor. (Proclus 1970, 142)

It is likely that Copernicus read this work by Valla and thus wished to communicate to his readers that some humanists call the mathematical principles "axioms" and that it is not a case of distinguishing the Euclidean meaning of postulates and axioms because he is using axioms as a generic term for a principle to be taken for granted. In addition, as a trainee of mathematics and logic in Italy and Poland, Copernicus was certainly aware of the Euclidean meaning.

All this confirms that Swerdlow's interpretation has aprioristically assumed that axioms must be intrinsically connotated with self-evidence. As mentioned, Copernicus does not adhere to any Hilbertian project, nor is he dealing with geometry in the fashion of Peano's program on the foundations of geometry (Rosen 1976). Therefore, there is good reason to leave aside the option of "self-evidence" and instead look for an interpretation in the literature that Copernicus had at his disposal.

What has gone unnoticed so far is that Copernicus could refer to the Greek tradition of mathematical sciences, which was keen on using the axiomatic approach. For instance, several Greek works of mathematical sciences often adopted seven unproved assumptions to open treatises, such as Euclid's *Optics* (Burton 1945) and

Archimedes's *On the Equilibrium of Planes* (Heath 2002, 189–190). It is likely that Copernicus arranged his treatises on seven principles in order to refer to that tradition, which could well have served as a model for him. Archimedes's *On the Equilibrium of Planes* was circulating in Italy thanks to the Latin translation of William of Moerbecke and that work opens with the word *petimus* (the verb linked to *petitio*, meaning "we require," "we state some principles"), which is followed by exactly seven principles (Clagett 1976, 116). Moreover, *Elementa Jordani* (*Elements of mechanics* by Jordan of Nemore) provides seven initial unproved assumptions (Clagett and Moody 1952, 154–155). Further research will clarify whether Copernicus could have had access to these sources, but it is likely that he could have assimilated, if not read, this literature in excerpts or classes at the universities he attended in Poland and Italy.

3.2 Logic

The second point of criticism aimed at Copernicus's axiomatic approach deals with the internal logic of his assumptions. According to Swerdlow:

> He [i.e. Copernicus] has, however, made more assumptions than were necessary. Since his adoption of heliocentric theory followed from his study of possible rearrangements of planetary models, only postulates 3 and 6 are temporally prior to his planetary theory. In fact only postulates 3 and 6 are logically prior while postulates 2, 4, 5, and 7 are consequences of 3 and 6, and postulate 1 stands by itself. Thus, if 3 and 6 are true, then 2, 4, 5, and 7 can be proved. Postulates 3 and 6 cannot be proved, and the evidence for their truth is the sense of the heliocentric theory itself, that is, the fixing of the order and distances of the planets, and the explanation of the second anomaly, gratuitously given here as the seventh postulate. It is also worth noting that the postulates, with the exception of the first, have no connection with the objections to Ptolemy's representation of the first anomaly stated earlier nor with Copernicus's own model for the first anomaly, but are concerned only with the heliocentric theory. Since Copernicus has raised no objection to Ptolemy's representation of the second anomaly, the introduction of these postulates at this point appears unmotivated. Perhaps this flaw and the logical error of stating postulates 2, 4, 5, and 7 as postulates rather than deductions from postulates 3 and 6 are intelligible if one considers that the *Commentariolus* may well have been written in haste with no revision. (Swerdlow 1973, 437)

It is true that some axioms can be derived from others, but the whole problem cannot be comprehended if one does not advance another question, which in the logic of historical inquiry precedes Swerdlow's question: Is Copernicus seeking internal rigor in his assumptions? And what kind of rigor is required in the sixteenth century for a brief text of mathematical astronomy? Is he striving to give a list with as few assumptions as possible?

There is indeed a logic in Copernicus's axioms, although they are – strictly logically speaking – redundant, and this logic was brilliantly exposed by Edward Rosen (1976). The passage is reported here in full, and axioms are mentioned by numbers between parentheses.

> (1) undermines Eudoxus' principle of homocentricity by denying that everything in the universe is centered on the earth: "There is no one center." Consequently there are multiple centers, of which the earth is one. (2) puts the earth at the center, not of the universe, but of the moon's motions. Another center is the sun, placed by (3) near the center of the universe. The distance between these two centers, earth and sun, is said by (4) to be imperceptible in comparison with their distance from the stars. (5) The stars do not move, their apparent daily motion being due to the earth's axial rotation. (6) The sun does not move, its apparent motion being due to the earth's orbital revolution. (7) The apparent retrogression of the planets is likewise only an appearance due to the motion of the terrestrial observer. By common consent, (7) is Copernicus' single most valuable contribution to technical astronomy. (Rosen 1976, 47)

Later, the examination of Copernicus's logical background will allow André Goddu to convincingly describe the Copernican heliocentrism as an attempt toward envisioning the universe as an integral whole, thus in need of a "unique, non-arbitrary, and commensurable structure benefitting the exquisite craftsmanship of its architect" (Goddu 2009, 332). Copernicus was indeed trained in *mereology*, the branch of medieval logic which deals with the relations between the whole and the parts (Goddu 2009).

Moreover, as anticipated above, the need for axioms in Copernicus's times was not seen as a logical necessity; indeed, it could even be seen to signal a lack of rigor, since a science should be justified without principles. The main axiom of astronomy was that celestial bodies move along uniform and circular motions, and Copernicus did not need to mention it in his list. In sum, Copernicus and his contemporaries would not deem the "redundancy" of axioms to be a mistake or a problem in the context of *Commentariolus* and the literature to which it refers (Ptolemy and ancient astronomers).

By considering all terminological and logical concerns in relation to Copernicus's assumptions, it is evident that Swerdlow (1973) was moving his objections from applying a part of the developments of modern axiomatics to a sixteenth-century context. It is therefore expedient to explore what contemporary studies on axiomatics might have to say about Copernicus's mathematical practice. Moreover, it is worth examining whether one should look for an external source or to study the axioms in *Commentariolus* as an integral part of Copernicus's creativity, a key feature in his development of the mathematical theory for heliocentrism.

4 Copernicus's Axiomatic Approach

Historians have focused on the terminology and the logic that the seven assumptions bring to the fore, but nobody has yet stressed that there is another axiom, which precedes the seven axioms and which was the common assumption for all scholars dealing with astronomy since antiquity. The additional axiom runs as follows: all celestial bodies move according to uniform and circular motions (Taub 1993; Feke 2018). In this instance, reflections on axiomatics in mathematical practice provide new tools to build an interpretive framework which unifies the shared fundamental axiom along with the seven axioms and their historical and philosophical aspects.

Dirk Schlimm (2013) has analyzed the use of axioms in contemporary mathematical practice and argued that "axioms can play many roles in mathematics and that viewing them as self-evident truths does not do justice to the ways in which mathematicians employ axioms" (Schlimm 2013, 73). As the ancient world was not monolithically Euclidean in assuming principles in mathematical sciences, so the contemporary mathematical, philosophical, and logical disciplines are similarly not attached to the notion of self-evidence and perfect rigor when it comes to issues of axiomatics. As for questions of redundancy and internal logic among axioms, Schlimm's study offers a significant remark: "Popular criteria for axioms, e.g., that they should be as few, simple, and self-evident as possible, are highly idealized desiderata and by no means necessary conditions for systems of axioms" (Schlimm 2013, 39).

Most importantly, before the time of Hilbertian and formalistic interpretations of Copernicus's axioms, professional mathematicians were also raised on the axiomatic approach. Notably, the renowned collective of mathematicians under the name of Nicolas Bourbaki stated that the essential aim of the axiomatic method "is exactly that which logical formalism by itself can not supply, namely the profound intelligibility of mathematics" (Bourbaki 1950, 223). The same Bourbaki argued that formalization is "but one aspect of this [axiomatic] method, indeed the least interesting one" (Bourbaki 1950, 223).

All this speaks in favor of considering Copernicus's method as axiomatic even from a contemporary perspective. On this account, nothing should prevent historians from regarding Copernicus's *petitiones* as proper axioms.

As noted previously, historians of astronomy have inquired into Copernicus's possible sources in his use of axioms. By dealing with this puzzle in mathematical practice, another question arises: Does Copernicus need a source for establishing axioms? On the one hand, the historian can answer that Copernicus is likely seeking to contribute to the Greek tradition of mathematical sciences, with which he could have come into contact via Latin translations of Archimedean works, in which axioms are used in a somewhat generic sense and as true statements about the universe and nature. Thus, Copernicus is striving to give a list of seven assumptions to engage with that tradition, conforming to a model. On the other hand, Copernicus knows his mathematics: his intuition might well form part of his practice, that is, from the experience he assimilated with ancient astronomical mathematics and the observations he conducted. Copernicus's axiomatic approach might be the outcome of his mathematical practice, not necessarily the derivation from philosophical notions or other scientific sources he could have encountered during his career. This hypothesis is further supported by an idea that several professional mathematicians have shared on the intelligibility of mathematics and its intrinsic creativity. For instance, André Weil has claimed that:

> The views of Greek philosophers about the infinite may be of great interest as such; but are we really to believe that they had great influence on the work of Greek mathematicians?

> Some universities have established chairs for “the history and philosophy of mathematics”: it is hard for me to imagine what those two subjects can have in common. (Weil 1980, 230)

5 An Interpretive Framework

It is clear that Copernicus employs a non-Euclidean approach in axiomatics, driven by the needs of reforming Ptolemy and his predecessors toward a new vision of the universe as an integral whole. Interestingly, Copernicus would refine his heliocentric theory after completing *Commentariolus*, outlining his definitive theory only in 1543 (in *De revolutionibus orbium coelestium – On the revolutions of the heavenly spheres*). The latter work does not list the seven axioms but presents their content along with theorems and proofs. On this account, it is likely that the seven axioms fed Copernicus’s creativity and were engines for new discoveries during the composition of the final heliocentric theory. Further research will surely shed light on this issue.

The details of Copernicus’s axiomatic approach could be described by adopting the guidelines set out by Dirk Schlimm in his study on the use of axioms in contemporary mathematical practices (Schlimm 2013). Such an attempt toward a new interpretation is also needed to counteract a Copernican literature which has been almost Euclidean or has tended to see Copernican axioms in a single dimension, that is, as dependent on an external source, and not in terms of their intrinsic mathematical and practical dimension – here one could well recall Bourbaki’s and Weil’s words on mathematical intuition and creativity. According to Schlimm, there are four main points in a comprehensive study on the use of axioms (Schlimm 2013, 39–40; points quoted in the following).

(a) “Axiomatics is an epistemic and methodological tool that can be employed in various ways. In other words, there is no single role that axioms play in mathematical practice; rather, the same set of axioms can be employed in different roles.”

This is in accordance with the non-monolithically Euclidean axiomatics since Greek Antiquity, which is useful to recall in order to understand why Copernicus considered “postulate” as being equivalent to “axiom.”

(b) “Popular criteria for axioms, e.g., that they should be as few, simple, and self-evident as possible, are highly idealized desiderata and by no means necessary conditions for systems of axioms.”

This point has been extensively developed in treating the logic of Copernican axioms.

(c) "The practical usefulness of axioms goes well beyond the context of justification and the aim of clarifying and providing foundations for mathematical theories; they are also engines for discovery in mathematics."

This point can be seen in the fact that Copernicus continued to develop his theory after redacting *Commentariolus*. It took him around 30 years to publish his final heliocentric theory.

(d) "The dimensions of *presentation*, *role*, and *function* allow us to characterize different uses of axioms and to compare them along these three axes, with the purpose of clarifying discussions of axiomatics in mathematical practice."

Schlimm provides this framework for one interpretation among many, but this is in keeping with the historical case presented above. While it is evident how points a, b, and c have been already touched on in Copernicus's case, point d needs to be further articulated. It is necessary to recall that the Copernican axioms are seven plus the common fundamental axiom of ancient astronomy. On this account, the *function* of the axioms is to set the foundations for the geometry describing a heliocentric universe and to develop it further; the Copernican axioms *present* the main points of a new theory, marking a strong break from the astronomers of the past. The axiom of uniform and circular motion and the seven others have different *roles*. To describe this feature, it is worth introducing the distinction between *structural* and *foundational* axioms as suggested by Solomon Feferman (Feferman 1999) and adapt it to a context of sixteenth-century mathematical astronomy and logic. As for the structural axioms, Feferman claims:

> When the working mathematician speaks of axioms, he or she usually means those for some particular part of mathematics such as groups, rings, vector spaces, topological spaces, Hilbert spaces, etc. These axioms have nothing to do with self-evident propositions, nor are they arbitrary starting points. They are simply definitions of kinds of structures that have been recognized to recur in various mathematical situations. (Feferman 1999, 100)

This is an apt definition for the seven Copernican assumptions, for in Copernicus's case there are seven not self-evident propositions to define a heliocentric structure of the universe, and he conceives them as non-arbitrary assumptions about physical hypotheses to envision the universe as an integral whole (Goddu 2009). As such, it makes sense to define them as *structural* in their context.

Concerning foundational axioms, Feferman states that:

> in contrast to the working mathematician's structural axioms, when the logician speaks of axioms, he or she means, first of all, laws of valid reasoning that are supposed to apply to *all* parts of mathematics, and, secondly, axioms for such fundamental concepts as number, set, and function that underlie all mathematical concepts; I call the latter *foundational axioms*. (Feferman 1999, 100)

Commentariolus contains the fundamental axiom of ancient astronomy, which underlies all the other assumptions, namely, all motions of celestial bodies must be uniform and circular. This could well serve as a foundational axiom in Copernicus's composition, and it is also coherent with his vision of a whole and integral universe (Goddu 2009). On these grounds, adapting Feferman's distinction to *Commentariolus* would build a comprehensive interpretive framework for Copernicus's axiomatic approach.

A Schlimm-Feferman framework integrates aspects of mathematical practice with the historical questions noted above on terminology and logic. In such a framework, the foundational axiom of uniform and circular motion can be deemed the one which triggers Copernicus's creativity. As Copernicus would continue to further elaborate his theory leading up to the publication of his major work, circa 30 years later, the seven axioms can be considered engines of discovery and aids to mathematical creativity.

6 Conclusion

Copernicus's axiomatic approach toward a heliocentric theory of the universe, as exposed in his *Commentariolus*, has triggered a range of questions among historians, who have studied his axioms in their terminological and logical aspects. Terminologically, given the rarity of the term *axioma* in Latin literature before the first half of the fifteenth century, Copernicus likely encountered the term in some Latin humanist work (probably Giorgio Valla's *De expetendis*) and thus became aware that somebody was referring to *petitiones* as *axiomata*. As a result, he added the term *axioma* with the intention of conveying that some humanist authors may call mathematical principles *axiomata*. Humanism was indeed the cultural climate in which Copernicus was living during the years of his Italian apprenticeship. He did not need to enter the post-Euclidean debate on the difference between postulate and axiom because he was not writing a treatise on geometry but a draft on astronomy, that is, mathematics applied to a branch of natural philosophy: he considered his models as physically real, not instrumental.

It is notable that there was no controversy concerning Copernicus's axioms prior to 1973. What Swerdlow provided was a kind of formalistic criticism that morphed into a debunking of Copernicus's logic, which proved to be anachronistic: such criticism was based on a strict interpretation of axiomatics applied to history, which, in turn, does not reflect the current scenario of axiomatics in logic and mathematics. For instance, the rigor and logic for which Swerdlow was advocating with regard to axioms did not correspond to what professional mathematicians such as Nicolas Bourbaki and André Weil were thinking on axiomatics and the potential of the axiomatic method in the second half of the twentieth century.

Moreover, this case study has shed light on the similarities between axiomatics in early modern and contemporary times, a sort of bridge with the axiomatics of the present and the past, a task which was already undertaken some time ago by Patrick Suppes. Indeed, Suppes claimed that:

> There was certainly a sense of methodology deeply embedded in Euclid, Archimedes, and Ptolemy, but it was not a sense of methodology that was completely explicit or totally worked out, just as Aristotle's own general principles are never exemplified in any detailed and complicated scientific examples of an extended sort. The gap between philosophical analysis, canons of axiomatic method, and actual working practice was about the same order of magnitude that it is today. What is surprising, I think, from a philosophical standpoint is that the gap seems, if anything, to have widened rather than narrowed over the past 2000 years. (Suppes 1993, 40)

The gap between historical–philosophical, logical, and mathematical (practical) plans is what is evident in the history of the scholarship on Copernicus's *Commentariolus*. Studies on mathematical practice have proven beneficial to better comprehend the Copernican axiomatic approach and provide some order to such plans. In sum, to speak about Copernicus's assumptions as axioms and to consider the assumptions (*petitiones*) as synonymous with axioms (*axiomata*) is to give a correct translation of the Copernican text even from a contemporary and not mono-dimensional (self-evident) perspective on axiomatics, thereby avoiding the pitfalls of anachronism.

Unlike past considerations of Copernican axioms, this study has pointed out that the seven unproved assumptions set out by Copernicus are linked to the axiom of uniform circular motion. To better comprehend this feature in *Commentariolus*, an interpretive framework based on Schlimm and Feferman sheds new light on the different features of Copernican axioms and enables an understanding of them through the lens of pluralism. The utility of the Schlimm-Feferman framework is at least twofold: first, it holds together the plurality of the uses of axioms (by considering the use of the term in the literature before Copernicus) and the difference in the kinds of axioms (foundational and structural); second, it points to the potentiality of Copernicus's axioms as tools for mathematical creativity and engines of discovery in light of Copernicus's final elaboration of the heliocentric theory.

Ultimately, Copernicus's mathematical practice adds another nuance to the possibility of considering the importance and applicability of mathematics and its history: as Wittgenstein famously observed, "mathematics is a colorful conglomerate of techniques of proof." [Full original quotation: "Ich möchte sagen: Die Mathematik ist ein BUNTES Gemisch von Beweistechniken.-Und darauf beruht ihre mannigfache Anwendbarkeit und ihre Wichtigkeit."] (Wittgenstein 1956, p. 84, author's translation). Indeed, the Copernican axioms set out in *Commentariolus* would form the basis on which to draw proofs at a later stage, in *De revolutionibus* (1543), but the history of mathematical creativity toward heliocentrism – between the compositional stages of *Commentariolus* and *De revolutionibus* – remains to be written. At any rate, if the *Commentariolus* per se does not explicitly contain techniques of proof, historians dealing with the Copernican axioms have certainly given birth to a colorful conglomerate of interpretations. Taking a closer look at axiomatics, drawing on the work of mathematicians, logicians, and philosophers of mathematical practice, has offered a new interpretation and has hopefully brought some order and clarity to such a historical conglomerate.

7 Cross-References

▶ Euclidean and Non-Euclidean Geometry in the History and Philosophy of Mathematics Practice
▶ Historiography of Mathematics from the Mathematician's Point of View
▶ Implicitly Defining Mathematical Terms: A Path Toward Pluralism
▶ Introduction to the Origins of the History and Philosophy of Mathematical Practice
▶ Logic in the History and Philosophy of Mathematical Practice
▶ Non-deductive justification in mathematics
▶ One Mathematic(s) or Many? Foundations of Mathematics in 20th Century Mathematical Practice
▶ The Values of Mathematical Proofs
▶ What Mathematicians Do: Mathematics as Process and Creative Rationality

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