

Embracing monotonicity

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Abstract

A *non-embracing* consequence relation is one such that no set of wffs closed under it is equal to the set of all wffs. I prove that these relations have no deductive power if they are also extensive and monotonic.

Let α, β, \dots stand for sentences of a set \mathbf{L} of wffs. \mathbf{L} can be inductively defined in the usual way, although we are not going to presuppose any particular definition. Also let $\mathbf{A}, \mathbf{B}, \dots$ stand for subsets of \mathbf{L} , which may be finite or not. A *consequence relation* $\vdash: \wp\mathbf{L} \times \mathbf{L}$ has in its extension those pairs $\langle \mathbf{A}, \alpha \rangle$, such that α is a logical consequence of \mathbf{A} . The expression $\langle \mathbf{A}, \alpha \rangle \in \vdash$ is usually abbreviated by $\mathbf{A} \vdash \alpha$. Finally, the *consequence set* of \mathbf{A} , or \mathbf{A}^+ , is just the set of all the logical consequences of \mathbf{A} .

Definition 1. $\mathbf{A}^+ = \{\alpha \mid \mathbf{A} \vdash \alpha\}$

For now, let us only consider the following properties of \vdash .

Axiom 2 (Extension). $\mathbf{A} \subseteq \mathbf{A}^+$

Axiom 3 (Monotonicity). $\mathbf{A}^+ \subseteq (\mathbf{A} \cup \mathbf{B})^+$

Corollary 4. $\mathbf{A} \subseteq \mathbf{B} \Rightarrow \mathbf{A}^+ \subseteq \mathbf{B}^+$ {3}

Languages with negation usually have other properties concerning consistent and trivial sets, which I will define as follows:

Definition 5 (Consistency). \mathbf{A} is *consistent* $\Leftrightarrow \alpha, \neg\alpha \in \mathbf{A}$ holds for no α , and *inconsistent* otherwise.

Definition 6 (Triviality). \mathbf{A} is *trivial* $\Leftrightarrow \mathbf{A} = \mathbf{L}$.

To assert a trivial set of statements is the same as asserting everything and nothing, which justifies us in regarding any such assertion as absurd [1]. The *principle of explosion* transfers this absurdity to inconsistent sets since, according to it, *ex contradictione sequitur quodlibet* (ECQ), i.e. anything follows from a contradiction. Consequently, any formula whatsoever follows from an inconsistent set.

Postulate ECQ (Explosion). $\mathbf{A}^+ = \mathbf{L}$, for any inconsistent \mathbf{A} .

Postulate ECQ has been questioned both on philosophical and logical grounds. These criticisms reached their highest point with the formulation of formal systems of logic that restricted its validity: the so-called *paraconsistent logics*. Such logics may be characterised by the following postulate, which is just the negation of ECQ.

Postulate P (Paraconsistency). $\mathbf{A}^+ \neq \mathbf{L}$, for some inconsistent \mathbf{A} .

This postulate does not forbid that some inconsistent sets satisfy ECQ, for it only requires that at least one does not. And yet, we can define a class of logics where it is impossible to trivialise a set on grounds of it being inconsistent. Following Perzanowski's [3] terminology, these may be called *strongly paraconsistent logics*.

Postulate SP (Strong paraconsistency). $\{\alpha, \neg\alpha\}^+ \neq \mathbf{L}$

Postulate SP does not completely rule out trivialisation. In principle, for every set \mathbf{A} there could be some sophisticated wff α capable of trivialising it, that is, such that $(\mathbf{A} \cup \alpha)^+ = \mathbf{L}$. For example, in the paraconsistent calculus \mathbf{C}_1 of da Costa [2], a set \mathbf{A} such that $\mathbf{A} \vdash * \neg\alpha$ could be trivialised by α . Notwithstanding that, this postulate suggests an even stronger principle that would make it impossible to trivialise any set. Following Popper [4], I will call this the *principle of non embracingness*.

Postulate $\bar{\mathbf{E}}$ (Non embracingness). $\mathbf{A}^+ \neq \mathbf{L}$, for all $\mathbf{A} \subset \mathbf{L}$.

It clearly follows from this that no \mathbf{A} obtained by subtracting a singleton from \mathbf{L} can entail formula that is not already a member of \mathbf{A} .

Lemma 7. $\mathbf{L} - \mathbf{A}$ is a singleton $\Rightarrow \mathbf{A}^+ = \mathbf{A}$. {2, 3, $\bar{\mathbf{E}}$ }

Proof. Assuming that $\mathbf{L} - \mathbf{A}$ results in that $\mathbf{A} \subseteq \mathbf{L}$ and that $\mathbf{L} - \mathbf{A}$ is not empty. From these, it follows that $\mathbf{A} \subset \mathbf{L}$, which by postulate $\bar{\mathbf{E}}$ entails that $\mathbf{A}^+ \neq \mathbf{L}$. This, by extension, entails that $\mathbf{A}^+ = \mathbf{A}$. 👉

This lemma extends to every subset of \mathbf{L} .

Theorem 8. $\mathbf{A}^+ = \mathbf{A}$ {2, 3, $\bar{\mathbf{E}}$ }

Proof. In order for $\mathbf{A}^+ = \mathbf{A}$ not to hold in general, there needs to exist at least one \mathbf{A} for which there is at least one $\alpha \notin \mathbf{A}$ such that $\mathbf{A} \vdash \alpha$. If \mathbf{A} and α are such, then we have that $\mathbf{A} \subseteq \mathbf{L} - \{\alpha\}$. From this, corollary 4 guarantees that $\mathbf{A}^+ \subseteq (\mathbf{L} - \{\alpha\})^+$, which implies that $\mathbf{L} - \{\alpha\} \vdash \alpha$, contradicting lemma 7. Hence, no \mathbf{A} entails an α that is not already a member of it. 

This leads to the general result that a non-embracing relation of consequence satisfying monotonicity (and extension) has no deductive power, for it can only deduce as a theorem what already is a premise. In such case, what we may call the *principle of embracingness* would be essential in the realm of monotonic logics, where everything must follow from something.

We can, of course, restrict non-embracingness only to finite subsets of \mathbf{L} . This would make sense on intuitionistic grounds, since it does not seem plausible that a consequence relation, intended to formalise finitistic reasoning, can be applied to any arbitrary finite set. Whether intuitionism can offer a new opportunity for monotonicity and embracingness, or whether this theorem holds outside of the realm of monotonicity requires further research.

References

- [1] Newton Carneiro Affonso da Costa. *Ensaio sobre os fundamentos da lógica*. 2nd ed. São Paulo: Hucitec, 1994.
- [2] Newton Carneiro Affonso da Costa. “On the Theory of Inconsistent Formal Systems”. In: *Notre Dame Journal of Formal Logic* 15.4 (1974), pp. 497–510. EUCLID: ndjfl/1093891487.
- [3] Jerzy Perzanowski. “Parainconsistency, or inconsistency tamed, investigated and exploited”. In: *Logic and Logical Philosophy* 9 (2001): *Parainconsistency: Part III*. Proceedings of the Symposium of Parainconsistent Logic, Logical Philosophy, Informatics and Mathematics. On the occasion of the 50th anniversary of Stanisław Jaśkowski’s seminal talk. Ed. by Jerzy Perzanowski and Andrzej Pietruszczak. (Toruń University, 15–18 de jul. de 1998), pp. 5–24. ISSN: 1425-3305. DOI: 10.12775/LLP.2001.001.
- [4] Karl Raimund Popper. “Are contradictions embracing?” In: *Mind* 52 (1943), pp. 47–50. DOI: 10.2307/2267998.