# Make It So: Imperatival Foundations for Mathematics

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**Abstract**

This article articulates and assesses an *imperatival* approach to the foundations of mathematics. The core idea for the program is that mathematical domains of interest can fruitfully be viewed as the outputs of construction procedures. We apply this idea to provide a novel formalisation of arithmetic and set theory in terms of such procedures, and discuss the significance of this perspective for the philosophy of mathematics.

"*Philosophers have hitherto only* interpreted *the world: the point is to* change *it.*" Karl Marx, Theses on Feuerbach

## 1 Introduction

In contemporary philosophy of mathematics it is common to think of mathematical discourse as primarily if not entirely propositional. Propositions describe how things are in the world; on this picture, then, mathematics is primarily a descriptive enterprise, one that aims to work out and then clearly articulate what is the case with the mathematical 'parts of reality'.

This article will explore the prospects for an alternative view, one according to which mathematical discourse is at least in a significant part *imperatival*. Imperatives in the form of *instructions* or *commands* tell someone (how) to do something; on this picture, then, mathematics is at least in part concerned with action, with ways of changing the world rather than describing it.

There are hints of this idea in ancient Greek mathematics. In Euclid's elements, for example, one encounters two sorts of "proposition": on the one hand, one has declarative theorems (*theoremata*), results that show some relation between geometric objects must obtain; on the other, one has what came to be known as "problems" (*problemata*), which detail something approximating a recipe for the construction of a particular geometrical figure. As an example of

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the former, we have proposition 4: *that* two triangles with two similar sides and the same angles are similar. On the side of the latter, we have proposition 1, which shows *how* to construct an equilateral triangle about a given segment.[1]

The grammar of the solution to a problem is imperatival. In standard translations of Euclid, the solution to a problem takes the form of a series of "Let..." statements, where what follows is an assertion that one of the basic allowed actions has been performed. In such reasoning, Euclid will say: "Let $P$ have been done..." and reason thereafter as though it has in fact been done. But we can equally well conceive of such arguments being given without using "Let..." clauses. For example, we might give a solution to the problem "Make an equilateral triangle over a given segment PQ!" in a series of commands or instructions: "Create a circle with radius PQ centered at P! Then, create a circle with radius PQ centered at Q!", etc.

Although this way of arguing is still present in what you might call "informal" mathematics, it is generally translated away by formalisations on the grounds that the purely quantificational/propositional idiom seems sufficient. But there is no principled obstacle to giving a logic for such constructive processes and formalising constructive arguments in the Euclidean style. This article will explore a way of doing precisely this, offering imperatival formalisations of arithmetic and set theory in the Euclidean spirit.

Why bother with this grammatical reconstruction? One reason is simply that the Euclidean, activity-based conception of mathematics seems to us to be interesting and neglected. But to raise the stakes, there is also the possibility that such a formulation of mathematics may offer philosophical benefits, including a well-motivated solution to the paradoxes of set theory, and a new source of evidence for the consistency of mathematical theories. Claims of this kind have in fact been made on behalf of imperatival foundations of mathematics in the literature – most notably, by Kit Fine (2005) – but there has been little systematic work to develop relevant logical programs and assess their philosophical merits. This article will aim to do precisely this.

We begin (in sections 2 and 3) by developing an imperatival language and logic that allows us to reason rigorously about the effects of performing certain specified actions. We'll then (in section 4) use this logic to show that the execution of certain commands would result in structures satisfying axioms of standard mathematical theories – we focus on arithmetic and set theory. Finally, we turn (in section 5) to a philosophical assessment of the results, and the corresponding imperativalist foundation for mathematics they suggest.

## 2 Language

The logic we will employ is a higher-order logic, based on a modification of the standard (functionally) typed $\lambda$ calculus. In any such typed system, one first defines a set of grammatical types, designed to represent idealized grammatical

[1] For more on this distinction, see Sidoli (2018) and references therein.

categories familiar from natural language, and then provides a vocabulary of terms in the relevant type system.

Our approach will be no different, although we will have recourse to some types in our type system that do not feature in standard treatments.

The types are built up recursively: we first give some 'basic' types, and then specify means of constructing complex types from the basic ones. In standard typed $\lambda$ calculi, there are usually only two basic types: the type $e$, of entity denoting expression, is designed to model expressions like "Jack" and "Jill" and "the number 9" that serve to pick out objects; and the type $t$, of truth-evaluable expression, is designed to model expressions like "snow is white" and "grass is green" that are used to make complete declarative statements. But for our purposes, we will require in addition to these two the type $\iota$, of imperatival expression, which is designed to model expressions like "eat your greens!" and "plug in the TV, then turn it on!" that are used to make commands and give instructions.

In addition to these basic types, we also have means of generating complex types from simple ones. Here, the standard machinery in functional type theory is the $\to$ operator, which carries us from given types $\sigma$ and $\tau$ to the type of expression $\sigma \to \tau$, where expressions of the latter type are ones that make an expression of type $\tau$ when completed by one of type $\sigma$. These include e.g. the predicate-of-entities type $e \to t$, with corresponding expressions like "... is a dog". In addition to these complex types, we will also make use of plural types: where $\sigma$ is any given type, the type $\sigma\sigma$ will be the type of expression that ranges over multiples of things of type $\sigma$. For example, "Jack and Jill", "the planets", and "the numbers under 9000" correspond to terms of type $ee$.

For future reference:

**Definition 2.1** ($\iota$-types). *$\iota$-types, henceforth just types, are defined inductively as follows.*

- *$e$ is a basic type, the type of* entity denoting *expression;*
- *$t$ is a basic type, the type of* truth-evaluable *expression;*
- *$\iota$ is a basic type, the type of* imperatival *expression;*
- *whenever $\tau$ is a type, so too is $\tau\tau$, the type of pluralities at type $\tau$.*
- *whenever $\sigma, \tau$ are types, $\sigma \to \tau$ is the type of expression which, on completion by one of type $\sigma$, yields one of type $\tau$.*

There is room for discussion about whether our type classifications are in some sense "correctly" capture the grammatical classifications present in natural language. One might wonder, for instance, if imperative and declarative sentences should really be seen as having a different syntactic category, with corresponding semantic denotations, as opposed to being one and the same category associated with a different pragmatic force. At least one of us thinks such questions are misguided: there is always theoretical artifice in such divisions, and consequently one should feel free to divide up linguistic practices into syntactic and semantic buckets subject only to minimal requirements of accordance with practice. But even for those who think there are facts of this kind, evidence from language at the very least leaves the question open. To the extent that expressions like "the door is shut" and "someone is smoking" belong to a coherent, common grammatical category, there seems an equally good, if prima facie, case to be made that "shut the door!" and "smoke!" have their own, separate one. Such expressions are almost never, for example, substitutable for one another without loss of grammaticality, and in a way that seems somehow deeper than those failures of substitution known already for sentences and names in the traditional theory.

We can now turn to a discussion of the language we will be using. As usual, we allow an infinite stock of variables at every type, and will avail ourselves of standard propositional connectives, quantifiers, and the identity symbol at each type.

As to complex terms, we will make heavy use of the device of $\lambda$-abstraction: for each term $T$ of type $\tau$ and variable $x$ of type $\sigma$, $\lambda x.T$ is a term of type $\sigma \to \tau$. We also have terms generated by application: where $X$ is a term of type $\sigma \to \tau$, and $Y$ a term of type $\sigma$, $XY$ is a term of type $\tau$.

We will also officially have a term $\prec^{\tau}$ of type $\tau\tau \to \tau \to t$, which intuitively corresponds to membership among pluralities: $\prec Xx$ means $x$ is one of the $X$s. Instead of writing $\prec Xx$ we will generally just write $Xx$, though it should be borne in mind that strictly speaking $Xx$ is not application in the sense above. Also, we will adopt the convention throughout that upper-case latin variables $X, Y, Z$ will range over pluralities, and $F, G, H$ will range over intensional properties (i.e. will be variables of type $\sigma \to t$).

Finally, we can come to the more interesting parts of the language, namely that fragment which involves the distinctively imperatival apparatus.

Throughout the paper we will be focused on command that are in a natural sense "creative": they command the creation of new objects, or correspond to iterations or other complex combinations of such commands. Thus, our sole "atomic" command forming operator, which we will write as $!^{\sigma}$ (for various types $\sigma$), commands the creation of something with a certain type $\sigma$ feature. Slightly more precisely: it takes in a predicate $F$ of type $\sigma \to t$, and yields the command "Make something which is $F$!".

We will also need complex imperatival operators. These allow us to make more complex commands out of our basic ones. For example, given commands $i$ and $j$ we will have the command to do $i$ and then do $j$, which we will write $i; j$; similarly, $p \to i$ is the command to check and see if $p$ is true, and if it is do do $i$ (and otherwise, do nothing).

We also have quantificational commands: $\forall_i^{\sigma}$ is an operator that takes in an imperatival property, for example something like "Kick $x$!", and yields the quantified command to do the command to each thing: in this case, yielding the command to kick everything. (We will generally drop the subscripted $i$ and superscripts $\sigma$ from quantifiers, where it is clear from context what is meant.)

Finally, we have a collection of modal terms that serve to connect the imperatival and declarative parts of the language. Given an imperative $i$, $[\cdot]$ gives us a modal operator $[i]$. The primary interpretation for $[i]p$ is as "no matter how you do $i$, $p$", or "$p$ will be true after $i$ has been done". But one can also interpret it in other ways: in many contexts, it makes sense to read it along the lines of Euclid's "Let $i$ have been done", so that $[i]p$ means: "Let $i$ have been done; then $p$".

We also have a general notion of necessity, which we write with the plain $\Box$; this we take to express something like "absolute necessity", with $\Box p$ meaning $p$ must be the case no matter what, and so in particular will be true no matter what constructions we carry out, was true before we did any constructions, etc.

We sum up the language in Definition 2.2.

**Definition 2.2** ($\mathcal{L}$)**.** *The language $\mathcal{L}$ contains:*

1. *Variables:*
   - *If $\sigma$ is a type, then we have a countable stock of variables $x_1^\sigma, ..., x_n^\sigma, ...$ of type $\sigma$.*
2. *Propositional terms:*
   - $\wedge$ *of type* $t \to t \to t$*;*
   - $\neg$ *of type* $t \to t$*;*
   - $\forall^\sigma$ *of type* $(\sigma \to t) \to t$*;*
3. *$\lambda$-abstracts, plural and applicative terms:*
   - *the term $\lambda x.a : \sigma \to \tau$, whenever it contains $x : \sigma$ a variable and $a : \tau$;*
   - *the term $\prec^\tau$ of type $\tau\tau \to \tau \to t$*
   - *the term $ab : \tau$, whenever it contains $a : \sigma \to \tau$ and $b : \sigma$;*
4. *Imperatival terms:*
   - *the term $!^\sigma$ of type $(\sigma \to t) \to \iota$*
   - *the term $;$ of type $\iota \to \iota \to \iota$*
   - *the term $\to$ of type $t \to \iota$;*
   - *the term $\forall_i^\sigma$ of type $(\sigma \to \iota) \to \iota$;*
5. *Modal operators:*
   - *the term $[\cdot]$ of type $\iota \to t \to t)$*
   - *the term $\Box$ of type $t \to t$*

# 3 Logic

Now that we've laid down an imperatival language, and glossed the intended interpretations for its vocabulary, we describe the logical axioms and rules that will be assumed in our axiomatization of mathematical theories in §4.

§3.1 deals with the declarative fragment of the logic, which is more or less standard. §3.2 then discusses the central axioms relating to imperatives and modalities: these axioms provide a bridge between the imperatival and declarative terms, allowing one to reason about what effect the execution of commands of various kinds will have. §3.3 then shows how to use the logic of §3.2 to define the notion of a *chain of iterations* of a command, and then to define the notion of *indefinite iteration*, which will be one of our key tools in the sequel. Finally, §3.4 formalizes the notion of *executability* of a command.

## 3.1 Declarative Logic

The axioms for the "declarative" part of our logic, are more or less standard: we assume classical propositional logic, positive free logic for the quantifiers, and other principles governing application and $\lambda$-abstraction:

**PL** Every closed classical tautology

**QL** Rules for positive free quantifier logic

**Ex1** Existence for all the propositional and imperatival connectives and quantifiers, and identity

**Ex2** Closure of existence under function application

**Con** $\alpha$, $\beta$ and $\eta$ conversion rules

We also assume a strong form of the axiom of choice as part of our higher-order logic, along with some standard principles governing the plural terms:

**PlurExt** $\forall X, Y((Xx \equiv Yx) \supset X = Y)$

**PlurComp** $\exists X \forall x(Xx \equiv \Phi)$, no free $X$ in $\Phi$

**Choice** $\exists f^{(\sigma \to t) \to \sigma} \forall F^{\sigma \to t}(\exists x F x \supset F(fF))$

The Choice principle given here perhaps calls for some explanation, as it is not completely standard. It asserts the existence of a relevantly typed function which, given *any* non-empty property of the relevant type, returns something to which that property applies, if there is any such entity. Such an $f$ is naturally thought of as a "global" choice function for properties, as it is a single function that picks a witness for every instantiated property 'at once'. The existence of such functions is perhaps controversial, but the assumption is made here purely for convenience.[2]

[2] We could get by, for example, with just a restricted form of choice for pluralities.

## 3.2 Imperative Logic

With the declarative fragment of the logic in place, the business of this subsection is to give axioms governing the imperatival terms of $\mathcal{L}$. These work by connecting the imperatival terms $i$ to declarative ones $p$ through modal-imperatival claims $[i]p$.

The first axiom governs the ! commands. The idea behind these is that formulas of the form $!F$ correspond to commands to *make* an $F$. To capture this idea axiomatically, we first impose the requirement that no matter how you make an $F$, there is an $F$:

**Make** $[!F]\exists F$

Variable binding is, as usual, handled by $\lambda$ abstraction, with $!x.\Phi x$ abbreviating $!(\lambda x.\Phi x)$.

**Make** is a minimum requirement on the execution of the command to make an $F$; but there other questions about what's required by such a command that it does not settle. For example, if asked to make an $F$, what should one do if there is an $F$? And is one allowed to do anything besides making an $F$?

The answers to these questions don't matter tremendously, but answering them (in a certain way) will smooth out our theory considerably later on. In effect, our policy is to require that **Make** commands are always executed as frugally as possible: if there is an $F$, nothing should be done, and if there isn't, only a single $F$ should be made.

It turns out that we have the means to formalise this idea using the modal and plural apparatus we have, and that along the way we will encounter some concepts that will find frequent application in the sequel and that therefore warrant definition.[3]

First, we impose the following principle, designed to ensure that if there is an $F$, nothing should be done to comply with the command to make one.

**Eco1** $\exists F \supset \forall p([!F]p \equiv p)$

Notice how we have defined the idea of $!F$'s "doing nothing" here in terms of quantification over propositions. We will make heavy use of this notion, and so we will give it a memorable definition. Set:

😴 $:= \lambda i.\forall p([i]p \equiv p)$

then 😴$(i)$ holds exactly when anything you can make the case by doing $i$ is already true, and vice versa; i.e., $i$ does nothing.

Next, we want to impose the constraint that if there isn't an $F$ the only new thing you make is a thing that $F$s – you do "nothing other" than making a single $F$. In our logic, this requires a rather fiddly formulation, since we do not have readily available backtracking operators. But we can define the notion using our plural logic, and as with the case of **Eco1**, we will use this occasion

[3]In fact, **Make** is a consequence of these **Eco**nomy principles, but we leave the former in for expository reasons.

to introduce defined notions making such concepts more readily available to us down the line.

**Eco2** $\neg\exists F \supset (\exists X(\forall y(Xy) \wedge [!F]\exists y(Fy \wedge \neg Xy \wedge \forall x(\neg Xx \supset x = y))$

**Eco2** says that, if there are no $F$s, then no matter how you make an $F$, it should be true that there is an object that is $F$, and that is the sole object that did not exist before you made an $F$.

Note the use of plural logic in formalising this idea: in effect, we use plural comprehension outside the scope of $[!F]$ to pick out the things that exist before we make an $F$, and say that after making an $F$ we have an $F$ which is not one of *those*.[4]

We will make heavy use of this device. But to avoid the proliferation of plural quantifiers and instances of comprehension this definition involves, we will henceforth use a scheme of metalinguistic abbreviation to capture the same idea. Specifically, we introduce a meta-linguistic abbreviation $new(x)$ to mean that $x$ is an object that is "new" relative to the previously occurring modal, as just defined with the device used in **Eco2**. Thus for **Eco2** we can now write

$$\neg\exists F \supset [!F]\exists x(Fx \wedge new(x) \wedge \forall y(new(y) \supset y = x))$$

with the latter simply intended as an abbreivation of the former. We will also use *new* (no variable) to abbreviate the claim that there is something in the relevant modal context that is new. (A formal definition of the *new* pseudo-predicate is given in the next footnote.[5])

We now turn to the complex commands. For ;, our intended interpretation is that $i; j$ should mean "Do $i$, then $j$!". This we axiomatize as follows:

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[4] That we can use pluralities in this way—to pick out only those objects which existed "before" the command was executed—requires principles about the rigidity of plurality membership which we discuss at the end of the section. Also note that apparent simplifications of **Eco2**, such as replacing the $[!F]$ formula with $\exists!x.Fx \wedge \neg Xx$, fall short of their intended significance.

[5] We will use formulas $\varphi$ involving a predicate $new(x)$ as metalinguistic abbreviations for a formulas $\varphi^*$ in our official language, not containing $new(x)$, subject to the following rules:

- for atomic $\varphi$, $\varphi^* := \varphi[new/\lambda x.x \neq x]$;
- $*$ commutes with modals, connectives and quantifiers (with a restriction implicit in the next condition);
- if $\varphi$ is of the form $\circ\psi$, where $\circ$ may be any of $\Box, \Diamond, [i]$ or $\langle i\rangle$, and *new* only occurs in the scope of the instance of the $\circ$ in question, then $\varphi^* := \exists X\forall y(Xy \equiv y = y \supset \circ\psi[new/\lambda y.\neg Xy]^*)$.
- we will use *new* (no variable) as an abbreviation for $\exists x new(x)$.

This gives a systematic way of translating away uses of *new*. The following rules of inference hold given these definitions:

**r1** $\neg Ex, \circ Ex \vdash \circ newx$

**r2** $\circ newx \vdash \neg Ex$

**r3** $\circ newx \vdash \circ Ex$

Moreover, our definitions can be used to show that the expansion of our language by a predicate *new* satisfying these constraints is a conservative extension of our logic.

**Then** $[i;j]\varphi \equiv [i][j]\varphi$

This definition is standard in dynamic logic (see Harel et al. (2000), e.g.) and reflects the simple fact that something must result from doing $i$ then $j$ exactly when it must result from doing $i$, that it must result from doing $j$.

For $p \to i$, we want to get at the idea of “check and see if $p$ is true, if so do $i$”. This we axiomatize as follows:

**If1** $p \supset \forall q([p \to i]q \equiv [i]q)$

**If2** $\neg p \supset$ 😴$(p \to i)$

The second here says that if $p$ is false, $p \to i$ does nothing. The first says that, if $p$ is true, then you can make something the case by doing $p \to i$ exactly when you can make it the case by doing $i$. We can think of $p \to i$ as “indiscriminable” from $i$ in these circumstances.[6]

Next, we want principles governing the imperatival quantifier. There is perhaps more than one way to understand a quantified command like “Kick every ball!"—does one execute this command in “parallel" (i.e., kick all the balls simultaneously), or in series (“kick this ball, then that one, then that one, ...!"). We shall opt for for the former understanding. We are therefore looking for axioms connecting the action of $i$ on each individual entity with its “collective” result when applied to all (current) entities at once (i.e. in parallel). Since here we are interested only in commands that bottom out in “make” commands, i.e. that are fundamentally concerned with the introduction of entities with desired properties, the following axioms are natural and turn out to be sufficient. (Here and throughout, $Ey$ abbreviates the existence predicate $\exists z(z = y)$ or suitable alphabetic variant.)

**All1** $\exists x[i]Ey \supset [\forall x\, i(x)]Ey$

**All2** $\langle \forall x\, i \rangle Ey \supset \exists x \langle i(x) \rangle Ey$

**All3** $\forall x$😴$(i(x)) \supset$ 😴$(\forall x i)$

The first of these is a sort of “union” principle: it ensures that anything you must get by doing $i$ to some given $x$ is something you must get by doing $i$ to every $x$. This seems natural given that doing $i$ to all $x$s involves *at least* doing $i$ to each particular $x$.

The latter two more like “intersection” principles: **All2** says that if you can get something by doing $i$ to everything, then there is some $x$ such that you can get that thing by doing $i$ to it. **All3** says that if doing $i$ to each particular $x$ does nothing, then so too does doing $i$ to every $x$. These reflect the idea doing $i$ to every $x$ involves *nothing more than* doing $i$ to each particular $x$.[7]

[6] We note, as an aside, that necessary indiscriminability is a natural criterion of identity for imperatives.

[7] The status of the converses to these principles is subtle. We think that in the general logic the converses admit of intuitive counterexamples. But when we eventually turn to

The final component of the imperatival logic involves specifying the modal logic that should hold for the imperatival-modals $[i]$.

Here we assume that, at a minimum, the modal logic of each satisfies $\mathsf{K}$, has necessitation, and has the converse Barcan formula. The first two are intuitively just saying that after we've done a command – assuming we can in fact do it – we end up in normal circumstances where the laws of logic hold. The latter, again, may be construed as a stipulative restriction on the kinds of commands we will consider. Accepting CBF means that we can only create, and never destroy. This is really just a simplifying assumption we could do without, adopted because we are interested more in creation than destruction here.

While we *could* get by with only these assumptions on the logic of $[i]$, we can achieve a great simplification in future arguments and by requiring, in addition, what we call the *determinism axiom*:

**D** $\langle i \rangle p \supset [i]p$

Intuitively, axiom **D** tells us that there is at most one way to execute a command. Since it is so useful in simplifying arguments further down the line, we will henceforth accept axiom **D** for each $i$. Again, we emphasize we do not think of this as an axiom that somehow reflects a deep truth about the nature of commands, indeed we think it is pretty obviously false in general. (Consider "Bake a cake!" or "Draw a line!".) Nevertheless, it seems plausible that there are classes of "maximally specific" commands satisfying **D**, and we can always restrict our attention to those, just by stipulating that our commands are "filled in" maximally outside our explicit postulations.

We now consider the modal logic of the simple modality, $\Box$. The intended interpretation of $\Box p$, again, is that $p$ must be the case no matter what, and so in particular will be true no matter what constructions we carry out, was true before we did any constructions, etc. Accordingly, we make $\Box$ an $\mathsf{S5}$ modality and relate it to the $[i]$ as follows:

**ML** $\mathsf{S5}$ for $\Box$, with neither Barcan nor its converse

**Nec** $\Box\varphi \supset [i]\varphi$

Finally, we will need some principles governing the interaction between plural terms and $\Box$. We first have two principles stating the membership in pluralities is "rigid": pluralities can neither lose or gain members as we move across possibilities:[8]

**PlurStab** $Xx \supset \Box Xx$

deterministic imperatives—imperatives such that $\langle i \rangle p \supset [i]p$– then we think that the converses of all three principles are natural and reasonable. Indeed, **All1** and **All2** each yield the converse of the other in the deterministic setting. Since later on we will exclusively be concered with deterministic commands, we could strengthen these to biconditionals, but since we don't need to, and think the biconditionals generally questionable, we will leave these as they are.

[8] Compare the similarly named principles of Linnebo (2013). For a general discussion of the interaction between plural logic and modality, see also Roberts (2022).

**PlurInext** $\forall X(\forall x(Xx \supset \Box F) \supset \Box \forall x(Xx \supset F))$

Often, we'll invoke both of these principles simultaneously under the label of "plural rigidity".

Finally, we have a principle to the effect that whenever a plurality exists, all its members must as well:

**PlurD** $Xx \supset \Box(E(X) \supset E(x))$

That concludes our presentation of the basic logic.

## 3.3 On Iteration

Many of the operators we have axiomatized are considered in Fine (2005). But Fine also includes a device for forming complex imperatives that we have not: namely, an operator $\cdot^*$ which takes an imperative $i$ and returns its "indefinite iteration" $i^*$. The thought behind this command is that it should require repeatedly doing $i$ as many times as is possible.

Indefinite iterations of commands was central's to Fine's proposal for how to construct the natural numbers and sets—the construction of each required, roughly, the indefinite iteration of a more basic number- or set-introduction command. One advantage of our higher-order framework, in which quantification over imperatives and propositions is allowed, is that we do not need to take the operator as primitive, but rather can define it—or at least, can define something that has all the logical features needed of it.

We'll build up to the definition in stages. First, it is clear that with the resources we have in hand, we can define commands that, intuitively, correspond to the *finite* iterations of a given imperative $i$. Let

$$\heartsuit := (\exists y(y = y)) \to (!x.x = x)$$

and observe that $\heartsuit$ provably does nothing. (Argue by cases on whether there is anything or not.) Now, set

$$i_0 = \heartsuit$$

and

$$i_{n+1} = i_n; i$$

to get the various definable finite iterations of $i$.

We can define infinite iterations of $i$ too: for we may define a property $\Omega$ that picks out the finite iterations of $i$ using the ancestral

$$\Omega := \lambda i, j.\forall F(F\heartsuit \wedge \forall k(Fk \supset F(k; i))) \supset Fj$$

and if we then set

$$\alpha := \lambda F.\forall j(Fj \to j)$$

$\alpha(\Omega i)$ will then have the force of "do every finite iteration of $i$". By the **All** axioms and **D**, it produces exactly the things that are produced by the finite iterations of $i$.[9]

$\alpha(\Omega i)$, then, loosely corresponds to the $\omega$th iteration of $i$. But it is not the final definable iteration, since we also have $\alpha(\Omega i); i$, and so on. How long do these things go on? Is there a "maximal iteration"?

It turns out that, perhaps surprisingly, we do have a natural candidate for the longest iteration in this setting. Specifically may consider the command to do *all* the iterations of $i$, which we will call $i^*$ and construe as the command to repeat $i$ forever.

The definition works by extending the above trick with the ancestral so as to characterize all the iterations of $i$, since we may then consider the quantified command to do all of them and show this has the required force.

Towards this, we'll use the following definitions:

**Definition 3.1.** *Say $F$ is* weakly closed for $i$ *when, if it applies to some $j$, it applies to the command to do $j$ then do $i$.*

*Symbolically:*

$$Cl := \lambda i, F.\forall j(Fj \supset F(j; i))$$

**Definition 3.2.** *Say $F$ is* closed under limit procedures *iff, whenever it applies to all commands with a property $F$, it applies to the command to do all commands with property $F$. Symbolically,*

$$Lim := \lambda F.\forall G(G \subseteq F \supset F(\alpha(G)))$$

**Definition 3.3.** *Say that a property $F$ of commands is* strongly closed for $i$ *when it is closed for $i$, and is closed under limit procedures. Symbolically:*

$$Scl := \lambda i, F.Cl(F, i) \wedge Lim(F)$$

Strong closure captures a form of transfinite iteration. For, note that if a property is strongly closed, and it applies to some $j$, then it applies to $j; i$, and $j; i; i$, $j; i; i; i$ and so on by weak closure. It is clear also that any strongly closed property applying to $j; i$ must also apply to the command to do every finite $j$-iteration of $i$, and the command to do every finite $j$-iteration of $i$ and then do $i$, and so on.

We may then define the notion of an iteration by taking the least strongly closed property. Actually for technical reasons it is convenient to define the slightly more general notion of a $j$-iteration of $i$, which is an iteration of $i$ that begins with $j$.

**Definition 3.4.** *Say that a command $k$ is a* $j$-iteration of $i$ *iff $k$ has every property $F$ that (i) applies to $j$ and (ii) is strongly closed for $i$. Symbolically,*

$$It := \lambda j, i, k.\forall F(Scl(F, i) \wedge Fj \supset Fk)$$

[9]Note that there need be no sense in which $\alpha(\Omega i)$ actually involves some infinite task. Indeed there is no reason that there be any more than one iteration of $i$, as $i$ might equal $i; i$, and so on, in which case $\alpha(\omega i)$ will be just $i$.

We write $It^i_j$ for $Itji$, and $It^i$ for $It^i_\heartsuit$. The commands in $It^j_i$ are the iterations of $i$ beginning with $j$; the commands in $It^i$ are iterations of $i$ (starting by doing nothing).

It is easy to show that $j; i$, $j; i; i$, $j; i; i; i$, and even 'do every finite $j$-iteration of $i$' are all $j$-iterations of $i$. More generally:

**Lemma 3.5.** $\forall i(It^i_j j; i \wedge Scl(It^i_j, i))$

*Proof.* This follows immediately from the definitions. □

We also have a form of induction provable for iterations.

**Lemma 3.6** (Induction on Iterations.)**.** *For any $F, i, j$, if*

1. *$Fj$; and*
2. *whenever $Fk$ and $I^i_j(k)$, $F(k; i)$; and*
3. *for any $G$ with $G \subseteq F$ and $G \subseteq I^i_j$, $F(\alpha(G))$;*

*then for any $k$ with $It^i_j(k)$, we have $Fk$.*

*Proof.* Again, an immediate consequence of definitions. □

We may now define indefinite iteration, as suggested above, to be the command to do *all* the iterations of $i$.

**Definition 3.7.** *Given $i, j$, set $i^*_j := \alpha(It^i_j)$, and $i^* := i^*_\heartsuit$.*

This definition, though natural, has a curious impredicativity about it: for the command $i^*$ to do *all* iterations of is itself just one among the iterations of $i$.

**Lemma 3.8.** $\forall i(It^i i^*)$

*Proof.* Every strongly closed $F$ that applies to every iteration of $i$ must also apply to $i^*$, by the limit condition in strong closure. □

Note that the lemma implies that $i^*; i$, $i^*; i; i$, and so on, are all iterations of $i$. Our logic does not suffice to prove (we think) the generalisation to infinite iterations and to $i^*; i^*$ itself, but these are natural extensions of the system.[10]

Our imperatival logic is strong enough to ensure that $i^*_j$ tolerably well behaved. To close this section, we give one example of this and prove a lemma which will be important in what follows. The lemma tells us that, if something is made by an indefinite iteration of $i$, it must be made first by some particular application of $i$: nothing new gets generated 'in the limit'.

**Lemma 3.9.** *If $\neg Eo \wedge [i^*_j]Eo$, then either $[j]Eo$ or there is a $j$-iteration $k$ of $i$ such that $[k]\neg Eo \wedge [k; i]Eo$*

[10] There are various ways we could augment our logic so as to make such principles provable, for instance by adding the quantifier distribution principle $\forall x(i; jx) = i; \forall xjx$, no $x$ free in $i$.

*Proof.* Suppose $\neg[j]Eo$ and there is no such $k$, so that for any $j$-iteration $k$ of $i$:

$$[k]\neg Eo \supset [k;i]\neg Eo$$

Then by induction on iterations, we then have that for every $j$-iteration $k$ of $i$, $[k]\neg Eo$. It then follows by **All1** that $[i^*_j]\neg Eo$, contradicting the hypothesis. $\square$

### 3.4 Executability

The modal logic of the operators $[i]$ has been stipulated to contain $\mathsf{K}$ and be closed under necessitation. When combined with the axiom **Make**, this entails $[!x.x \neq x]p$ for every proposition $p$. Hence in the case of $!x.x \neq x$, everything is necessary and nothing is possible. But intuitively $!x.x \neq x$ is not something that can be done. This leads naturally to the following definition:

**Definition 3.10.** *$i$ is* executable *iff $\langle i\rangle\top$.*

Note that in the case of a non-executable command, $\neg\langle i\rangle p$ for all $p$. For non-executable $i$, then, everything is $i$-necessary and nothing is $i$-possible. ("There are no $i$-worlds.")

Since many of the commands in our imperatival approach to foundations of mathematics are indefinite iterations, it will be useful to prove some general results on what executability for indefinite iterations entails. The key result here is something we call the *fixed point lemma*: it says that, for commands that are "essentially creative" in that they effect changes always by making things, executability for an indefinite iteration $i^*$ is equivalent to the iterations of $i$ reaching a fixed point, in the sense that after doing $i^*$ doing $i$ does nothing more.

**Definition 3.11.** *Say that $i$ is* essentially creative *if and only if it makes a difference when and only when it makes a new object. Symbolically,*

$$EC := \lambda i.\forall p\big(p \equiv \langle i\rangle(\neg\text{new} \supset p)\big)$$

The following lemma shows that "Make!" commands are essentially creative, and that essential creativity is "transmitted upwards" along complex command forming operators.

**Lemma 3.12** (EC lemma)**.** *We have the following.*

1. *$!F$ is essentially creative for any $F$;*
2. *If $i$ and $j$ are essentially creative, then $i;j$ is essentially creative;*
3. *If $i$ is essentially creative, then $\phi \to i$ is essentially creative;*
4. *If, for all $x$, $i(x)$ is essentially creative, then $\forall xi$ is essentially creative.*

*Proof.* As follows:

1. Use **Eco2**, **Eco1**.

2. Suppose (for reductio) that $p \wedge \langle i; j\rangle(\neg\text{new} \wedge \neg p)$. Then for $U$ a plurality such that $\forall x U x$,

$$\langle i\rangle(\forall y U y \wedge \langle j\rangle(\forall z U z \wedge \neg p)) \tag{1}$$

   Since $j$ is essentially creative and $p$ is the case, we have $[j]\forall z U z \supset p$. Since $i$ is also essentially creative, $[i](\forall y U y \supset ([j]\forall z U z \supset p))$[11], which contradicts (1).

3. Immediate.

4. Suppose for reductio $p \wedge \langle \forall x i\rangle(\neg\text{new} \wedge \neg p)$. Then by determinism (**D**), $[\forall x i]\neg\text{new}$, and hence for all $x$, $[i(x)]\neg\text{new}$. Since each $i(x)$ is essentially creative, it follows that 😴$(i(x))$ for each $x$, and hence that 😴$(\forall x i)$ by **All3**, which contradicts $p \wedge \langle \forall x i\rangle \neg p$.

□

The EC Lemma helps us determine some useful properties of indefinite iterations. For instance, putting together parts 2. and 4. yields:

**Lemma 3.13.** *If $i$ is essentially creative, $i^*$ is essentially creative.*

We're now ready to establish that for essentially creative $i$, the executability of $i^*$ gets us $i$-fixed-points:

**Theorem 3.14** (Fixed point lemma)**.** *For any essentially creative $i, j$, no matter how you do $i^*_j$, doing $i$ again does nothing more. Hence, if $i^*_j$ is executable, you can do it so that doing $i$ again does nothing.*

*Proof.* The previous lemmas entail all $j$-iterations of $i$ are essentially creative.

Thus, we need only show that

$$[i^*_j][i]\neg\text{new}.$$

Suppose, for reductio, $\langle i^*_j\rangle\langle i\rangle Eo \wedge new(o)$ for some $o$. Then $\langle i^*_j; i\rangle Eo$, and hence $[i^*_j; i]Eo$ by **D**. Given that $i^*; i$ is itself an iteration of $i$, it follows from **All1** that $[i^*_j]Eo$, and hence that $[i^*_j][i]\neg new(o)$ by plural rigidity principles. But we now have that $\langle i^*_j; i\rangle Eo \wedge \neg new(o) \wedge new(o)$, which is a contradiction.

We thus conclude $[i^*_j][i]Eo \supset \neg new(o)$, as required. □

[11] Note we rely here on **Ex2**

# 4 Mathematics

With our language and logic in place, we now turn to the project of characterising mathematical domains of interest in imperatival terms, focusing on arithmetic and set theory. In each case, we will offer commands whose execution would demonstrably suffice to produce domains of numbers and sets sufficient to model standard axioms for number and set theory. Later, we will relate these results to the philosophical ideas surrounding Euclideanism about mathematics discussed in the first section.

The background structure to our imperatival formalizations of mathematical theories will always be the same – a template suggested already in Fine (2005). The idea is that we will introduce commands whose execution will construct a model for the mathematical theory of interest.

We approach this task in four stages. First, we introduce a stock of new basic predicates into the domain that are tailored to the mathematical domain we want to formalize. So in geometry they might be predicates for point and line, in set theory for set and member, etc. Next, we lay down what we will call (following Fine) "postulational constraints". These are universal rules governing the postulational predicates that "constrain" allowed courses of action. After that, we specify a particular command in the postulational predicates whose execution would produce a model for the relevant theory. Finally, we prove this is so in the logic.

Before moving on to the details, we should briefly mention a point about how the treatment of mathematics offered here in imperatival terms correlates with the broadly "Euclidean" conception of mathematics we discussed at the outset. In the Euclidean paradigm, there is an important role for imperatives in setting out arguments *that certain mathematical objects can be produced*, as distinct from arguments that the objects in question have certain properties or stand in certain relations (assuming they exist). This was the distinction between theoremata and problemata from the first section. Now, our imperatival logic enables us to reimpose this distinction and in fact to argue that certain canonical mathematical objects can be constructed (assuming the executability of certain basic commands sufficiently many times). This enables us, in principle, to put number theory and set theory in terms similar to Euclid's approach to geometry: we might argue, for example, that one can construct the sum $n+k$ for any given $n$ and $k$, much as Euclid shows that one can construct an equilateral triangle over a given base.

We have chosen, however, to focus on more powerful constructions – ones that not only enable us to construct particular numbers (or sets) of interest, but that enable us to construct all the natural numbers at once, or that enable us to construct initial segments of the cumulative hierarchy strong enough to satisfy axioms of standard set theory. That is, we focus on commands that enable us to produce *models* for the mathematical theory in question, rather than just to produce particular objects of interest in the theory; in this respect, we are doing Euclidean metamathematics as much as we are doing mathematics. Part of the reason for this is to enable a discussion (in the spirit of Fine (2005)) of the significance of the imperatival paradigm for the epistemology of consistency and the solution to the set-theoretic paradoxes; another part of the reason is that, as we will discuss in section 5, this approach contains within it, implicitly, the weaker and more direct Euclidean approach to the relevant domains. That said, we think that a direct presentation of number theory or set theory in the Euclidean spirit would be independently interesting, and is something we hope to pursue in future work.

## 4.1 Arithmetic

We will first consider the case of arithmetic.

### 4.1.1 Signature and Postulational Constraints

In line with the template just given, we begin by expanding our signature by primitive predicates $N$ and $S$, for *number* and *successor*, and impose the following postulational constraints.

| | |
|---:|:---|
| Successor Uniqueness | $Sxy \wedge Sxz \supset y = z$ |
| Predecessor Uniqueness | $Syx \wedge Szx \supset y = z$ |
| Number Stability | $Nx \supset \Box Nx$ |
| Successors are Numbers | $Sxy \supset Nx \wedge Ny$ |
| Predecessor Stability | $Sxy \supset \Box Sxy$ |
| Predecessor Inextensibility | $\forall y(Syx \supset \Box Fy) \supset \Box \forall y(Syx \supset Fy))$ |

### 4.1.2 The "Make Numbers!" command

With the postulational constraints in hand, we move on to the next step—devising a command $Num$ to produce the natural numbers.

**Definition 4.1.** *Let $\zeta$ be the command to make a number:*

$$!x.Nx$$

**Definition 4.2.** *Let $\sigma$ be the command to make a successor of each given number:*

$$\sigma := \forall x(Nx \to !y.Sxy).$$

**Definition 4.3.** *Let $Num$ be the command $\sigma^*_\zeta$.*

Thus, the "Make numbers!" command $Num$ is the instruction to do all $\zeta$-iterations of $\sigma$. By our **All** principles, doing this will get us anything you can get by doing $\zeta$ (effectively making zero), by doing $\zeta; \sigma$ (making zero then one), doing $\zeta; \sigma; \sigma$, etc.

### 4.1.3 Main Theorem

We will now show that if $Num$ can be done, then once it has been done every theorem of arithmetic holds of the numbers, assuming that there were no numbers to begin with.

Here and throughout we let $PA^N$ abbreviate the second-order Peano Axioms written in terms of $N$ and $S$.

**Theorem 4.4.** *If there are no numbers to begin with, then no matter how you do* $Num$*,* $PA^N$ *holds. Symbolically,*

$$\neg\exists x Nx \supset [Num]PA^N.$$

The assumption that there are no numbers to begin with is annoying, but needed. Without it, we might start out with an infinite descending chain of numbers, and would therefore be unable to prove $PA^N$ holds after we do $Num$, which only throws more numbers on the unruly pile.

Since issues of this kind will recur, we will introduce some new terminology, writing $[i]_0 p$ to mean $\neg\exists x Nx \supset [i]p$, where typically $i$ will be a $\zeta$-iteration of $\sigma$, and will say that $i$ has been done "starting from scratch" in such cases.

We now establish some auxiliary lemmas toward the main theorem. First, a lemma to the effect that the existence of a number entails the existence of its predecessors.

**Lemma 4.5** (Predecessor Lemma)**.**

$$\Box\forall x\Box\forall y(Syx \supset \Box(Ex \supset Ey))$$

*Proof.* Suppose $Ey$ and $Syx$. Necessarily, there is a plurality $X$ of all predecessors of $x$. Now, by Predecessor Rigidity, if $x$ exists, then any plurality applying only to predecessors of $x$ is such that necessarily it applies to any predecessor of $x$. Hence necessarily, if $x$ exists, then there is an $X$ such that necessarily any predecessor of $x$ is one of the $X$'s.

Since $y$ is actually a predecessor of $x$, it's necessarily possible that $y$ is so. Hence it is necessary that if $x$ exists, then there is an $X$ such that it possibly it applies to $y$. But since plurality membership is rigid, necessarily if $x$ exists, then there is an $X$ which applies to $y$. But then we have that necessarily, if $x$ exists, $y$ exists by **PlurD**.

$\Box$

Next, we show that $Num$ has the fixed point property, using results established earlier.

**Lemma 4.6.** *$Num$ is essentially creative.*

*Proof.* Follows straightforwardly from the EC lemma (lemma 3.12). $\Box$

**Corollary 4.7.** $[Num]$😴$(\sigma)$.

We are now in a position to prove the *Main Theorem.*

*Proof.* We assume throughout that $Num$ is executable, since otherwise the result is trivially true.

Assume there are no numbers. Then, letting $Z := \lambda x.\neg\exists y(S(y,x))$, we have:

(1) $[Num]\exists x(Nx \wedge Zx)$;

(2) $[Num]\forall x(Nx \supset \exists! y(Ny \wedge S(x,y)))$;

(3) $[Num]\forall x, y, z((S(x,z) = S(y,z) \supset x = y)$

(4) $[Num]\forall x(Nx \supset Z(x) \vee \exists y[S(y,x)])$

(5) $[Num]\forall F(F0 \wedge \forall n(Fn \supset Fn+1) \supset \forall n(Fn))$.

in (5), we make use of certain obvious notational conventions.

To establish (1), note that in order to do $Num$ we must do $\zeta$. Since there are no numbers, $\zeta$ does something, and introduces a number, which we may call 0.[12] By **All1**, 0 also exists after doing $Num$ from scratch. Note, moreover, that 0 is also the unique number existing after $\zeta$. If after $Num$ 0 did have a predecessor, that predecessor would have to exist also after $\zeta$, contradicting the uniqueness claim.

For (2), suppose to the contrary that $\langle Num\rangle\exists x(Nx \wedge \neg\exists y[Ny \wedge S(x,y)])$. Then $\langle Num\rangle\langle\sigma\rangle$new, since $\sigma$ will then make a successor $y$ of any such $x$, and that $y$ could not have existed before, else it would have been a successor of $x$ by Predecessor Rigidity. But this contradicts 4.7. Uniqueness of such $y$ for each $x$ also follows by the postulational constraints.

(3) follows from predecessor uniqueness.

We show (4) and (5) at once by an induction on iterations. To begin, define the transitive closure of 0 under $S$ in the standard Fregean way, and say that a number is *good* when it is in $TC(0,S)$. Note that the postulational constraints imply that if a number $x$ is good after some $\zeta$-iteration $i$ of $\sigma$, then $x$ is still good after all further iterations.[13]

Now, plainly any good number is either 0 or the successor of some number, so it suffices to show that for every $\zeta$-iteration $i$ of $\sigma$, $[i]\forall x[Nx \equiv \mathit{good}(x)]$ to get (4). (5) follows by the Fregean definition of transitive closure.

For the base case, let us show that $[\zeta]\forall x[Nx \equiv \mathit{good}(x)]$. Since there are no numbers, the effect of $\zeta$ is to introduce a single number, 0, which is clearly good. So we have the result for $\zeta$ alone.

For the successor-inductive step, suppose some $\zeta$-iteration $j$ of $\sigma$ only makes good numbers. Now either $\sigma$ does something, or it does nothing. If $\sigma$ does nothing, then all numbers are still good after $j;\sigma$. If $\sigma$ does do something, it must introduce the successor of a good number, which will then be good as well.

[12] By our determinism axiom/ the postulational constraints, this entity is the only thing we can get by doing $\zeta$, so introducing a name for it is fine.

[13] More formally: if there are no numbers, then after any $\zeta$-iteration $i$ of $\sigma$, any good number $x$ is such that, necessarily, if a $\zeta$-iteration $j$ makes $x$ from scratch, then $x$ is good after $j$ also. The proof is by induction on good numbers.

For the limit-inductive step, if all $\zeta$-iterations of $\sigma$ in some $X$ only produce good numbers, by **All2** any number produced by $\alpha X$ must be produced by one of the $X$s, and the result follows. $\square$

## 4.2 Set Theory

We now turn to the case of set theory. Our plan, just as in the case of arithmetic, is to lay down some postulational constraints, devise a command for creating the sets, and then to prove that after this command is executed, $ZFC$ holds of the sets. Here, however, matters are complicated by issues relating to Russell's paradox.

### 4.2.1 Signature and Postulational Constraints

We assume the language to contain a predicate $S$ for sethood and a relation symbol $\in$ for membership. We use $Set(x, X)$ to abbreviate

$$\forall y(y \in x \equiv Xy).$$

The postulational constraints for sets parallel those for numbers:

| | |
|---|---|
| Extensionality | $\forall x, y(Set(x, X) \wedge Set(y, X)) \supset x = y)$ |
| (Plurality-Uniqueness) | $\forall X, Y(Set(x, X) \wedge Set(x, Y) \supset X = Y)$ |
| Set Stability | $Sx \supset \Box Sx$ |
| Set-member | $x \in y \supset Sy$ |
| Membership Stability | $x \in y \supset \Box x \in y$ |
| Membership Inextensibility | $\forall y(y \in x \supset \Box Fy) \supset \Box \forall y(y \in x \supset Fy)$ |

Plurality Uniqueness, the set-theoretic parallel for Predecessor Uniqueness, is given in brackets, because it in fact follows from the definition of $Set$ and the rest of our logic.

### 4.2.2 The "Make sets!" command

It is common to describe an iterative set construction process when motivating standard axioms of set theory. Start with some things, possibly none; introduce all possible sets of things you currently have; once finished, do the same thing again; and then again; and so on, continuing indefinitely in this fashion, taking unions at limits.

We will axiomatize such a process directly here, in the form of instructions for an iterative procedure. The procedure to be iterated is: make sets of all things you currently have.

**Definition 4.8.** *Let $\rho$ be the command to take each plurality $X$ and to make a set with those things as its elements. Symbolically:*

$$\forall X!x.S(x) \wedge \forall y(y \in x \equiv Xy)$$

**Definition 4.9.** *Let Set be $\rho^*$ – the command to repeatedly introduce all sets of things in the domain.*

Our earlier lemmas on essential creativity and fixed points then imply:

**Lemma 4.10.** *Set is essentially creative.*

**Corollary 4.11.** *No matter how you do Set, doing $\rho$ will do nothing after. Symbolically, $[Set]$😴$(\rho)$.*

One might hope to leverage this fixed-point result, just as we did in the case of arithmetic, to derive the axioms of $ZFC$. But there is a problem here:

**Theorem 4.12** (Generative Russell)**.** *The Set command is not executable. Symbolically, $\neg\langle Set\rangle\top$.*

*Proof.* $\langle Set\rangle\top$ iff $\langle Set\rangle$😴$(\rho)$, by the fixed point lemma. But if $\langle Set\rangle$😴$(\rho)$ then by **All2** and **Eco1** it follows that $\langle Set\rangle\forall X\exists x Set(x,X)$, something which generates a contradiction by the familiar Russell-reasoning. □

The generative Russell shows that one cannot have an executable command that iterates $\rho$ (the set introduction command) indefinitely. Since our goal is to have devise a command for producing the sets which is plausibly executable, or at least not provably inexecutable, we will need a more subtle approach.

### 4.2.3 Hedging one's sets

There is in fact a general moral to the Russell paradox. Say that a command $i$ is a *necessary difference maker* iff

$$\Box(\langle i\rangle\top \supset \neg\text{😴}(i)).$$

Then:

**Fact 4.13.** *No essentially creative necessary difference maker has an executable indefinite iteration.*

However, what we can do (following a suggestion of Fine (2005)) is employ "hedged" versions of necessary difference makers. These are commands of the form $(p \to i)^*$, which have the intuitive force of "While $p$, do $i$!": the command says to repeatedly check and see if $p$ is true and to stop only when $p$ is false (or $i$ fails to run).

The lemmas below demonstrate the that, where $i$ is an essentially creative necessary difference maker, "While $p$, do $i$!" is executable just if one can iterate $i$ in such a way that $\neg p$ eventually results.

**Definition 4.14.** *Set*

$$\mathbf{w} := \lambda p, i : (p \to i)^*.$$

**Lemma 4.15.** *If $i$ is essentially creative, $\mathbf{w}(p,i)$ is essentially creative.*

**Lemma 4.16** (while-do lemma)**.** $[\mathbf{w}(p,i)]\neg p$ *for any essentially creative NDM* $i$*.*

*Proof.* This is trivial if $\mathbf{w}(p,i)$ isn't executable. Otherwise, if you can do $\mathbf{w}(p,i)$ so that $p$, $p \to i$ behaves like $i$, which must do something. The result follows by essential creativity of $\mathbf{w}(p,i)$. $\square$

**Definition 4.17.** *Redefine Set to be an operation of type* $t \to \iota$ *that takes in a hedge condition* $h$ *and returns the command to iterate set introduction until* $h$ *is violated. Formally,*

$$Set := \lambda h.\mathbf{w}h\rho.$$

*We will generally write* $Set_h$ *for* $Set(h)$*.*

By choosing suitable $h$, we may then use these hedged commands to generate indefinitely large universes of sets without running into the contradictions involved in $\rho^*$ – or at least, without *obviously* running in to them. (We will discuss these issues of consistency/executability in more detail later on.)

### 4.2.4 Main Theorem

Our main result is that there is a plausibly executable hedged command by which one can postulate the sets in such a way that they satisfy second-order Zermelo-Fraenkel set theory with Choice (we'll denote this with $ZFC$). As before, we use $[i]_0 p$ here in a manner analogous to its use in the arithmetic case, to mean $\neg\exists x Sx \supset [i]p$ (i.e. starting from scratch).

The requirement that our command be 'plausibly executable' perhaps requires some explanation. The reason we need the restriction is that the foregoing results how that it is actually really easy to find hedges $h$ for which we can prove that if there are no sets, no matter how you do $Set_h$, second-order $ZFC$ holds of the sets: just take $h$ to be a tautology, so that $Set_h$ is just $\rho^*$. But this case is uninteresting, since one demonstrably cannot follow this command. Thus although no matter how you do it, ZFC holds, it is equally true that no matter how you do it, the negation of ZFC holds.

There are other $h$ for which the result is non-trivial, however, assuming consistency of standard set theories. To establish this, we'll show that:

**Theorem 4.18.** *There is a hedge* $h$ *for which:*

*(i) We can prove that, if there are no sets, no matter how you do* $Set_h$*, second-order* $ZFC$ *holds of the sets (i.e.* $[Set_h]_0 ZFC^S$*), and*

*(ii) If* $ZFC$ *is possibly true, then* $Set_h$ *is possibly executable (i.e. if* $\Diamond ZFC$*, then* $\Diamond\langle Set_h\rangle\top$*).*

Note that $\rho^*$ violates (ii).

What should our "hedging" $h$ be? Our ultimate proposal for $h$ is a somewhat unwieldy four-part disjunction characterizing various structural features of the (von Neumann) ordinals. In effect, these will assert that the ordinals form an *accessible* plurality, in a sense intimately related to the familiar notion of inaccessibility from large cardinal theory. The idea is that the process of making sets will have to iterate until the ordinals no longer have any of these features, at which point the process will have gone on long enough to ensure that the axioms of $ZFC$ hold.

By way of motivation for our hedge, and in the spirit of logical modularity, it will be helpful to proceed by considering the axioms one by one, and investigating what sort of hedge is needed for deriving each of them. As we'll see, some axioms will hold no matter the hedge, while others require the hedge to have certain properties.

It will be useful to establish some terminology before we properly get going.

**Definition 4.19.** *Let $\eta$ abbreviate $h \to \rho$, leaving context to determine the relevant $h$.*

**Definition 4.20.** *When $\langle j \rangle h$ for some iteration $j$ of $\eta$, say that the hedge is* in effect at *$j$.*

We now turn to proving that the axioms of set theory follow on the assumption our hedge entails certain propositions.

**Foundation, Separation, Union, Choice: No Hedging Required!** We first tackle the axioms of Foundation, Separation, Union, and Choice. Deriving these axioms requires nothing specific of the hedge; no matter what $h$ is, if there are no sets, then after $Set_h$, these axioms are true.

First up, the axiom of foundation, which we prove in the following series of lemmas. The first will see many applications in the sequel.

**Lemma 4.21** (Elements Lemma)**.** *If $\neg Ex \wedge [\eta]Ex$, then there is a plurality $X$ with $[\eta]Set(x, X)$.*

*Proof.* The assumption entails $\eta$ is indiscriminable from $\rho$. So use **All2** and **Eco1**. □

Say that a set $x$ is *irregular* iff it has no member from which it is disjoint. Say also that a plurality $X$ is *infinitely descending* iff for each $x \prec X$, there is a $y \prec X$ such that $y \in x$.

**Lemma 4.22.** *An application of $\rho$ (or $\eta$) can make an irregular if and only if there exists an infinitely descending $X$.*

*Proof.* The right to left is obvious (and superfluous to our purposes). For the left to right, observe that if $\eta$ produces an irregular $x$, then the plurality $X$ of $x$'s members, which exists before $\eta$ by the Elements Lemma, is infinitely descending. □

**Lemma 4.23** (Foundation)**.** *Let $h$ be any hedge. Then, for any iteration $i$ of $\eta$ (starting from scratch), if there are no sets and you can get a set $x$ by doing $i$, either $x$ is empty or else it has a member with which it has disjoint intersection.*

*Proof.* By the previous lemma, it is enough to show that after each iteration $i$ of $\eta$, there is no infinitely descending plurality. But this is a routine induction on iterations. $\square$

This result is pleasing, in that it shows the standard axiom of foundation does not need to be assumed in this setting but may be proved from more basic features of the logic of set construction. (More cautiously: it follows if we assume we start from scratch.)

Before we can prove the others, we need some more technical lemmas. The first uses results already established on indefinite iterations to show that any set that is produced by $Set_h$ is 'first' produced by doing $\eta$ after doing some iteration $i$ of $\eta$.

**Lemma 4.24** (Birthday Lemma)**.** *When starting from scratch,*

$$[Set_h]Ex \wedge S(x)$$

*implies there is an iteration $i$ of $\eta$ such that $[i]\neg Ex$ and $[i;\eta]Ex$.*

*Proof.* This is just lemma 3.9.

$\square$

The next lemma parallels the Predecessor Lemma (Lemma 4.5) from before.

**Lemma 4.25** (Member Dependence Lemma)**.** *For any possible set $x$ and any possible member $y$ of $x$, necessarily, if $x$ exists then $y$ does as well and is a member of $x$. Symbolically:*

$$\Box\forall x\Box\forall y(y \in x \supset \Box(Ex \supset (Ey \wedge y \in x)))$$

*Proof.* By parallel argument to the Predecessor 4.5. $\square$

And finally, we can use the Birthday and Member Dependence lemmas to prove a lemma to the effect that whenever you form a set $x$ by doing $\eta$, the "stage" at which you form it also contains pluralities corresponding to all possible combinations of elements of $x$.

**Lemma 4.26** (Stage Lemma)**.** *For any plurality $X$ of elements of a given set $x$, then for any possible member $z$ of $X$, necessarily: if $x$ does not exist but can be attained by doing $\eta$, then $z$ exists.*

*Symbolically:*

$$\forall X\forall x(\forall y(Xy \supset y \in x) \supset \Box\forall z(Xz \supset \Box(\neg Ex \supset ([\eta](Ex) \supset Ez))))$$

*Proof.* Suppose $E(X)$ and $E(x)$, and every $y$ in $X$ has $y \in x$. Now suppose for reductio that it is possible that there is a $z \in X$ such that possibly $\neg Ex \wedge [\eta](Ex) \wedge \neg Ez$. Since $\Diamond Xz$, $Xz$ by **PlurStab**. Then by **PlurD** $z$ exists, since $X$ does, and hence is a member of $x$.

Now, since it is possible that $\neg Ex \wedge [\eta]Ex \wedge \neg Ez$, we have

$$\Diamond\exists Y([\eta](Ex \wedge Set(x, Y)) \wedge \neg Ez) \qquad (2)$$

Since (in fact) $z \in x$, it is necessary that if $x$ exists, then $z$ exists and is a member of $x$, by the Set Dependence Lemma. Hence it is possible that $\neg Ez$ and that there is an $Y$ such $[\eta](Ex \wedge z \in x \wedge Set(x, Y))$. Hence $\Diamond Yz$, so by **PlurStab**, $\Box Yz$. But now we may derive

$$\Diamond\exists Y([\eta](Ex \wedge Set(x, Y)) \wedge Yz \wedge \neg Ez)$$

using (2), which in turn yields a contradiciton by **PlurD**.

□

We are now in a position to derive Separation and Union.

**Lemma 4.27** (Separation)**.** *For any hedge h:*

$$[Set_h]_0\forall x\exists y\forall z(z \in y \equiv z \in x \wedge \phi(z))$$

*Proof.* Suppose $x$ exists after $Set_h$. By Plural Comprehension, there is a plurality $Y$ of all those members $z$ of $x$ such that $\phi(z)$.

Now by the Birthday Lemma, there is an iteration $i$ of $\eta$ such that after $i$, $x$ does not exist, but after doing $\eta$ additionally it does. Now form, after $i$, $Y'$, the plurality of all and only those things which belong to $Y$. Since $\eta$ does something, after $\eta$ there is a set $y$ whose members are exactly those of $Y'$.

Now, after $Set_h$, $y$ exists by **All1**. We need only show $Set(y, Y)$. Obviously, any member of $y$ belongs to $Y$, by the rigidity of plural membership. Suppose $Yz$: by the Stage Lemma, $z$ exists after $i$, and hence belongs to $Y'$, and so is a member of $y$.

□

**Lemma 4.28** (Union)**.** *For any hedge h,*

$$[Set_h]_0\forall x\exists y(Sy \wedge \forall z(z \in y \equiv \exists w(w \in x \wedge z \in w)))$$

*Proof.* Suppose $x$ exists after $Set_H$. Let $Y$ be the plurality of elements of elements of $x$ (which exist after $Set_h$, that is).

By the Birthday Lemma, we may find an iteration $i$ of $\eta$ with $[i]\neg Ex$ but $[i;\eta]Ex$. Form after $i$ the plurality $Y'$ of things belonging to $Y$. Then after doing $\eta$ on top of $i$ there is a set $y$ with $Set(y, Y')$.

By **All1**, $y$ exists after $Set_h$. We need only show $Set(y, Y)$. Obviously, any member of $y$ belongs to $Y$, by the rigidity of plural membership. Suppose $Yz$: then $z \in z'$ for some $z' \in x$, and by the Subset Lemma, $z'$ exists after $i$, and hence, by the Set Dependence Lemma, $z$ exists after $i$. Thus $z$ belongs to $Y'$, and so is a member of $y$.

□

Lastly, the Axiom of Choice follows no matter the hedge, given our background plural logic.

**Lemma 4.29** (Choice)**.** *If there are no sets, then no matter how you do $Set_h$, the axiom of choice will be true.*

*Proof.* We take the form of choice involving disjoint non-empty sets.

Suppose there are no sets, and that after $Set_h$, $x$ is a set $X$ of non-empty disjoint sets.

Then by the Birthday Lemma, there is some iteration $i$ of $\eta$ such that $[i](\neg Ex \wedge [\eta]Ex)$. After $i$, then, all the members of $x$ exist. Let $\mathsf{X}$ be the rigid property applying to a plurality $X$ just in case there is a member $y$ of $x$ whose members are exactly the $X$'s. Then by Plural Choice, there is a functional relation $R$ such that $\exists y R(X, y) \supset \mathsf{X}(X)$ and if $R(X, y)$, then $y$ is among the $X$'s. Let $Z$ be the plurality consisting of those objects $z$ such that $\exists X R(X, z)$. Then after $i; \eta$, there is a set whose members are exactly those in $Z$, and this set witnesses the Choice principle. □

Our results so far have not depended on particularities of the hedge $h$: if there are no sets, then no matter the hedge $h$, after $Set_h$, the sets satisfy Union, Separation, and Foundation, and Choice. To derive the remaining axioms of $ZFC$, however, we have to make more demands on our hedge. We first discuss pair, power and infinity together before moving on to replacement.

**Pair, Power, Infinity: Bounding Hedges** The axioms of pair and power require us to place non-trivial constraints on $h$. For example, set $h := \neg\exists x Sx$; clearly $[Set_h]_0\neg$"pair".

The fix here is to choose $h$ so that it fails only when there is no "final round" of applications of $\eta$. We can concoct propositions with this property by observing that certain features of the outputs of iterations of $\eta$ tell you how long the iteration has been going on: in particular, the length of (von Neumann) ordinals that have been produced. Thus, by choosing $h$ to fail only when the von Neumann ordinals produced are unbounded, we acheive the affect of stopping only when closed under iterations of $\rho$.

Let us be a bit more preicse. Say that a plurality $X$ of ordinals is *bounded in* $Y$ iff some ordinal in $Y$ has every ordinal in $X$ (other than itself) as a member:

$$\mathsf{B} := \lambda XY.\exists x[Yx \wedge \forall y(Xy \wedge y \neq x \supset y \in x)]$$

We will call $X$ *bounded in itself* when $\mathsf{B}XX$.

We then offer

**Definition 4.30** (Ordinal Bound)**.** *Let $\Omega$ be the plurality of all ordinals (in whatever context is ambient*[14]*), and let $\mathsf{o}$ be the proposition that the ordinals are bounded in themselves.*

$$\mathsf{o} := \mathsf{B}(\Omega, \Omega).$$

*Say that a hedge* requires unboundedness *iff $\mathsf{o}$ (strictly) entails $h$, i.e. iff*

$$\Box\mathsf{o} \supset h.$$

[14] Technically, we are appealing to plural comprehension on the formula saying that $x$ is an ordinal.

Our claim now is that if $h$ requires unboundedness, then $Set_h$ yields a model for pair and power (starting from scratch).

Toward proving this, we first sharpen the idea that we can use the ordinals to measure how long iterations have been going on in a series of lemmas.

**Lemma 4.31.** *For any iteration $i$ of $\eta$, no matter how you do it, the resultant plurality of (von Neumann) ordinals is transitive in the sense that $Xy$ and $x \in y$ implies $Xx$ and well ordered by membership in the sense that $\lambda x, y.x \in y \wedge Xx \wedge Xy$ is a well-order.*

*Proof.* A routine induction on iterations. □

**Lemma 4.32.** *Suppose that $h$ requires unboundedness. Then, for any iteration $i$ of $\eta$,*

$$[i](\neg Ex \wedge [\eta]Ex) \supset [i;\eta]h.$$

*Proof.* Let $X$ be the plurality of ordinals existing after $i$. Since doing $\eta$ after $i$ creates something new, after $i$, the while-condition obtains, and so, after $\eta$, there is an ordinal $\alpha$ whose members are exactly the $X$'s. But $\alpha$ cannot itself be among the $X$'s, else $\alpha \in \alpha$, contradicting foundation. So $\alpha$ only exists after the application of $\eta$.

Moreover, $\alpha$ contains any other ordinal that exists after doing $\eta$ (after $i$): hence the ordinals are bounded in themselves and so the hedge obtains. □

We are now in a position to establish Pairing and Power Set, conditional on using a suitable hedge condition.

**Lemma 4.33** (Pair)**.** *Suppose that $h$ requires unboundedness. Then,*

$$[Set_h]_0 \forall x, y \exists z \forall w(w \in z \equiv w = x \vee w = y).$$

*Proof.* Let $x$ and $y$ be sets which exist after $Set_h$. Then there are iterations $i_x$ and $i_y$ of $\eta$ such that $[i_x]\neg Ex \wedge [i_x;\eta]Ex$ and $[i_y]\neg Ex \wedge [i_y;\eta]$. Then after $[i_x;\eta]$ there is an ordinal $o_x$ containing all other ordinals existing after $i_x;\eta$, and similarly an ordinal $o_y$ containing all other ordinals existing after $i_y;\eta$.

Now consider the plurality $X$ containing just $i_x;\eta$ and $i_y;\eta$, and the corresponding $\alpha(X)$, which is an iteration of $\eta$. Then either $o_y \in o_x$ or $o_x \in o_y$, and hence either $o_x$ or $o_y$ contains all other ordinals existing after $\alpha(X)$, since all such ordinals exist either after $i_x;\eta$ or $i_y;\eta$. Hence after $\alpha(X)$, the hedge is in effect, and so $\alpha(X);\eta$ will then create a set of $x$ and $y$. □

**Lemma 4.34** (Subset Lemma)**.** *For any sets $x$ and $y$, if $y \subseteq x$, then necessarily: if $x$ does not exist but exists after $\eta$, then there exists after $\eta$ a $y'$ necessarily coextensive with $y$:*

$$\forall x \forall y((Sx \wedge Sy \wedge y \subseteq x) \supset \Box((\neg Ex \wedge [\eta]Ex) \supset [\eta]\exists y'(\Box \forall z(z \in y' \equiv z \in y))))$$

*Proof.* Because $\Box$ is an S5 modality, it suffices to show:

$$\Diamond(Ex \wedge Ey \wedge y \subseteq x) \supset ((\neg Ex \wedge [\eta]Ex) \supset [\eta]\exists y'(\Box\forall z(z \in y' \equiv z \in y)))$$

So suppose it is possible that $Ex \wedge Ey \wedge y \subseteq x$, and suppose $\neg Ex \wedge [\eta]Ex$. Let $Y$ be the plurality of all $z \in y$. Then after $\eta$, there is a $y'$ with $\forall z(z \in y' \equiv Yz)$.

We aim to show that, after $\eta$, $y'$ is necessarily coextensive with $y$. Suppose not: then either it is possible that there is a $z \in y$ but $z \notin y'$, or $z \notin y$ but $z \in y'$. Suppose the first. Since it is possible $z \in y$, it is necessarily so, and hence possibly $z \in x$. It then follows by the Stage Lemma that $z$ exists, and so $z \prec Y'$. Hence after $\eta$, $z \in y'$ and so necessarily so. If, on the other hand, it is possible that $z \in y'$, then $z \in y'$ after $\eta$, and so $z \prec Y'$. Again by the Stage and Membership Dependence lemmas, $z'$ must exist before $\eta$, and so $z \in y$, and so necessarily so. $\Box$

**Lemma 4.35** (Power)**.** *Suppose that $h$ requires unboundedness. Then:*

$$[Set_h]_0 \forall x \exists y \forall z(z \in y \equiv z \subseteq x).$$

*Proof.* Let $x$ exist after $Set_h$, and let $X$ be the plurality of all its subsets. By the Birthday Lemma, there is some iteration $i$ such that $[i](\neg Ex \wedge [\eta]Ex))$. By plural comprehension, after $i;\eta$, there is a plurality $X$ of all subsets of $x$. Since the hedge is in effect after $i;\eta$ (Lemma 4.32), after $i;\eta;\eta$, there is a set $y$ such that $\forall z(z \in y \equiv Xz)$.

By **All1**, $y$ exists after $Set_h$, since $i;\eta;\eta$ is an iteration of $\eta$, and moreover by the rigidity of membership, $\forall z(z \in y \equiv Xz)$ as well. By the Subset Lemma, for every subset $x'$ of $x$, there is a necessarily coextensive set $y'$ which exists after $i;\eta$, and hence which belongs to $y$. But then since $y'$ and $x'$ both exist after $Set_h$ and are coextensive, $y' = x'$. Hence $y$ contains every subset $x'$ of $x$. $\Box$

Infinity, likewise, is handled by asserting the ordinals have an unbounded proper initial segment.

**Definition 4.36.** *Say that a hedge* requires infinitude *iff the proposition* $\mathsf{f}$ *that* $\Omega$ *is finite (strictly) entails $h$.*

**Lemma 4.37** (Infinity)**.** *If $h$ requires infinitude, then no matter how you do $Set_h$, the axiom of infinity obtains.*

**Replacement: Traversability and weakness.** We will build up to Replacement in two steps. First, we introduce a hedge condition that ensures Replacement holds when restricted to ordinals. Then, we introduce a further hedge condition that ensures every set is equinumerous with some ordinal, from which replacement follows in its full form.

Say that a plurality $X$ of ordinals is *traversable* iff there is a map that has as domain some member of $X$ and unbounded range in $X$:

$$\mathsf{T} := \lambda X.\exists y, F[Xy \land F : y \to X \land \neg \mathsf{B}(ran(F), X)]$$

Here, $F$ is a second-order variable: we can take it to be a relation on sets, for instance.

**Definition 4.38.** *Let* $\mathsf{t}$ *be the proposition that the ordinals are traversable:*

$$\mathsf{t} := \mathsf{T}\Omega$$

*Say that a hedge* requires intraversability *iff it is (strictly) entailed by* $\mathsf{t}$*:*

$$\Box(\mathsf{t} \supset h)$$

**Lemma 4.39** (Ordinal Replacement)**.** *Let $h$ be any hedge that requires intraversability. Then, if there are no sets, after doing $Set_h$, if $\alpha$ is an ordinal and $F(\beta)$ is a set for each $\beta \in \alpha$, there is a set whose members are exactly $F[\alpha]$, the image of $\alpha$ under $F$.*

Since the proof of this lemma is a little involved we will start by outlining the strategy: we will first show that any set made from scratch by an iteration of $\eta$ can be associated (after the iteration) with a certain ordinal, corresponding to its rank in standard set theory; we will then show that, if the supremum of the successors of the rank ordinals associated with the sets in some plurality $X$ is created (from scratch) by some iteration of $\eta$, then the set whose elements are the $X$ is also created by $\eta$. With those lemmas in tow, proving ordinal replacement from the relevant hedge condition is a simple matter.

On to the details. First, let us agree to call a relation $R$ between sets and ordinals a *rank relation* when the following two conditions obtain:

1. $R(x, 0)$ holds iff only non-sets are members of $x$;
2. If $x$ has any set-elements, then $R(x, \alpha)$ holds only if every set-element of $x$ is related to an ordinal $\beta < \alpha$.[15]

Note that the 'union' of rank relations is always a rank relation.

Next, say $x$ *has a rank* if it is related by a rank relation to some ordinal; and that it is *of rank* $\alpha$ if the least ordinal related by a rank-relation to $x$ is $\alpha$. Note that if $x$ has a rank, then it has rank $\alpha$ for some $\alpha$. We will write $Rk(x, \alpha)$ to mean that $\alpha$ is the least ordinal related by $x$ in a rank relation, and $Rk_{\leq}(x, \alpha)$ to mean that $\alpha$ and $x$ are related to by some rank relation.

**Lemma 4.40.** *If there are no sets, then after any iteration $i$ of $\eta$, every set $x$ has a rank.*

[15] A set-element is an element that is a set; we leave open the possibility that a set has non-set-elements, to allow for impure set theory.

*Proof.* By induction on iterations. The base and limit cases are immediate. For the successor, in the non-trivial case $x$ is made by an application of $\eta$, and all its elements exist after $j$. By the IH they all have ranks. The union of the rank relations of its elements is a rank relation, $R$, and exists by comprehension. Moreover, the set of all ordinals after $j$ is made after $\eta$ and is a new ordinal. The relation like $R$ but that assigns that new ordinal to $x$ is then the desired rank relation. $\square$

**Lemma 4.41.** *For any $x$ and $\alpha$, $Rk_{\leq}(x, \alpha)$ if and only if $[\eta]Rk_{\leq}(x, \alpha)$. Consequently, the same holds for $Rk(x, \alpha)$.*

*Proof.* Immediate. $\square$

**Lemma 4.42.** *If there are no sets, then after any iteration $i$ of $\eta$, if the supremum of the successors of the rank ordinals of sets in some plurality $X$ exists, then so does the set whose elements are the $X$.*

*Proof.* First, note that it is easy to show that after any iteration $i$ of $\eta$ (from scratch), the rank ordinal of a set is always at least the supremum of the successors of the rank ordinals of elements, with the proof an induction on sets.

For $X$ a plurality of set let $R_X$ be the plurality of the rank ordinals of elements of $X$; and for a plurality $O$ of ordinals, let $\sigma O$ be the supremum of the successors of ordinals in $O$. Finally, let us abuse notation and write $Set(X)$ for that object $x$ (if it exists) with $Set(x, X)$. We therefore seek to prove that

$$[i]_0 \forall X(E\sigma R_X \supset ESet(X)) \tag{3}$$

which we do by induction on iterations.

For the successor case, suppose $[j;\eta]EX \wedge E\sigma R_X$. Suppose first $\eta$ does nothing; then by the IH there is a set of the $X$'s. Suppose second it does do something. Then form after $j$ the plurality $X'$ of all and only those things belonging to $X$. Then since $\eta$ does something, there is after $\eta$ a set $x$ with $Set(x, X')$. We need only show now that $Set(x, X)$ as well. Obviously, if $y \in x$, then $Xy$ by the rigidity of plural membership.

Now suppose $Xy$. Note that $z$ must exist after $j$. For, since it exists after $j;\eta$, we know that after $j$ the plurality $Y$ of its elements exists, by the Elements Lemma. But since $\sigma R_Y \in \sigma R_X$ exists after $j$, by the IH, $y$ must already exist after $j$ as well. Hence after $j$ $Ey$ and $X'y$, and so $y$ belongs to $x$.

For the limit case, suppose $[\alpha(F)]EX \wedge E\sigma R_X$. Then there is an iteration $i \in F$ such that $[i]E\sigma R_X$. We need only establish that $[i]EX$. This we do by induction on sets after $\alpha F$. In fact it will be easier to prove something slightly stronger. For a plurality $Z$, say that a plurality $T_Z$ is the transitive closure of $Z$ under membership when every $Z$ is a $T_Z$, and whenever $T_Z x$ and $y \in x$ then $T_Z y$. (Defining this in our logic is a simple application of Frege's ancestral trick.) Then we show

$$[\alpha(F)]\forall x(T_X x \supset \Box(\neg\exists z Sz \supset [i]Ex)) \tag{4}$$

which implies the result by plural comprehension within $i$. We prove (4) by induction on membership. Suppose every member of $x$ has the property indicated, and suppose $x$ itself is in $T_X$. Since every member $y$ of $x$ is in $T_X$ we may conclude that $\Box\neg\exists x Sx \supset [i]Ey$. Hence the plurality of members of $x$ exists after $i$, and so by the $IH$ we may conclude that $x$ itself exists after $i$. The result follows.

$\square$

We proceed to prove Ordinal Replacement (Lemma 4.39).

*Proof.* We know every set after $Set_h$ has a rank. So after $Set_h$, we have $F$ as given but also $F'$ associating each set with its rank. By intraversibility, the supremum of the range of (the successors of) $F'$ exists. The result now follows from the previous lemma.

$\square$

To establish Replacement proper, we need a hedge that requires every set to be isomorphic to an ordinal. This will allow us to take any function on set to a function on the corresponding ordinal, and hence appeal to ordinal replacement to secure full replacement.[16]

First, the condition on hedges. Say that some ordinals $X$ are *weak* just in case, for some $\alpha$ among them, $\alpha <_c X$ but the plurality of subsets of $\alpha$ is cardinally at least as great as $X$. (Here $<_c$, the relation of being less than in cardinality, and other cardinal comparison notions are defined in higher-order terms.)

**Definition 4.43.** *Say that a hedge* requires strength *iff it is (strictly) entailed by the proposition,* we*, that the ordinals are weak.*

Next, we will show how if our hedge requires infinitude, strength and intraversibility, then after doing it every set is isomorphic to a Von Neumann ordinal; connecting this with ordinal replacement then yields a quick proof of full replacement.

Toward that, we need still a further supporting lemma, one reflecting a central part of the import of the standard axiom of replacement: namely, that if our hedge requires intraversibility, then ordinal addition is everywhere defined after doing $Set_h$.

Say a relation $R$ is *summative* when

- $R(\alpha, 0, \gamma)$ only if $\alpha = \gamma$;
- $R(\alpha, \beta + 1, \gamma)$ only if $\gamma$ is $\delta + 1$ and $R(\alpha, \beta, \delta)$;
- $R(\alpha, \lambda, \gamma)$ only if $\gamma$ is the limit of $\delta$ with $R(\alpha, \beta, \delta)$ for $\beta < \alpha$.

[16]The situation here is a little subtle, in that how things play out will depend on whether we take ourselves to be concerned with pure sets or impure sets as well. We will focus on the case of the pure sets here, and indicate in footnotes how that of impure sets ought to be handled.

Now, let $\Sigma(\alpha, \beta, \gamma)$ iff every summative relation relates them; and note that $\Sigma$ is functional in $\alpha$ and $\beta$. We will write $\alpha+\beta$ for the $\gamma$ (if it exists) with $\Sigma(\alpha, \beta, \gamma)$. Note also that, if defined, $\alpha + \beta$ satisfies the standard recursion equations for ordinal addition: $\alpha + (\beta + 1) = (\alpha + \beta) + 1$, etc.

If there are some ordinals, then $\Sigma$ is always *somewhere* defined: it is easy to see that $0 + 0 = 0$ for example. On the other hand, after an arbitrary iteration of $\eta$ there is no reason to believe that $\Sigma$ is *everywhere* defined, and indeed it is not hard to see that this is not generally the case. However, this will hold after $Set_h$ for suitable $h$:

**Lemma 4.44** (Ordinal Addition)**.** *If $h$ requires infinitude and intraversibility, then after $Set_h$, $\alpha+\beta$ exists for all ordinals $\alpha$ and $\beta$ (if we started from scratch).*

*Proof.* Induction on ordinals $\beta$ in parameter $\alpha$. For the successor step, use infinitude, and for the limit, use intraversibility. □

We may now show:

**Lemma 4.45.** *Suppose $h$ requires infinitude, strength and intraversibility. Then: if there are no sets, then after $Set_h$, every set injects into an ordinal.*

*Proof.* Let us say that a set or plurality *has a cardinality* if it injects into an ordinal. (Note that if a set injects into an ordinal, then it is trivial to find a bijection between that set and an ordinal, by taking the least ordinal into which it injects.)

For each ordinal $\alpha$, let $V(\alpha)$ be the plurality of $x$ with rank $\alpha$. Since the elements of any set are all contained among some $V(\alpha)$, it suffices to show that (after $Set_h$) $V(\alpha)$ has a cardinality for each $\alpha$. This we show by induction on ordinals.

Base case: when $\alpha = \varnothing$, $V(\alpha)$ is also empty.

Successor step: By the IH, $V(\alpha)$ has a cardinality, $\beta$. By strength, the subsets of $\beta$ are cardinally smaller than the ordinals, and so we may choose $\gamma$ greater in size than the subsets of $\beta$. But clearly $V(\alpha + 1)$ is bounded by $\gamma$.[17]

Limit step: if it holds for every $\alpha < \lambda$, then every $V(\alpha)$ has a cardinality $\gamma_\alpha$. By intraversibility, these have a supremum, $\gamma$; we show that this supremum is the carindality of $V(\lambda)$.

Let the $F_\alpha$ be injections from each $V(\alpha)$ into ordinals $\gamma_\alpha$ for $\alpha < \lambda$. Then, we may produce a relation $R$ that associates each $\alpha < \lambda$ with an injection $G_\alpha$ into $\gamma_\alpha$ which, in addition, agrees with all $G_\beta$ for $\beta < \alpha$ in the common domain.

To do so, we prove by induction on ordinals that for each $\alpha < \lambda$, a relation exists that relates all ordinals up to $\alpha$ to such an injection. The base case is

[17] If we are dealing with impure sets, this argument will not do. We may instead argue as follows.

Say that the ordinals $X$ are *big* just in case, for any $\alpha$ among them, $X_\alpha <_c X$ and $U <_c X$, where where $X_\alpha$ is the plurality of members of $\alpha$, and $<_c$ is "strictly smaller in cardinality than" (which is to be defined in higher-order terms), and $U$ is the plurality of non-sets. We then modify to lemma to require that if the ordinals are traversible and neither big nor strong, then the hedge obtains.

easy. In the successor $\beta+1$, just take the injection $F_{\beta+1}$ on $V_{\beta+1}$, add $\gamma_\beta$ to the values of $F_{\beta+1}$ in $V(\beta+1) - V(\beta)$, and use the values of the inductively given $G_\beta$ for arguments on $V(\beta)$. The limit case is immediate.

But now the unions of the injections $F_\alpha$ for $\alpha < \lambda$ is the desired injection for $V(\lambda)$.

□

We may now establish replacement.

**Lemma 4.46** (Replacement)**.** *Suppose that the hedge requires intraversability and weakness. Then, after doing $Set_h$, Replacement holds.*

*Proof.* Since traversability implies the hedge obtains, Ordinal Replacement holds. Since the traversability and the weakeness of the ordinals implies the hedge obtains, for every set there is some ordinal no smaller in cardinality to that set. Given a function $F$ on a set, we can then define a function $F'$ on an ordinal than which it is no smaller in cardinality, and argue by Ordinal Replacement to the desired set. □

**Corollary 4.47.** *Suppose $h$ is a hedge that requires unboundedness, infinitude, intraversability and strength. Then the main theorem holds for $h$.*

*Proof.* We have already established that (i) of the main theorem holds (Lemmas 4.23, 4.27, 4.29, 4.33, 4.35, 4.37, 4.46 together with the PCs). It remains to establish (ii). It is enough to show that $ZFC$ entails $Set_h$ can be done; but this is easy to see, since if $ZFC$ is true the ordinals violate all our hedging conditions and hence $Set_h$ amounts to doing nothing. □

That concludes our derivation of the axioms of set theory from the assumption of the executability of $Set_h$. We now turn to philosophical discussion.

# 5 Discussion and Assessment

In this paper, we have developed a logic that enables one to reason about the effects certain iterative, constructive procedures would have on a given domain if carried through; and we have showed how to use that logic, together with the assumption that certain commands are executable, to argue that domains of mathematical interest in number theory and set theory can be produced. In this section, we turn to a brief discussion of the potential for philosophical significance in this program.

We begin by discussing the broadly Euclidean paradigm in the foundations of mathematics, according to which arguments that certain objects can be constructed form a legitimate part of mathematics, in the light of the technical developments we have set out; in particular, we will discuss certain objections one might raise to the application of these imperatival ideas to contemporary mathematics. Ultimately we find these wanting: it is our belief that there is a legitimate conception of contemporary mathematics underlying our formalisation, indeed one that is well worth scrutiny by philosophers of mathematics.

After that we will move on to a discussion of some potential advantages offered by such imperatival views, advanced in a similar setting by Fine: in particular, we will discuss the question of whether the imperatival approach yields benefits for the epistemology of consistency, or for our understanding of the set-theoretic paradoxes.

## 5.1 Mathematics in the imperatival idiom?

We set out this investigation by appeal to a broadly 'Euclidean' paradigm for giving mathematical arguments. In this paradigm, one assumes 'axiomatically' that one has the ability to perform certain acts, and then argues that certain states of affairs can be brought about by iterating said acts. Mathematics, so understood, is at least in part a science of possible construction, in which one describes means for bringing about states of affairs, and not just a description of fixed states of affairs obtaining for antecedently given abstract objects.

One might have various sorts of questions about and objections to this general picture of the nature of mathematics, and the extent to which our logical work manifests it. In this subsection, we discuss such issues.

We should begin at the outset with a simple, scholarly concern: how "'Euclidean" is all this? The short answer is we aren't Euclid experts and can't really say, but that equally we don't think it a very important question. Although reading translations of Euclid is certainly suggestive of the idea we are discussing, we are not adamant that it or anything like it was going on in the Elements.

Another, slightly more difficult issue is one that we already raised briefly in section 4: one might object that we have not really shown how to do contemporary *mathematics* in the imperatival idiom, but rather only to do *metamathematics*, that is, we have only shown how to *build models* for arithmetic, but not how to *do arithmetic* in these terms. The construction of a Euclidean triangle is properly part of Euclidean geometry, in a way that the construction of a model for arithmetic is not properly part of arithmetic.

We don't find this objection too concerning. For a start, let's note that an approach to arithmetic proper is already implicit in what we have. (The situation is analogous with set theory, but we'll focus on the simpler arithmetic case here.) That is, if we assume that there are no numbers, and we assume that $Num$ is executable, then we can view theorems of arithmetic as describing states of affairs that can be brought about by doing various $\zeta$-iterations of $\sigma$, or (in quantificational cases) as results describing states of affairs that can always be brought about by continuing $\zeta$-iterations of $\sigma$ by further iterations of $\sigma$. For example, consider the theorem of arithmetic that says there are infinitely many primes. We have $[Num]_0$"for every prime there is a greater one"; by our **All** clauses, this reduces to the claim that for every $\zeta$-iteration of $\sigma$, one can iterate $\sigma$ further to produce a new prime. Thus, implicit in the characterisation for models of arithmetic given here is a fundamentally different approach to arithmetic in general, one in the genuinely Euclidean idiom.

A still more serious concern runs as follows. Even if one concedes that we can naturally think Euclidean geometry and even other sorts of geometry as "sciences of construction", in which recipes and procedures play an important role and are relevant items in arguments, this is simply not going to be so for the cases of interest to us, in particular in arithmetic and set theory. There are two salient points of difference that one might point to: first, that while we might have a handle on what it is to construct a triangle or a line, we have no corresponding idea of what it could mean to make a set or a number; and second, that were the constructions of Euclidean geometry require at most finitely many iterations of a process, our constructions require transfinitely many.

On the first point, about not knowing what it means to construct sets or numbers, there are two reasonable lines of reply, the one concessive and the other agressive.

The concessive line says: sure, it is nonsense to talk about literally constructing sets and numbers. But one can also play along, talking as if one could construct sets and numbers, and perhaps get some useful results by doing so. We will elaborate the useful results you might hope for much more below, but roughly one might think that be explicitly describing recipes for constructing the sets or the numbers one might get evidence for the consistency of these theories.

The more aggressive line would argue that we can make sense of processes in which sets and numbers are made. In the case of set theory, one can think of a set as something one gets if one by introducing an object as a representative for a fixed plurality of other objects; then to make a set is just to make a thing and fix it once and for all as the representative of those others. Or again in the case of number theory, one might think of a number as an entity introduced as a representative of a finite cardinality quantifier. All that is required of such an object (in either case) is that it be an object, and that it be uniquely determined by the plurality/cardinality quantifier, meaning that we do not assign it to any other plurality or cardinality quantifier in the course of further constructions – something that is required by the postulational constraints anyway. While this may be a non-standard metaphysics of sets and numbers, it does not seem to us to be incoherent, and is well-suited to making sense of the idea that sets or numbers might be literally created: just make an object, say a paper hat, and then assign it to the relevant entity. Now of course this sort of view does not immediately square with the logic we have presented, which requires for example that anything which can be the set of some things necessarily is their set; but we can view this sort of postulational constraint equally well as a restriction on the rules for the construction sequence we are interested in – once you've made something the set of some things, never make a different thing the set of them or it the set of something else – rather than reflecting an essentialist feature of the nature of sets.

That leaves the second point, about the need for transfinite iteration. We don't dispute, of course, that infinity is present in our theory in a way that it is not in Euclid: the instructions to produce a model for set theory or arithmetic essentially involve transfinite iterations, whereas nothing like that is involved in Euclidean geometry. This does not detract however from those respects in which our proposal is Euclidean.

But perhaps one thinks that the Euclidean paradigm is stronger philosophically because it only requires finite iterations. For example, maybe one has a view of mathematics as the science of constructions possible, not for some god-like creatures, but for finite creatures like us. Then Euclid's constructions, being finite, have a validity ours lack. But is it so clear Euclid is any more human than we are? The Euclidean idiom taken seriously for geometry already involves a reasonable amount of idealisation: one of Euclid's already-cited postulates was "to produce a finite line continuously in a straight line", but as Simplicius already sagely observed in his 6th century commentary on the Elements, the postulation that a straight line can be extended from Aries to Libra is, when taken literally, somewhat "foolhardy".[18] Indeed we know that the production of a genuine equilateral triangle in nature is impossible.

Any reasonable imperatival formulation of mathematics will therefore involve idealisation and abstraction. In this then our approach is common with what we attribute to the Euclidean, even if our approach has more of the flavor of idealisation about it given its infinitary nature. We certainly differ from Euclid in brooking infinity; but it's a difference in degree, not kind.

There are other ways the finite and infinite come apart in our system, and some of these ways may have philosophical consequences—in particular, in ways with consequences to what light our system might shed on the consistency of theories like $\mathsf{PA}_2$ and $ZFC_2$.

Fine (2005) suggests that a broadly imperatival or procedural (or, to use his term, "postulational") approach to mathematics might provide some new evidence for the consistency of mathematical theories like $PA_2$ and $ZFC_2$. The idea is that if we can show that some command $i$ is executable, and that $\langle i \rangle PA_2$, for instance, then we have some good evidence that $PA_2$ is consistent.

Although the link between executability and consistency is subtle in various ways, to be discussed below, nevertheless there is clearly some scope for an argument of this kind. From the exectuability of $Num$, for example, it follows that $\Diamond PA_2$, something which in turn (given model-theoretic consistency of our logic[19]) entails model-theoretic consistency of arithmetic.

But what grounds do we have for thinking $Num$ and $Set_h$ *are* executable? A natural thought here (a form of which can also be found in Fine (2005)) is that we might hope to provide an inductive argument from the premises that the basic commands are executable – say to introduce successors or to introduce sets – to the conclusion that the iterative commands in question are also executable. Such an argument would necessarily appeal to the idea that executability somehow "flows upward" from the basic commands to the complex ones. Indeed, as mentioned previously, our logic already proves some links between the executability of complex commands and their more basic constituents. For instance: if both $i$ and $j$ are necessarily executable, then so is $i; j$ (this is a

[18] Translation from Al-Nayziri. See final page of Hodges (2013).

[19] We have not provided a model theory here; its provision is for another time. But we take it to be pretty clear that a modification of a possible worlds model theory, along the lines of (Harel et al. 2000), will do the trick.

quick consequence of **Then**). In other words, if you string two or even finitely many necessarily executable commands together, the result itself is necessarily executable. If we assume that each of Euclid's basic construction procedures is necessarily executable, then the whole complex procedure, being but a finite sequence of basic ones, is provably so as well.[20]

It was with an eye exactly to give such proofs of executability that we designed $Num$ and $Set_h$ in the way we have, as complex imperatives based on more basic ones of whose good standing we can more easily convince ourselves. There are, after all, other imperatives we could have written down that would have produced the natural numbers or the sets. Consider for instance, $i = !x.\mathsf{PA}_2$; by **Make**, it follows that $\langle i \rangle \mathsf{PA}_2$. If $i$ is executable, then it follows that $\mathsf{PA}_2$ is consistent, since we will have built a model for the axioms. But we have no real grounds to assert that $i$ is executable, perhaps excepting some antecedent belief in the consistency of $\mathsf{PA}_2$ itself. On the other hand, the more fine-grained structure of $Num$ and $Set_h$ leaves open the possibility of getting an inductive proof of executability going.

But at this point we run into what is perhaps the most significant difference made by our appeal to infinitary commands. As just stated, our logic proves that necessary executability is preserved by composition by $\ulcorner ; \urcorner$. We do not, though, have a parallel theorem for composition by $\forall$: even if $i(x)$ is executable for every $x$, it does not follow that $\forall x i(x)$ is executable. And this is a serious obstacle in arguing for the consistency of $Num$ and $Set_h$. Even if we assume the necessarily executability of $\sigma$ and $\eta$, we cannot prove with our logic as it stands that their indefinite iterations $\sigma^* = Num$ and $\eta^* = Set_h$ are executable. (This point will receive much more discussion in the next section, on consistency.)

One might think, then, that the Euclidean adherence to finitude is a wisely cautious one: it ensures that, if we assume some basic commands are legitimate, that we will not stray out of the realm of legitimacy. It should be noted, of course, that we might be interested anyway in commands whose executability we cannot prove, so that the Euclidean restriction would still be too demanding. But the point remains: if we are to make some real epistemological gains via this imperatival logic, we need to examine the prospects for principles relating the executability of basic imperatives to infinite assemblages of them. In the next section, we consider how we might do so.

## 5.2 Consistency

We've mentioned a few times that (following Fine (2005)) one might hope to marsh al our imperatival constructions for models of standard mathematical theories in to some sort of evidence for the consistency of the theories in question. This subsection is devoted to discussing the extent to which that is so.

[20] One might worry that our system is ill suited for this purpose because it proves already that some commands are executable which, intuitively, ought not to be. But so far as we can tell, our system only proves that commands which do nothing at all are executable. For consistency proofs of more robust commands, one will need to make substantive assumptions—exactly as it should be.

We have offered arguments from the assumption that certain commands are executable to the existence of structures satisfying standard axioms for arithmetic and set theory. Here, by a structure for the theory in question, we mean a structure in the 'internal' sense (cf Button and Walsh (2018)): we have pluralities $X$, after doing the commands, such that the second order (single sentence) axiomatizations for these theories hold in $X$. This gives us a sort of relative consistency argument: if we understand consistency of $T$ as $\Diamond\exists X T^X$, then we may argue from the executability of certain commands to the consistency of the theories.

There is a subtlety here that needs addressing, however: that $\Diamond PA_2$ or $\langle i\rangle \mathsf{PA}_2$ for some executable $i$ is evidence for the consistency, in its more familiar sense, of $\mathsf{PA}_2$ is a claim that needs some defending. After all, we cannot *always* infer the consistency or logical possibility of some axioms (or sentences expressing them) from a claim that those axioms (or sentences expressing them) are possible in some other sense. We think that $\mathsf{PA}_2$ *might* be true, but from that epistemic possibility we surely can't infer that $\mathsf{PA}_2$ is consistent. This yields a challenge for the imperativalist: how might possibility, in the sense at issue, bear upon consistency?

There are many ways it might, depending on the intended interpretation of our logic. If you think that imperatives like $Num$ and $Set_h$ are commands that we can actually carry out (if executable), just as we can carry out the command: "Shut the front door!", then the connection between executability and consistency is clear. If it is metaphysically possible to execute $Num$, and if, having done so, we will have made a structure satisfying $\mathsf{PA}_2$, then it's metaphysically possible that some structure satisfies $\mathsf{PA}_2$. And whatever the status of postulational possibility, the connection between metaphysical possibility and logical consistency is tolerably clear. (Note that this line of thought is available whether or not we take the metaphysically possible entities to be produced in the course of executing $Num$ to really be *numbers*.

There are also less realist interpretations still of the logic which might be attractive. One natural thought about why we believe set theory and arithmetic to be consistent involves appeal to an ability we seem to have to describe structures in which these theories are true. Focusing on set theory, for example, we have the cumulative hierarchy structure, which is often offered as part of the support for believing the consistency of $ZFC$ – contrasted negatively with cases like $NF$, where no such intuitive structure is available. But of course, we might be wrong in our descriptions, and the more confident we can be we aren't the more confident we can be in consistency. Here is where our imperatival logic can enter the frame: for we might secure this confidence by not only describing the finished model, but also describing a *process* for *building* the model, especially one where we build it up in a series of more basic steps, whose consistency or legitimacy are easier to grasp.

An example: suppose you were in doubt about whether the definition of an equilateral triangle was logically consistent. You could convince yourself of the consistency by going through Euclid's construction. You have no doubt that a line can always be drawn between two points, and that a circle with any radius can always be drawn about any point. Euclid's proof shows how from these two obviously legitimate procedures, a procedure for constructing an equilateral triangle can be assembled. If the basic parts are legitimate, the thought then goes, then so is the procedure for constructing the whole—with the result that you have convinced yourself that an equilateral triangle can be legitimately constructed and hence that the definition of one is consistent.

One can think of our imperatival logic as a system for making these constructive intuitions precise, in the obvious way. One can fashion complex imperatives using $\ulcorner ; \urcorner$, $\ulcorner \rightarrow \urcorner$, and $\ulcorner \forall \urcorner$ out of more basic ones, and then, using our logic, indeed prove that the construction procedure results in the structure desired. We can take Euclid's basic postulated procedures as basic commands; and then code up his construction, a sequence of commands, as one complex command via $\ulcorner ; \urcorner$; and then finally prove that the complex command results in an equilateral triangle.

So we now have various models for thinking about how executability ties in with consistency. But what about executability itself? How confident should we be that $Num$ or $Set_h$ *are* executable?

As we've already briefly mentioned, there is an idea suggested in Fine (2005) that we might hope to attain inductive proofs of executability. The natural way to do so would be to show that executability 'flows upward' from basic commands to more complex ones. As we've said, this is easy in the case of ;, and it is equally easy for $\rightarrow$. More exactly, we have

**Theorem 5.1** (Upwards Transmission Theorem)**.**

$$\langle p \rightarrow i \rangle \top \equiv \neg p \vee \langle i \rangle \top$$

$$\langle i; j \rangle \top \equiv \langle i \rangle \langle j \rangle \top$$

$\forall$, however, is a different story. In that case, the natural analogue would be

**Definition 5.2** (Universal Upwards Transmission)**.**

$$\langle \forall x I x \rangle \top \equiv \forall x \langle I x \rangle \top$$

but it is easy to see that UUT will make big trouble. If we assume for example that $\rho$ is necessarily executable, it is clear (by induction on iterations) that UUT implies $\rho^*$ is too, landing us back with the Russell paradox. Indeed, more generally:

**Theorem 5.3.** *If any essentially creative necessary difference maker is necessarily executable, then UUT fails.*

So the imperativalist cannot hope to give an inductive consistency argument of the type hoped for. For if the basic commands in question are necessarily executable, then the needed upwards transmission principles simply cannot hold.

A natural fallback position would be to consider restricted forms of UUT. For example, in arguing for the exectuability of $Num$, one can actually show that it is sufficient to assume only that if each particular *finite* $\zeta$-iteration of $\sigma$ is executable, then the command 'do all finite $\zeta$-iterations of $\sigma$!' is executable too. More generally, we can consider the family of principles:

**Definition 5.4** (Restricted Universal Upwards Transmission)**.** *For a given property $P$, let $P$-UUT be the principle:*

$$\forall x(Px \supset \langle Ix \rangle \top) \equiv \langle \forall x(Px \to Ix) \rangle \top$$

In case $x$ ranges over imperatives, we may set $I := \lambda i.i$, attaining

$$\forall i(Pi \supset \langle i \rangle \top) \equiv \langle \forall i(Pi \to i) \rangle \top$$

and where $P$ expresses the property of being a finite $\zeta$-iteration of $\sigma$, this will have the desired effect for $Num$.

One could also hope to provide restricted forms of UUT that would enable the inductive proofs of executability to go through for fragments of set theory. But we think there are reasons not to be so excited about this project when it comes to advancing issues with respect the epistemology of consistency. Given that the restricted versions of UUT are partial approximations to the inconsistent UUT, we surely need special reasons to believe that the fragmentary versions *are* themselves consistent. And it is hard to see what independent evidence we could have for this.

It thus seems that the prospects for finding evidence for consistency of standard theories in the form of an inductive consistency proof look dim. This is not to say, though, that there is nothing of value here in the project presented. What we have shown is that the consistency of the theories can be reduced to the executability of these commands, which in turn can be intuitively supported by appeal to executability of the basic commands together with the executability of sufficiently long iterations. We thus have a kind of ontological analogue of the Gentzen style consistency proofs for arithmetic. Even if it does not hugely increase our confidence in consistency, it does provide a new angle on it, and hopefully contributes in some small way to our reasons for belief.

## 5.3 Paradox and Potentialism

Another potential point of value in considering imperatival foundations for mathematics concerns the set-theoretic paradoxes. In particular, there is a prima facie case to be made that the imperativalist framework may yield a new kind of 'potentialist' solution to them.[21] The purpose of this section is to discuss the extent to which this is the case. We focus here on Russell's paradox, though analogous points stand to be made about the other set-theoretic paradoxes as well.

Russell's paradox arises from the simple result that there is no set of all non-self-membered sets, on pain of contradiction. When coupled with the view that there is a definite totality of all possible sets, the simple point of logic implies that there is an absolute dichotomy between those pluralities that are the elements of sets and those that aren't; but the dividing line between these pluralities is apt to seem arbitrary.

[21] (Fine 2005, p101ff) suggests something of this kind.

A key idea behind 'potentialist' solutions to this problem is to reject the idea that there is a definite totality of all possible sets in favor of a view on which any things can form a set, and on which any possible sets are therefore not necessarily all the sets.

There is a natural sense in which the procedural postulationist framework as we have developed it manifests a version of a potentialist solution to the Russell paradox, and one with an interestingly different profile to most extant proposals – but of course, on that has considerable overlap with that of Fine (2005), and to a lesser extent also that of Hellman Hellman (1989b).

In particular, on the assumption that $\rho$ is necessarily executable, an assumption we have largely taken up throughout this paper, it will be true that necessarily there can always be more sets. Indeed, if we make the stronger assumption that $Set_h$ and $\rho$ are necessarily executable, we attain a picture on which a never ending series of universes of sets can be produced, each of height some inaccessible cardinal; a picture with some motivation in the extant set-theoretic literature (see here especially Zermelo (1930b)). For, on the assumptions mentioned, execution of $Set_h$ would produce a universe of height $\kappa$, where $\kappa$ is the least inaccessible; but it is not hard to see also that execution of $Set_h; \rho; Set_h$ will yield a universe of height $\kappa_1$, where $\kappa_1$ is the second inaccessible, and so on. There will clearly be no largest universe that can be generated in this way, and the engendered picture is a form of potentialism with striking similarities to Zermelo's. (The question: what is the least cardinal one cannot get by chaining together iterations of $Set_h$ and $\rho$ in this way? strikes us as interesting, but we have not been able to resolve it.)

One of the distinctive features of the resolution to the paradoxes embodied here is in its requirement of a stopping point, set in advance, for any set construction process that can be deemed as legitimate. In Øystein Linnebo's and James Studd's modal systems for set theory ((**?**), (Linnebo 2010), (Linnebo 2018)), axioms for set theory are interpreted in modal terms, and the modals are given a procedural gloss (albeit a metaphorically intended one) in terms of steps of individuation of objects / reinterpretations of language. But in both cases, domains for set theory are taken to emerge when such acts are iterated indefinitely, as it were, with no specified stopping point. On the other hand, for reasons already covered here, any instructions for the construction of a domain of sets in our logic will have to contain within them an implicit stopping point, in the form of a hedge, in order to be executable.

This makes some difference to the conceptual lay of the land here. One natural 'revenge' worry for potentialists is: why can't we just bring together all the sets you can ever possibly get by individuating or reinterpreting, and make them into a new set? In reply to this, potentialists of the Linnebo/Studd variety must appeal either simply to the 'big stick' of contradiction, holding that the contradiction that arises from this is sufficient reason to believe it can't happen, or else to indefinite extendibility of the modal notions themselves, allowing that we can possibly* bring together all possible sets in this way only in some expanded sense of possibility. But our procedural potentialist offers a third way out: we can simply point to the fact that there is necessarily no hedge that can lead to the production of all possible sets, since (under the assumption of the necessary exectuability of $\rho$) any hedge that leads to an executable iteration will fall short of producing every set. In contrast, e.g., to Linnebo's program, the need for hedging conditions was something we were assured of from the get-go in procedural postulationism, rather than an afterthought needed to avoid revenge; prima facie at least, the procedural postulationist may therefore have an advantage here.

It is worth mentioning though that the framework here has considerable flexibility, and is not even thoroughly tied to potentialism. The versions of potentialism we have just discussed were all premised on the necessary executability of $\rho$, something which leads to a form of potentialism insofar as it implies there could always be more sets. But we could also work in a procedural postulationist framework that denied this. Indeed, to return to issues raised in the last section, denying the necessary executability of $\rho$ will in fact allow us to keep UUT in its full generality: whenever some iterations of $\rho$ are in fact executable, the command to do all of them will be too. Indeed, there will be a command that will produce all the sets there could ever be, namely the hedged command to do all the executable iterations of $\rho$. The output of this command would be the universe of sets, as envisaged by non-potentialists in set theory.

This situation is reminiscent of the one in declarative mathematics. Two popular responses to Russell's Paradox available in that context are (a) to admit that there are some extensionally determinate collections (e.g. the universe of all sets) that cannot form sets, or (b) to hold that any extensionally determinate collection can be formed into a set (in a larger domain). The former approach leads to so called (height) actualist pictures of sets, and the latter leads to (height) potentialism.[22]

We are faced with an analogous choice here. We can accept the necessary executability of $\rho$, something which leads naturally to an imperatival potentialism about sets, one that requires us to give up UUT. But we could also deny that $\rho$ (and any other essentially creative necessary difference maker) is necessarily executable, and hold fast to the idea that there is a single, fixed, maximal universe of all sets. Where the declarative mathematician faces the choice of which conditions do or do not form sets, the imperativalist must answer the question of which (pluralities of) imperatives are executable.

Of course, it may be that imperatival foundations are more *suggestive* of potentialism. Perhaps the nature of $\rho$ just convinces the imperativalist that it is always doable. If this route is taken, then we are owed an explanation of the intuitive pull of **UUT**, but this isn't *forced* on you. And vice versa, if you decide to reject the necessary executability of $\rho$. Just as in declarative mathematics, we have to make an *choice* about how you want to resolve the paradoxes, and the choice does not seem *obvious* to us.

[22]Of course for each picture there's many more questions to ask about the nature of classes and/or potentialism. One popular answer for the actualist (e.g. Kanamori (2009)) is to treat class talk as shorthand for first-order formulas, but others treat them as plurals (e.g. Uzquiano (2003)) or mereological sums (e.g. ). And there's a wide variety of potentialist approaches (e.g. Zermelo (1930a), Hellman (1989a), Linnebo (2010)).

## References

Button, T. and Walsh, S. P.: 2018, *Philosophy and Model Theory*, Oxford, UK: Oxford University Press.

Ewald, W. B. (ed.): 1996, *From Kant to Hilbert. A Source Book in the Foundations of Mathematics*, Vol. I, Oxford University Press.

Fine, K.: 2005, Our knowledge of mathematical objects, *in* T. Z. Gendler and J. Hawthorne (eds), *Oxford Studies in Epistemology*, Clarendon Press, pp. 89–109.

Harel, D., Kozen, D. and Tiuryn, J.: 2000, *Dynamic Logic*, MIT Press.

Hellman, G.: 1989a, *Mathematics Without Numbers*, Oxford University Press.

Hellman, G.: 1989b, *Mathematics Without Numbers: Towards a Modal-Structural Interpretation*, Oxford, England: Oxford University Press.

Hodges, W.: 2013, Modality in mathematics, *Logique et Analyse* **56**(221), 5–23.
**URL:** *http://www.jstor.org/stable/44085245*

Kanamori, A.: 2009, *The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings*, 2nd edn, Springer.

Linnebo, Ø.: 2010, Pluralities and sets, *Journal of Philosophy* **107**(3), 144–164.

Linnebo, O.: 2013, The potential hierarchy of sets, *Review of Symbolic Logic* **6**, 205–228.

Linnebo, O.: 2018, *Thin Objects: An Abstractionist Account*, Oxford University Press, Oxford.

Roberts, S.: 2022, Pluralities as nothing over and above, *Journal of Philosophy* **119**, 404–424.

Sidoli, N.: 2018, Uses of construction in problems and theorems in euclid?s elements i?vi, *Archive for History of Exact Sciences* **72**(4), 403–452.

Uzquiano, G.: 2003, Plural quantification and classes, *Philosophia Mathematica* **11**(1), 67–81.

Zermelo, E.: 1930a, On boundary numbers and domains of sets, *Ewald (1996)*, Vol. 2, Oxford University Press, pp. 1208–1233.

Zermelo, E.: 1930b, Über grenzzahlen und mengenbereiche, *Fundamenta Mathematicae* **16**(1), 29–47.
**URL:** *http://eudml.org/doc/212506*