Is (un)countabilism restrictive?

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Abstract

Let's suppose you think that there are no uncountable sets. Have you adopted a restrictive position? It is certainly tempting to say yes—you've prohibited the existence of certain kinds of large set. This paper argues that this intuition can be challenged. Instead, I argue that there are some considerations based on a formal notion of restrictiveness which suggest that it is restrictive to hold that there *are* uncountable sets.

Keywords. set theory; restrictive theory; countabilism; uncountabilism

Introduction

This paper is directed towards the debate concerning whether or not there are uncountable infinite sets. The 'standard' answer (at least post-Cantor) is that there are, and we know this by Cantor's Theorem. To get going, let's define the following two positions:

Uncountabilism is the position that there are uncountable infinite sets.

Countabilism is the position that every infinite set is countable.

One way of proceeding is to provide plausible philosophical reasons to argue *directly* for one or other position. There is a separate but

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related question of whether either position is *restrictive*, which supports a position *indirectly* if we assume that restrictiveness is a theoretical vice. And it's tempting to think that countabilism is the restrictive position—after all, doesn't the uncountabilist just straightforwardly claim that there are *more* sets than the countabilist? Let's dub this thought the **Restrictiveness Intuition**.

It is the objective of this short paper to argue that the **Restrictiveness Intuition** can be challenged. In fact, we will see that there is a reasonable position on which the opposite is true. I'll start by identifying the following:

Main Aim. I will present a formal analysis of restrictiveness, based on Maddy's analysis in [Maddy, 1998], on which it is the *uncountabilist* and *not* the countabilist that makes restrictive claims. Furthermore, many of the natural responses on behalf of the uncountabilist achieve only parity with the countabilist.

On this basis, I will conclude that it is not obvious that countabilism is a restrictive position and that the opposite may well be correct.

Here's the plan: In §1 will explain why the **Restrictiveness Intuition** isn't obviously correct, and ignores some important considerations about models of set theory. §2 will explain Maddy's account of restrictiveness as it appears in [Maddy, 1998]. §3 will provide a modification of Maddy's definition to fit in the countabilist/uncountabilist debate, and show that given this modification it is the *uncountabilist* who makes restrictive claims. §4 will provide some philosophical examination of the results and consider some objections. We'll note that whilst the uncountabilist can respond, the **Restrictiveness Intuition** is still shown to be flawed. §5 concludes that the countabilist position is not clearly restrictive, and presents some open questions.

1 Why the Restrictiveness Intuition isn't obviously correct

The reason the **Restrictiveness Intuition** isn't obviously correct stems from the observation that although an uncountabilist can view the countabilist as saying correct things about some restricted domain(s), the countabilist can make exactly the same move. Let' make this clearer with some formal details.

The uncountabilist's theory is normally taken to include at least ZFC, although any theory that implies that there is at least one uncountable set would do. For the countabilist on the other hand, there are several options (depending on one's background motivations). One could, for instance, hold a variety of predicativism (e.g. [Feferman and Hellman, 1995]). Whilst interesting, we'll set these views aside—our interest in this article will be what happens when we merely drop the Powerset Axiom and assume that every set is countable. Whilst our discussion won't address philosophical motivations for this idea beyond restrictiveness, it bears mentioning that there are pictures that provide such grounds (cf. [Scambler, 2021, Builes and Wilson, 2022]).

As is now known in set theory, dropping the Powerset Axiom is not a trivial move. Simply deleting it results in a theory weaker than one would like. We therefore need some distinctions:

Definition 1. We distinguish between the following theories:

(1.) $\mathsf{ZFC}-$ is ZFC with the Powerset Axiom Removed and the Axiom of Choice (AC) formulated as the claim that every set can be well-ordered.

(2.) ZFC^- is $\mathsf{ZFC}-$ with the Collection and Separation Schema substituted for the Replacement Scheme.

(3.) ZFC^-_{Ref} is ZFC^- with the following schematic reflection principle added (for any ϕ in the language of set theory):

$$\forall x \exists A(x \in A \wedge \text{“}A \text{ is transitive”} \wedge \phi \leftrightarrow \phi^A)$$

i.e. for any set x there is a transitive set A such that $x \in A$ and ϕ is absolute between A and the universe. We will refer to this principle as the *First-Order Reflection Principle* (or just '*Reflection*').

(4.) By $\mathsf{NBG}-$, NBG^-, and NBG^-_{Ref} we mean the corresponding versions of NBG, with two sorts of variables and any corresponding schema replaced by single second-order (predicative) axioms.

It's known that these theories don't have the same models (given mild consistency assumptions).[1] Generally speaking $\mathsf{ZFC}-/\mathsf{NBG}-$ is regarded as too weak for many purposes (see [Gitman et al., 2016] for some discussion of this point).

Since we're going to be discussing countabilism/uncountabilism, it will be useful to provide the following abbreviation:

[1]See here [Zarach, 1996] and [Gitman et al., 2016].

Definition 2. We will abbreviate the axiom "Every set is countable" by Count.

With these definitions in hand, let's consider how uncountabilists and countabilists might interpret one another. Since we want the Axiom of Foundation to have its intended interpretation, to get the ball rolling we'll just insist that each of the countabilist and uncountabilist can adopt a theory that lets them produce a transitive model of the base theory their opponent advocates (assuming that it is consistent to do so). So, for example, the countabilist might adopt ZFC^-_{Ref} plus "There is a transitive model of ZFC" whereas the uncountabilist might just stick with ZFC for now (since ZFC proves that there is a transitive model of ZFC^-_{Ref} + Count). For example, in any model of ZFC, the hereditarily countable sets (often denoted $H(\omega_1)$) provides a model of ZFC^-_{Ref} + Count. What do these models *look like* from each perspective under these assumptions?

Well, the uncountabilist can find very 'nice' transitive models of the countabilist theory. To do so, she needs to interpret the countabilist as leaving out a bunch of sets. In particular, she needs to forget about all the uncountable sets that live in her ZFC world.

However, given how we've set things up, the same is true for the *countabilist*. They can interpret the uncountabilist as talking about some transitive model or other that misses out functions witnessing the countability of various sets. There is thus a kind of 'duality' or 'symmetry' between the two positions. From the perspective of the uncountabilist, the countabilist always misses a whole bunch of sets from their picture (viz. everything that is not hereditarily countable). However from the countabilist's perspective it is the *uncountabilist* who misses a bunch of sets—namely they have to miss out all the bijections that should exist between some 'uncountable' set and the natural numbers. It's thus not clear that either view postulates 'more' sets than the other. Whilst the uncountabilist has 'more' in the sense that if uncountabilism is true then there's no bijections between certain sets, from the countabilist's perspective this is achieved by an artificial restriction. In other words, what counts as 'more' depends on which of the two is true. Thus, without further argument there is no particular reason to prefer one position over the other (at least insofar as restrictiveness is concerned).[2]

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[2]Similar observations were made by [Skolem, 1922] and are in the background of much of the literature on the indeterminacy of reference (see [Button and Walsh, 2018]). But as we'll see there's no need to hold some kind of 'indeterminacy of reference' to get the ball rolling here—we'll just be looking at how theories can provide interpretations and what this might say about restrictiveness.

2 Maddy-restrictiveness

Is there a way to break the deadlock? In this section and the next, we'll suggest that by modifying Maddy's analysis of restrictiveness (cf. [Maddy, 1998]) we can obtain results on which it is the *uncountabilist* that makes restrictive claims.

In order to make this out, we will build on the idea considered in the last section of providing 'good' or 'nice' interpretations, and what this might tell us about restrictiveness. That 'good' interpretations are central to the ability of a theory to interpret mathematics without restriction has been advocated informally by [Steel, 2014] and formally by [Maddy, 1998] and [Meadows, F] (often this is referred to as the *interpretive power* of a formal theory). Maddy's core idea is that a theory T maximises over another T' when one can find a 'good' interpretation of T' in T, but not the other way around (and no extension of T' could do the job either).

Maddy makes this out using the following definitions:

Definition 3. [Maddy, 1998] T *shows* ϕ *is an inner model* iff:

(i) For all $\sigma \in \mathsf{ZFC}$, $\mathsf{T} \vdash \sigma^{\phi}$.

(ii) $\mathsf{T} \vdash \forall \alpha \phi(\alpha)$ or $\mathsf{T} \vdash (\exists \kappa \text{“}\kappa \text{ is inaccessible”} \wedge \forall \alpha[\alpha < \kappa \rightarrow \phi(\alpha)])$

(iii) $\mathsf{T} \vdash \forall x \forall y([x \in y \wedge \phi(y)] \rightarrow \phi(x))$.

Definition 4. [Maddy, 1998] ϕ is a *fair interpretation of* T_1 *in* T_2 (where T_1 extends ZFC) iff:

(i) T_1 shows ϕ is an inner model, and

(ii) for all $\sigma \in \mathsf{T}_1$, $\mathsf{T}_2 \vdash \sigma^{\phi}$.

Definition 5. [Maddy, 1998] T_2 *maximizes* over T_1 iff there is a ϕ such that:

(i) ϕ is a fair interpretation of T_1 in T_2.

(ii) $\mathsf{T}_2 \vdash \exists x \neg \phi(x)$.[3]

Definition 6. [Maddy, 1998] T_2 *properly maximizes over* T_1 iff T_2 maximizes over T_1 but T_1 does not maximize over T_2.

[3]Maddy actually has a slightly more complicated definition, but (as she) notes, this condition suffices for current purposes in the presence of Foundation (which we've assumed).

Definition 7. [Maddy, 1998] T_2 *inconsistently maximizes over* T_1 iff T_2 properly maximizes over T_1 and T_2 is inconsistent with T_1.

Definition 8. [Maddy, 1998] T_2 *strongly maximizes over* T_1 iff T_2 inconsistently maximizes over T_1 and there is no consistent extension of T_1 that properly maximizes over T_2.

Definition 9. [Maddy, 1998] T_1 is *restrictive* iff there is a consistent T_2 that strongly maximizes over T_1.

The rough idea of Maddy's proposal is that a theory T_2 extending ZFC is maximises over another T_1 just in case:

1. T_2 is consistent,
2. T_2 inconsistent with T_1,
3. T_2 can represent T_1 in an appropriately 'nice' context (either an inner model, truncation at an inaccessible, or an inner model of a truncation at an inaccessible), and
4. There's no way of extending T_1 to get a 'nice' context in which to interpret T_2.

Maddy then uses her definition to show that $V = L$ is restrictive in her sense. Informally speaking, this is because ZFC+"There exists a measurable cardinal" is inconsistent with $V = L$, can produce an inner model satisfying $V = L$ (namely L), and no extension of ZFC + $V = L$ can ever find a nice model for ZFC + "There exists a measurable cardinal" (roughly, this is because L is the smallest possible inner model under inclusion[4]). With Maddy's account in hand, let's see how we might apply a similar notion to the current context.

3 Countabilist maximisation

We will need to modify Maddy's definition slightly, since she is considering extensions of ZFC and we want to leave it open that countabilism is true. In this section I'll explain the relevant modification, before pointing out that it leads to maximisation of the countabilist perspective over the uncountabilist one (and hence uncountabilism is restrictive, given the definition).

[4]One also needs to handle truncation at an inaccessible, but this is easy to check—see [Maddy, 1998] for details (the basic point is that if you've got a measurable cardinal, you've got to have $0^\sharp$ floating around).

We want to consider cases where we don't have ZFC as the base theory, but rather allow every set to be countable. For this, we need our base theory to be one not including the Powerset Axiom. For the purposes of this paper we'll consider ZFC^-_{Ref}. We then define:

Definition 10. In Maddy's definition, replace every occurrence of ZFC with ZFC^-_{Ref}. We say that a theory T_2 extending ZFC^-_{Ref} *modified Maddy maximizes* over T_1, iff T_2 *strongly maximizes* over T_1 in this new sense. For the sake of ease, we will simply refer to this phenomenon as strong maximisation from hereon out.

But with this definition in hand, we can immediately identify the following simple fact:

Fact 11. Let A be one of the usual large cardinal axioms (of course this is somewhat imprecise, so the reader should feel free to substitute their favourite e.g. "There is an inaccessible cardinal", "There is a measurable cardinal" if they wish to have a precise result). Let T be a consistent extension of ZFC^-_{Ref}+"Every set is countable" capable of producing a definable inner model for A (we'll see some concrete examples of such theories soon). Then T always strongly maximises over ZFC + A.

The proof is short and instructive, and so we include it here:

Proof. For brevity, let's denote the theory ZFC^-_{Ref}+"Every set is countable"+"There is a definable inner model for ZFC + A" by $\mathsf{T}^{\mathsf{A}}_{\mathsf{Count}}$. We have simply assumed that there is an inner model of ZFC + A in $\mathsf{T}^{\mathsf{A}}_{\mathsf{Count}}$, so the existence of a fair interpretation is handled from the off. Clearly $\mathsf{T}^{\mathsf{A}}_{\mathsf{Count}}$ proves that there are sets outside this interpretation, in particular the relevant collapsing functions witnessing the actual countability of the 'uncountable' sets in the fair interpretation, so we have maximisation. Clearly also ZFC + A does not maximise over $\mathsf{T}^{\mathsf{A}}_{\mathsf{Count}}$. In particular, no extension of ZFC + A can *ever* produce a fair interpretation of any theory including the statement "Every set is countable", since this statement can only be true in a transitive model of height at most ω_1 (i.e. the ordinals of the model can be at most ω_1). To get inconsistent (and hence strong) maximisation, we now only need note that $\mathsf{T}^{\mathsf{A}}_{\mathsf{Count}}$ is trivially inconsistent with ZFC + A. □

The core point, given ZFC^-_{Ref}, is that when every set is countable we can still have ZFC in an *inner model*, where an inner model contains all ordinals. However the same isn't true within ZFC (even when we restrict to an inaccessible rank) since the notion of countability is upwards absolute. One is always restricted to models of at most height

ω_1 in transitive interpretations of ZFC^-_{Ref} + "Every set is countable". In this sense, the countabilist has an interpretive *advantage*, they can have all ordinals in interpretations of ZFC plus large cardinals by simply leaving out the subsets that witness countability, but going back the other way is not possible.

Already in [Maddy, 1998], Maddy was aware that there could be trivial counterexamples to her definition in the context of ZFC based on 'dud' theories—those that are simply 'cooked up' to strongly maximise over others but for the 'wrong' reasons. One considered by Maddy is ZFC + $V \neq L$ + $\neg Con(\mathsf{ZFC})$, which strongly maximises over ZFC + $V = L$ but for clearly 'bad' reasons.

A theory of the form $\mathsf{T}^{\mathsf{A}}_{\mathsf{Count}}$, whilst not clearly dud, is certainly gerrymandered to get the result by just throwing in the assumption that we can find an inner model of the required kind. As it stands then, the result given above is not especially strong. Certainly it would be substantially strengthened if there were 'natural' theories compatible with ZFC^-_{Ref} + Count that yielded the required inner models. I contend that there are at least two candidates (in ascending order of naturalness): (i) axioms of definable determinacy, and (ii) inner model hypotheses. Let's examine each in turn.

Inner models for large cardinals and determinacy. Axioms of definable determinacy are claims about the existence of strategies for certain kinds of game. For the purposes of this paper the details are not so critical, the main point is that they reverse to the existence of large cardinals in inner models and can be formulated within ZFC^-_{Ref}.[5] This has been known for a long time in the context of ZFC (see [Koellner, 2014] for a survey). However, many of these equivalences still hold in ZFC^-_{Ref}.[6] A clear exposition and survey of several results appears in Regula Krapf's thesis [Krapf, 2017]. In particular, within ZFC^-_{Ref}, if one has $\utilde{\Pi}^1_1$-Determinacy and the $\utilde{\Pi}^1_2$-Perfect Set Property one can obtain inner models of ZFC+"Every set of ordinals has a sharp" (and indeed this implication can be reversed from a model of ZFC+"Every set of ordinals has a sharp").[7] With Projective Determinacy one gets inner models with Woodin cardinals (in particular n-many for every $n \in \mathbb{N}$).[8]

[5]A folklore result (see §5.1 of Regula Krapf's PhD thesis [Krapf, 2017]) shows that second-order arithmetic and ZFC^- + "Every set is countable" are bi-interpretable, and many axioms of definable determinacy are formalisable as statements of second-order arithmetic.

[6]Roughly speaking, so long as one can construct well-founded ultrapowers, you can build the inner models, and ZFC^-_{Ref} suffices for this.

[7]See [Krapf, 2017], Ch. 5 for a proof of this equivalence.

[8]See [Koellner and Woodin, 2010] for a description of how to get models of large cardinal axioms from determinacy hypotheses.

In all of these cases, ZFC^-_{Ref} + Count, when augmented with definable determinacy, will strongly maximise over the relevant theories extending ZFC of weaker consistency strength. So, for example:

Fact 12. Let X be a set of ordinals. ZFC^-_{Ref} + Count + $\utilde{\Pi}^1_1$-Determinacy + $\utilde{\Pi}^1_2$-Perfect Set Property strongly maximises over ZFC + "$X^\sharp$ exists".

or:

Fact 13. Fix some natural number n. Then ZFC^-_{Ref} + Count + PD strongly maximises over the theory ZFC + "There are n-many Woodin cardinals".

Thus there are countabilist perspectives, based on natural axioms, that do strongly maximise over the uncountabilist perspective axiomatised by ZFC.

Whilst this result goes some way towards providing 'natural' axiom compatible with countabilism, there are still a couple of disadvantages to the approach. First whilst axioms of definable determinacy are natural enough from a mathematical standpoint, they are not obviously directly justifiable from any known conception of set. Martin, for example, writes:

> Is PD true? It is certainly not self-evident. ([Martin, 1977], p. 813)[9]

Most justifications for axioms of definable determinacy therefore appeal to 'extrinsic' justifications, or try to justify other axioms that imply them.[10] It would thus be preferable if there were axioms providing inner models of large cardinals for the countabilist that are more clearly intuitively plausible and/or respond to a particular conception of set.

A second difficulty is that whilst many axioms of definable determinacy can be formulated as statements of second-order arithmetic, they are not obviously *countabilist* in any sense. That is, they are still (believed to be) consistent with the existence of uncountable sets and ZFC. From the countabilist's perspective, it would thus be preferable to have an axiom that is *countabilist* (i.e. implies that every set is countable) *and* yields large cardinal strength. This brings us on to:

[9]There are many similar examples, such as [Martin, 1976], p. 90 [Moschovakis, 1980], p. 610. For a survey of some of the (lack of) motivation for determinacy axioms, see [Maddy, 1988].

[10]For an attempt within ZFC, see [Welch and Horsten, 2016] and [Roberts, 2017]. The distinction between intrinsic and extrinsic justification is far from controversial, see [Barton et al., 2020] for discussion.

Inner Model Hypotheses. Recently, [Barton and Friedman, S] have developed such an axiom drawing on previous work in [Friedman, 2006] and [Friedman et al., 2008]. Their axiom is based on *absoluteness*—the idea that anything that 'could' be true in an outer model is true in the universe. It reads as follows:

Definition 14. [Barton and Friedman, S] *Ordinal Inner Model Hypotheses.* The *Ordinal Inner Model Hypothesis for* T or $\mathsf{OIMH}^{\mathsf{T}}$ states that if a first-order sentence $\phi(\vec{a})$ with ordinal parameters $\vec{a}$ in V is true in a definable inner model $I^* \models \mathsf{T}$ of an outer model $V^* \models \mathsf{T}$ of V obtained by a definable pretame class forcing, then $\phi(\vec{a})$ is already true in a definable inner model $I \models \mathsf{T}$ of V. We shall use OIMH^- and OIMH^-_{Ref} to denote the OIMH for ZFC^- and ZFC^-_{Ref} respectively.

The axiom is somewhat technical to state, and a few remarks are in order. Firstly, a class forcing is simply a forcing notion that can have a proper class of conditions. Such a forcing is 'pretame' exactly when it preserves ZFC^-.[11] Secondly, the restriction to ordinal parameters is desirable since the introduction of arbitrary real parameters yields an inconsistency with ZFC^-_{Ref}.[12] Thirdly, the axiom is not first-order expressible, but can be expressed in some extensions of NBG^-_{Ref}.[13]

These complications aside, the thought behind the axiom is that anything realisable in an inner model of an outer model is already realised in an inner model. In this way, the universe has been maximised with respect to what could be true. Our focus will be on the OIMH^-_{Ref}.

Clearly the OIMH^-_{Ref} implies that every set is countable, since one can collapse the cardinality of any set in an outer model. Moreover, it is consistent relative to ZFC with Projective Determinacy added, partially assuaging any worries concerning its consistency.[14] Importantly for us:

Theorem 15. Suppose that the universe satisfies $\mathsf{ZFC}^-_{Ref} + \mathsf{OIMH}^-_{Ref}$. Then the universe satisfies "$0^\sharp$ exists".[15]

[11] For the details of the definition, see [Friedman, 2000].

[12] See [Barton and Friedman, S] Theorem 25 for the result.

[13] The core problem is that we can't express that something is a model of ZFC^- or ZFC^-_{Ref} with one sentence (in the powerset context, the V_α hierarchy handles this issue). Rather, in order to express the axiom, one needs to quantify over models existentially, so the conclusion is an infinite disjunction. The natural formulation of the OIMH^- and OIMH^-_{Ref} are thus not first-order and is not even given by a first-order scheme (i.e. infinite conjunction of first-order sentences). Instead it is an infinitary Boolean combination of first-order sentences of low infinitary rank. However if one adopts a theory with more class comprehension—such as MK^-_{Ref}—the axiom is formalisable using the technique of coding outer models given in [Antos et al., 2021].

[14] See [Barton and Friedman, S], Theorem 28.

[15] See [Barton and Friedman, S], Theorem 31.

Here we use "the universe" to denote any structure of the required form, recalling that some higher-order resources are needed to formulate the axiom. We can now note:

Fact 16. Let T be a recursive first-order fragment of the theory of such a universe satisfying ZFC^-_{Ref} + Count + "$0^\sharp$ exists" (e.g. ZFC^-_{Ref} + Count + "$0^\sharp$ exists" itself will do). Then T strongly maximises over ZFC.

Proof. Since "$0^\sharp$ exists" implies that L satisfies ZFC, this is an immediate consequence of Fact 11. □

The existence of $0^\sharp$ in fact goes substantially beyond ZFC. We can identify:

Fact 17. Let ϕ be a large cardinal axiom that holds in L when $0^\sharp$ exists under the theory ZFC^-_{Ref} + Count. Let T be a a recursive first-order fragment as in the above Fact 16. Then T strongly maximises over ZFC + ϕ.

Proof. Exactly as in Fact 16, the result is immediate by Fact 11. □

Examples of theories strongly maximised over by such T include ZFC with any of "There is an inaccessible cardinal", "There is a Mahlo cardinal", "There is a proper class of inaccessible cardinals", and "There is a proper class of Mahlo cardinals".[16] So there are reasonable countabilist theories that modified-Maddy-maximise over not just ZFC, but also ZFC extended with many large cardinal axioms. Note that for large cardinals stronger than $0^\sharp$ no such maximisation result is known (measurable cardinals, for example, remain out of reach). Nonetheless, it may be that there are other unknown consequences of the OIMH^-_{Ref} or other axioms that do yield further maximisation results.

Thus, at least as far as our modified version of restrictiveness goes, it is the *uncountabilist* rather than *countabilist* who seems to have the restrictive theories. What should we make of all this?

4 Philosophical analysis

Before we move on to philosophical discussion, let's sum up the above mathematical observations. We've seen that:

[16] In fact, this strong maximisation will hold as we move up through the hierarchy of indescribable cardinals. Since the details will be obscure to non-specialists and known to specialists, I omit them here.

(1.) If we move to a modification of Maddy-maximisation on which we take ZFC^-_{Ref} to be the base theory, then if ϕ is a large cardinal axiom and an extension T of ZFC^-_{Ref} + Count is able to produce an inner model of ZFC + ϕ, then T will modified-Maddy-maximise over ZFC + ϕ (this was Fact 11).

(2.) There are natural axioms (e.g. axioms of definable determinacy) that produce such T when added to ZFC^-_{Ref} + Count.

(3.) There is at least one natural *and* intuitively plausible axiom (namely OIMH^-_{Ref}) that both implies that every set is countable and yields recursive first-order theories that strongly Maddy maximise over ZFC with many large cardinals added.

These observations imply that there is at least one sense in which the uncountabilist perspective is restrictive in contrast to the countabilist one. The idea that missing out subsets is 'worse' than failing to produce uncountable sets is vindicated if we assume the characterisation of restrictiveness given. There are, however, numerous reactions one might have to this state of affairs. I do not take any stand on which is 'correct', but rather I am merely aiming to articulate the space of possibilities.

Reaction 1: Be strong. One response we might put forward on behalf of the uncountabilist begins by noting that in order for a theory T_1 to provide a fair interpretation of another T_2, T_1 must have at least as high consistency strength as T_2. This is because if one assumes $Con(\mathsf{T}_1)$, one can always obtain a model of T_2 by restricting to the fair interpretation available in any model of T_1. A way of dodging the restrictiveness results for the natural theories we considered (i.e. the ones based on determinacy and the OIMH^-_{Ref}) then would simply be to move to an uncountabilist theory that has stronger consistency strength. For instance ZFC + "There is a supercompact cardinal" more than suffices.

Response. Let's first remark that even for the *weakest* results we've considered here, we already have $0^\sharp$, and $0^\sharp$ suffices to yield a fair interpretation of ZFC plus many large cardinals (in L). In fact, $0^\sharp$ is often seen as beyond the limit of 'intrinsic' justification for uncountabilist theories.[17] Justificatory subtleties aside, I fail to see what the uncountabilist could appeal to here that the countabilist couldn't also make use of. For, all that the countabilist needs to justify to gain strong maximisation (and hence restrictiveness) is the claim that the uncountabilist's

[17]See here [Koellner, 2009]. See also [Roberts, 2017] and [Welch and Horsten, 2016] for dissenting voices.

theory is true in some definable inner model. But this seems to be a weaker requirement that justifying the *truth* of the theory, whatever base one is working with (ZFC vs. ZFC^-_{Ref} has very little role to play in this question). So, if ϕ is a large cardinal axiom, it seems like if the uncountabilist can justify acceptance of ZFC + ϕ then the countabilist can easily justify the acceptance of ZFC^-_{Ref} + Count + "There is a (definable) inner model of ZFC + ϕ". Perhaps there are responses here in terms of something like 'the absolute' (e.g. justification via reflection principles), but any such argument will have to make it clear why the countabilist cannot simply 'piggy back' off this justification to obtain consistency in a definable inner model.

Reaction 2: Reject the modification to Maddy restrictiveness. A different response would be to object to the modification of Maddy-restrictiveness I made. A rough intuition behind Maddy's characterisation of restrictiveness is to look at cases where one theory can interpret another in a 'nice' model. One might think, however, that a shift in base theory can precipitate a shift in the class of models that are considered especially 'nice'. This is particularly salient when moving from ZFC to ZFC^-_{Ref}. The natural models of the former theory are normally understood to be of the form V_κ for κ inaccessible,[18] whereas natural models of the latter are usually understood to be $H(\kappa)$—the hereditarily κ-sized sets. This might affect what one considers to be a 'fair interpretation'. If, for example, we allow inner models of $H(\kappa)$ to be fair interpretations, then we will lose strong maximisation since ZFC^-_{Ref} + Count + "There is a definable inner model for ϕ" can be true in models like $H(\omega_1)$, and hence it is not true that there is no extension of ZFC and its cognates that has a fair interpretation for our countabilist theory.

Response. This, I think, is in one sense a strong objection. Once we have adopted ZFC^-_{Ref} as the base theory, the whole notion of what should be a fair interpretation has plausibly shifted from the ZFC context. Really, discussion of Maddy-style restrictiveness should incorporate consideration of a wide variety of possible 'nice' interpretations, and what comes out as restrictive dependent upon the class of interpretations allowed.[19]

However there's a sense in which the uncountabilist isn't *vindicated* here. Recall that we started our discussion with the **Restrictiveness**

[18]Although Maddy insists that κ be inaccessible, it is an interesting question whether it need be. One could consider worldly cardinals (i.e. where $V_\kappa \models$ ZFC) for example.

[19]Indeed this is already so in the ZFC-context. See, for example, [Incurvati and Löwe, 2016].

Intuition that the uncountabilist straightforwardly postulates the existence of more sets. Given ZFC + ϕ, a move to a modified version of restrictiveness that allows for a fair interpretation of ZFC^-_{Ref} + Count can only every achieve *parity* with ZFC^-_{Ref} + Count + "There is a definable inner model for ZFC + ϕ", in the sense that neither will strongly maximise over the other because of the existence of possible fair interpretations going each way in extensions of the relevant theories.[20] So, even if we accept that it's not restrictive to say that there's uncountable sets, the **Restrictiveness Intuition** is still misguided since it's not *restrictive* to say that there are only countable ones.

Reaction 3: Reject Maddy-style restrictiveness altogether. If one wishes to obtain a vindication of uncountabilism over countabilism, it's thus preferable to come up with a notion of restrictiveness that *does* legislate in favour of the uncountabilist. It should be noted that there are such notions out there. [Meadows, F], for example, considers a category-theoretic analysis of restrictiveness based on a notion of *retraction* (this builds on work by Visser in [Visser, 2004]). Assessing the details would take us too far afield, however we should note that given this notion of restrictiveness ZFC^- + Count is restrictive relative to ZFC^-, where ZFC is not.[21] So there is at least one notion of restrictiveness on which it is the countabilist, and not the uncountabilist, who makes restrictive claims.

Response. This response strikes me as the most attractive for the uncountabilist, and might be a good place to push. However, I also think that it's unlikely that just *one* account of restrictiveness provides the whole story here. Instead, I think of different formal accounts of restrictiveness—including the ones put forward by Maddy, Meadows, and myself—as making different intuitions about restrictiveness formally precise. So, as a formalisation of the intuition behind the countabilist's position, our account performs well, even if there are other accounts of restrictiveness based on different intuitions that pull in other directions. Moreover, our analysis shows that interpretations based on 'niceness of interpretation' are unlikely to tell against the countabilist since inner models are some of the 'nicest' interpretations we have. We thus see that the **Restrictiveness Intuition** cannot just be naively held, since there are at least *reasonable* positions on which countabilism is not restrictive. If one wants to support the **Restrictiveness Intuition**, there is pressure to make precise the relevant sense(s) of restrictiveness

[20] Of course one could muscle a restrictiveness result by prohibiting inner models as fair interpretations, but this is a sufficiently unattractive prospect as to not merit serious consideration.

[21] See Proposition 6 of [Meadows, F].

on which it comes out as vindicated.

Reaction 4: Accept countabilism. The final reaction we shall consider is to simply accept countabilism.

Response. Whilst I personally find this somewhat attractive, it should be noted that this is not a 'magic bullet' that closes the issue. Apart from the fact, noted above, that there are other notions of restrictiveness that legislate in the opposite direction, it should be noted that countabilism comes with substantial costs. There is a reason that ZFC has become the 'standard' set theory: It is theoretically simple, provides us with all the usual mathematical objects, and can be given a robust underlying conception (in the form of the iterative conception of set). Whilst ZFC^- + Count and its extensions are perfectly theoretically simple, its adoption results in a substantially different picture of mathematics; in particular the reals make up a proper class and it is unclear how one should think of objects of third-order arithmetic. Moreover, the iterative conception of set as normally articulated depends on power set, and thus is not automatically available in ZFC^-.[22] The point to be emphasised is simply that restrictiveness is just one theoretical virtue among many. A proper decision between countabilism and uncountabilism (or an explanation of why no such decision is desirable) should consider restrictiveness as part of a wider examination and weighing of various theoretical virtues. This suggests that analysing possible underlying conceptions of countabilism and the lie of mathematical landscape when it is adopted are important tasks for philosopher examining the foundations of mathematics.

5 Conclusions

In this article, we've seen that there are perspectives on which countabilism is very far from restrictive, and indeed can have an interpretive advantage over uncountabilism. Of course the issues are subtle, and the arguments here are not meant to be conclusive. The point is simply that countabilism is not a mathematically sterile and restrictive position, but rather admits of pleasant theoretical features.

In the end, we will have to ask: *What do we want/expect from a (set-theoretic) foundation?* Do we really *need* a hierarchy of uncountable sets? Or is it enough to find 'good' interpretations of our mathematical theories and results?

[22] For some discussion of how we might incorporate mathematics and the iterative conception under countabilism, see [Scambler, 2021] and [Barton and Friedman, S].

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