# Exploring the potential of philosophy of mathematics and of science to enhance mathematics education from an epistemological perspective

research project

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## Background and significance

The complexity of education resides in its interdisciplinary nature and in the interaction of its social with its theoretical aspects in the phenomenon of education. A special kind of complexity is exhibited by mathematics education, which can be fairly associated with the nature and content of mathematics as a special discipline.

A kind of 'mathematics anxiety' has always been detected and reported in the empirical realm of mathematics education by all participants in the phenomenon – students, parents, teachers, policy makers, and researchers. Studies within mathematics education have confirmed the existence of this so-called "mathematics anxiety" among secondary and high school students as a global phenomenon and have related it to various social and psychological aspects of the processes of teaching and learning mathematics. The concept of *understanding* of mathematics has been employed in the analysis of both the causes and the effects of this anxiety: a poor or improper understanding of the taught mathematics leads to anxiety and conversely, anxiety affects understanding.

Passolunghi et al. (2016), Foley et al. (2017), and Núñez-Peña & Bono (2019) found that mathematics anxiety has a significant negative impact on secondary education students' achievement in mathematics due to improper or poor understanding of the taught mathematics. Several authors in the educational psychology field, including Holmes & Hwang (2016), Guita & Tan (2018), and Choi-Koh & Ryoo (2019), found that cooperative and active learning methodologies ameliorate students' mathematics anxiety and positively impact their performance in mathematics. These findings have been supported by classical authors like Stodolsky (1985), who attributed students' high levels of mathematics anxiety to a lack of social support related to specific teaching methodologies and strategies.

In general, past research on the relationship between understanding of mathematics and 'mathematics anxiety' has been conducted with the concepts and methods of educational sciences and psychology; the causes, the effects, and the proposed methodologies for the enhancement of mathematics teaching with respect to this relationship have been analyzed relative to various social and psychological aspects of the processes of teaching and learning mathematics, that is, in a theoretical framework specific to social sciences.

In the empirical realm of mathematics education, traditional teaching methods as well as curriculum development have proved ineffective with respect to the detected mathematical anxiety and its effects for the majority of students worldwide. Many students who do not have a proper understanding of the mathematical concepts still learn mathematical procedures and come to apply them correctly – and so many times their teachers are pleased with such results. This is possible due to the very nature of mathematics, which has applicative and algorithmic features seemingly independent of its conceptual base; that is, we can "do" mathematics without going deeper into the understanding of concepts involved. On the basis of such sufficiency, the idea is strengthened in the student's system of beliefs that mathematics is something merely instrumental and computational, which is far from the complexity of mathematics as an epistemic-theoretical construct.

There is a consensus that mathematical knowledge is of two distinct kinds: conceptual and procedural knowledge and the relations between conceptual and procedural knowledge are often bi-directional and iterative (Rittle-Johnson & Schneider, 2015). The distinction traces back to (Hiebert & Lefevre, 1986, pp. 1 – 27). Accordingly, we have conceptual and procedural understanding of mathematics and conceptual and procedural approaches toward teaching mathematics. This distinction maintains a tension between the two teaching approaches, not that they cannot collaborate or be unified, but rather, relative to the practical circumstances of the teaching practice, which actually impose instructors to teach in one way or another.

In practice, the inclination toward the procedural approach is largely the result of policy making, which imposes a "socially useful" mathematics to be taught in schools (SSMR, 2016; Bahr & Bossé, 2008; NCTM,

2000, 2014; Leonelli & Schmitt, 2001; AAMT, 2002). This approach is somehow justified by the constraints of the school setup: the limited time frame of a class, the pragmatic goal of passing tests and exams (which are based on problem solving), and the applicative nature of mathematics itself.

Our proposed theoretical framework of implementing philosophical knowledge in mathematics education targets for its application the *conceptual* understanding and approach of teaching.

There is general consensus that conceptual knowledge in mathematics should be defined as knowledge of mathematical concepts, the ways they can be known through their relationships (Hiebert, 1986, pp. 3-4; Star, 2005), knowledge about facts, generalizations and principles (Baroody, Feil, & Johnson, 2007, p.107). However, this view does not extend knowledge much beyond the mere content of mathematics as a discipline, its formalism, principles, and methods. In this intensional view, conceptual knowledge is just a *mathematical* perquisite of procedural knowledge. Knowledge about mathematics as a whole, its nature and specificity among other disciplines, and methods of acquiring knowledge do not fall within this view: further, the complex epistemology of mathematics does include this kind of knowledge, both intensional and extensional (we are talking here about language, semantics, symbolism, structure, epistemic virtues, truths, motivations and goals, empirical influence, applicability, and cognitive and anthropocentric aspects, among others).

Given the special place mathematics and its nature hold as an object of study in an intersecting zone of epistemology, philosophy of mathematics, and philosophy of science as theoretical-philosophy disciplines, and adjacently, fundamentals of mathematics and history of mathematics, the necessary theoretical contributions for such framework are expected to come from these domains. Their potential in this respect has already been established in the works of Ernest (1989, 1991, 1994), Ernest et al. (2016, pp. 3-17), (Godino & Batanero, 1998), Skovsmose (2013/1994), Kitcher (1983), and others, but it was not explored from a practical-applicative perspective in mathematics education, nor applied in the policies of education.

It is not surprising that we place philosophy of science in the list of the potential contributors, since *applicability* of mathematics and its constitutive role in sciences is a field investigated within this discipline and accounts for the nature of mathematics. Besides, both philosophy of mathematics and philosophy of science have an inner educational dimension that potentially can be exploited in mathematics and science education.

It should be noted that fundamentals and history of mathematics, although having tight connections with philosophy of mathematics and of science, cannot *alone* provide the extensional knowledge that the philosophical disciplines can provide in order to enrich traditional conceptual knowledge. From a historical perspective, fundamentals of mathematics provide the *formalisms* of the various theories as attempts to overcome contradictions, paradoxes, or inadequacies, as a foundation of mathematics. But the content of this sub-discipline can be assimilated with the traditional mathematical content taught in schools, since it reflects how mathematics is constructed and has the same formal nature; however, it does not provide the *whole* picture of the nature and features of mathematics in relation to general and scientific knowledge. Insights from fundamentals and history of mathematics are present to some extent in the curriculum of several systems of high school education worldwide, mostly as notes or optional content; they are taught as disciplines in many mathematics departments of colleges or universities instead.

The general goal of this project is to provide the primary conceptual and theoretical framework for the adequate implementation of philosophical knowledge in mathematics education through a double-sided approach: from philosophy to mathematics education and conversely.

The research aims at crystallizing a primary conceptual and theoretical framework necessary for the application of these contributions in mathematics education, from an epistemological perspective, by arguing for and making applicable the general principle that teaching *about* mathematics along with traditionally teaching formal mathematics and 'doing' mathematics does not render the process more difficult; rather, it enhances mathematics education, with effects on decreasing mathematics anxiety and stimulating performance. We can fairly hypothesize that a philosophically enhanced conceptual understanding of mathematics makes one a better problem solver eventually, and the current research will argue for this. This hypothesis is already supported by past research on the relations between the conceptual[1] and procedural learning of mathematics (see for instance Baroody, 2003; Rittle-Johnson & Siegler, 1998; Rittle-Johnson et al., 2001).

The theoretical research on the proposed topic will be followed by a research design for a further phase of implementation and assessment, to be conducted within mathematics education research, with the methods of this discipline.

[1] In the traditional non-philosophical view.

**Understanding, meaning, and holistic approach**

The necessity of turning to philosophy for mathematics-education needs was motivated in the works of Ernest, Godino & Batanero, Skovsmose, Kitcher (cited in the previous section) and others, who provided arguments based on conceptual clarification, and the cultural and social dimension of mathematics. A similar though differently motivated necessity was advanced for the case of science education (Hills, 1992; Matthews, 1994; Mellado et al., 2006; Höttecke and Silva, 2011; and others), which can stand as an adjacent argument for the mathematical case.

We will also propose two new arguments, related this time to mathematics itself as an epistemic construct: First, the 'foundational-dependence' argument, based on the premise that philosophy has influenced dramatically the foundations and development of mathematics through its critical questions, conceptual debates, analysis of paradoxes, consistency, and adequacy. Further, contribution to the foundation of mathematics will be shown to imply the potential for contribution in understanding of mathematics. Second, the 'nature-of-mathematics' argument, based on philosophical investigations into the nature of mathematics (including open questions), which runs on the line of thought that philosophical inquiry on the mathematical concepts, and the concept of mathematics as a whole, adds knowledge on the subject even through its open questions, while stimulating critical thinking, which goes hand in hand with both conceptual understanding and the analytical skills required in mathematical learning and practice.

In the research of mathematical education, the problem of defining a concept of *mathematical* understanding has been given a central role. Skemp (1976) distinguished between relational and instrumental understanding in mathematics; Nickerson (1985, pp. 229 - 236) identified understanding through its results and the relationships of these results to established knowledge. Hiebert & Carpenter (1992, p. 67) advanced a similar definition for understanding as involving the building of a conceptual structure based on mental representations. Sierpinska (1994, Ch. 2 - 3) distinguished between the 'act of understanding', 'understanding' as the result of that act, and 'processes' of understanding as further cognitive activity. More recent views (such as Dreyfus & Eisenberg, 2012; Pino-Fan et al., 2015; Greeno, 2017; Newton & Sword, 2018) either offer refined versions of these definitions or advance other relationships between the same basic constituent concepts. Let us note that all of these views are more intensional than extensional and draw on the idea that mathematical understanding should be defined structurally.

In theoretical philosophy, understanding is tightly related to the epistemic concepts of *meaning* and *context,* not only in what concerns language, but as a general epistemological concept that involves valuable and distinguishable knowledge (see Kvanvig, 2003, 2009; De Regt & Dieks, 2005; Grimm, 2014). But the meaning of mathematical concepts, statements and theories, and context of the creation, development and application of mathematics – all these are investigated within epistemology, fundamentals of mathematics, philosophy of mathematics and of science, and even of language. Importance of meaning has already been established in research as concerns prospecting a theory of mathematical education, still in relation to the philosophical aspects of mathematics (see Godino, 1996; Godino & Batanero, 1998); context, on the other hand, has not been sufficiently researched and one task would be to identify exhaustively the contexts of mathematical understanding (among which are the logical, theoretical, applicative, epistemic, historical, and perhaps the anthropocentric) and incorporate them into the new notion of holistic mathematical understanding that we propose, which is intended to be both intensional and extensional, and to reflect the *entire* specificity of mathematics. It is the context component of understanding that can reflect this specificity more than the meaning can reflect: Mathematics is by its nature *both* abstract and concrete, theoretical and applicative, discipline and language, discipline and method, mental activity and propositional construct; mathematics is self-generative and self-applicative, and includes second- and higher-order predications.

In the study on contexts, we also propose to incorporate the latest findings in the philosophy of *applicability* of mathematics, which has shaped a new view on the nature of mathematics. Starting from the unexplained 'miracle' of applied mathematics – see (Wigner, 1960), for the original formulation of this philosophical puzzle and (Bangu, 2012), for an overview of the solutions provided – mathematical creation, reasoning, and successful application were related to the biological-cognitive structures of humans (see the studies in perceptual mathematics, such as Teissier, 2005; Ye, 2010; Mujumdar and Singh, 2016).

In this epistemological framework, we will investigate the hypothesis that many teaching methods and strategies used for and assumed to enhance explanation of mathematical concepts are not enough in ensuring a *full* understanding (in particular, example-based and metaphorical explanation – that is, making mathematics look "friendlier" or "fun", including through new video and web technologies, is hypothesized to provide only mental-cognitive comfort for the student and not to add new active knowledge on the topic).

The classical definitions of mathematical understanding put forward a plurality of "understandings", suggesting that we cannot talk about understanding *per se*. But such separations make difficult any attempt to unify the conceptual and procedural approaches of teaching, and maintain a tension between them. A *holistic* concept of understanding defined on epistemological grounds complies with the aim of unifying the conceptual and procedural approaches of teaching and would also accommodate with the alleged psychological nature of understanding as a 'mental state' (which is meant as singular).
The theoretical framework we aim for is based on old and new arguments for the contribution of philosophy in mathematics education and is developed around a holistic concept of mathematical understanding, which should be defined so as to accommodate – as an epistemological concept – with the conceptual frameworks of educational psychology and mathematics education. It focuses rather on conceptual clarification *ab initio*, which is needed in respect to the subject, and investigates it with the methods of theoretical philosophy. The theoretical product of the investigation is then proposed for application in mathematics education where its application can be further studied with the methods of this discipline, including empirical ones.
The general methodological novelty of this research is that it inverts the usual direction of inquiry within social sciences (from empirical to theoretical), even though its object of investigation belongs traditionally to mathematics education.
This project brings new evidence for the general principle – unexpectedly for some people – that theoretical philosophy can be as practical and applicative as it is perceived as abstract and foundational, and social sciences would benefit by its methods and content.

**Goals, methods, and foreseen outcomes**

The specific goals and main steps of this prospected research are these: 1) to develop two additional arguments for the potential of philosophy in mathematics education, called 'the foundational-dependence argument' and 'the-nature-of-mathematics argument'; 2) to define a concept of holistic understanding of mathematics from the educational perspective, on epistemological grounds; 3) to delimit clearly the concepts and knowledge zones from philosophy that can contribute to the proposed theoretical framework; 4) to sketch a primary theoretical framework able to yield criteria of application of the philosophical contributions in mathematics education and to explore the ways to unify the conceptual and procedural approaches of teaching mathematics; 5) to draw general directions of further theoretical and empirical research within mathematics education in regard to implementation of the results and effectiveness in the educational practice.
Once we clarify the holistic concept of mathematical understanding and its motivations, this clarification would provide the primary criteria for *delimiting* the knowledge zones, concepts, and topics of the philosophical disciplines called to contribute to the proposed topic and framework. This is the first task of step 3.
The taxonomy of the contexts of a holistic mathematical understanding – once completed – is expected to identify clearly all those philosophical zones so as to make their content amenable to synthesis, reduction, and adaptation in regard to further implementation in mathematics education. The possibility of limiting the content of these zones on criteria of relevance should be analyzed for the pragmatic goal of not overloading curricular content and class time.
In step 4, we shall gather the results of the previous steps and sketch a primary theoretical framework applicable in mathematics education. The framework is supposed to provide the theoretical base and all criteria for practical implementation.
The methods used in this research belong largely to theoretical philosophy, such as conceptual analysis, criticism, doubt about sense, and theoretical modeling; however, methods specific to educational sciences will be also used occasionally, such as analysis of existing data, reports, and feedback from mathematics teaching practice. The research will run exclusively in the theoretical philosophy area, even though the object of investigation belongs to mathematics education (as a social science) as *locus classicus* and the application is targeted to this latter area.

Once we crystallize a theoretical framework around an adequate concept of holistic mathematical understanding, we can provide the general criteria for optimizing the curricular content and teaching methodology with contributions from the theoretical-philosophical disciplines. In a subsequent stage of implementation (in collaboration with researchers in mathematics education), the research designed in step 5

will be proposed in the academic realm of that area, within which various hypotheses can be tested empirically and assessments can be made with the tools of educational sciences.
As a final theoretical-applicative product of this project, we foresee the development of a course addressed to mathematics instructors with the topic of how to use effectively the philosophical knowledge in the content and methodology of teaching mathematics.

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