

# Extension, Translation, and the Cantor-Bernstein Property

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## Abstract

The purpose of this paper is to examine in detail a particularly interesting pair of first-order theories. In addition to clarifying the overall geography of notions of equivalence between theories, this simple example yields two surprising conclusions about the relationships that theories might bear to one another. In brief, we see that theories lack both the Cantor-Bernstein and co-Cantor-Bernstein properties.

## 1 Introduction

Consider the following two theories.<sup>1</sup>

**The theory  $\mathbf{T}_1$ .**  $T_1$  is formulated in the signature  $\Sigma_1 = \{\sigma, p_0, p_1, p_2, \dots\}$  where  $\sigma$  is a sort symbol and each of the  $p_i$  is a unary predicate symbol. We define the theory as follows:

$$T_1 = \{\exists_{=1}x(x = x)\}$$

$T_1$  says that there is exactly one thing, but it is silent on whether or not that thing is  $p_i$ .

**The theory  $\mathbf{T}_2$ .**  $T_2$  is formulated in the signature  $\Sigma_2 = \{\sigma, q_0, q_1, q_2, \dots\}$  where  $\sigma$  is a sort symbol and each of the  $q_i$  is a unary predicate symbol. We define the theory as follows:

$$T_2 = \{\exists_{=1}x(x = x), \forall x(q_0(x) \rightarrow q_1(x)), \forall x(q_0(x) \rightarrow q_2(x)), \dots\}$$

$T_2$  says that there is exactly one thing, and that if that thing is  $q_0$ , then it is  $q_i$  for all of the other  $i$  too. One can think of the predicate  $q_0$  as a kind of ‘light switch’ that turns on all of the other predicates  $q_i$ .

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<sup>1</sup>For preliminaries on model theory the reader is encouraged to consult Hodges (2008). For additional discussion of these two theories see Halvorson (2012) and Barrett and Halvorson (2016b).

The aim of this paper is to examine these theories and the relationships between them. In particular, we will demonstrate the following five claims.

1.  $T_1$  and  $T_2$  are not definitionally equivalent.
2.  $T_1$  and  $T_2$  are not Morita equivalent.
3. There are conservative translations  $F : T_1 \rightarrow T_2$  and  $G : T_2 \rightarrow T_1$ .
4. There are essentially surjective translations  $H : T_1 \rightarrow T_2$  and  $K : T_2 \rightarrow T_1$ .
5. There is no essentially surjective and conservative translation between  $T_1$  and  $T_2$  (in either direction).

It is natural to separate these claims into two kinds. The first two claims are about what kinds of extensions exist for  $T_1$  and  $T_2$  (i.e. they have no common definitional nor Morita extension), while the last three are about what kinds of translations exist between  $T_1$  and  $T_2$ .

After proving these claims, we will draw some conclusions from this example that are related to current debates concerning the structure and equivalence of theories. In addition to clarifying the overall geography of notions of equivalence between theories, this simple example yields two surprising conclusions about theories. First, it can be that one theory is embeddable in another and the other is embeddable in the one, but the two theories are not equivalent. And second, it can be that one theory ‘has all of the structure’ of another and the other ‘has all of the structure’ of the one, but the two theories are nonetheless do not have the same structure. In brief, one might put these two conclusions as follows: Theories lack both the Cantor-Bernstein and the co-Cantor-Bernstein properties.

## 2 Extension

Let  $\Sigma \subset \Sigma^+$  be signatures with  $p \in \Sigma^+ - \Sigma$  an  $n$ -ary predicate symbol. Recall that a **definitional extension** of a  $\Sigma$ -theory  $T$  to the signature  $\Sigma^+$  is a  $\Sigma^+$ -theory

$$T^+ = T \cup \{\delta_s : s \in \Sigma^+ - \Sigma\},$$

such that for each predicate symbol  $s \in \Sigma^+ - \Sigma$ , the sentence  $\delta_s$  is an explicit definition of  $s$  in terms of  $\Sigma$ . One can also define new function and constant symbols, but for our purposes this will not be important.

Recall that two theories are **definitionally equivalent** if they have a ‘common definitional extension’. More precisely: if  $T_1$  is a  $\Sigma_1$ -theory and  $T_2$  is a  $\Sigma_2$ -theory,  $T_1$  and  $T_2$  are definitionally equivalent if there is a definitional extension  $T_1^+$  of  $T_1$  to the signature  $\Sigma_1 \cup \Sigma_2$  and a definitional extension  $T_2^+$  of  $T_2$  to the signature  $\Sigma_1 \cup \Sigma_2$  such that  $T_1^+$  and  $T_2^+$  are logically equivalent (in the sense that they have the same class of models).

We now have the following simple proposition.

**Proposition 1.**  $T_1$  and  $T_2$  are not definitionally equivalent.

*Proof.* Suppose for contradiction that  $T$  is a common definitional extension of  $T_1$  and  $T_2$ . Since  $T$  defines each of the predicate symbols of  $T_2$ , there is a  $\Sigma_1$ -sentence  $\phi$  such that  $T \models \forall y q_0(y) \leftrightarrow \phi$ . Now one can verify that the sentence  $\phi$  has the following property: If  $\psi$  is a  $\Sigma_1$ -sentence and  $T_1 \models \psi \rightarrow \phi$ , then either (i)  $T_1 \models \neg\psi$  or (ii)  $T_1 \models \phi \rightarrow \psi$ . But  $\phi$  cannot have this property. Consider the  $\Sigma_1$ -sentence

$$\phi \wedge \forall x p_i(x)$$

where  $p_i$  is a predicate symbol that does not occur in  $\phi$ . We trivially see that  $T_1 \models (\phi \wedge \forall x p_i(x)) \rightarrow \phi$ , but neither (i) nor (ii) hold of  $\phi \wedge \forall x p_i(x)$ . This is a contradiction, so  $T_1$  and  $T_2$  are not definitionally equivalent.  $\square$

It has recently been suggested that definitional equivalence is too strict a standard of equivalence between theories, in the sense that it judges theories to be inequivalent that we have good reason to consider equivalent. For example, Euclidean geometry can be formulated with only a sort of points (Tarski, 1959), with only a sort of lines (Schwabhäuser and Szczerba, 1975), or with both a sort of points and a sort of lines (Hilbert, 1930).<sup>2</sup> Since these formulations use different sort symbols, and we have so far provided no way of defining new sort symbols, definitional equivalence does not capture any sense in which they are equivalent. In order to address this shortcoming of definitional equivalence, a more liberal standard of equivalence has been proposed. It has been called “Morita equivalence” (by Barrett and Halvorson (2016b, 2017a,b)) and “generalized definitional equivalence” (by Andr eka et al. (2008)).<sup>3</sup>

The precise details of Morita equivalence are not important for our purposes here, but the basic idea is simple. Morita equivalence allows one to define new *sort* symbols — in addition to new predicate, function, and constant symbols — using some basic construction rules. Two theories are then said to be Morita equivalent if they have a ‘common Morita extension’, which is just like a common definitional extension except that it might define new sorts. One can show that geometry formulated in terms of points is Morita equivalent to geometry formulated in terms of lines; one uses the sort of lines to build the sort of points, and vice versa (Barrett and Halvorson, 2017a).

One might wonder whether our theories  $T_1$  and  $T_2$  are Morita equivalent. The following simple result has already been demonstrated.<sup>4</sup>

**Proposition 2.**  $T_1$  and  $T_2$  are not Morita equivalent.

*Proof.* The proof proceeds in precisely the same manner as the proof of Proposition 1. Barrett and Halvorson (2016b, Theorem 5.2) give the precise details.  $\square$

<sup>2</sup>See Szczerba (1977) and Schwabh user et al. (1983, Proposition 4.59, Proposition 4.89).

<sup>3</sup>See also Hudetz (2017a,b) and Tsementzis (2015).

<sup>4</sup>The result goes through even if one formulates  $T_1$  and  $T_2$  using two different sort symbols, instead of the same sort symbol  $\sigma$ .

### 3 Translation

Our next three claims are about the kinds of ‘translations’ that exist between these theories. We need some basic preliminaries.

Let  $\Sigma_1$  and  $\Sigma_2$  be signatures. A **reconstrual**  $F$  of  $\Sigma_1$  into  $\Sigma_2$  is a map from the predicates in the signature  $\Sigma_1$  to  $\Sigma_2$ -formulas that takes an  $n$ -ary predicate symbol  $p \in \Sigma_1$  to a  $\Sigma_2$ -formula  $Fp(x_1, \dots, x_n)$  with  $n$  free variables.<sup>5</sup> One can think of the  $\Sigma_2$ -formula  $Fp(x_1, \dots, x_n)$  as the ‘translation’ of the  $\Sigma_1$ -formula  $p(x_1, \dots, x_n)$  into the signature  $\Sigma_2$ . We will use the notation  $F : \Sigma_1 \rightarrow \Sigma_2$  to denote a reconstrual  $F$  of  $\Sigma_1$  into  $\Sigma_2$ .

A reconstrual  $F : \Sigma_1 \rightarrow \Sigma_2$  extends to a map from arbitrary  $\Sigma_1$ -formulas to  $\Sigma_2$ -formulas in the natural recursive manner. In the case where one is only considering signatures with predicate symbols (as we are here), this map is particularly easy to describe. Let  $\phi(x_1, \dots, x_n)$  be a  $\Sigma_1$ -formula. We define the  $\Sigma_2$ -formula  $F\phi(x_1, \dots, x_n)$  recursively as follows.

- If  $\phi(x_1, \dots, x_n)$  is  $x_i = x_j$ , then  $F\phi(x_1, \dots, x_n)$  is the  $\Sigma_2$ -formula  $x_i = x_j$ .
- If  $\phi(x_1, \dots, x_n)$  is  $p(x_1, \dots, x_n)$ , where  $p \in \Sigma_1$  is an  $n$ -ary predicate symbol, then  $F\phi(x_1, \dots, x_n)$  is the  $\Sigma_2$ -formula  $Fp(x_1, \dots, x_n)$ .
- If  $F\phi$  and  $F\psi$  have already been defined for  $\Sigma_1$ -formulas  $\phi$  and  $\psi$ , then we define the  $\Sigma_2$ -formula  $F(\neg\phi)$  to be  $\neg F\phi$ ,  $F(\phi \wedge \psi)$  to be  $F\phi \wedge F\psi$ ,  $F(\forall x\phi)$  to be  $\forall xF\phi$ , etc.

Suppose that  $T_1$  and  $T_2$  are theories in the signatures  $\Sigma_1$  and  $\Sigma_2$ , respectively. We say that a reconstrual  $F : \Sigma_1 \rightarrow \Sigma_2$  is a **translation**  $F : T_1 \rightarrow T_2$  if  $T_1 \models \phi$  implies that  $T_2 \models F\phi$  for every  $\Sigma_1$ -sentence  $\phi$ . A translation  $F$  gives rise to a map  $F^* : \text{Mod}(T_2) \rightarrow \text{Mod}(T_1)$ , which takes models of the theory  $T_2$  to models of the theory  $T_1$ . For every model  $A$  of  $T_2$  we first define a  $\Sigma_1$ -structure  $F^*(A)$  as follows.

- $\text{dom}(F^*(A)) = \text{dom}(A)$ .
- $(a_1, \dots, a_n) \in p^{F^*(A)}$  if and only if  $A \models Fp[a_1, \dots, a_n]$ .

A straightforward argument demonstrates that  $F^*(A)$  is indeed a model of  $T_1$  (Barrett and Halvorson, 2016a, §4).

A translation  $F : T_1 \rightarrow T_2$  is **conservative** if  $T_2 \models F\phi$  implies that  $T_1 \models \phi$  for any  $\Sigma_1$ -sentence  $\phi$ . We make two brief remarks about this concept. First, one can think of a conservative translation as a kind of ‘injection’ or ‘embedding’ on sentences. Indeed, one can easily verify that the following simple condition is equivalent to a translation  $F : T_1 \rightarrow T_2$  being conservative.

- For any  $\Sigma_1$ -sentences  $\phi_1$  and  $\phi_2$ , if  $T_2 \models F\phi_1 \leftrightarrow F\phi_2$  then  $T_1 \models \phi_1 \leftrightarrow \phi_2$ .

<sup>5</sup>This notion naturally extends to signatures that contain function and constant symbols, but that will be unimportant for our purposes. See Hodges (2008), Button and Walsh (2018), and Barrett and Halvorson (2016a) for details.

This condition is clearly capturing a kind of injectivity on sentences: If a translation maps two  $\Sigma_1$ -sentences to equivalent  $\Sigma_2$ -sentences, then they must have been equivalent to begin with. Second, one can easily verify that if a translation  $F : T_1 \rightarrow T_2$  is such that  $F^* : \text{Mod}(T_2) \rightarrow \text{Mod}(T_1)$  is surjective, then  $F$  is conservative.

We now have our first simple result concerning the existence of translations between our theories  $T_1$  and  $T_2$ .

**Proposition 3.** *There are conservative translations  $F : T_1 \rightarrow T_2$  and  $G : T_2 \rightarrow T_1$ .*

*Proof.* Consider the reconstruals  $F : \Sigma_1 \rightarrow \Sigma_2$  and  $G : \Sigma_2 \rightarrow \Sigma_1$  defined by

$$F : p_i \mapsto q_{i+1} \qquad G : q_i \mapsto p_0 \vee p_i$$

It is trivial that  $F : T_1 \rightarrow T_2$  is a translation. Since  $G$  maps the  $\Sigma_2$ -sentence  $\forall x(q_0(x) \rightarrow q_i(x))$  to  $\forall x((p_0(x) \vee p_0(x)) \rightarrow (p_0(x) \vee p_i(x)))$  and  $T_1$  entails this latter sentence, it follows that  $G$  is a translation too.

It remains to show that  $F$  and  $G$  are conservative. One does this by showing that  $F^*$  and  $G^*$  are surjective. Suppose that  $M$  is a model of  $T_1$ . Then  $M$  is completely determined by which of the  $p_i$  hold of the one thing. We let  $N$  be the model of  $T_2$  defined as follows:  $N$  has the same domain as  $M$ ,  $q_0$  does not hold of the one thing in  $N$ , and  $q_{i+1}$  holds of the one thing in  $N$  if and only if  $p_i$  holds of the one thing in  $M$ . One trivially sees that  $F^*(N) = M$ , so  $F^*$  is surjective. A similar argument shows that  $G^*$  is surjective.  $\square$

Since they are conservative, the translations  $F$  and  $G$  from Proposition 3 can be thought of as injections or embeddings between the theories  $T_1$  and  $T_2$ . But there is a natural sense in which neither is ‘surjective’. For example,  $F$  does not map any  $\Sigma_1$ -formula to any logical equivalent of the  $\Sigma_2$ -formula  $q_0(x)$ . The following property makes this thought precise. We say that a translation  $F : T_1 \rightarrow T_2$  is **essentially surjective** if for every  $\Sigma_2$ -formula  $\psi$  there is a  $\Sigma_1$ -formula  $\phi$  such that  $T_2 \models \forall x_1 \dots \forall x_n(\psi(x_1, \dots, x_n) \leftrightarrow F\phi(x_1, \dots, x_n))$ . One can easily verify that neither  $F$  nor  $G$  from Proposition 3 are essentially surjective. But we do have the following simple result.

**Proposition 4.** *There are essentially surjective translations  $H : T_1 \rightarrow T_2$  and  $K : T_2 \rightarrow T_1$ .*

*Proof.* Consider the reconstruals  $H : \Sigma_1 \rightarrow \Sigma_2$  and  $K : \Sigma_2 \rightarrow \Sigma_1$  defined by

$$H : p_i \mapsto q_i \qquad K(q_i) = \begin{cases} p_0 \wedge \neg p_0 & \text{if } i = 0 \\ p_{i-1} & \text{otherwise} \end{cases}$$

It is trivial that  $H$  is an essentially surjective translation. Since  $K$  maps the  $\Sigma_2$ -sentence  $\forall x(q_0(x) \rightarrow q_i(x))$  to  $\forall x((p_0(x) \wedge \neg p_0(x)) \rightarrow p_{i-1}(x))$  and  $T_1$  entails this latter sentence, it follows that  $K$  is a translation. It is easy to see that  $K$  is essentially surjective.  $\square$

One can verify that the theory  $T_1$  does not entail the sentence  $\forall x(p_0(x) \rightarrow p_1(x))$ , but  $T_2$  does entail  $H(\forall x(p_0(x) \rightarrow p_1(x)))$ . Similarly, the theory  $T_2$  does not entail  $\neg\forall xq_0(x)$ , but  $T_1$  does entail  $K(\neg\forall xq_0(x))$ . This means that neither  $H$  nor  $K$  is conservative.

One therefore wonders whether there is any translation between  $T_1$  and  $T_2$  that is essentially surjective and conservative. One might consider a translation like this to be an ‘isomorphism’ between the two theories. The following proposition settles this issue.

**Proposition 5.** *There is no essentially surjective and conservative translation between  $T_1$  and  $T_2$  (in either direction).*

*Proof.* Suppose that there is a translation  $F : T_1 \rightarrow T_2$  that is essentially surjective and conservative. We show that this implies that  $T_1$  and  $T_2$  are definitionally equivalent, which contradicts Proposition 1. (The same argument demonstrates that there is no essentially surjective and conservative translation  $T_2 \rightarrow T_1$ .)

Let  $q_i$  be one of the predicate symbols in  $\Sigma_2$ . Since  $F$  is essentially surjective, there is some  $\Sigma_1$ -formula  $\phi_i$  such that  $T_2 \models \forall x(q_i(x) \leftrightarrow \phi_i(x))$ . This allows us to define a reconstrual  $G : \Sigma_2 \rightarrow \Sigma_1$  by setting  $G(q_i) = \phi_i$ . One then uses the fact that  $F$  is conservative to verify that  $G : T_2 \rightarrow T_1$  is a translation, and to verify that both  $T_1 \models \phi \leftrightarrow GF\phi$  and  $T_2 \models \psi \leftrightarrow FG\psi$  for all  $\Sigma_1$ -formulas  $\phi$  and  $\Sigma_2$ -formulas  $\psi$ . This means that  $T_1$  and  $T_2$  are intertranslatable, and so (Barrett and Halvorson, 2016a, Theorem 2) implies that they are definitionally equivalent.  $\square$

## 4 Conclusion

We conclude by mentioning three payoffs of this discussion.

### Mutual faithful interpretability and Morita equivalence

The first concerns the recent debate about when two theories should be considered equivalent. A significant amount of attention has been devoted to better understanding the relationships between different standards of equivalence of theories and to applying these standards to various cases of interest.<sup>6</sup> The simple example that we have considered here allows us to further clarify the overall geography of standards of equivalence between theories.

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<sup>6</sup>For discussion of these relationships, see for example Barrett and Halvorson (2016a,b) and (Button and Walsh, 2018, p. 118). In addition to the recent articles on equivalence already cited, see Barrett (2017), Coffey (2014), Curiel (2014), Halvorson (2013), Glymour (2013), Hudetz (2015, 2017a), Knox (2011, 2014), North (2009), Rosenstock et al. (2015), Rosenstock and Weatherall (2016), Teh and Tsementzis (2017), Van Fraassen (2014), and Weatherall (2016, 2017), and the references therein

Recall that  $T_1$  and  $T_2$  are **mutually faithfully interpretable** if there are conservative translations  $F : T_1 \rightarrow T_2$  and  $G : T_2 \rightarrow T_1$ .<sup>7</sup> It is already well-known that definitional equivalence is a stricter standard of equivalence than mutual faithful interpretability is. More precisely, any pair of theories that are definitionally equivalent are mutually faithfully interpretable, but there are mutually faithful interpretable theories that are not definitionally equivalent (Andréka et al., 2005; Button and Walsh, 2018). Morita equivalence, however, is a more liberal notion of equivalence than definitional equivalence. So it is natural to ask the following question.

*If two theories  $T_1$  and  $T_2$  are mutually faithfully interpretable, then are  $T_1$  and  $T_2$  also Morita equivalent?*

If the answer to this question were yes — that is, if mutual faithful interpretability entailed Morita equivalence — this would be a mark against Morita equivalence as a plausible standard of equivalence between theories. It is widely accepted that mutual faithful interpretability is too liberal a standard of equivalence; it considers theories to be equivalent that we have good reason to consider inequivalent (Szczerba, 1977). So if the answer to the above question were yes, this would immediately imply that Morita equivalence is too liberal a standard as well.

The answer to the above question, however, is no. It is not the case that Morita equivalence is entailed by mutual faithful interpretability. Our discussion here has demonstrated precisely this. It follows from Proposition 3 that our theories  $T_1$  and  $T_2$  are mutually faithfully interpretable, but as was stated in Proposition 2, they are not Morita equivalent.

The second and third payoffs of this discussion are of a more straightforwardly philosophical nature. This example yields the following two surprising conclusions about theories. It can be that one theory is embeddable in another and the other is embeddable in the one, but the two theories are not equivalent. And it can be that one theory ‘has all of the structure’ of another and the other ‘has all of the structure’ of the one, but the two theories do not have the same structure.

## The Cantor-Bernstein property

We begin by discussing the former. Proposition 3 captures a sense in which  $T_1$  can be embedded in  $T_2$  and  $T_2$  can be embedded in  $T_1$ . The existence of the conservative translation  $F : T_1 \rightarrow T_2$  shows us that  $T_1$  can be ‘viewed as a part of’  $T_2$ . It is easy to see how this is the case:  $T_1$  can be viewed as the part of  $T_2$  that is constructed from the predicate symbols  $q_1, q_2, \dots$  and so on. And similarly the existence of the conservative translation  $G : T_2 \rightarrow T_1$  shows us that one can view  $T_2$  as a part of  $T_1$ ; it can be viewed as the part of  $T_1$  that is constructed from the  $\Sigma_1$ -formulas  $p_0, p_0 \vee p_1, p_0 \vee p_2, \dots$  and so on. Despite the

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<sup>7</sup>See Button and Walsh (2018, p. 117). Our presentation here is slightly less general, since we have not discussed translations with parameters.

fact that each of these theories can be embedded in the other, they are not the same theory. Propositions 1, 2, and 5 substantiate this intuition. Our theories  $T_1$  and  $T_2$  are not definitionally equivalent, Morita equivalent, nor does there exist an essentially surjective and conservative translation between them.

What we have essentially shown is that theories lack the Cantor-Bernstein property. The Cantor-Bernstein theorem famously says that if there are injections  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  between sets  $X$  and  $Y$ , then there is a bijection between  $X$  and  $Y$ . Because of this theorem we say that sets have the Cantor-Bernstein property. And indeed, it makes sense to talk about the Cantor-Bernstein property for any category. One says that a category  $C$  has the Cantor-Bernstein property if for any objects  $c$  and  $d$  of  $C$  whenever there is a monomorphism (i.e. a generalization of the concept of an injection) from  $c \rightarrow d$  and a monomorphism  $d \rightarrow c$ , the objects  $c$  and  $d$  are isomorphic. The property captures a basic intuition one might have about the notion of ‘being a part of’: If  $X$  is a part of  $Y$  and  $Y$  is a part of  $X$ , then  $X$  and  $Y$  are the same. Our example here demonstrates that the category of theories does not have the Cantor-Bernstein property.<sup>8</sup> In other words, the analogue of the Cantor-Bernstein theorem does not hold of theories:  $T_1$  and  $T_2$  can be embedded into one another, but nonetheless they are not the same.<sup>9</sup>

## The co-Cantor Bernstein property

We now turn to our final payoff. It can be that one theory ‘has all of the structure’ of another and the other ‘has all of the structure’ of the one, but the two theories do not have the same structure. One might put this payoff more suggestively as follows: It can be that one theory ‘is as ideologically rich as’ another and the other ‘is as ideologically rich as’ the one, but the two theories are not ideologically equivalent. The ideology of a theory is usually thought of as the range of concepts that are expressible in the language in which the theory is formulated (Quine, 1951). It takes a moment to explain why our example yields this surprising conclusion about structure and ideology.

The existence of an essentially surjective translation  $H : T_1 \rightarrow T_2$  captures a sense in which  $T_1$  has all the structure of (or is as ideologically rich as)  $T_2$ . The idea here is simple, when  $H$  is essentially surjective, any formula  $\psi$  that one can formulate in the language of  $T_2$  is expressible using the language of  $T_1$ ; that is precisely what the essential surjectivity of  $H$  guarantees. There is some formula  $\phi$  in the language of  $T_1$  that  $H$  translates to a logical equivalent of  $\psi$ . Intuitively, this means that the theory  $T_1$  can define all of the structures that  $T_2$  has. Or in other words,  $T_1$  can express all of the same concepts that  $T_2$  can. Quine (1951, p. 15) himself endorses the basic idea that one can investigate the ideology of a

<sup>8</sup>There are various ways to ‘categorify’ the collection of first-order theories. The reader is encouraged to consult Visser (2006) for details.

<sup>9</sup>Another example of the failure of the Cantor-Bernstein property is the relation between classical and intuitionistic logic. Obviously, intuitionistic logic can be embedded in classical logic. The famous Gödel translation shows that intuitionistic logic can be embedded in classical logic. But it is nonetheless unnatural to think of the two logics as equivalent, as was recently pointed out, for example, by Dewar (2018).

theory by examining what kinds of translations exist between theories: “Much that belongs to ideology can be handled in terms merely of the translatability of notations from one language into another; witness the mathematical work on definability by Tarski and others.”

One can grasp the basic idea here by considering the following two simple examples. Suppose that  $T$  is a theory in the signature  $\Sigma$  and consider the signature  $\Sigma^+ = \Sigma \cup \{p\}$ , where  $p$  is a new unary predicate symbol

1. Suppose that  $T^+$  is the  $\Sigma^+$ -theory that has precisely the same axioms as  $T$ . There is a natural sense in which  $T^+$  has all of the structure of  $T$ , but not vice versa;  $T^+$  has the new piece of structure  $p$  that  $T$  lacks. One can capture this basic idea by looking to facts about essentially surjective translations between these two theories.

First, there is an essentially surjective translation  $T^+ \rightarrow T$ . The translation simply maps each symbol in  $\Sigma$  to itself and maps  $p$  to any  $\Sigma$ -formula with one free variable. It is trivial to verify that this is indeed an essentially surjective translation. This makes precise our basic intuition that  $T^+$  is as ideologically rich as  $T$ . Second, since there is no  $\Sigma$ -formula that is logically equivalent to  $p$ , the translation  $T \rightarrow T^+$  that maps every element of  $\Sigma$  to itself is not essentially surjective. This makes precise our intuition that  $T$  is *not* as ideologically rich as  $T^+$ .

2. Now suppose instead that  $T^+$  is a definitional extension of  $T$  to the signature  $\Sigma^+$ . In this case there is a strong sense in which  $T$  and  $T^+$  have precisely the same structure and ideological richness;  $p$  is not a piece of structure that is new to  $T^+$ . Rather, it is explicitly definable in terms of those structures that  $T$  posits, so there is a strong sense in which  $T$  already has the structure  $p$ .

This basic intuition can be made precise by noticing that there are essentially surjective translations  $T \rightarrow T^+$  and  $T^+ \rightarrow T$ . It follows from our discussion in point 1 above that there is an essentially surjective translation  $T^+ \rightarrow T$ . And since  $T^+$  is a definitional extension of  $T$ , the translation  $T \rightarrow T^+$  that maps every element in  $\Sigma$  to itself is in this case essentially surjective. The  $\Sigma$ -formula  $\phi$  that defines  $p$  is (obviously) logically equivalent to  $p$ .

The existence of an essentially surjective translation from  $T_1$  to  $T_2$  captures a strong sense in which  $T_1$  has all of the structure or ideological richness of  $T_2$ . The two examples above show that this idea is a direct generalization of the kinds of simple examples that formed our intuitions about amounts of structure and ideological richness in the first place.

In addition, the tools that have recently been used to compare amounts of structure between theories cohere well with the idea that essentially surjective translations capture facts about structure. It has been suggested — by Earman (1989), and more recently by North (2009), Swanson and Halvorson (2012), and Barrett (2015a,b) — that the size of an object’s automorphism group can be used

as a guide to the amount of structure that the object has. An automorphism of an object is a structure-preserving bijection from the object to itself. If an object has more automorphisms, therefore, that suggests that the object has less structure that these automorphisms are required to preserve. The important fact for our purposes here is the following: If there is an essentially surjective translation from  $T_1$  to  $T_2$ , this captures a sense in which the models of these two theories have automorphism groups of precisely the same size.

It only takes a moment to make this claim precise. An **automorphism** of a  $\Sigma$ -structure  $M$  is a bijection  $f : M \rightarrow M$  that satisfies  $M \models p[a_1, \dots, a_n]$  if and only if  $M \models p[f(a_1), \dots, f(a_n)]$  for any predicate symbol  $p \in \Sigma$  and elements  $a_1, \dots, a_n \in M$ . One can easily prove the following.<sup>10</sup>

**Proposition 6.** *Let  $H : T_1 \rightarrow T_2$  be an essentially surjective translation with  $N$  a model of  $T_2$ . Then  $H^*(N)$  and  $N$  have the same automorphism group.*

*Proof.* If  $f : N \rightarrow N$  is an automorphism of  $N$ , one can easily verify using the definition of  $H^*$  that  $f$  is also an automorphism of  $H^*(N)$ . Suppose that  $f : H^*(N) \rightarrow H^*(N)$  is an automorphism and let  $q \in \Sigma_2$  be a predicate symbol in the signature of  $T_2$ . We immediately see that for any  $a \in N$ :

$$N \models q[a] \iff H^*(N) \models \phi[a] \iff H^*(N) \models \phi[f(a)] \iff N \models q[f(a)]$$

Here  $\phi$  is a  $\Sigma_1$ -formula such that  $H\phi$  is logically equivalent to  $q$ , whose existence is guaranteed by the essential surjectivity of  $H$ . The first and third biconditionals follow from this choice of  $\phi$  and the definition of  $H^*$ , while the second follows from the fact that  $f$  is an automorphism of  $H^*(N)$ . This means that  $f$  is an automorphism of  $N$ .  $\square$

Our current best methods of comparing amounts of structure therefore lend further support to the following claim: The existence of an essentially surjective translation from  $T_1$  to  $T_2$  captures a sense in which  $T_1$  has all of the structure or ideological richness of  $T_2$ .

Now that we have this idea on the table, we can return to consider our pair of theories  $T_1$  and  $T_2$ . As we have seen in Proposition 4, there are essentially surjective translations  $H : T_1 \rightarrow T_2$  and  $K : T_2 \rightarrow T_1$ . This captures a sense in which  $T_1$  has all of the structure of (or is as ideologically rich as)  $T_2$  and  $T_2$  has all of the structure of (or is as ideologically rich as)  $T_1$ . But Propositions 1, 2, and 5 imply that  $T_1$  and  $T_2$  do not have the *same structure* or the *same ideology*. This is completely intuitive:  $T_2$  has a kind of ‘light switch’ predicate  $q_0$  that turns on all of the other predicates  $q_i$ , while  $T_1$  contains no such piece of structure.

This shows that theories lack what one might call the ‘co-Cantor-Bernstein property’. It is easy to verify that if there are surjections  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  between sets  $X$  and  $Y$ , then there is a bijection between  $X$  and

<sup>10</sup>A similar argument actually demonstrates an even stronger result: If  $F$  is essentially surjective translation, then  $F^*$  is a full functor between the categories of models of  $T_2$  and  $T_1$  (Halvorson, 2019; Barrett, 2018).

Y. One might call this the ‘co-Cantor-Bernstein theorem’, and therefore say that sets have the co-Cantor-Bernstein property. Once again, it makes sense to talk about the co-Cantor-Bernstein property for any category. We can say that a category  $C$  has the co-Cantor-Bernstein property if for any objects  $c$  and  $d$  of  $C$  whenever there is an epimorphism (i.e. a generalization of the concept of an surjection) from  $c \rightarrow d$  and an epimorphism  $d \rightarrow c$ , the objects  $c$  and  $d$  are isomorphic. Our example here demonstrates that the category of theories does not have this property. Our two theories  $T_1$  and  $T_2$  can be surjected onto one another, but nonetheless, as Propositions 1, 2, and 5 tell us, they are not equivalent. These concepts of amount of structure and ideological richness are therefore more subtle than one might have initially thought.

One might wonder how these results come to bear on the current debates about structure and equivalence in philosophy of science. Real scientific theories are, of course, much more complicated than theories in first-order logic. We therefore should be careful not to assume that the ‘category of all scientific theories’ has all the features that the category of first-order theories has. Nonetheless, we should take ‘no go’ results like the ones we have presented here very seriously. If the category of first-order theories fails to have some nice, simple feature — like the Cantor-Bernstein or co-Cantor-Bernstein properties — then that gives us reason to doubt that the category of all scientific theories will have that feature. For example, it would seem natural to think that if  $T_1$  is reducible to  $T_2$ , and  $T_2$  is reducible to  $T_1$ , then  $T_1$  and  $T_2$  are equivalent theories. This is exactly the kind of inference, however, that the results presented here about first-order theories call into question.

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