Gauge Invariance for Classical Massless Particles with Spin

Jacob A. Barandes[1, *]
[1] *Jefferson Physical Laboratory, Harvard University, Cambridge, MA 02138*
(Dated: October 10, 2020)

Wigner's quantum-mechanical classification of particle-types in terms of irreducible representations of the Poincaré group has a classical analogue, which we extend in this paper. We study the compactness properties of the resulting phase spaces at fixed energy, and show that in order for a classical massless particle to be physically sensible, its phase space must feature a classical-particle counterpart of electromagnetic gauge invariance. By examining the connection between massless and massive particles in the massless limit, we also derive a classical-particle version of the Higgs mechanism.

I. INTRODUCTION

The ingredients of classical physics are usually simpler to visualize and understand than those of quantum theory. It is therefore worthwhile to investigate which seemingly quantum phenomena turn out to have classical realizations, if only to provide the kind of intuition that can lead to discoveries.

As an important example, intrinsic spin is often regarded as fundamentally quantum in nature, but there exists a fully classical description of relativistic point particles with arbitrary masses and fixed spin. With the eventual goal of describing and extending this framework,[1] we begin in Section II by suitably generalizing the usual Lagrangian formulation of classical physics to a more expressly Lorentz-covariant form. In Section III, we review the classification of particle-types in terms of transitive group actions (also known as homogeneous spaces) of the Poincaré group, expanding on earlier work [2–4] and paralleling Wigner's classification [5] of quantum particle-types in terms of irreducible Hilbert-space representations of the Poincaré group. We will be most interested in the massless case, for which we present new results that include the emergence of a classical-particle form of electromagnetic gauge invariance. In Section IV, we revisit this appearance of gauge invariance from the perspective of the massive case in the massless limit, along the way deriving a classical-particle version of the Higgs mechanism, another novel result.

II. THE MANIFESTLY COVARIANT LAGRANGIAN FORMULATION

Consider a classical system with time parameter t, degrees of freedom q_α, Lagrangian L, and action functional

$$S[q] \equiv \int dt\, L(q, \dot{q}, t), \tag{1}$$

where dots here denote derivatives with respect to the time t. Before we apply this framework to classical relativistic point particles, we will find it useful to recast these ingredients in a form that is more manifestly compatible with relativistic invariance.

To do so, we begin by replacing t with an arbitrary smooth, monotonic parameter λ. Letting dots now denote derivatives with respect to λ, we can rewrite the action functional in the reparametrization-invariant form[6]

$$S[q,t] \equiv \int d\lambda\, \mathscr{L}(q, \dot{q}, t, \dot{t}), \tag{2}$$

where

$$\mathscr{L}(q, \dot{q}, t, \dot{t}) \equiv \dot{t}\, L(q, \dot{q}/\dot{t}, t). \tag{3}$$

We introduce a raised/lowered-index notation according to

$$\begin{aligned} q^t &\equiv c\,t, \qquad q_t \equiv -c\,t, \\ q^\alpha &\equiv q_\alpha, \\ p^t &\equiv H/c, \quad p_t \equiv -H/c, \\ p^\alpha &\equiv p_\alpha. \end{aligned} \tag{4}$$

where p_α are the system's usual canonical momenta, H is the system's usual Hamiltonian derived from the original Lagrangian L in (1), and c is a constant with units of energy divided by momentum. The quantities p^t and p^α are then expressible in terms of the function (3) as

$$p^t = \frac{\partial \mathscr{L}}{\partial \dot{q}_t}, \quad p^\alpha = \frac{\partial \mathscr{L}}{\partial \dot{q}_\alpha}, \tag{5}$$

and one can show that the Euler-Lagrange equations take the symmetric-looking form

$$\dot{p}^t = \frac{\partial \mathscr{L}}{\partial q_t}, \quad \dot{p}^\alpha = \frac{\partial \mathscr{L}}{\partial q_\alpha}. \tag{6}$$

Moreover, the action functional (2) now takes a form that resembles a Lorentz-covariant dot product involving a square matrix $\eta \equiv \mathrm{diag}(-1, 1, \dots)$ that naturally generalizes the Minkowski metric tensor from special relativity,

$$S[q] = \int d\lambda \left(p_t \dot{q}^t + \sum_\alpha p_\alpha \dot{q}^\alpha\right) = \int d\lambda \begin{pmatrix} p^t & p^\alpha \end{pmatrix} \eta \begin{pmatrix} \dot{q}^t \\ \dot{q}^\alpha \end{pmatrix}, \tag{7}$$

despite the fact that the degrees of freedom q_α are not assumed at this point to have anything to do with physical space. The action functional is then invariant under transformations

$$\begin{pmatrix} q^t \\ q^\alpha \end{pmatrix} \mapsto \Lambda \begin{pmatrix} q^t \\ q^\alpha \end{pmatrix}, \quad \begin{pmatrix} p^t \\ p^\alpha \end{pmatrix} \mapsto \Lambda \begin{pmatrix} p^t \\ p^\alpha \end{pmatrix} \tag{8}$$

for square matrices Λ satisfying the condition $\Lambda^{\mathrm{T}}\eta\Lambda = \eta$.

Thus, this reparametrization-invariant Lagrangian formulation motivates the introduction of phase-space variables $q^t, q^\alpha, p^t, p^\alpha$ that transform covariantly under a generalized notion of Lorentz transformations. We therefore refer to this framework as the *manifestly covariant Lagrangian formulation* of our classical system's dynamics.

III. TRANSITIVE GROUP ACTIONS OF THE POINCARÉ GROUP

Wigner showed in [5] that classifying the different Hilbert spaces that provide irreducible representations of the Poincaré group yields a systematic categorization of quantum-mechanical particle-types into massive, massless, and tachyonic cases.[7] As shown in various treatments, such as [2–4], there exists a classical analogue of this construction, one version of which we review here. Toward the end of this section and in the next section, we will present fundamental new results concerning previously unexamined features of the massless case.

A. Kinematics

We start by laying out a formulation of the kinematics of a system that we will eventually identify as a classical relativistic particle.

Given a classical system described by a manifestly covariant Lagrangian formulation, we say that its phase space provides a transitive or "irreducible" group action of the Poincaré group (or serves as a homogeneous space of the Poincaré group) if we can reach every state (q, p) in the system's phase space by starting from an arbitrary choice of reference state (q_0, p_0) and acting with an appropriate Poincaré transformation $(a, \Lambda) \in \mathbb{R}^{1,3} \rtimes O(1, 3)$, where a^μ is a four-vector that parametrizes translations in spacetime and $\Lambda^\mu{}_\nu$ is a Lorentz-transformation matrix. The Poincaré group singles out systems whose phase spaces consist of spacetime coordinates

$$X^\mu \equiv (cT, \mathbf{X})^\mu \equiv (cT, X, Y, Z)^\mu \tag{9}$$

and corresponding canonical four-momentum components

$$p^\mu \equiv \frac{\partial \mathscr{L}}{\partial \dot{X}^\mu} \equiv (E/c, \mathbf{p})^\mu, \tag{10}$$

where we identify $H \equiv E$ as the system's energy. We will see that such a system formalizes the notion of a classical relativistic particle.

To be as general as possible, we allow the system to have an intrinsic spin represented by an antisymmetric spin tensor,

$$S^{\mu\nu} = -S^{\nu\mu}, \tag{11}$$

in terms of which we can define a proper three-vector $\tilde{\mathbf{S}}$ and a three-dimensional pseudovector $\mathbf{S}$ according to

$$S^{\mu\nu} \equiv \begin{pmatrix} 0 & \tilde{S}_x & \tilde{S}_y & \tilde{S}_z \\ -\tilde{S}_x & 0 & S_z & -S_y \\ -\tilde{S}_y & -S_z & 0 & S_x \\ -\tilde{S}_z & S_y & -S_x & 0 \end{pmatrix}^{\mu\nu}. \tag{12}$$

Hence, the system's phase space consists of states that we can denote by (X, p, S) and that, by definition, behave under Poincaré transformations (a, Λ) according to

$$(X, p, S) \mapsto (\Lambda X + a, \Lambda p, \Lambda S \Lambda^{\mathrm{T}}). \tag{13}$$

Taking our reference state to be

$$(0, p_0, S_0) \tag{14}$$

for convenient choices of p_0^μ and $S_0^{\mu\nu}$ that will be made on a case-by-case basis later, we can therefore write each state of our system as

$$(X, p, S) \equiv (a, \Lambda p_0, \Lambda S_0 \Lambda^{\mathrm{T}}), \tag{15}$$

so a^μ and $\Lambda^\mu{}_\nu$ effectively become the system's fundamental phase-space variables.

To keep our notation simple, we will refer to a^μ as X^μ in our work ahead, keeping in mind that these variables are independent of the Lorentz-transformation matrix $\Lambda^\mu{}_\nu$. We will therefore express the functional dependence of the system's manifestly covariant action functional as $S[X, \Lambda]$.

It is natural to introduce several derived tensors from the system's fundamental variables $X^\mu, p^\mu, S^{\mu\nu}$. The system's orbital angular-momentum tensor is defined by

$$L^{\mu\nu} \equiv X^\mu p^\nu - X^\nu p^\mu = -L^{\nu\mu}, \tag{16}$$

and $L^{\mu\nu}$ together with $S^{\mu\nu}$ make up the system's total angular-momentum tensor:

$$J^{\mu\nu} \equiv L^{\mu\nu} + S^{\mu\nu} = -J^{\nu\mu}. \tag{17}$$

Defining the four-dimensional Levi-Civita symbol by

$$\begin{aligned} \epsilon_{\mu\nu\rho\sigma} &\equiv \begin{cases} +1 & \text{for } \mu\nu\rho\sigma \text{ an even permutation of } txyz, \\ -1 & \text{for } \mu\nu\rho\sigma \text{ an odd permutation of } txyz, \\ 0 & \text{otherwise} \end{cases} \\ &= -\epsilon^{\mu\nu\rho\sigma}, \end{aligned} \tag{18}$$

the system's Pauli-Lubanski pseudovector is

$$W^\mu \equiv -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}p_\nu S_{\rho\sigma} = (\mathbf{p}\cdot\mathbf{S},\ (E/c)\mathbf{S}-\mathbf{p}\times\tilde{\mathbf{S}})^\mu. \quad (19)$$

The following quantities are then invariant under proper, orthochronous Poincaré transformations, and therefore represent fixed features (or Casimir invariants) of the system's phase space:

$$-m^2c^2 \equiv p_\mu p^\mu, \quad (20)$$

$$w^2 \equiv W_\mu W^\mu, \quad (21)$$

$$s^2 \equiv \frac{1}{2}S_{\mu\nu}S^{\mu\nu} = \mathbf{S}^2 - \tilde{\mathbf{S}}^2, \quad (22)$$

$$\tilde{s}^2 \equiv \frac{1}{8}\epsilon_{\mu\nu\rho\sigma}S^{\mu\nu}S^{\rho\sigma} = \mathbf{S}\cdot\tilde{\mathbf{S}}. \quad (23)$$

In the analogous quantum case, the third of these invariant quantities, the spin-squared scalar s^2, would be quantized in increments of $\hbar$ (or, more precisely, $\hbar^2$). In our classical context, we are essentially working in the limit of large quantum numbers, in which the correspondence principle holds and these quantities are free to take on fixed values from a continuous set of real numbers. Note, in particular, that the invariance of s^2 is entirely separate from issues of quantization, just as the invariance of m^2 does not require quantization.

B. Dynamics

We now turn to the system's dynamics.

In the absence of intrinsic spin, $S^{\mu\nu}=0$, the system's manifestly covariant action functional is, from (7), given by

$$S_{\text{no spin}}[X,\Lambda] = \int d\lambda\, p_\mu \dot{X}^\mu = \int d\lambda\,(\Lambda p_0)_\mu \dot{X}^\mu. \quad (24)$$

We will eventually need to establish a definite relationship between the system's four-momentum p^μ and its four-velocity $\dot{X}^\mu \equiv dX^\mu/d\lambda$.

First, however, we will extend the action functional (24) to include intrinsic spin. We begin by introducing the standard Lorentz generators:

$$[\sigma_{\mu\nu}]^\alpha{}_\beta = -i\delta^\alpha_\mu \eta_{\nu\beta} + i\eta_{\mu\beta}\delta^\alpha_\nu. \quad (25)$$

Using the composition property of Lorentz transformations applied to the case of infinitesimal shifts $\lambda \mapsto \lambda + d\lambda$ in the parameter λ,

$$\begin{aligned}\Lambda(\lambda+d\lambda) &= \Lambda(d\lambda)\Lambda(\lambda)\\ &= (1-(i/2)d\theta^{\mu\nu}(\lambda)\sigma_{\mu\nu})\Lambda(\lambda),\end{aligned} \quad (26)$$

where $d\theta^{\mu\nu}$ is an antisymmetric tensor of infinitesimal Lorentz boosts and angular displacements, we have

$$\begin{aligned}\dot{\Lambda}(\lambda) &\equiv \frac{\Lambda(\lambda+d\lambda)-\Lambda(\lambda)}{d\lambda}\\ &= -\frac{i}{2}\dot{\theta}^{\mu\nu}(\lambda)\sigma_{\mu\nu}\Lambda(\lambda).\end{aligned} \quad (27)$$

Invoking the following trace identity satisfied by antisymmetric tensors $A^{\mu\nu} = -A^{\nu\mu}$,

$$\frac{1}{2}\mathrm{Tr}[\sigma^{\mu\nu}A] = iA^{\mu\nu}, \quad (28)$$

we can express the rates of change $\dot{\theta}^{\mu\nu}(\lambda)$ according to

$$\dot{\theta}^{\mu\nu}(\lambda) = \frac{i}{2}\mathrm{Tr}[\sigma^{\mu\nu}\dot{\Lambda}(\lambda)\Lambda^{-1}(\lambda)]. \quad (29)$$

By an integration by parts, we can then recast the action functional (24) (up to an irrelevant boundary term) as

$$S_{\text{no spin}}[X,\Lambda] = \int d\lambda\,\frac{1}{2}L_{\mu\nu}\dot{\theta}^{\mu\nu}. \quad (30)$$

With the alternative form (30) of the action functional in hand, we can straightforwardly introduce intrinsic spin into the system's dynamics by making the replacement $L_{\mu\nu} \mapsto J_{\mu\nu} \equiv L_{\mu\nu} + S_{\mu\nu}$. Converting the term involving $L_{\mu\nu}$ back into the form (24), we thereby obtain the new action functional

$$S[X,\Lambda] = \int d\lambda\,\mathscr{L} = \int d\lambda\left(p_\mu\dot{X}^\mu + \frac{1}{2}\mathrm{Tr}[S\dot{\Lambda}\Lambda^{-1}]\right), \quad (31)$$

which now properly accounts for intrinsic spin.

The equations of motion derived from this action functional are

$$\dot{p}^\mu = 0, \quad (32)$$

$$\dot{J}^{\mu\nu} = 0, \quad (33)$$

and respectively express conservation of four-momentum and conservation of total angular momentum, in keeping with Noether's theorem and the symmetries of the dynamics under Poincaré transformations. It follows that the Pauli-Lubanski pseudovector (19) is conserved, $\dot{W}^\mu = 0$, and that the scalar quantities $-m^2c^2$ and w^2 defined in (20)–(21) are guaranteed to be constant, as required.

As shown in [8], constancy of the spin-squared scalar s^2 defined in (22) requires the imposition of an important Poincaré-invariant condition on the system's phase space. To see why, we make use of the equation of motion (33) to compute the rate of change of s^2:

$$\frac{d}{d\lambda}\left(\frac{1}{2}S_{\mu\nu}S^{\mu\nu}\right) = S_{\mu\nu}\dot{S}^{\mu\nu} = 2\dot{X}^\nu p^\mu S_{\mu\nu} = 0.$$

Keep in mind that without a definite relationship between the four-momentum p^μ and the four-velocity $\dot{X}^\mu$, this condition is nontrivial. Because it establishes a constraint on all solution trajectories in the particle's phase space, we conclude that the following Lorentz-invariant condition must hold:[9]

$$p_\mu S^{\mu\nu} = 0. \tag{34}$$

Combined with the system's equations of motion (32)–(33), this condition yields a pair of basic relationships between the system's four-momentum p^μ and its otherwise-unfixed four-velocity $\dot{X}^\mu$:

$$p \cdot \dot{X} = \pm mc^2 \sqrt{-\dot{X}^2/c^2}, \tag{35}$$

$$m\sqrt{-\dot{X}^2/c^2}\, p^\mu = \mp m^2 \dot{X}^\mu. \tag{36}$$

The equations (32)–(36) complete our specification of the system's dynamics.

C. Classification of the Transitive Group Actions

Specializing to the *orthochronous* Poincaré group, classifying the different systems whose phase spaces give transitive group actions is a straightforward exercise that parallels Wigner's approach in [5]. As derived in detail in [10], one finds that each such system can describe a massive particle $m^2 > 0$ or a massless particle $m^2 = 0$ with either positive energy $E = p^t c > 0$ or negative energy $E = p^t c < 0$, or a tachyon $m^2 < 0$, or the vacuum $p^\mu = 0$. Furthermore, the relations (35)–(36) imply that for each of these cases, the four-momentum is parallel to the four-velocity, $p^\mu \propto \dot{X}^\mu$. It then follows immediately from the equations of motion (32) and (33) that $L^{\mu\nu}$ and $S^{\mu\nu}$ are separately conserved.

For a massive particle, we can take the reference state (14) to describe the particle at rest, with reference four-momentum

$$p_0^\mu = (mc, \mathbf{0})^\mu. \tag{37}$$

The condition (34) then eliminates unphysical spin degrees of freedom and implies that the particle's spin tensor (12) reduces to the three-dimensional spin pseudovector $\mathbf{S}$, whose possible orientations fill out a compact, fixed-energy region of the particle's phase space.

On the other hand, for massless particles and tachyons, the little group of Poincaré transformations that preserve the particle's reference four-momentum p_0^μ dictates that the particle's phase space at any fixed energy is seemingly noncompact, leading to infinite entropies and other thermodynamic pathologies, besides problems that arise in the corresponding quantum field theories.[11] For a tachyon, the only way to eliminate this noncompactness is to require that the spin tensor vanishes, $S^{\mu\nu} = 0$, meaning that tachyons are naturally spinless.

For a massless particle, by contrast, the story is more interesting. We can take the massless particle's reference four-momentum to be

$$p_0^\mu = (E/c, 0, 0, E/c)^\mu, \tag{38}$$

and the condition (34), $p_\mu S^{\mu\nu} = 0$, then implies the corresponding reference spin tensor

$$S_0^{\mu\nu} = \begin{pmatrix} 0 & S_{0,y} & -S_{0,x} & 0 \\ -S_{0,y} & 0 & S_{0,z} & -S_{0,y} \\ S_{0,x} & -S_{0,z} & 0 & S_{0,x} \\ 0 & S_{0,y} & -S_{0,x} & 0 \end{pmatrix}^{\mu\nu}. \tag{39}$$

The most general little-group transformation preserving the reference four-momentum (38) consists of a Lorentz-transformation matrix Λ of the form[12]

$$\Lambda(\theta, \alpha, \beta) = R(\theta) L(\alpha, \beta), \tag{40}$$

where

$$R(\theta) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{41}$$

is a pure rotation by an angle θ around the z axis and where

$$L(\alpha, \beta) \equiv \begin{pmatrix} 1+\zeta & \alpha & \beta & -\zeta \\ \alpha & 1 & 0 & -\alpha \\ \beta & 0 & 1 & -\beta \\ \zeta & \alpha & \beta & 1-\zeta \end{pmatrix} \tag{42}$$

is a complicated combination of Lorentz boosts and rotations. One can show that

$$R(\theta_1) R(\theta_2) = R(\theta_1 + \theta_2), \tag{43}$$

$$L(\alpha_1, \beta_1) L(\alpha_2, \beta_2) = L(\alpha_1 + \alpha_2, \beta_1 + \beta_2), \tag{44}$$

so rotations $R(\theta)$ around the z axis and the Lorentz transformations $L(\alpha, \beta)$ respectively form a pair of commutative subgroups of the particle's little group. Noting that

$$R(\theta) L(\alpha, \beta) R^{-1}(\theta) = L(\alpha\cos\theta + \beta\sin\theta, -\alpha\sin\theta + \beta\cos\theta), \tag{45}$$

we identify the little group as $ISO(2)$, which is the noncompact group of rotations and translations in the two-dimensional Euclidean plane.

These little-group transformation act nontrivially on the particle's reference spin tensor (39):

$$L(\alpha, \beta) S_0 L^{\mathrm{T}}(\alpha, \beta) = S_0 + \begin{pmatrix} 0 & -\beta S_{0,z} & \alpha S_{0,z} & 0 \\ \beta S_{0,z} & 0 & 0 & \beta S_{0,z} \\ \alpha S_{0,z} & 0 & 0 & -\alpha S_{0,z} \\ 0 & -\beta S_{0,z} & \alpha S_{0,z} & 0 \end{pmatrix}. \tag{46}$$

Hence, the only way to ensure that the massless particle has a compact phase space at fixed reference energy while still allowing for nonzero spin is to impose the following equivalence relation on the particle's phase space:

$$(X, p, S) \cong (X, p, S'). \tag{47}$$

This equivalence relation is a new result. It is a classical-particle manifestation of the gauge invariance that arises for the gauge potential A_μ in electromagnetism, and it cuts the particle's phase space at fixed energy down to a compact extent. The distinct physical states of the massless particle are then characterized by a spacetime position X^μ, a four-momentum p^μ, and a helicity $\sigma \equiv (\mathbf{p}/|\mathbf{p}|) \cdot \mathbf{S}$.[13]

IV. THE MASSLESS LIMIT

We can better understand the origin of the novel equivalence relation (47) by starting with the massive case $m > 0$ and then taking the massless limit $m \to 0$.

Our original reference state (37) degenerates for $m \to 0$, so we instead take the massive particle's reference four-momentum to be

$$\bar{p}^\mu \equiv (\bar{p}^t, 0, 0, \bar{p}^z)^\mu = \left(\sqrt{(\bar{p}^z)^2 + m^2c^2}, 0, 0, \bar{p}^z\right)^\mu. \tag{48}$$

This choice has the correct $m \to 0$ limit (38):

$$\lim_{m\to 0} \bar{p}^\mu = (E_0/c, 0, 0, E_0/c)^\mu, \quad E_0 \equiv \bar{p}^z c. \tag{49}$$

Moreover, (48) is related to our original choice (37) of reference four-momentum for the massive particle by a simple Lorentz boost $\bar{\Lambda}$ along the z direction,

$$\bar{p}^\mu = \bar{\Lambda}^\mu{}_\nu p_0^\nu, \tag{50}$$

and the new reference value $\bar{S}^{\mu\nu}$ of the massive particle's spin tensor is related to its original reference value $S_0^{\mu\nu}$ according to

$$\bar{S}^{\mu\nu} \equiv (\bar{\Lambda} S_0 \bar{\Lambda}^{\mathrm{T}})^{\mu\nu} = \begin{pmatrix} 0 & \frac{\bar{p}^z}{mc} S_{0,y} & -\frac{\bar{p}^z}{mc} S_{0,x} & 0 \\ -\frac{\bar{p}^z}{mc} S_{0,y} & 0 & S_{0,z} & -\frac{\bar{p}^t}{mc} S_{0,y} \\ \frac{\bar{p}^z}{mc} S_{0,x} & -S_{0,z} & 0 & \frac{\bar{p}^t}{mc} S_{0,x} \\ 0 & \frac{\bar{p}^t}{mc} S_{0,y} & -\frac{\bar{p}^t}{mc} S_{0,x} & 0 \end{pmatrix}^{\mu\nu}. \tag{51}$$

For $m \to 0$, we have $\bar{p}^t, \bar{p}^z \to E_0/c$, so the components of $\bar{S}^{\mu\nu}$ involving $\bar{p}^t/mc$ or $\bar{p}^z/mc$ diverge. Furthermore, there is a discrete mismatch in the particle's spin-squared scalar (22) between the massive case and the massless case:

$$\begin{aligned} s^2 = S_{0,x}^2 + S_{0,y}^2 + S_{0,z}^2 \quad &(\text{massive}) \\ \neq S_{0,z}^2 \quad &(\text{massless}). \end{aligned} \tag{52}$$

These discrepancies are hints that the massive case includes spin degrees of freedom that need to be removed before taking the massless limit.

Our approach for removing these ill-behaved spin degrees of freedom is motivated by a corresponding procedure in quantum field theory that was originally developed by Stueckelberg in [14]. We start with the redefinition

$$\begin{pmatrix} \bar{S}_x \\ \bar{S}_y \end{pmatrix} \mapsto \frac{mc}{\bar{p}^t} \begin{pmatrix} \bar{S}_x + \bar{p}^t \varphi_x \\ \bar{S}_y + \bar{p}^t \varphi_y \end{pmatrix} = \frac{mc}{\bar{p}^t} \begin{pmatrix} \bar{S}_x \\ \bar{S}_y \end{pmatrix} + mc \begin{pmatrix} \varphi_x \\ \varphi_y \end{pmatrix}, \tag{53}$$

where $\varphi_x(\lambda)$ and $\varphi_y(\lambda)$ are arbitrary new functions on the particle's worldline. The particle's spin tensor (51) then has the decomposition

$$\bar{S}^{\mu\nu} = \begin{pmatrix} 0 & \frac{\bar{p}^z}{\bar{p}^t} S_{0,y} & -\frac{\bar{p}^z}{\bar{p}^t} S_{0,x} & 0 \\ -\frac{\bar{p}^z}{\bar{p}^t} S_{0,y} & 0 & S_{0,z} & -S_{0,y} \\ \frac{\bar{p}^z}{\bar{p}^t} S_{0,x} & -S_{0,z} & 0 & S_{0,x} \\ 0 & S_{0,y} & -S_{0,x} & 0 \end{pmatrix}^{\mu\nu} + \begin{pmatrix} 0 & \bar{p}^z \varphi_y & -\bar{p}^z \varphi_x & 0 \\ -\bar{p}^z \varphi_y & 0 & 0 & -\bar{p}^t \varphi_y \\ \bar{p}^z \varphi_x & 0 & 0 & \bar{p}^t \varphi_x \\ 0 & \bar{p}^t \varphi_y & -\bar{p}^t \varphi_x & 0 \end{pmatrix}^{\mu\nu}, \tag{54}$$

and the spin-squared scalar (22) becomes

$$s^2 = \left(1 - \left(\frac{\bar{p}^z}{\bar{p}^t}\right)^2\right)\Big((S_{0,x} + \bar{p}^t \varphi_x)^2 + (S_{0,y} + \bar{p}^t \varphi_y)^2\Big) + S_{0,z}^2. \tag{55}$$

The particle's spin tensor (54) is now invariant under the simultaneous transformations

$$\begin{pmatrix} \bar{S}_x \\ \bar{S}_y \end{pmatrix} \mapsto \begin{pmatrix} \bar{S}_x \\ \bar{S}_y \end{pmatrix} - \bar{p}^t \begin{pmatrix} f_x \\ f_y \end{pmatrix}, \tag{56}$$

$$\begin{pmatrix} \varphi_x \\ \varphi_y \end{pmatrix} \mapsto \begin{pmatrix} \varphi_x \\ \varphi_y \end{pmatrix} + \begin{pmatrix} f_x \\ f_y \end{pmatrix}, \tag{57}$$

where $f_x(\lambda), f_y(\lambda)$ are arbitrary functions on the particle's worldline.

Our massive particle's original phase space, with states labeled as (X, p, S), is therefore equivalent to a *formally enlarged* phase space consisting of states (X, p, S, φ) under the equivalence relation $(\bar{X}, \bar{p}, \bar{S}, \varphi) \cong (\bar{X}, \bar{p}, \bar{S} - \bar{p}^t f, \varphi + f)$, suitably generalized from the reference state $(\bar{X}, \bar{p}, \bar{S}, \varphi)$ to general states (X, p, S, φ) of the system. Indeed, one can check that the specific choice $(f_x, f_y) \equiv -(\varphi_x, \varphi_y)$ yields $(\bar{X}, \bar{p}, \bar{S} + \bar{p}^t \varphi, 0)$, which gives back the state $(\bar{X}, \bar{p}, \bar{S})$ after undoing the redefinition (53) of $\bar{S}^{\mu\nu}$.

We can now safely take the massless limit of the system's redefined spin tensor (54):

$$\lim_{m\to 0}\bar{S}^{\mu\nu} = \begin{pmatrix} 0 & S_{0,y} & -S_{0,x} & 0 \\ -S_{0,y} & 0 & S_{0,z} & -S_{0,y} \\ S_{0,x} & -S_{0,z} & 0 & S_{0,x} \\ 0 & S_{0,y} & -S_{0,x} & 0 \end{pmatrix}^{\mu\nu} + \frac{E}{c}\begin{pmatrix} 0 & \varphi_y & -\varphi_x & 0 \\ -\varphi_y & 0 & 0 & -\varphi_y \\ \varphi_x & 0 & 0 & \varphi_x \\ 0 & \varphi_y & -\varphi_x & 0 \end{pmatrix}^{\mu\nu}, \tag{58}$$

and

$$\lim_{m\to 0} s^2 = S_{0,z}^2. \tag{59}$$

The degrees of freedom describing spin components perpendicular to the particle's reference three-momentum $\bar{\mathbf{p}}$ no longer contribute to the particle's spin-squared scalar s^2. If we remove these ancillary degrees of freedom by setting φ_x, φ_y equal to zero, then the particle's spin tensor (58) reduces correctly to the reference spin tensor (39) for a massless particle, and our equivalence relation (56) reduces to the gauge invariance (47). We have therefore completed our recovery of the massless case from the $m \to 0$ limit of a massive particle.

Furthermore, if we run these arguments in reverse, then we see that we can transform a massless particle with nonzero spin into a massive particle by introducing additional spin degrees of freedom, a classical counterpart of the celebrated Higgs mechanism.

ACKNOWLEDGMENTS

J. A. B. has benefited from personal communications with Howard Georgi, Andrew Strominger, David Griffiths, David Kagan, David Morin, Logan McCarty, Monica Pate, and Alex Lupsasca.

* barandes@physics.harvard.edu

[1] For a more comprehensive treatment of the results in this paper, see [10].

[2] A. P. Balachandran, G. Marmo, B.-S. Skagerstam, and A. Stern, *Gauge Symmetries and Fibre Bundles - Applications to Particle Dynamics*, 1st ed. (Springer-Verlag Berlin Heidelberg, 1983) arXiv:1702.08910 [quant-ph].

[3] J.-M. Souriau, *Structure of Dynamical Systems*, 1st ed. (Birkhäuser, 1997).

[4] M. Rivas, *Kinematical Theory of Spinning Particles* (Springer, 2002).

[5] E. P. Wigner, Annals of Mathematics **40**, 149 (1939).

[6] For an early example of this technique, see [15]. For a more modern, pedagogical treatment, see [16].

[7] See [17] for a pedagogical review.

[8] B.-S. Skagerstam and A. Stern, Physica Scripta **24**, 493 (1981).

[9] This condition is a classical-particle analogue of the Lorenz equation $\partial_\mu A^\mu = 0$ that appears both in the Proca theory of a massive spin-one bosonic field and as the Lorenz-gauge condition in electromagnetism. As in those field theories, the condition (34) serves to eliminate unphysical spin states.

[10] J. A. Barandes, (2019), arXiv:1911.08892.

[11] See, for example, , but also [18] for a more optimistic take.

[12] For a derivation, see, for example, [10, 17].

[13] Note that if we permit parity transformations, which map $\sigma \mapsto -\sigma$, then we must require that the equivalence relation (47) hold only for states that share the same helicity σ.

[14] E. C. G. Stueckelberg, Helvetica Physica Acta **11**, 225 (1938).

[15] P. A. M. Dirac, Proceedings of the Royal Society A **111**, 405 (1926).

[16] A. Deriglazov and B. Rizzuti, American Journal of Physics **79**, 882 (2011), arXiv:1105.0313 [math-ph].

[17] S. Weinberg, *The Quantum Theory of Fields*, 1st ed., Vol. 1 (Cambridge University Press, 1996).

[18] P. Schuster and N. Toro, Journal of High Energy Physics **2013** (2013), 10.1007/JHEP09(2013)104, arXiv:1302.1198 [hep-th].