Language, Models, and Reality: Weak existence and a threefold correspondence

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Abstract

How does our language relate to reality? This is a question that is especially pertinent in set theory, where we seem to talk of large infinite entities. Based on an analogy with the use of models in the natural sciences, we argue for a threefold correspondence between our language, models, and reality. We argue that so conceived, the existence of models can be underwritten by a *weak* notion of existence, where weak existence is to be understood as existing in virtue of *language*.

Introduction

This paper addresses issues concerning our relationship between *language*, *models*, and *reality*. Much of the philosophy of language, logic, and metaphysics attempts to understand the relationship between the linguistic devices we employ and the underlying ontology. This is often done by providing a *semantics* that accounts for the *truth* of our claims about the world—we explain the truth of various claims by explaining how the language we use is interpreted.

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Under one account, we just bluntly point to reality: What do the truth and falsity of our claims depend on? On the way the world is of course! A problem with this view is that often we make claims about reality that seem to capture some aspect of it, but are not *obviously true*. Some claims of theoretical physics, for example, do not clearly map on to *exactly* the way the world is, requiring idealisation. So, how should we account for this discrepancy?

An attractive response to this problem is to understand claims in the context of *models*. A claim is true in some class of models, and these models can correspond better or worse to the way the world actually is. So the realist about the nature of physical reality can consider the class of models for some physical theory, and can then try to come up with an account of ″closeness to truth″ or ″verisimilitude″ of particular models. Of course, saying what this ″verisimilitude″ amounts to is a difficult philosophical problem, but the shape of the account is at least plausible.

This problem and the proposed response are especially interesting in the case of set theory. There, the structure of the underlying ontology for set theory is especially controversial. This is particularly so in light of the independence phenomenon—there are many sentences of set theory that are neither provable nor refutable from the axioms, with no agreed upon solution. When assessing a set-theoretic claim for its ″truth″, it is thus hard to know whether and how it conforms to set-theoretic reality.

However the modelling perspective is available here too. On this view (which we'll discuss in §1) we are able to view *models* of set theory as providing better or worse pictures of the way the sets might be. So perhaps there is a similar kind of view: Even if our set-theoretic discourse doesn't perfectly match up with set-theoretic reality, it is non-vacuous and informative for understanding the sets. In slogan form: *We can still learn about the sets by studying models of set theory.*

One problem with this position is that models of set theory are often taken just *to be* particular kinds of set. There seems then to be a kind of circularity in the above picture (see §2). For, we want to use the models to talk about the sets, but the models themselves *just are* kinds of set. Our intermediary—the models—seems to depend on the very thing we're trying to understand—the sets.

We think a response can be obtained by employing a notion of *weak existence* (see §4). Call the notion in which entities exist in fundamental reality (whatever it may be) *strong* existence. *Weak existence* is the sense in which something exists merely in virtue of the language we employ. This is a notion with a rich history and an increasing body of work studying it (see §3). We will argue that weak existence can be leveraged to obtain the existence of models for our set-theoretic claims. This, in turn, provides an account of enough models of set theory to do the modelling work and, we think, stands free of set theory itself. Putting all this together achieves our:

Main Aim. Propose an account on which the relationship between set-theoretic claims and reality is understood as a *threefold* correspondence between (1.) the language we use, (2.) models for our theories of sets, and (3.) the world itself.

We will consider some objections in §5. We end the paper in §6 with a concluding summary, and suggest a few applications of this approach to the philosophy of set theory for future development.

1 An analogy: The use of models in science and set theory

As we mentioned above, we will use models to explain the relationship between our language and the world. In order to do so, we therefore need to understand the role of the models in set-theoretic investigation. While the importance of a model-theoretic approach in set theory needs little or any justification (see [Antos, F] for a thorough substantiation of this point), it will be useful for us to be clear on our theoretical and methodological background.

The starting point of our analysis is searching for cohesion between the natural and formal sciences. More concretely, we want to consider set theory via analogy with other scientific investigations. Of course any particular field has its peculiar tools, tricks, and rules of thumb, but if we step back enough to blur these specific features we can notice a substantial and fruitful unity within the different sciences. Important for us will be the notions of *model* and *modelling*. Let us now take a look at some general traits of models, outlining the aspects that will be of relevance.

Reality is complex and there is no better representation of it beyond reality itself. This is why science needs models: simplified versions of reality that can be more easily studied. We will argue that by assuming a similar theoretical background, we can more easily understand important ideas and debates in the philosophy of set theory.

Let us first introduce the main reason that pushed the set-theoretic community to focus on the study of models, namely the widespread presence of independence.

Independence. There are many statements in the language of set theory that are neither provable nor refutable from our "standard" axioms ZFC.

Examples of sentences independent from ZFC include variants of the Continuum Hypothesis (e.g. GCH, CH), axioms of definable determinacy (e.g. PD, $\mathsf{AD}^{L(\mathbb{R})}$), and large cardinal axioms (e.g. "There is an inaccessible cardinal", "There is a proper class of measurable cardinals").

The plethora of models that set-theorists produced in showing the independence of the above sentences has motivated something of a model-theoretic turn in set theory over the last century. The study of models was already percolating beneath the surface in the 1920s and 1930s (for example in Zermelo's work on models of second-order set theory in [Zermelo, 1930]), but it was after the discovery of the independence of CH via [Gödel, 1940] and [Cohen, 1963] that the contemporary turn to models really took off. As Hamkins writes:

> Set theorists build models to order. As a result, the fundamental objects of study in set theory have become the models of set theory, and set theorists move with agility from one model to another. [Hamkins, 2012, p. 418]

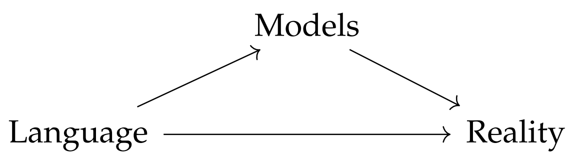
The abundance of the models set theorists have obtained was made possible by the sharpening of model-theoretic techniques like forcing and inner model constructions.[1] Although mathematically fruitful, more and more independent sentences were discovered in a wide variety of different models, and the prospects for discovering truths about the universe of set theory became less and less clear. As a matter of fact, different models of ZFC were offering different mutually incompatible pictures of the way set-theoretic reality might be. One might ask: Which should we regard as (more) faithful descriptions?[2] How to reconcile the different pictures provided by the different models? What sentences are in fact true? What is the relationship between the models and set-theoretic reality? Do they really help us to describe it?

[1] Of course these are not the only techniques available, but the most well-known. Moreover, there is a rich body of work examining independence in theories weaker than ZFC. For example, the study of the independence of the axioms of ZFC from one another has produced interesting results, such as (i) within ZFC without Powerset Axiom there can be no uncountable cardinals, and (ii) neither Replacement nor Foundation are provable from the version of ZFC with these axioms removed). And this phenomenon is very general, it is instantiated everywhere from the highest reaches of large cardinal theory down to weak systems of arithmetic.

[2] See [Barton, 2020b] and [Antos et al., 2021] for some discussion of this question.

The point that we would like to stress here is that these are deep and important questions both for mathematicians and philosophers. The attempt to answer them, and the consequent way of phrasing the problem of independence in terms of the study of the models of set theory, is a theoretical move that should not be underestimated. And here we find a resemblance with the question of the relationship between models and reality that we find in the natural sciences. Far from being the only way to conceptualise this model-theoretic shift in set theory, this analogy suggests that we use the models of set theory to study the sets themselves.

On the basis of this intuition we would like to sketch a threefold relation between language, models, and reality and to suggest that this can help clarifying theoretical and methodological questions related to the study of set theory. The relation we want to describe can be depicted as follows.



Language can be used to describe reality directly, in term of sentences expressing some true or false propositions. This function of language is purely descriptive. However, if we utter or believe some false sentence(s), it is not the case that we simply *fail* to talk about reality (or can only be interpreted in some restricted interpretation). Rather we can view our theorising as *partially* corresponding to reality. By viewing models as intermediaries, we can lay the groundwork for such a proposal—we talk about models which can conform better or worse to the way the world is.

A further advantage of using models as an intermediary consists in allowing us to test our hypotheses about reality not in isolation, but in connection with other hypotheses and ideas. Many models (and model-theoretic constructions) provide us with ”information-rich” contexts in which we can analyse particular principles (on the assumption that the principles themselves are consistent). For example, if one can construct an ”L-like” model of ZFC + ϕ for some suitable ϕ, then one knows that many statements of set theory are consistent with ϕ (e.g. as done with GCH and large cardinal axioms). In a similar vein, on the assumption that ZFC + ϕ has a model, we can force over the model to obtain many other interrelations of set-theoretic principles (possibly not including ϕ, depending on the nature of the forcing). So a model does not just give us *one* picture of the way the world might be, but can allow us to generate multiple such pictures, each of which can be viewed as a possible candidate for representing reality more or less faithfully.[3]

In this sense models provide a medium that allows language to meaningfully correspond to reality beyond simple truth or falsity. The gain of this intermediate step is given by the structured way in which we can depict reality. As any model in natural sciences, also models of set theory capture some aspects of reality, while neglecting others. For example, any countable model clearly misses many reals and, because of its cardinality, offers a very pale image of the full richness of the sets. Although countable models do not provide fully faithful pictures, they nonetheless offer interesting insights on how the sets could or could not be.

Similarly, many entities of interest to the scientist—infinitely deep oceans, frictionless planes, miniature wind tunnel prototypes and so forth—are clearly not perfect representations of reality. Others, perhaps, get closer (e.g. a suitable candidate mathematical model for the structure of the universe under general relativity). All, however, are useful given particular purposes—a model for general relativity isn't going to help much with trying to predict the behaviour of ocean waves for generating tidal power, the former is just too mathematically complex to provide useful predictions in that context. In the end, though, we would like to view each as a description of an underlying subject matter and we would like some cohesion between the different models of reality.[4] And sometimes a slight dissimilarity from reality is the price we pay for the tractability of the relevant model.

Analogically, some models of set theory are more tractable than others. For example we can have models that are pointwise-definable (like the Shepherdson-Cohen minimal model—the smallest transitive model of ZFC under inclusion). Such models plausibly don't capture the full richness of set-theoretic reality, which we normally take to include non-definable objects (e.g. some real numbers) and many satisfy false statements (e.g. the Shepherdson-Cohen model satisfies $V = L$, which is usually taken to be false). Others might get closer, but might be more intractable, such as when we (schematically) assert that there is a V_δ elementary in the universe. One main goal of set-theorists and

[3]Some similar ideas are given in [Arrigoni and Friedman, 2013], but with the target of studying maximality principles rather than as a way of talking about the world.

[4]Some scholars don't think full cohesion is desirable, for example Nancy Cartwright's 'patchwork' account of the philosophy of science (cf. [Cartwright, 1999]).

philosophers is to produce a theory of how these models are together able to provide a coherent description of reality, despite their varying degrees of tractability and disagreements on many statements of set theory.

Of course, the nature of this modelling relation is responsible for some of the hardest problems in the philosophy of science. What does this *simplification* amount to? What is the *relation* between reality/phenomena and models thereof? How *faithful* are models? What conclusions can we draw from the study of *different* models? There is an enormous literature on the topic (see [Frigg and Hartmann, 2020]). On the one hand, these represent a cluster of challenges for the approach we outline here. On the other, it suggests that there are a wealth of resources that might be applied to the set-theoretic context.

Now is not the time to fully articulate the nature of the modelling relation between models and reality. We hope to have convinced the reader that there is a fruitful project to be examined here, and we leave the details to future work (we will consider some suggestions in §6). Whilst the approach is not yet fully developed, one can see how studying models in order to understand reality can be attractive. By the independence phenomenon, we know that ZFC fails to get much traction on set-theoretic reality. By studying models, we can hope to understand better the ways the world might be. As we'll now see, there's a powerful objection that threatens to scupper this project before it has had chance to begin its voyage. The rest of the paper will be devoted to outlining this problem and providing a response, opening the door to further projects in this direction.

2 The Circularity Problem

From a philosophical perspective, we face an immediate challenge before even embarking on such a programme. This is because we often take the models of set theory to be *themselves* sets. As Koellner puts it (concerning Hamkins' claim that the fundamental objects of study in set theory are models):

> To this one is bound to protest that the fundamental objects are still what they always were, namely, *sets*. Some of those sets happen to be models of certain theories. [Koellner, 2013, p. 12]

If we follow Koellner in thinking that models just are particular kinds of set, we can generate a problem for the person who wants to use models of set theory in providing pictures of the way sets might be: We are using the very things we are meant to be clarifying in providing said clarity. Call this the **Circularity Problem**.

Let's develop this in a little more detail. To this end, it's helpful to consider an analogy. One purpose of considering models in science (rather than reality itself) is to find a way of representing the world that can be agreed on by all parties. A realist about unobservable entities and a constructive empiricist can agree on the intrinsic facts about the models employed. What they disagree about is how those models and reality interrelate. But this shared point of contact allows both to consider the nature of the modelling relation.

If we have a simple identification of models with certain sets, the same does not apply. There *the very nature of the models* is dependent upon the nature of the sets. This has the unwelcome feature that we do not have the same kind of ontological neutrality as in the sciences. For a simple example, consider a realist and a nominalist about sets (and let us assume that neither has any complaints about set theory as a mathematical area of study, even if they disagree sharply on the ontology). It would be preferable if, ceteris paribus, the nominalist and realist could have some models as a point of contact. But they can't if we identify models with sets—the nominalist thinks that such things don't exist, so the question of how models and reality interrelate is moot. This problem is very general; any time there is significant disagreement on the nature of the underlying ontology (as there so often is in set theory) we run the risk of losing a shared grip on the nature of models. A slightly more complicated example involves contrasting the set-theoretic universist (who believes that there is an all-encompassing universe of sets) and a multiversist (who believes there is no such universe). How do theorists with such radically different views on the nature of sets come to agreement on the models? How much can be shared between the two views? (We shall provide a tentative proposal in §6.2.)

This metaphysical problem comes attached with an epistemological one. Even supposing that we're happy being set-theoretic realists, we may still worry about our epistemological access to the sets. They are, all things considered, pretty abstract and complicated objects. But now we have identified models with sets, it is unclear if there is any epistemological gain to be had. Contrast the case of models of relativistic theoretical physics. There we substitute one intractable object (the physical universe) with a much more tractable one (a particular kind of manifold). But given the identification of models with sets, we have substituted the study of one class of objects with the study objects of the very same kind, and so any epistemological problems of access with the former extend to the latter.

Our response will use the notion of *weak existence*—existence in virtue of language—to ground enough models to do the work we want. Models that exist in virtue of weak existence need not be ontologically dependent on set theory, and there are reasons to be hopeful that we have better epistemological access to them. We devote the next section to the task of showing that a close relation between language and existence is not new. On the contrary, weak existence is part of a rich philosophical tradition.

3 Existing accounts of weak existence

In order to get a better handle on weak existence, we will start by considering some ideas as they appear in philosophy more broadly. Because of the many and heterogeneous ideas involved, it is hard to provide a comprehensive survey. However, we want to show that the possibility of a *direct* link between language and existence is an idea that has been advanced in many different contexts. Existence *merely in virtue of language* is what makes the resulting notion of existence "weak" rather than "strong", the latter is a notion concerning true or false propositions about existence in the *world as a whole*, weak existence is *merely linguistic*. This has substantial implications; language is a human tool, while existence, in a strong sense, is often understood as being independent from our cognitive, linguistic, or mental abilities.

We will later see how this plays out in the specific case of set theory. For now, in order to delimit the field of investigation, we would like to review positions of mathematicians and philosophers that can be seen as implying such a notion of weak existence. We will discuss a cluster of accounts that, on closer inspection, are quite philosophically and chronologically disparate. If anything, this will show that this notion of weak existence is compatible with many different views concerning strong existence, and indeed provides an undercurrent of much philosophical thought. This neutrality will be an important feature of our account of the relation between language and reality.

3.1 Consistency implies existence

An ideal starting point is Hilbert's view on existence. As is well known from the Frege-Hilbert controversy, Hilbert held the position that consistency implies existence.

> I found it very interesting to read this very sentence in your letter, for as long as I have been thinking, writing and lecturing on these things, I have been saying the exact reverse: if the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist. [Gabriel et al., 1980, p. 39]

Because of the syntactic character of the notion of consistency, we can safely consider his notion of existence as directly linked to language. Notice that, although Hilbert is often described as the champion of formalism, this direct connection between consistency and existence is compatible with very different views about mathematical objects. Indeed, consistency could be seen as constitutive of existence (as a formalist/nominalist reading would suggest) or as indicative of existence (as Cantor claimed, considering the perfect harmony between the immanent and the transcendent realities described in [Cantor, 1883])[5].

Another important aspect of this Hilbertian notion of existence, besides its linguistic root, is its semantic realisation. Indeed, the independence results that Hilbert proved in the *Grundlagen der Geometrie* rest on the possibility of constructing different models for (non-Euclidean) geometries from one another. In this way the existence provided by consistency is realised in a model-theoretic way and, therefore, this notion of weak-existence is assimilated to that of existence in a model.[6]

3.2 Abstraction principles

Another interesting case of a direct connection between language and weak existence is given by the use of abstraction principles, or abstraction more generally in the foundations of mathematics.

We can (of course) find very different views on abstraction that are compatible with different view on the ontology of mathematics. For example, Dedekind held that abstraction is able to create new mathematical objects. According to his view natural numbers are free creation of the human mind, while reals are abstracted from Dedekind's cuts. This creative use of abstraction found the fierce opposition of

[5]A similar position has been proposed in modern times, with the name of full-blooded platonism, in [Balaguer, 1998], as arguably the most tenable form of mathematical platonism.

[6]Indeed, some people have argued that this is notion of existence in mathematics that should be attributed to Hilbert. See, for example, [Doherty, 2017].

Frege, who could not accept any psychological interference in the platonic realm of thought. Nonetheless, Frege viewed the practice of definition in mathematics as still yielding something new, namely the attribution of sense and reference to a new symbol through an act of declaration. The important difference with respect to other accounts of definition, however, lay in the possibility to constrain this form of creativity with clear criteria. In 1903, in the *Grundgesetze der Arithmetik*, Frege wrote:

> Can our procedure be called a creation? The discussion of this question can easily degenerate into a quarrel about words. In any case, our creation, if one wishes so to call it, is not unconstrained and arbitrary, but rather the way of proceeding, and its permissibility, is settled once and for all.[Ebert and Rossberg, 2013, §147][7]

Whilst Frege did not have a theory/meta-theory distinction, the forms of neo-logicism that originated from the failure of Frege's foundational project have managed to show that abstraction principles weaker than Basic Law V have models witnessing their consistency [Heck, 2011].

In this tradition we find also positions like that of Øystein Linnebo on thin objects. Linnebo, inspired by Frege, proposes a realist view about mathematics in which our linguistic practices contribute directly to the semantics of our mathematical sentences.

> In short, the assertibility conditions make a twofold contribution: to the determination of the semantic content of beliefs (and thus also to the truth of these beliefs), and to the formation of the beliefs. [Linnebo, 2018, p. 201]

The notion of mathematical object that emerges from this picture is that of thin objects, on which an object is the possible referent of a singular term. In this context, it is by reflecting on mathematical language and the various assertibility conditions that govern it that we get to know that mathematical objects exist. The epistemological twist of Linnebo's account makes it possible for him to hold a realist view of mathematical ontology. However, the semantic properties that emerge from our linguistic practices are of philosophical interest in themselves; they are used to back up existential claims.

[7]See [Ebert and Rossberg, 2019] for a discussion and an interpretation of these lines in the context of Frege's Platonism.

3.3 Internal questions and easy arguments

Another influential position that connected (weak) existential claims and linguistic practice(s) is that of Carnap. Specifically, the distinction between internal and external existential claims.

> If someone wishes to speak in his language about a new kind of entities, he has to introduce a system of new ways of speaking, subject to new rules; we should call this procedure the construction of a linguistic *framework* for the new entities in question. And now we must distinguish two kinds of questions of existence: first, questions of the existence of certain entities of the new kind *within the framework*; we call them *internal questions*; and second, questions concerning the existence or reality *of the system of entities as a whole*, called, *external questions*. [Carnap, 1950, pp. 21–22, original emphasis]

The standard reading of this distinction takes the internal question as motivating a notion of existence that is weaker than that investigated by metaphysicians (interested in external questions) and that can be simply answered on the base of the linguistic resources of the framework from which it emerges. Therefore, following Carnap's anti-metaphysical stance, the syntactic criteria provided by theoremhood offer clear criteria for a satisfiable answer to internal mathematical questions. For example, the question on whether there is a prime number between a n and $2n$ (for $n > 1$) is completely answered by the proof that this is indeed the case. On the contrary, whether natural numbers exist in general is an external question that arguably does not possess a clear (and relevant) answer. The weak existence provided by internal mathematical questions is, in Carnap's view, an anti-metaphysical perspective that collapses linguistic interpretation for singular terms and existence (in a weak sense).

More recently, a similar position has been developed by Amie Thomasson. In [Thomasson, 2014] she argues in favour of a deflationist notion of existence that, although compatible with realism, is rooted on our linguistic practices. The lightness of this new form of realism is attributed, by Thomasson, to the easy form that the corresponding ontological arguments assume. For example, if someone asserts that she ate two bagels, she can automatically commit to the existence of a natural number: the number two. There are affinities here with Carnap's ideas; to commit to some entity existing is just to commit to a linguistic framework in which such a commitment can be made. Space does not permit a full analysis of Thomasson's position, but one can see how there is an articulation of a close connection between language and its semantics; a connection able to account for a weak form of existence.

So, we see that weak existence is an idea with a rich history in philosophy. And as the reader may be able to guess, the idea that linguistic practice can underwrite existential claims may be useful in obtaining models for set theory, independent of set theory itself. It is our contention that we can do so, dissolving the **Circularity Problem**.

4 Underwriting models with weak existence

Let's start by recalling the **Circularity Problem**. If we intend to study set-theoretic reality in terms of the models of set theory (partial descriptions of it), how are we to avoid the circularity brought about by the fact that models are (allegedly) just sets? In other terms, is there a way to account for models of set theory that does not view them as sets and that does not use features of mathematical objects that would be hard to justify outside set theory?

We will argue that one can do so by casting the existence of some models in a linguistic light. If our strategy is successful it will provide a linguistic approach to models that does not depend on set-theoretically loaded assumptions regarding the nature of set-theoretic reality. Notice that even if our strategy succeeds, we do not have to argue that models are *not* sets. What models "are" is, we think, ultimately unimportant. Our reasons for thinking so are roughly Benacerrafian; as far as models go it doesn't really matter what they're composed of, so long as one can think meaningfully of objects together satisfying certain theories, we are in the clear. And clearly, if we *assume* set theory, we *can* encode models of set theory using sets. What we *do* want to do is identify a class of models that are sufficient for the needs of set theory and also obtainable via methods compatible with weak existence. In general, you can still view models as sets *if you like*, there is just no *obligation* to do so.

We will argue that models can be represented in linguistic terms and this allows us to separate enough models ontologically from the sets in order to answer the **Circularity Problem**.[8] The key observation

[8]One proposal that is parallel to our own, but we will only consider parenthetically, is Carolin Antos' recent idea that models should be regarded as fundamental entities in their own right (cf. [Antos, F]). Drawing on Penelope Maddy's naturalism/second philosophical approach (see [Maddy, 1997] and [Maddy, 2007]) Antos argues that viewing models of set theory as fundamental (in addition to sets) provides an "effective means to particular desirable ends" (Maddy, 1997, p. 194) and on

is that models of set theory can *themselves* be conceived of as composed of syntax. This idea (that models can be thought of as composed of pieces of language) will be familiar to anyone who has studied proofs of the Completeness Theorem. We revisit some core moves made there to motivate this claim.

How does one prove the Completeness Theorem? As is well known, its standard formulation (i.e. every valid sentence has a proof) admits of an equivalent formulation that every consistent set of sentences is satisfiable. The textbook proof of this latter form proceeds by forming a *term model*. Given a consistent set of sentences Γ, we first add infinitely many constants $c_0, ..., c_n, ...$ to the language. Next, we carefully extend Γ to a maximal infinite consistent set Γ^* (in which quantified sentences are linked to constants via conditionals). We can then build a model for Γ by letting the domain of the model be the constant symbols and the predicates be interpreted via the atomic sentences in Γ^*.[9]

Notice that we can think of this proof as specifying a model via weak existence—the model we end up with is composed of *syntactic objects*. This yields a way of obtaining models of set theory. For, suppose we are considering some set Γ of sentences of ZFC. Then, so long as Γ is consistent, we can think of a model of Γ as composed of constants.[10]

The above strategy is sufficient to get us *a* model for any particular

these grounds their fundamentality should be accepted. Antos' arguments are very helpful given our predicament, since they provide practice-based reasons to holds that models are sui generis entities. This is a possible response to the **Circularity Problem**, but our project is different. We want to show how (some) models can be postulated to exist in the light of weak existence. The advantage of this is that it is plausibly less ontologically demanding than the Antos picture, and requires no recourse to Second Philosophy. Moreover, we claim that there is epistemic gain in viewing models as providing us with weak existence in set theory, since the models are explicitly constructed (rather their existence simply abductively inferred). But our proposal and Antos' need not be viewed as *competing* (indeed we view them as *complementary*).

[9]Things are a scooch more complicated if one has terms in the language. Since we're just working in the language of set theory (that has a single non-logical predicate symbol '$\in$') we don't have to worry about these subtleties, but since we think weak existence still suffices for other contexts too, it's worth at least providing a footnote on the matter. Where terms are involved, one has to instead think of the domain of the final term model as composed of sets of closed terms that are provably equal given Γ. As far as weak existence goes, the move from constants to these sets of terms strikes us as innocent, though we need not take a stand on the matter for the purposes of this paper.

[10]The eagle-eyed and fox-cunning reader may ask themselves at this point whether the fragment of the Completeness Theorem needed has reverse mathematical strength. We will discuss this later (§5).

set theory we take to be consistent, but does it get us *enough* models for set theory to underwrite the *study of models of set theory* as it actually occurs in practice? There it is not just important that *there is* a model (for some theory T), but that the model bears the appropriate relationship to other models (e.g. that given a model M, we can move to the forcing extension $M[G]$ etc.). Can we get models bearing the appropriate relationships in this manner?

Our response will be two-pronged. First we'll note a cheap kind of response—by just increasing the strength of the theory we take to be consistent in the beginning, we can obtain weak-existence models that are sufficient to do any particular construction we like. However, we'll then note a less cheap response, pointing out that the most important model-theoretic constructions in set theory—forcing and inner models—can be interpreted in weak existence terms.

Starting with the 'cheap' strategy, it bears mentioning that nothing in our construction of a weak existence model (e.g. as in the Completeness Theorem) depended upon the use of any particular theory or other. So long as we take a strong enough theory T that suffices to produce the models we need for some construction or other, we can turn Koellner's observation on its head. Since *can* do the model theory of sets within set theory, a model of T obtained by weak existence will suffice for all model theory that can be done in T. We just live in this model of T and do all the required model theory purely set-theoretically there. To illustrate this, suppose I consider the model theory of ultrapowers using a measurable cardinal κ. Instead of getting a weak existence model for ZFC, I could simply argue (howsoever I choose to—now is not the time to adjudicate such matters) that I have good reason to believe that ZFC + "There is a measurable cardinal" is consistent. Let this be our theory T. By building a weak existence model for T (as above), I obtain a model M that satisfies ZFC and contains (what M takes to be) a measurable cardinal. I then just run the usual model-theoretic constructions in M. And this method is very general, applying to many model-theoretic constructions (e.g. inner models and forcing) that can be done within a particular consistent extension of ZFC.

Whilst this strategy is fine so far as it goes, we think that much stronger things can be said about weak existence in relation to many of the model building constructions present in set theory. In this way, the above response masks some important features of set-theory in relation to weak existence that are more apparent if we approach the problem directly.

What we want to point out is that there's a kind of weak existence *relative* to a starting model that can be leveraged in favour of obtaining models for forcing extensions. Let's suppose we have some model or other M (which could, for example, be obtained via weak existence). We want to force over M. How does this proceed? One way of doing so is to start with a particular kind of Boolean-algebra $\mathbb{B}$. We then (within M) define the model $V^{\mathbb{B}}$ as follows:

(i) $V_0^{\mathbb{B}} = \emptyset$

(ii) $V_{\alpha+1}^{\mathbb{B}} = \{f | \text{"}f \text{ is a function"} \wedge dom(f) \subseteq V_\alpha^{\mathbb{B}} \wedge ran(f) \subseteq B\}$ (for successor ordinal $\alpha + 1$).

(iii) $V_\lambda^{\mathbb{B}} = \bigcup_{\beta<\lambda} V_\beta^{\mathbb{B}}$ (for limit ordinal λ)

Another (equivalent) method is to start with a partial order $\mathbb{P}$ in M, and consider the class of $\mathbb{P}$-names—these are functions whose elements' first coordinate is another $\mathbb{P}$-name and whose second coordinate is some condition $p \in \mathbb{P}$ (the whole thing gets off the ground because $\emptyset$ is trivially a $\mathbb{P}$-name). Then, when presented with a generic filter G for $\mathbb{P}$ and M, we can recursively evaluate the $\mathbb{P}$-names to obtain the forcing extension $M[G]$. This is done for some $\mathbb{P}$-name σ by including the values of a name τ such that $\langle \tau, q \rangle \in \sigma$ if $q \in G$.

The important point that we wish to note is that the construction of the forcing language, and assessment of what the forcing extension satisfies, can be viewed as a particular kind of weak existence construction in M. We introduce pieces of *language* via the $\mathbb{P}$-names, and within $M^{\mathbb{B}}$ the $\mathbb{B}$-names are all viewed as sets existing side by side (though $M^{\mathbb{B}}$ might view some of them as identical). In this sense, we have weak existence *relative* to M.

One salient objection here is that the forcing language is *enormous*, since it has a constant for every name, it will be proper-class-sized. In the present context though this should not deter us. The background in which we are considering these extensions is just a countable model M obtained weakly via the same idea as the Completeness Theorem. So, though 'proper-class-sized' relative to M, the forcing language we end up discussing is really just another countable language, if we can obtain every element of M as weakly existing, then so can we obtain every element of $V^{\mathbb{B}}$.

So, we can obtain weak existence models for a theory we take to be consistent, but we can also think of forcing as obtaining particular kinds of weak existence models given the specification of an initial background. We can thus obtain enough models to do the model theory of sets via weak existence (at least as far as forcing and inner models are concerned). We conjecture that many model-theoretic constructions (e.g. ultrapowers in general) can be conceived via weak existence methods, but we need not take a stand on this here.

Before we continue, we should pause to reflect on the exact strength of our claims. On the one hand, our claims are quite strong—we claim that models satisfying a wide class of set-theoretic constructions can be obtained via methods that are compatible with weak existence. However, in a different respect our claims are quite modest—we only claim that one *can* find such models in terms of methods that yield objects that weakly exist. We certainly do *not* claim that *every* model of set theory can be found by weak existence. But we get enough to give us 'pictures' of the way the world might be and respond to the **Circularity Problem**.

5 Objections

Thus far, we've sketched an account of how our utterances regarding the sets might relate to reality. We speak of *models* (which in turn can be viewed as linguistically constructed), and then these models may conform better or worse to the way the world is. We now discuss a few objections that our account might face.

Why not just use the natural numbers? First of all one might object that our appeal to weak existence is not necessary, since we could have just done everything using natural numbers. Indeed (so this objection runs) since we can always view the syntax of a countable recursive language as encoded within a suitable theory of arithmetic, then any talk about weak existence is (at the end of the day) just a way of talking about the natural numbers. And we know (by the Completeness Theorem) that we can build a model using natural numbers for any consistent countable theory.

Our response is epistemological in nature. Whilst (of course) we *could* view a consistent countable theory as modelled using natural numbers if we so desired, having a weak existence model is more epistemologically direct. In particular the process of forming a term-model is one of composing a model via syntax. Of course this is all *countable* and so can be *encoded* by natural numbers. But why go through this epistemic detour? By noting that languages and names for objects are employed as soon as one is discussing any subject matter, and that these can be mobilised in the service of generating models, we shorten the gap between our language and the representations we employ.

Moreover our view is able to account for the practice of several model-theoretic constructions—such as the forcing constructions—in syntactic terms. We thus get a similarity between the construction *of* a model, and the constructions done *over* a model. Whilst the natural number interpretation is fine insofar as it goes (and indeed, given our purposes here—the more models of different kinds the better!) it blurs epistemically relevant and interesting features.

Too many constants? A different line of objection is to criticise our approach by accusing it of smuggling in infinitary resources beyond those licensed by the epistemologically safe environment provided by language. One might complain that in order to build the model, we had to add *infinitely* many constants to our language. On these grounds, one might complain that this stretches the use of "existence in virtue of the language employed" beyond sensible bounds.

We can provide a reasonably quick answer here that helps clarify our position. We do not claim that we obtain this language by ordinary usage as it occurs in day-to-day life. However there *is* a kind of existence appealed to across philosophy that allows a degree of idealised linguistic resources to underwrite existence claims (this was the point of surveying various accounts in §3). And it is this idealised notion, we argue, that can deliver the models required and has benefits beyond regarding models as only obtained within set theory.

Did we smuggle in infinitary resources? There is a more subtle objection lurking here, however. If it could be further shown that there is an implicit use of some infinitary assumption in constructing our models, one might undermine our claim that we have divorced the existence of models of set theory from set theory itself.

We can go further by observing that from the Completeness Theorem and the Soundness Theorem, one can prove the Compactness Theorem. And in turn, we can note that the Compactness Theorem is equivalent (modulo ZF) to various weak but non-trivial Choice-like principles; e.g.: Tychonoff's Theorem for Hausdorff spaces, the Boolean Prime Ideal Theorem, and the principle according to which every filter on a Boolean algebra can be extended to an ultrafilter.[11] One might press the point that we have used a non-negligible amount of the Axiom of Choice in supporting the existence of weak existence models, and thereby undermine our own argument.

We can respond to this by noting that for our purposes (getting models for ZFC with some consistent set of sentences of set theory added) we only need the Completeness Theorem for *countable* languages. This version of the Completeness Theorem is equivalent (modulo RCA_0) to Weak König's Lemma—the principles that every infinite subtree of the binary tree has an infinite branch. So we do not depend on a strong set-theoretic background, but rather only require

[11] For a survey, see [Paseau and Leek, 2022].

a small fragment of second-order arithmetic, well within the bounds of what is often regarded as Hilbertian reductionism, and equivalent (over RCA_0) to many natural statements of ordinary mathematics (e.g. the Jourdan Curve Theorem).[12] Whilst we acknowledge that some mathematical background is needed to run our arguments, it is too strong to say that the construction of weak existence models is 'really' underwritten by set theory, rather than our ability to talk about the small amount of mathematics required for weak existence more generally.

6 Conclusions and open questions

We started this paper with a remark about the nature of using models to talk about reality. In the context of set theory we face a challenge—how to account for the existence of models when they themselves are meant to be sets? Appealing to weak existence, we think, guarantees enough models of set theory to do the job without appeal to any further assumptions regarding set-theoretic foundations. This then suggests that we can think of our set-theoretic talk as involving a threefold correspondence between our utterances, models of those utterances, and the world itself (in a manner consonant with many areas of science).

On the one hand, we think we've provided a satisfactory solution to the challenge we proposed. On the other hand, our proposal is merely *programmatic*, in that it establishes the possibility and attractiveness of pursuing a particular idea, rather than providing a full account of the details of how it is to be carried out. There are thus *many* questions left open by the approach. We wish to identify some of them here (and hope that the reader will allow the conclusion to be slightly longer than is usual).

6.1 Verisimilitude of a model

The question of "verisimilitude" helps to clarify exactly what we've achieved in this paper. We have merely substantiated the claim that the threefold-correspondence view is attractive and plausible. However, more needs to be done to provide a full account of the view. In particular, a natural project is try to ascertain which models of set the-

[12]See [Simpson, 2009] for a discussion of the reverse mathematics here, and in particular §IV.3 for a proof of the equivalence between the Completeness Theorem for countable languages and Weak-König's Lemma.

ory are closer to truth than others. This approach is not without precedent in the philosophical literature. Saharon Shelah (in [Shelah, 2002]) has advanced a conception of set-theoretic truth which should be conceived of *measure-theoretically*, with different sentences of set theory assigned different measures. He writes:

> I do not agree with the pure Platonic view that the interesting problems in set theory can be decided, we just have to discover the additional axiom. My mental picture is that we have many possible set theories, all conforming to ZFC. I do not feel "a universe of ZFC" is like "the sun", it is rather like "a human being" or "a human being of some fixed nationality".... So my meaning in saying "why the hell should [$V = L$] be true", is not that it is provably false, just as "the national lottery in the last ten years was won successively in turn by the nephews of the manager, so we know that there was cheating" is mathematically not proved. Clearly L is very special, to some extent unique, thus, the statement $V = L$ should get probability zero (thought not being impossible). So L is certainly a citizen with full rights but a very atypical one. Also a typical citizen will not satisfy $(\forall\alpha)(2^{\aleph_\alpha} = \aleph_{\alpha+\alpha+7})$ but probably will satisfy $(\exists\alpha)(2^{\aleph_\alpha} = \aleph_{\alpha+\alpha+7})$. However, some statements do not seem to me clearly classified as typical or atypical. You may think "does CH, i.e., $2^{\aleph_0} = \aleph_1$ hold?" being like "can a typical American be Catholic". More reasonably CH has a small measure, still much much more than $V = L$. [Shelah, 2002, p. 12].

Shelah's suggestion, whilst tantalising, has not been worked out in full mathematical and philosophical detail. We therefore ask:

Question. How should we think of *verisimilitude* for the advocate of the threefold-correspondence approach in set theory?

6.2 Characterising debates and communication

Recently there has been a lot of focus on the question of how many universes of sets there are. Some authors think (often motivated on the basis of the set-theoretic paradoxes and various model-theoretic constructions like forcing) that there is not one universe of sets but rather many universes of sets that are ontologically on a par (even if they might not be equal in other ways). Others think that there is just one maximal universe of sets.

Using the notion of weak existence proposed here, we can make sense of this difference. These authors, we contend, are disagreeing about *strong* (rather than *weak*) existence. The latter is agreed upon by all parties. It is the former that is not—one group thinks that weak existence corresponds *exactly* to strong existence, the other thinks they radically come apart. Weak existence thus provides another perspective on this debate.

We conjecture that this way of looking at what is different between the two views helps to explain the (surprising!) levels of *agreement* between the groups. Each party to the debate is able to agree on much, for example the correctness of proofs. Often such reasoning is presented in very semantic terms (e.g. by reasoning about the structure of "the universe"); set theorists are by and large not churning out first-order ZFC-derivations.[13] This presents a puzzle (along similar lines to the **Circularity Problem**): How is so much agreement possible when the reasoning seems to be semantic and their respective pictures of set-theoretic reality are so radically different? We can give an attractive answer: The parties agree on the nature of weak existence, and this provides a shared semantic content. What is disagreed upon is the philosophical claim of how this weak existence maps on to strong existence.

Really substantiating this response we leave to other work. There is much to this idea that, while promising, remains mere speculation. For example the idea that there is significant semantic (as opposed to proof-theoretic) content to what set theorists are doing is highly controversial. Thus the nature of the puzzle itself, as well as the possible response, needs to be made out in more detail.

6.3 The access problem

In the philosophy of mathematics, there is the general problem of access. This is a problem with a long history, but is perhaps most famously pressed by Paul Benacerraf in [Benacerraf, 1973]. How do we gain knowledge of mathematical objects, if they are non-causal, and non-spatiotemporal etc?

A natural line of thought here is is that we can think of various kinds of mathematical property as *universals* and we gain knowledge

[13] There is, of course, a deep question about the nature of this reasoning, and its relationship to derivations. See [Rav, 1999], [Azzouni, 2004], [Larvor, 2016], and [Tanswell, F] for discussion and further references.

of the universals by studying their *instances* which *can* be concrete. For example this idea is pressed by [Giaquinto, 2017] and [Barton, 2020a] regarding natural number cognition, drawing on work in the cognitive sciences (e.g. [Dehaene, 1997], [Nieder and Dehaene, 2009]).

This strategy perhaps works with number theory; we have cognitive access to instances of small numbers and the general rules and processes that would generate larger ones. But it is very unclear how such a strategy could work in the case of set theory—the operations there seem too infinitary, too ontologically prodigious, to be so tractable.

Our work suggests a response to this issue, at least insofar as first-order set theory is concerned. We can have access to large infinitary objects via our access to particular linguistic representations of those objects in some weak existence model of ZFC. In other words, as concrete instances of number-theoretic phenomena can help accessing the abstract rules of number theory, in the same way concrete linguistic models of first order theory can help accessing more abstract areas of mathematics, like, for example set theory. Therefore, the properties of abstract mathematical objects whose strong existence make them hard for us to reach can be better studied, understood, and known by means of the concrete character of the model-theoretic representation we can offer them through language. Again though, this suggestion, whilst temping, would require serious further work to be made out in detail.

6.4 Modelism

Next we consider *moderate modelism*. This is a view that again has a long history, but has been neatly isolated by Tim Button and Sean Walsh in [Button and Walsh, 2018]. It comprises the following two theses:

(1.) Talk of particular mathematical structures should be understood as claims about *isomorphism types* (this is the "modelism").

(2.) No appeal to mathematical intuition is allowed in specifying our knowledge of these isomorphism types (this is the "moderate" part).

In particular Button and Walsh consider the idea that we know the isomorphism types via description. *Very* roughly put, they argue that the moderate modelist's position is self-undermining; in order to fix a logic with sufficient expressive resources to do the descriptive work the moderate modelist wants, they must appeal to resources they themselves have prohibited. Moderate modelism *requires* one be *immoderate*.

Whilst we agree with Button and Walsh's argument, we think that weak existence can inform this debate. Whilst we concur that some degree of immoderation is necessary, perhaps not a huge amount is needed. All we need to do in order to be able to talk about certain first-order theories is have confidence that they are consistent. Weak existence will then give us a structure in which that theory is realised, and give us access to that structure. Of course this is a long way off isomorphism types, but for first-order structure the modelist can appeal to weak existence.

It's helpful here to consider a remark from Hilary Putnam who writes:

> "Models are not lost noumenal waifs looking for someone to name them; they are constructions within our theory itself, and they have names from birth." [Putnam, 1980, p. 482]

Our arguments suggest that for a significant class of models (in particular the ones compatible with a weak form of existence) Putnam was on the right track. Indeed, models are constructions from within a theory. But they *can* be given names (indeed every element of a model can be viewed as a name) and this can provide an account of how we can be assured of their existence.

6.5 Weak existence as a motivation for further axioms?

Another possible gain of focusing on models of set theory as part of that threefold relation between language and reality is a clearer direction in the justification of new principles extending ZFC.

As a matter of fact, if models are only partial descriptions of a (possibly) independent reality, the way they represent that reality might influence the theory we want to build for it. In other words, if we assume that our theory of sets is meant to describe the universe, and that the construction of models of set theory is a constitutive part of our theorising, then the way we use models can influence directly the theory we choose. Accepting this might mean that model-theoretic considerations become relevant for extending ZFC with new axioms—we want axioms that give us the best pictures of the way the world might be.

To give a concrete example of how model-theoretic considerations might influence the choice and the justification of new axioms, consider the case of principles that ensure forms of generic absoluteness (i.e. invariance across the different models obtainable by set forcing—Bounded Forcing Axioms are one source of examples here, as are inner model hypotheses).[14] The justification of these axioms and of the axiomatic approach they suggest to the problem of independence, can be justified from a model-theoretic perspective by asking for the different (partial) perspectives (i.e. the different models of set theory) to be cohesive/coherent and exhibit agreement. If we take different models to provide different pictures of a unique underlying reality, then we would like these different pictures to cohere. Axioms that postulate exactly this phenomenon might be supported on these grounds.

This is just an example, but suggests that a clear separation between models and reality might help in a process of theory choice, also in the case of set theory.

6.6 Summing up

To sum up: We think that the weak existence is a powerful tool for philosophy and mathematics. Viewing these kinds of models as imperfect but helpful "depictions" of strong existence helps to understand the central place of model theory in our reasoning about sets. But this strategy, whilst promising, raises a host of open questions that need to be tackled. In particular: What kinds of applications can we find for weak existence? How does the cornucopia of models we can obtain help us in our philosophical and mathematical theorising?

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[14] See [Bagaria, 2005], [Bagaria, 2008], and [Friedman, 2016].

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