# Non-mathematical dimensions of randomness: Implications for problem gambling

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## Introduction

Randomness is an essential concept in gambling and problem gambling. On the one hand, randomness is the primary concept upon which probability theory and mathematical statistics are grounded, and these mathematical disciplines govern the mathematical models on the basis of which games of chance are conceived and function for the gambling industry. For the players, randomness (as they perceive or understand it) is what qualifies games as being "of chance" and what ensures certain fairness of them.

On the other hand, in problem gambling, randomness was also found to be involved essentially in the causes and correction of some math-related gambling cognitive distortions, especially the Gambler's Fallacy (GF): The widely used syntagma labeling this involvement is that gamblers subject to the GF have an 'incorrect *perception* of the *notion* of randomness' or similar wordings reflecting cognition and understanding of this concept (Clark, 2010; Fortune & Goodie, 2012; Goodie & Fortune, 2013; Williams & Vokey, 2015; Goodie, Fortune, & Shotwell, 2019). The incorrect perception of randomness was associated in most studies dedicated to this subject with erroneous or subjective probability estimates and other math-related cognitive distortions (such as the conjunction and disjunction fallacies). With such approaches and conceptual framing, the tendency has been to categorize randomness in the group of *mathematical* notions whose inadequate understanding or application is responsible for the knowledge-based component of the causes of the math-related cognitive distortions. A well documented wide-scope review by Keen & al. (2019) on the studies that evaluated gambling-education programs reveals that randomness was placed in the category of gambling-related mathematical concepts, and learning *about* randomness was associated with math classes or gambling-math courses and with all the mathematical information or curricular content sometimes taught in such educational interventions. Overall, in problem gambling research, the concept of randomness has been artificially granted a strong mathematical dimension and has even been qualified as mathematical in nature. In this article I will challenge this view, by arguing that randomness is a complex concept whose mathematical dimension is weak, and its non-mathematical dimensions are important in accounting for the nature of and solutions to the randomness-related issues in gambling and problem gambling, including the correction of the misconceptions and fallacies about probability and statistical concepts applied in gambling.

In the first section, I present a brief description of the concept of randomness from the perspective of philosophy of science and epistemology, by pointing out the exact relationship of this concept with probability theory. I show from an epistemological and historical standpoint that randomness is not a mathematical concept and that its

methodological-theoretical dimension accounts for its foundational role in sciences and mathematics.
In the second section, I show that in the gambling industry, randomness has two main interdependent roles – to ensure the functioning and fairness of the games – reflecting the functional and ethical dimensions of randomness. In regard to the traditional concerns about fairness, I argue that on the one hand, although the PRNGs (pseudorandom number generators) provide a (mathematical) algorithmic randomness, the ethical and functional dimensions of randomness are involved decisively in the concerns for fairness to a much greater degree than its mathematical dimension; on the other hand, the concern for fairness does not cancel the randomness of a game, but at most, alters it, and players would fare better to direct their concerns toward correcting their cognitive distortions rather than fairness.
In the third section I show – based on existent reviews – how the studies about education in gambling and educational programs that referred to or employed the concept of randomness in their content and methodology, did so by qualifying it artificially as a mathematical notion or concept, and implicitly considered learning about randomness a component of learning about the mathematics of gambling and the statistics curricula.
I argue that there are non-mathematical dimensions of randomness that essentially account for the involvement and roles of this concept in the cognitive zone of problem gambling rather than its mathematical dimension. I take the GF as a theoretical case study to make my point, and I show how randomness is involved in this cognitive distortion in its epistemic dimension; I argue further that forcibly "mathematizing" randomness affects dramatically the perception and conceptualization of this distortion and renders the implied correction self-contradictory.
Finally, I draw conclusions and bring three theoretical arguments to suggest that the artificial qualification of the concept of randomness as mathematical is not just a harmless lexicographical convention for simplifying things, but has implications in both the research in this field and the constitution and effectiveness of educational programs based on the results of the research. Therefore, problem gambling research in the cognitive zone and any prospected cognitive model of educational programs should incorporate the conceptual distinctions and the non-mathematical dimensions of randomness in their framework.

## 1. **Mathematical and non-mathematical dimensions of randomness**

The lexicographic definition of randomness across all major dictionaries is that of a quality or state of something of being *random*; further, the word 'random' is defined in dictionaries as an adjective meaning 'happening/made/determined/chosen by chance/accident/guess rather than design/method/plan/determination/pattern/purpose' (in all sort of choices and wordings). Such lexicographic definitions suffice in fixing a unique meaning for random and randomness in everyday speech, and thus avoid any semantic conflict in common communication. However, they are insufficient for reasoning with the concept of randomness in scientific and philosophical discourses where fine-graining this concept and distinguishing clearly between its constituents and

their mutual relationships are necessary for the consistency of the theories and meta-theories dealing with or related to randomness.
I will limit and simplify the conceptual analysis of randomness to the case of gambling. The phenomenon of gambling (including both games and players' behavior and activity) is scientifically approached and investigated by the disciplines of mathematics and psychology (and in some respects by economy, but this relationship does not fall within the scope of this research). The mathematical branches providing functional models for games of chance and gambling are probability theory, mathematical statistics, and game theory (the latter being more concerned with strategy). The relationship between gambling and randomness exists through the involvement of the first two branches of mathematics in the conception and description of the games, the analysis of their outcomes and functioning, and of the players' strategies. This relationship exists simply because randomness is a primary concept for probability theory and grounds the concept of probability. With such a relationship, we can obviously talk about a mathematical dimension of randomness, yet it is not the only dimension of this concept – for instance, just within the industrial side of gambling, randomness also has an ethical dimension, since the concerns about the fairness of the games are directly formulated in terms of randomness. I will come back to this ethical dimension in a later section.
Let's see what the mathematical dimension of randomness and the relationship between randomness and probability theory are exactly.
The widely accepted lexicographic and wiki definition of 'random' in the context of probability theory is relative to the word association 'random process' or 'random experiment': A random experiment is a well-defined procedure that obeys two conditions: 1. It can produce more than one outcome; 2. The possible outcomes are unpredictable.
Probability theory adopted this common definition of a random experiment and defined a 'random event' as a set of one or more outcomes of a random experiment.
With randomness defined as the quality of an experiment in being random, probability theory proceeded by defining elementary events as the random events consisting of one outcome and assuming that all elementary events are *equally possible*. It is this basic assumption on which all classical probability theory is built. In order to apply probability theory in any experiment or domain, we necessarily make an equally-possible assumption prior to application, which is a *conceptual* (not mathematical) consequence (but not the only one) of the randomness of the experiment; in other words, the equally-possible assumption falls within the concept of randomness.
It is said that 'random experiment' (and implicitly randomness) is a *primitive notion*[1] for probability theory (as are the notions of number, set, function, relation, variable, and so on, for the whole of mathematics). Primitive notions – especially those from mathematics – became objects of philosophical debate and investigation, and many times received descriptions, characterizations, and conceptual definitions within theoretical-philosophy disciplines. The topic of randomness is classical for such debates in philosophy of science

[1] Within a theoretical discipline like mathematics, a *primitive notion* is a notion that is not defined further with the methods of that discipline and by using other notions already defined within that discipline. Any defined notion regresses finitely to one or more primitive notions, which are just undefined concepts. (Otherwise, assuming all notions are defined through others, the regression would be infinite.)

and epistemology, within the same framework as the concept of probability. But a philosophical conceptual characterization of randomness was not established before mathematicians struggled and failed to assign it a mathematical definition.

### 1.1 **No mathematics of real randomness**

One of the features characterizing mathematics and mathematical work is the search for perfection. In our case, perfection would mean changing the status of randomness from primitive notion to mathematically defined notion. The several attempts to define randomness mathematically failed as either inconsistent, circular, or not adequately reflecting the described concept.

Mathematical work on defining randomness can be traced back before the probability theory was conceived; however, the work at the beginning to middle of the 20th century was influenced fruitfully by both the development of measure theory and modern probability theory, and the advancements in the science of algorithms and information theory. There are two important landmarks across these attempts, which proved once and for all that randomness cannot be mathematized, whatever the future development of mathematics:

The endeavor of defining randomness within a mathematical formal system started with the simplest instance of it – a random sequence – that is, a sequence for which any term is randomly generated, as not dependent upon the previous terms.

Richard von Mises (1919) has formulated an axiomatic definition of a random sequence by calling it a *collective*: A sequence of elements $a_1, a_2, a_3, ..., a_n, ...$ is called a *collective* if, given a property *f* of its terms and denoting by $n(f)$ the number of elements from the first *n* having the property *f*, the following two axioms are obeyed:

1) The ratio $n(f)/n$ converges toward a limit when the number of terms increases to infinity; and 2) If we remove, by a place selection, a part of the terms, the limit of the remaining sequence $n(f)/n$ is the same, whatever the place selection is.

Axiom 2, according to von Mises, expresses the random character of the sequence[2].

Von Mises shows that randomness for individual (infinite) sequences could be defined in general by requiring that 'collectives' consisting of elements appearing at positions specified by any procedure should exhibit equal average frequencies. This is how von Mises formalized the intuition that a random sequence should be unpredictable.

The 'collective' in von Mises's definition and its axiomatic context extends a more particular previous instance of randomness considered by Emile Borel (1908, 1909), namely that of a random sequence formed by only two symbols (0 and 1). Borel proposed an inductive logical proof for the fact that such a random sequence (0001101000111…) cannot be built: If we assume the first *n* terms of the sequence were built and it follows to write the $n + 1$ -th term, then two options are available: the first *n* results are taken into account or they are not. The first option cancels the random attribute of the construction. The second option brings us back to the same situation as we have at the beginning of the construction: How to choose one of the symbols 0 or 1 without taking into account any difference between these? Such a choice would be equivalent to a draw, but a draw

[2] It is called the irregularity axiom or the principle of impossibility of finding a playing system.

assumes an *experimental* intervention, which cancels the mathematical character of the construction. Far from being a convincing proof, it points out that we cannot state a constructive mathematical definition of a random sequence, and the theoretical difficulties of stating one seem insurmountable. Yet this would not exclude an axiomatic–descriptive definition, which von Mises provided.

Von Mises's definition is mathematically consistent and can be integrated in probability theory; however, it has not passed foundational and conceptual criticisms (Van Lambalgen, 1987; Siegmund-Schultze, 2006). Briefly, the main issue is that the *place selection* notion (even considered a primitive notion) cannot be defined without coming back to the definition of *randomness,* which creates an inacceptable circularity given that the aim is a mathematical definition. Moreover, von Mises did not provide a formal definition of admissible selection rules. Another unsolved issue is related to the proposed *random* attribute: the convergence proposed by axiom 2 describing randomness is not sufficient for the randomness to be *absolute*; it must include *properties* of the convergence – a sequence starting with, say, one thousands terms of 1 is not "as random as" a sequence starting with ten terms of 0 or 1 and so on.

While any axiomatic definition can be improved upon with respect to description of the defined concept (and such attempts have been numerous for randomness), one cannot ignore the *nature* of the issues raised with the previous attempts, which is fairly supposed to cause the issues also manifesting further in more complex axiomatic systems waiting to be devised.

The remarkable move in resolving the axiomatic objections raised to von Mises's definition was done by Alonzo Church (1940). He proposed that we take the admissible place selections to be the computable functions, for the subsequences included in a collective to be selected *effectively* and not in a kind of experimental or non-defined way. This idea aligned with the intuition that it is reasonable to regard prediction as a computational process, and hence restrict ourselves to computable selection functions. Thus, von Mises's definition was improved by Church to reach mathematical consistency – both internal and with probability theory – also with the contribution of Abraham Wald (1936, 1937), who showed that von Mises's notion of collective is nonempty if we restrict to any countable set of selection functions. The Mises-Wald-Church definition advanced the concept of *algorithmic randomness* and seemed to be the best possible axiomatic product by which to change the status of randomness from a primitive notion to a mathematical one and make an elegant fit into probability theory. Yet objections have been raised again, as expected: First, Jean Ville (1939) showed that von Mises's notion of collective is flawed in the sense that there are basic statistical laws that are not satisfied by this notion. He showed the existence a sequence which is random in the Mises–Wald–Church sense, but has too much regularity to be called random; for this sequence, the law of iterated logarithm is not fulfilled, although this law is natural for random sequences.

Independently of the objections to the mathematical system itself, both Church and von Mises raised the theoretical and conceptual concern of *incompleteness*. In brief, the endeavor of defining random sequences was aiming to arrive at a notion of probability that attained the resolutory ideal of completeness – that is, a notion of probability by means of which we could solve all problems of probability calculus; and given that probability theory is supposed to apply to all random events in the real world, the mathematical concept of randomness should reflect the nature of randomness in real life. However, restricting the notion to *some* class of place selections would prevent attaining completeness:

> "It is not possible to build a theory of probability on the assumption that the limiting values of the relative frequencies should remain unchanged only for a certain group of place selections, predetermined once and for all." (von Mises, 1981/1928, p. 90)

> "Their use [of the admissible place selections] for this purpose, however, is open to certain objections from the point of view of completeness of the theory, as has been forcibly urged by von Mises, and it is therefore desirable to consider further the question of finding a satisfactory form for the definition of a random sequence." (Church, 1940, p.133)

The mathematical effort to define randomness was stimulated by the results of von Mises and Church and produced in the next three decades other valuable axiomatic systems claimed to describe randomness such that the concept would be integrable into probability theory. But all the results eventually provided a form of algorithmic randomness and faced the same conceptual issue – that of incompleteness with respect to the real (general) randomness.
By their effort, mathematicians provided us with four basic mathematical notions of *algorithmic* randomness, each reflecting one of the following main features:
- stochasticity (or stability of frequencies, in terms of computable functions) – contributors: R. von Mises, A. Wald, A. Church, A. Kolmogorov, D. Loveland;
- typicality (in terms of measure theory) – contributors: P. Martin-Löf, L. Levin, C. Schnorr;
- chaoticity (in terms of Turing machines) – contributors: A. Kolmogorov, L. Levin, C. Schnorr;
- nonpredictability (in terms of game theory and gambling) – contributors: J. Ville, V. Uspensky.

I have limited this historical incursion to include the most influential landmarks, while not entering the technicalities of the mentioned definitions and theories and the variety of the mathematical theories. Well-organized historical presentations of the topic, packed with accessible descriptions of the theories, can be found in (Dasgupta, 2011), (Landsman & Wolde, 2016), or, in a more philosophical framework, (Eagle, 2021).
The conclusion to draw with respect to our topic is that all mathematical constructive or axiomatic attempts to define randomness (as a general concept extracted from reality to ground the mathematical concept of probability) either failed in consistency or were found incomplete. The "real" randomness is more complex than the mathematically defined algorithmic randomness and could not be captured in a mathematical definition, in the light of Borel's (1908) contemplation that "reason cannot reproduce the randomness."
It is worth noting that the concept of algorithmic randomness was found problematic or weak even for the needs of physics, especially quantum mechanics (see for instance Calude, 2004; Zak, 2016; Landsman, 2020).

Therefore, randomness is not a mathematical concept (unlike algorithmic randomness), where 'mathematical concept (or notion)' is defined as one having a *mathematical* definition. There is no definition to capture *together* even the four mathematical features caught in the different axiomatic systems (stochasticity, typicality, chaoticity, and nonpredictability), let alone other features perceived or intuited commonly as characterizing randomness in the real world.

The non-mathematical nature of randomness should not be too surprising – the concept of *set*, the main brick in the foundation of mathematics, remains a non-defined primitive notion[3].

Not qualifying randomness as a mathematical concept is not just a semantic opportunity resulting from any narrowing of the meaning of 'mathematical' down to notions of pure mathematics – as I will argue further, the concept does have a mathematical dimension, which I will explain in the next section.

Even from an applied-mathematics perspective, any prospected "mathematization" of the concept seems to be somewhat paradoxical at conceptual level:

Mathematical modeling starts with a process of *idealization* of the concepts or structures of the physical or empirical object under investigation, in order for them to be embeddable in the *mathematical* structures of the specific theory or theories. These theories govern the application and make possible the mathematical deductions and then the inferences about the investigated object in the real world (Bueno & Colyvan, 2011). Such idealization assumes keeping only those features of the concept or object that are relevant for the mathematical application and removing those that are unnecessary or cannot be expressed mathematically. That is, idealization eliminates or reduces pragmatically the *complexity* of the physical concept or object under investigation, which otherwise would not be suitable for the mathematical application. However, in the case of randomness, it is precisely the *full* complexity of the concept that any mathematical application would have as target for making inferences *about* randomness in the real world. That is because complexity does fall within the concept of randomness more than for other concepts.

Hence, an assumption on idealizing randomness for applying mathematics to it in order to find something about randomness in the real world becomes self-contradictory: By removing its complexity, we can infer truths about randomness through mathematical modeling; however, those truths will lose relevance in the real word, as complexity characterizes randomness decisively, and what we want to infer is about that specific complexity and not a "reduced" one.

Randomness is such a special concept for science and human reason that mathematicians and philosophers even asked whether randomness really exists. And the answer is not straightforward. One may fairly ask whether the non-mathematical nature of randomness has any significance for science beyond the philosophical, and in particular how the philosophical dimension of randomness can be related to or prove relevant for a narrow field of study such as problem gambling. I will answer this question after clarifying the non-mathematical dimensions of randomness and the exact relationship of randomness with problem gambling.

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[3] Yet *set* has a rigorous characterization in second-order logic (due to Bertrand Russell and his followers), and as such, has a logical nature, which is not the case with *randomness*.

## 1.2 **The mathematical dimension of randomness**

There is a mathematical dimension of randomness not because mathematicians struggled to define it or because we refer to it in our discourses along with other mathematical notions, but because of its relationship with the mathematical theory of probability.

Randomness is a primitive notion for theory of probability as the quality of an experiment being random. A random experiment is in turn a primitive notion, and we should note that it is not this notion that is effectively employed in probability theory, but rather that of *random event*. This latter notion is a primary mathematical concept for probability theory – the structural unit used to define the basic mathematical structure needed for the function of probability to be defined on it as a particular type of measure, namely the *field of events*. Both the random experiment and the random event are notions that keep probability theory interpretable or applicable in the empirical world, even from the theory's conceptual foundation. The concept of (real) randomness is embedded in these two primitive notions with respect to that interpretation or application; however, it is the notion of random event and its associated mathematical structure that "do the mathematical job" for probability theory, starting from the definition and properties of the probability function.

| *randomness* → | *random experiment* → | *random event* → | *field of events* |
|---|---|---|---|
| general concept and primitive notion | primitive notion | primary mathematical notion | structural mathematical notion |

In the scheme above, we can see the nature of each discussed concept in the mathematical context. We can see that the relationship of the general concept of randomness with probability theory is a three-step one, that is, it is not a direct relationship of the constitutive, applicative, or inferential kind. Moreover, the notion of random event is used in probability theory merely in its set-theoretic nature – random events are seen as just abstract sets with atomic elements of no content or any empirical nature (their union forms the *sample space* of the experiment).

For probability theory, randomness remained a marginal primitive notion. The Kolmogorovian account of probability (widely accepted as the standard account in mathematics and applied mathematics) is integrated within measure theory and based on mathematical structures whose properties do not depend on any of the features of randomness. The word "randomness" even disappeared from the language of probability theory and statistics. However, this is not a reason for denying the relationship of randomness with probability theory (even if it is a weak one, from a mathematical standpoint) and hence its mathematical dimension.

Even if the notion of random event is mathematically free of the concept of randomness (in the sense that the notion of event is mathematically identified through the structure it belongs, that is, through the axioms of a field of events), the concept is still detectable from an epistemological, non-mathematical perspective: An event as a set consists of some possible outcomes of the sample space, and each outcome can be identified with an *elementary event*, that is, an event that can no longer be decomposed in other events with respect to union. Probability theory starts with the premise that all elementary events associated with an experiment are *equally possible* and can be assigned the same probability. For pure mathematics, this is a mere convention of an axiomatic nature, but if we interpret it in empirical terms, it would sound as though the outcomes are equally possible just because there are no evidences that favor one outcome or another, or obtaining such evidences is actually impossible; whatever the wording, its meaning falls within the concept of randomness – we cannot talk about 'equally possible' without assuming randomness in the real world, even as a measure of our ignorance.

Both the indirect relationship within pure mathematics and the relationship via the empirical interpretation (or applied mathematics), of randomness with probability theory is of a theoretical-foundational nature and resides in the conceptual framework of the very foundation or probability theory.

Things are not much different in science. History and philosophy of science proved that absolute determinism is not operational for science, whether we talk about physics or natural sciences in general. Randomness, as a methodological-conceptual substitute for absolute determinism, is postulated or works as a strong idealization for scientific theories. Viewing randomness as a kind of convenient gizmo that we don't even know exists, or objecting to it as being a measure of our ignorance, has never led to questioning the success of science. Physics postulated the so-called fictional entities (electrons, protons, quarks, and so on) to reach its most prolific, elegant, and confirmed theories that explain the natural phenomena, and no scientist is concerned with the methodological-foundational fact that these theories are grounded on concepts with unclear ontological status. So why would things be different for randomness? Randomness is postulated in scientific theories for the probability theory to be applicable to them. The success of quantum mechanics and relativistic physics is a revolutionary achievement in science, and one of its merits is due to postulating randomness. For the physicist, the probability for a radium nucleus to decay during its half life is a constant like gravity acceleration.

In life and social sciences, the role of randomness is similar: The main mathematically-based methods[4] of these sciences (sampling and statistical inference, hypothesis testing, or measuring tendencies and attributes) rely on probability theory and statistics; when applying them, randomness is assumed in various ways in the study of populations.

Therefore, given the importance of randomness as a methodological necessity in our scientific theories, one should also accept its philosophical dimension, which has both ontological and epistemological components.

Taking stock, we have established a weak mathematical dimension of randomness (after arguing that it does not have a mathematical nature), and a strong theoretical-methodological dimension of an epistemic nature, in what concerns both mathematical and scientific theories and methods.

The mathematical dimension of randomness comes from its relationship with probability theory and is weak because of the indirect attribute of this relationship and the lack of axiomatic involvement in that theory. The theoretical-methodological dimension comes from the foundational and methodological roles of randomness in our well-established theories and is strong because of its necessity.

[4] In the sense of an instrumental role of mathematics, not a modeling one.

## 2. **The dimensions of randomness in gambling**

For mathematics and science, randomness is simply a convenient conceptual perquisite for probability theory itself and applied probability, for making the probabilistic/stochastic method operational and effective in scientific reasoning. This objective yet utilitarian and pragmatic feature of randomness grants it the quality of some *order* (for our reasoning). The infinite feature of randomness also adds to the qualification as order. Indeed, infinity is present in the concept of randomness, since we cannot think of something random without imagining it in an infinite context and instances[5]. Infinity has a homogenizing, completing, and ordering role in mathematics, and also in our understanding of the randomness.

On the other hand, randomness is conceptualized as a *disorder* (of the occurrences of events for which causes are not known in their entirety), given that concepts like 'no law', 'no purpose', indeterminacy, irregularity, independence, non-homogeneity fall within the concept. However, it is a special type of disorder, not just a chaotic one. It is a sort of *total* disorder, where the 'total' attribute can be expressed through 'equally possible', 'equally unknown', or just 'totally independent'.

Accepting randomness as both an order and a disorder should not twist our mind in any way, as this is not an inconsistency at all, since each of these two attributes was described in a different conceptual framework and context, and we do not even have precise theoretical definitions for them. This is just in the mere nature of randomness (Bar-Hillel & Wagenaar, 1991; Azcárate & al., 2006; Batanero, 2015).

This dualistic characterization (order-disorder) of randomness transcends the theoretical domains and applies beyond mathematics and science. Apparently simplistic, the characterization becomes powerful in non-theoretical empirical realms such as gambling.

Obviously gambling is a complex phenomenon, but whatever the complexity of the expert or non-expert knowledge associated with it, the use of and reference to the concept of randomness in the characterization of gambling and reasoning about gambling should not, on the one hand, carry the complexity of this concept unless necessary, and on the other hand, should employ the key conceptual distinctions between the constituent concepts and features of randomness.

How is randomness employed and how does it manifest in gambling, in the industrial meaning of the concept (the games of chance and playing them)?

First, games of chance function by two main designed components: a set of rules (including rules of playing and payout schedule) and a set of technical processes that generate the outcomes. The latter component is based on a primary process devised so as to cancel any possibility for information about causes and initial conditions that determine the outcome. This process may be spinning (wheels or reels), shuffling (cards or balls), rolling (dice), tossing (coins), or mixing (tickets or something else) and fulfills its role through physical features such as high speed, intensity, non-equilibrium, and/or non-visibility. This role is equated with *unpredictability* and *equal chances*, both conceptual components of randomness. We *need* the technical processes of generating the outcomes of the games to be random (in the sense of a random experiment), but such

[5] Think of Borel's simple notion of random sequence. If it were finite, one could come up at any time with a personal rule of generating a term from the previous terms.

need does not carry the entire epistemic complexity of the concept of randomness, not even its physical complexity. The need for randomness in gambling is far less pretentious than that of randomness in quantum mechanics, for instance.

Unpredictability and equal chances are essential conditions for a game of chance to be *functional* and *fair*. Functional, because it is a game of chance and chance should not be something predictable, but at most measurable. Fair, because no participant in the game should be favored with respect to chance. Functional and fair are not independent attributes of a game of chance. An unfair game cannot be functional as game of chance.

Observe that 'unpredictable' alone is not sufficient to characterize randomness needed in gambling. For instance, a roulette wheel that is not in a horizontal plane will not make any outcome predictable in a given spin; however, certain numbers will be favored cumulatively in the long run, which means unequal chances for the outcomes and implicitly unequal chances for the players (those knowing that information will be favored).

Therefore, randomness is employed in the physical processes of the games with the role of ensuring fairness of the games, and this fairness has two facets:

1) Fairness between players: No player should have any advantage over the others with respect to the possibility of determining or predicting the outcomes of the game; and 2) Fairness of the operator: Outcomes that are theoretically possible in the same measure should remain equally possible in practice. These two principles say that chance has to be effectively and fairly served, and the luck factor has to be decisive in the games of chance, as their name suggests.

In conclusion, randomness in gambling has functional and ethical dimensions reflected by its roles, which are interdependent. For games of chance, having their elementary outcomes unpredictable *and* equally possible in one play of a game is what is essentially required from randomness; this requirement matches the simplistic characterization order-disorder for this concept. Another requirement – this time relative to several plays of the same game – would be statistical independence, that is, the outcomes of a game should not depend statistically on the previous outcomes. Statistical independence also falls within the concept of randomness as a different component, although it is related to unpredictability and equal chances; it can itself be qualified as both order and disorder.

### 2.1 **Concerns about fairness. A tale-paradox.**

Fairness in gambling entails the *concerns* of all people involved in the phenomenon for ensuring fairness, and such concerns have concrete implications in the industry, down to the conception and technical design of the games. One such implication is that the hardware design of the games has been continuously enhanced in this respect – spinning, shuffling, and throwing devices benefited by advanced technology so as to provide "sufficient randomness" to ensure fairness. This sufficiency is meant still in terms of general randomness – while there will be always physical factors to be quantified and employed in the equations of prediction for those pursuing such an attempt, sufficient randomness is equated with the *physical* prevention of any deterministic approach to predicting outcomes.

The outcomes of the games of chance are of "a higher degree of randomness" than many other physical phenomena which we try to investigate deterministically, simply as a result of those technical efforts to enhance the random processes involved in the functioning of the games; in gambling, randomness is sought and strengthened. Nobody will even try to calculate how a die will fall or where the roulette ball will land in unbiased devices due to the complexity of the physical factors involved in the phenomenon, among which some are external or circumstantial (such as the air current and heat in the room, players' breath, vibrations of the floor, etc.); it is exactly the complexity or chaotic feature of randomness that is responsible for this attitude (the disorder).

Despite the sufficiency of randomness in classical (hardware) gambling, concerns about fairness have been always raised by players, experts, and policy makers, as it is human nature to guard fairness by assuming the possibility of cheating, and this is also a principle in economy. While such concerns do not raise any logical, ethical, or technical issues, their priority given by players may be problematic from a psychological-cognitive standpoint, as I will argue further.

Imagine a roulette player is concerned that the roulette they play might be biased (intentionally or accidentally from faulty construction, it does not matter). Assume that our player is subject to the GF. The player fairly estimates that it is more likely for the roulette to be unbiased than biased. After observing a number of spins, the player notices some numbers occurring much more frequently than others. With this *new information*, the player updates their belief that the roulette is biased, say evenly (50% biased, 50% unbiased). The player wants to make an *informed* decision on placing a bet, by asking themselves whether to bet on the observed frequent numbers or on other numbers: If the roulette is biased, they should bet on the frequent numbers, as it is likely that being favored by the bias, they will occur again. If the roulette is unbiased, they should bet on the other numbers, which according to the Gambling Fallacy, are more likely to occur than the others (in the player's fallacious belief). Therefore, given the concern for the fairness of the game, it is impossible for that player to make an informed decision, although information (both feeding their initial concern and in regard to previous outcomes) is what the player believes to qualify their decision as objective.

While this imaginary example can be qualified as a gambling tale or recreational paradox (call it *The concern for fairness and Gambler's Fallacy* paradox), and obviously its argument does not apply to any game (for instance, biased slots would not favor one player or another, like biased roulette, but just the house), it is just illustrative for the point I want to make regarding the concern for fairness.

Before making this point, let's play a bit with the possible hypotheses of the paradox to see how the conclusion changes with them:

- Assume the player is concerned about fairness, but is not subject to the GF. Then, given that there are equal chances for the roulette to be biased or unbiased, the player should bet on the observed frequent numbers (in case it is biased, even uncertain). We have an informed decision of the player.
- Assume the player is not concerned about fairness, but is subject to the GF. Then, the player should bet on the other numbers. This is also an informed decision of the player, even if faulty.

- Assume the player is not concerned about fairness and is not subject to the GF. In this case, an informed decision does not make any sense, as additional information is lacking. The player may bet at random; however, this is no longer an informed decision.

The only case in which an informed decision is impossible despite the presence of information is as the paradox has been initially formulated, where the concern for fairness is combined with the cognitive distortion of the GF.

We may also analyze a simpler formulation of the tale: Let's keep the initial hypothesis and assume that the player does not observe the previous outcomes and intends to play for only one or a few spins. In this case, even if the roulette is biased, the player can make no objective prediction for the next outcome or outcomes since they don't know which numbers are favored by the bias, so their concern is useless for their play.

While there are important cognitive aspects of the paradox that deserve theoretical treatment, this is not the place for them, so let's stay with the involvement of the concept of gambling randomness in relation with that of fairness. As stated, randomness is a condition for fairness in gambling. A (proven) unfair game implies losing from the randomness of its outcomes (or "altering" it); however, *concern* about a game being unfair does not necessarily entail losing randomness, just because that concern assumes uncertainty. Although randomness cannot be equated with uncertainty, randomness is still present in the outcomes of a game suspected to be unfair, as a sort of *combined* randomness – the possibly altered randomness of the outcomes combined with the randomness of the thought experiment 'being or not being unfair. The latter randomness is present because it is assumed that we have no evidence to determine one option or another (even if we know that the latter option is more probable).

For a better understanding of this principle, we have first to admit that randomness is not a feature of the uncertain world, which is essentially deterministic, but of the way we reason about this world. The randomness of the outcomes of a suspected game can be simply illustrated through our imaginary example above: It assumes two random possibilities for the player's concern – the roulette being biased or unbiased (denote them by *B* and *U*). The sample space of the combined experiment (the thought experiment per the concern and the roulette spin) gets twice as large as that where only the spin on an unbiased and unconcerned roulette was involved: instead of roulette numbers 1, 2, 3, …., we have U1, B1, U2, B2, U3, B3, …. Are these new outcomes random? Per the lexicographic definition of randomness, they are, since they are more than two and are unpredictable. Per the more complex concept of randomness in its epistemic and mathematical dimensions, they are "less random" than the unbiased, unconcerned case. First, the new outcomes are not equally possible – since the biased roulette favors certain numbers, those B-numbers are more likely to occur per the deterministic evidences. They are still unpredictable in single spins, but their average frequencies can be predicted cumulatively. Yet the random character of the U/B attributes remains embedded in the combined outcomes. We still have randomness, an altered randomness which is characterized not by 'order and disorder' somehow evenly (as we qualified randomness in general), but by disorder and less order (than the non-altered randomness). The alteration is the effect of the fairness concern. In simple gambling terms, gambling while being concerned for fairness is still gambling if no proof is available that the game is unfair or fraudulent.

I conclude here the discussion on the provided imaginary example by recalling that the paradox of the impossibility of an informed decision works only under the assumption that the concerned player is subject to the Gambling Fallacy. The moral of the story would be that cognitive distortions and suspicion do not play well together. While suspicion, even if justified or confirmed true, does not entirely cancel the real randomness of the game, randomness in cognitive distortions may be cancelled just by faulty thinking of it, and I refer of course to perception and understanding. I will come back to this matter in a future section. Therefore it's fair to ask whether players should be more concerned about the fairness of the games than about their own inclination to gambling cognitive distortions. Before I answer this question, I will discuss Pseudorandom Number Generators from the perspective of the concern for fairness, with respect to the nature of randomness.

### 2.1.1 **Pseudorandom Number Generators**

The physical processes of generating the outcomes of the games under the condition of randomness were replaced in modern games by electronic devices and software called Random Number Generators (RNG) which simulate the original processes. A generator of outcomes using software that incorporates a special algorithm which generates numbers associated with the game's outcomes is called Pseudorandom Number Generator (PRNG). PRNGs are present today in the construction of all electronic versions of casino games.

The algorithm of a PRNG outputs a distribution of the elements (numbers, in particular) in a given set or interval in the form of a sequence with special properties, after inputting a random number, called 'the seed.' The algorithm is in the form of computable mathematical expressions. The main two properties of a sequence generated by PRNG are: any term of the sequence is independent of the previously generated terms (by no rule of determination), and the terms are uniformly distributed[6] over the obtained sequence. There are also other properties that the PRNG is required to have (large period, reproducibility, portability, and so on); however, independence and uniformity are the main requirements for a PRNG to be qualified as effectively random relative to its domain of application or *good* for the application. The other properties count when assessing the various degrees of which a PRNG is good (Bhattacharjee & Das, 2022).

These two conditions – independence and uniformity – are widely accepted as effective features of randomness in real life, especially in gambling. They are attained through the mathematical properties of the computable functions used in the PRNG algorithm.

I will not go into the technicalities of the PRNG, as they fall outside the scope of this paper. I will focus just on the relationship between the nature of the randomness produced by PRNGs and the concern for fairness.

Obviously, concern for fairness of the operator is what motivated the development of the PRNGs, and the ethical dimension of randomness accounts for this fairness. The new level of the technology of generating random outcomes for the games can be motivated

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[6] The property of uniformity should be understood in terms of probability, that is, if we make a partition of the set of generated numbers into equal intervals, any generated number is equally probable in each interval.

through concrete facts and goals: 1) classical (technical) devices can be subject to intentional or accidental faulty construction, circumstantial damage, or fraudulent intervention; 2) the existence itself of the concerns related to these abnormal situations creates an unwanted atmosphere of suspicion in gambling which affects the relationships between the persons involved; 3) the processes of generating the outcomes must keep pace with the technological advancement of the games; and 4) the games should provide the best or most effective form of randomness available for their outcomes, as long as this aim is attainable technologically. Besides these concrete motivations, the electronic versions of the games, incorporated mostly into the online casinos, require PRNG as a necessity for their functionality.

Regarding the motivation for number 4, the nature of the particular type of randomness provided by PRNGs is not in contradiction with the superlative of the aim, because the aim is relative to technological availability. The PRNGs generate their random numbers by using *mathematical algorithms*, which are successions of steps of a recursive nature. The randomness provided by PRNG can be qualified as a form of algorithmic randomness, in the sense we have used the term in the section describing the non-mathematical nature of general randomness. Obviously this particular form of randomness is mathematical in nature, but the input of a seed means an experimental intervention that weakens the mathematical nature. In addition, both the seed and the recursive nature of the algorithm are in contradiction with the core feature of the concept of general randomness – that of absolute independence – a feature that mathematicians were not able to reproduce in a prospected mathematical definition of randomness and which would characterize the so-called 'true random generators.' These are also the reasons this form of random number generators is named with the prefix "pseudo,"

The concerns raised for the fairness of the PRNGs do not refer to the theoretical fact that PRNG-generated sequences are not purely (truly) random (per the abstract description of the general concept). The mathematical constitution of the PRNG algorithms is widely accepted as a guarantee that the randomness provided by them is "sufficiently" good or applicative for the field of gambling. Legislation also supports this principle and answers the concerns by regulations that establish the obligation of every casino to have their software tested and audited by an independent expert third party, including in what concerns their PRNG. The independent audits assume empirical and theoretical tests and ensure that the outcomes of the games are not influenced by variables such as the number of credits in play, size of the potential payout, VIP cards, and other subjective factors (Gambling Commission, 2021). The PRNGs passing such tests and evaluations are certified as fair, which means they are sufficiently random and properly implemented into the game's software.

And yet gamblers maintained their concerns about fairness despite both the virtues of the algorithmic randomness produced by PRNGs and the care of the regulating bodies for fairness by certification. An extensive empirical study about players' attitudes toward internet gambling in what concerns consumer protection and regulation, carried on 10,838 online casino and poker players from 96 countries, showed that only about half of the participants agreed or strongly agreed that online gambling software was fair. Moreover, 37.6% of respondents either agreed or strongly agreed that operators can manipulate the software in their favor (Gainsbury & al., 2013). These results were also confirmed by other recent studies (for instance, Konietzny & Caruana, 2019), but the concerns are also detectable from the gamblers' community in the content of gambling web portals, forums, and blogs.

Hence, aside from technological needs, the development of PRNGs was motivated by the concerns that their algorithmic randomness is better and fairer than mechanical randomness; however, the concerns were maintained for the former, mainly reflected in the suspicion that the operators have the possibility of cheating with their software. It is fair to hypothesize that the constructive element causing suspicion is the *seed*, without which a PRNG cannot work and which the generated sequence of numbers depends on, and as such is suspected to be manipulated as an input element.

New technological advancements responded to such concerns with three modifications: high-speed multiple generation (meaning that the PRNG generates hundreds or thousands of numbers per second, of which only one is chosen by the electronic processes of the game); the possibility of adding to the PRNG an initial sub-algorithm for randomly generating the seed (which, of course, needs another initial seed); and, as a cutting-edge innovation in online gambling, using the so-called *provably fair algorithms*. Provably fair algorithm is a PRNG algorithm using three elements as inputs instead of one – the operator's seed, a player's seed, and an integer variable (called nounce, which increases by one with every new hand or play). These elements are inputted into a secure hashing algorithm (SHA), which will combine them and output a hexadecimal string that is used by the PRNG to generate numbers. This conception is meant to ensure that all the bets are fair, and the player can verify their fairness.

While player's inputting their own seed seems to be a decisive factor in preventing any possible cheating with the PRNG, theoretical flaws of the provably fair algorithms themselves and their actual use are already being discussed in the gambling-expert community and argued in terms of tracking players' habits and strategies, running the algorithms in circumstantial forced conditions, altering the code, and intervening in game's functioning associated with the PRNG inputs (Butler, 2020). It's just a matter of time for new theoretical approaches to PRNGs to emerge in order to overcome such concerns.

### 2.1.2 Conclusion about the concerns for fairness

Let's take historical stock: The PRNGs replaced classical mechanical devices because of the concern about an altered randomness of the latter. The PRNGs were improved in their mathematical algorithms and implementation technology due to the concern about the possibility of the operators' cheating with them. The implementation of the PRNGs was improved by new seeding processes involving players' choices due to the same concern. New concerns for fairness have been raised, and perhaps the process will no doubt continue with any new technology also because it is human nature to doubt and be suspicious.

In the course of this advancement – stretching over decades – serious research and technological resources have been allocated in the industry. As for the players, who are actually the end-customers benefiting from the fairness of the games and the main objectors, their concerns also assumed effort and consumption of resources – track-recording, debating, reading and participating in community discussions, getting informed about the RNG and its issues, and so on. The question is to what extent is this worth the effort it consumes from the players' perspective.

While the issue of the games' fairness submits to the more general case of product fairness in the economy as to consumer protection – games are audited and certified like any other product and the cheating company naturally comes to self-eliminate from the market, given the competition – I will not take this argument for an answer to the previous question, but I will try to answer it from a problem-gambling perspective.

During the decades of concerns about fairness, there were no studies reporting any decrease of the problem gambling phenomenon in the global population. It is known that not only the programs of awareness, prevention, and counseling have a role in keeping players on the track of non-problematic gambling, but also their own efforts, since assimilation of knowledge reverts to their own will and cognition. However, when such efforts are directed in a tangential or even opposite direction, resources may not be allocated for what needs priority. I suggest that of course the concern for fairness should be the problem of the industry and legal bodies and not of the players, as otherwise their efforts would be detoured from contributing to more important concerns – those for not developing problem gambling. Instead of debating and learning about the RNGs and their technological flaws in regard to randomness, players could instead get better informed about the randomness-related cognitive distortions in gambling, in the form of the common misconceptions, irrational beliefs, and fallacies, which are considered risk factors in problem gambling (Orlowski & al., 2020; Philander & Gainsbury, 2023). Players could also be concerned about the transparency of the inner design of the games they play, which also falls within the ethical side of gambling (Bărboianu, 2014) and is related to general fairness.

I have argued that the concern for fairness does not cancel the randomness of a game; at most, it alters it. However, it can be hypothesized that an altered randomness has less harmful effects for the gambler than the effects of a cognitive distortion, in both the cognitive and the money-losing respects, in either long-, medium-, or short-run: With a cheating operator, the gambler would face losses that probably would not occur at a much different frequency than with a fair game in a given session; in the long-run, such an operator would be either out of business or converted to a fair one, or the player will quit their game due to the losses. Instead, playing under, say, the Gambling Fallacy or subjective estimations of probabilities of winning, the player may raise their stakes due to the fallacious premises, and as such may come to lose in higher amounts, not to mention that the distortion would affect their play a long time ahead, not only with the assumed operator, but also anywhere else. It's just a theoretical example suggesting that the concept of randomness is directly involved in the inclination of the balance 'concern for fairness – concern for responsible play': for the player, the perception and understanding of randomness influences the weight of both sides of the balance, while for the problem-gambling expert, the inclination toward the latter side is a fair theoretical hypothesis deserving additional research.

I suggest that new empirical studies on the gambler population, designed for assessing the inclination of the balance above relative to the gamblers' perception of the nature of randomness and on the concept of "sufficiently random" in gambling, would be fruitful for supporting the theoretical hypothesis and implementing it in prevention and awareness programs.

What is important to retain is that the ethical and functional dimensions of randomness, and much less the mathematical dimension, were involved decisively in our discussion about the concern for fairness. The fact that the PRNG algorithms have a mathematical nature does not seem to pose problems of fairness, nor of insufficient randomness (except perhaps for the extreme conspiracists), but rather, their actual application in the functioning of the games. Besides the PRNG, the relationship between randomness and probability theory (as accounting for its mathematical dimension) was not employed in any of the arguments of the discussion.

## 3. **The dimensions of randomness in problem gambling**

We have already entered the domain of problem gambling in the last section when discussing the concerns for fairness relative to the concept of randomness and found that the ethical and functional dimensions, and less so the mathematical dimension, of randomness are mainly involved in the mentioned issues.

In this section I will show that there are again non-mathematical dimensions of randomness that essentially account for the involvement and roles of this concept in the cognitive zone of problem gambling and not its mathematical dimension.

Randomness was employed and discussed in problem gambling as directly related to the mathematics of gambling (more precisely to probability theory and statistics applied in gambling), in the context of education of the gamblers and particularly that of the math-related cognitive distortions in the form of misconceptions, fallacies, and irrational beliefs.

A current of empirical research starting at the beginning of the 2000s tested the hypothesis that teaching gamblers the mathematics applicable to gambling would change their gambling behavior (Hertwig & al., 2004; Steenbergh & al., 2004; Williams & Connolly, 2006; Lambros & Delfabbro, 2007; Pelletier & Ladouceur, 2007; Peard, 2008; Turner & al., 2008; Costello & Fuqua, 2012; recently, Primi & Donati, 2022). Overall, these studies have yielded contradictory, non-conclusive results, and some of them unexpectedly tended to answer “no” to the hypothesis that gamblers receiving specific mathematical education show a significant change in gambling behavior after the intervention. Regardless of the criticism of many of these studies with respect to both their experimental setup and lab-specific methodology (Ladouceur & al., 2013; Keen & al., 2017), I will focus on the employment of the concept of randomness in such studies.

The studies that referred to or employed the concept of randomness (as a component of the educational intervention or program they discussed) in either the experimental setup, hypotheses, or conclusions or those programs/interventions, did so by qualifying randomness as a *mathematical* ‘notion’ or ‘concept’ and implicitly considered that learning about randomness falls within the learning about mathematics of gambling and the statistics curricula. This wide-spread approach is also confirmed by two extensive reviews (of 19 such studies) by Keen & al. (2017, 2019). Based on their reviews, Keen & al. (2019) advanced the idea that educational programs in problem gambling should shift away from messages about gambling harms and instead develop a cognitive-developmental model, where correction of the cognitive distortions through gambling-math education should have a central role. I fully endorse this thesis.

I will eventually argue that the artificial qualification of the concept of randomness as mathematical (despite its non-mathematical nature) is not just a harmless lexicographical convention for simplifying things, but has implications for both the research in this field and the constitution and effectiveness of the educational programs based on the results of the research.

My argument will be developed around the GF, which is the most representative distortion related to the concept of randomness, and iconic for the cognitive aspects of problem gambling.

### 3.1 **The Gambler's Fallacy and the knowledge about randomness**

The conceptual framework in which the gambling cognitive distortions have been investigated in problem gambling was that of the program of heuristics and biases (Kahneman & Tversky, 1974). In particular, the psychological nature of the GF was widely accepted as a representativeness heuristic, and the primary cognitive process invoked as determinant for this distortion was the "inadequate"/"erroneous"/"incorrect"/"faulty"/"biased" *perception* of the concept of randomness (Ayton & Fischer, 2004; Goodie & Fortune, 2013). The most frequently used description for this erroneous perception was that people have a biased concept or notion of randomness, which deviates from the *statistical* one (Hahn & Warren, 2009). In this approach to the GF, the concept of 'subjective randomness' has been advanced (Ayton & Fischer, 2004).

In this general description of the GF in the cognitive psychology and problem gambling fields, it is suggested that the reference mark for the pathological attribute of the perception of randomness (that is, relative to the "correct" or "good" or "objective" perception) would be the statistical (hence mathematical) notion of randomness. While I have shown that randomness does not have a mathematical nature, it is still fair to assume that the mathematical dimension of randomness was invoked and not its nature – or more precisely, the relationship between randomness and the statistical (probability) theory that underpins the GF. But is this really the case? Should one refer to a kind of mathematical randomness as pertaining to the correction of the GF?

When describing how the erroneous perception of randomness breaks down in concrete flaws in reasoning in terms of statistics and probability theory of those affected by the GF, three interdependent fallacies and misconceptions are detected to be possible, individually or combined:

*a* - not believing that the outcomes as elementary events or similar events associated with the same trial are equally probable (given the experience of the previous trials);

*b* - misunderstanding the notion of statistical independence of two events produced by two different trials of the same random experiment;

*c* - equating relative frequency with probability by incorrectly applying the Law of Large Numbers on finite intervals of trials (the so-called Law of Small Numbers).

(Bărboianu, 2022, p. 70-77)

Issue *c* falls within the effects of an inadequate understanding and application of a mathematical theory, where randomness is not explicitly employed as essential for any educational task. Applying the theory correctly does not assume understanding randomness in depth, but only following the math. It is more about knowing than perceiving, and correcting issue *c* would revert to fulfilling cognitive-educational tasks based on *mathematical* knowledge[7]. Therefore I will not treat issue *c* here with respect to randomness; I will focus instead of issues *a* and *b*, where 'perception' is more involved.

I think, though, that the distinction between perception and knowledge is not that relevant in the matter of GF with respect to the dimensions of randomness, since the general concept of randomness does not have a theoretical definition; therefore, I will not invoke such a distinction. I will instead argue that equating the "correct" perception of randomness with a kind of mathematical concept (and as such forcibly or artificially strengthening its mathematical dimension) is inconsistent with the implicit mathematical relationship assumed to exist between the "statistical" randomness and the mathematical notions and results involved in issues *a* and *b;* thus, correcting those issues would be just the result of having a good grasp of the associated mathematical knowledge. Let's take them one at a time:

***a***) The correct version of issue *a* is that similar outcoming events of a game (such as one number or another, or red or black in roulette) in the same trial are equally probable. Here we have to make the distinction between 'equally probable' and 'equally possible.' If talking about the *elementary* events of a random experiment (such as each number in roulette, or each combination of stops in slots), they are equally possible just because randomness is assumed in the conceptual framework of the applied theory. The 'equally possible' attribute of randomness is actually a theoretical idealization, making it epistemically equivalent with 'equally balanced evidences' or 'equally unknown' or 'lack of evidences' (recalling the methodological-theoretical dimension of randomness). If talking about similar *compound* events (such as red and black, in roulette, or some specific combinations of symbols in slots), they are equally probable, as sets of elementary events, according to the properties of the probability function (for those latter events their probability is conventionally assumed the same as in the virtue of the 'equally possible' idealization). Mathematically speaking, this concerns the distinction between the sample space and the field of events of a random experiment; in the sample space, the events (outcomes) are equally possible, while in the field of events, two events can be equally probable.

Schematically, for the same trial, the following chain of determination reflects the relationships between the discussed concepts:

$$\textit{randomness} \xrightarrow{\text{conceptual implication}} \textit{equally possible} \xrightarrow{\text{mathematical implication}} \textit{equally probable}$$

The first relationship is conceptual, as 'equally possible' falls within the concept of randomness (a particular to general determination), while the second relationship is merely mathematical (the derivation of a property within the mathematical theory). Therefore, there is no resulting mathematical relationship between randomness and

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[7] Perception is still involved in such an educational framework if talking about application and interpretation of that mathematical theory – it is about the perception of the concept of potential infinity present in the Law of Large Numbers, which is responsible for the in-depth understanding of the notion of statistical average.

'equally probable,' since the relationships are not both mathematical and no rule of transitivity applies. Randomness is neither identical nor equivalent, nor does it stand in an implication relationship with 'equally probable,' in a mathematical sense.
An individual affected by the GF by issue *a,* and being influenced by previous outcomes, would not believe in the "no evidence" or "equally balanced evidences" assumption (taking randomness to be disorder and not order in our dualist description), and thus would have no rational reason to deny the mathematical implication in the scheme.

**b**) The mathematical notion of statistical independence of two events *A* and *B* associated with two different trials of the same random experiment is defined by the relation $P(A \cap B) = P(A) \cdot P(B)$. This relation is mathematically equivalent with the conjunction of two relations: $P(A|B) = P(A)$ and $P(B|A) = P(B)$ (this equivalence is derived by employing the formula of conditional probability and doing the algebra).
The two latter relations express the independence in terms of conditionality – the probability of event *A* does not depend on event *B* and conversely – however, this (non)dependence falls within the concept of randomness as both unpredictability and no rule of determination (Event *B* does not *physically* depend on event *A* and conversely, although they are produced by the same experimental setup). This assumption is justified by the experiment having been qualified as random. Therefore, we have a scheme of determination similar to issue *a*:

*randomness* $\xrightarrow{\text{conceptual implication}}$ $P(A|B) = P(A)$ and $P(B|A) = P(B)$ (*unpredictability & no determination*) $\xrightarrow{\text{mathematical implication}}$ $P(A \cap B) = P(A) \cdot P(B)$ (*statistical independence*)

The different kinds of relationships between the three concepts render impossible a mathematical relationship between randomness and statistical independence, per the same argument as in the case *a*.
An individual affected by the GF by issue *b* would not believe in the independence described in terms of conditional probability (as believing in a sort of physical dependence of the two events produced by the same device; taking randomness to be order and not disorder) and would be less concerned or not concerned at all about the mathematical implication in the scheme.

Let's draw the conclusions about the GF case in what concerns the qualification of the "good"/"objective"/"unbiased" randomness as mathematical (statistical). Beyond the epistemic conceptual-theoretical arguments presented in the first sections that randomness is not a mathematical concept although it has a (weak) mathematical dimension, from the analysis of the two issues *a* and *b,* it follows: If randomness involved in the GF was a mathematical concept, then the equally-probable and statistical independence properties would be derived mathematically from randomness. If the correction of the GF assumed employing a mathematical concept of randomness, then any cognitive-educational intervention aiming at this correction would be based on mere mathematical knowledge (about mathematical implications involving defined concepts and properties). The two schemes of determination show that this is not the case – no mathematical relations can be established between randomness (however perceived as subjective or objective) and the two properties. It's the epistemic dimension and not the mathematical dimension of randomness that is decisively involved in the GF and its correction. This epistemic dimension is related to perception and cognition and reflects the *kind* of relationship of the concept with the mathematics of gambling.

Then, talking in terms of correction of the GF, any cognitive-educational tools, interventions, or programs based on the qualification of "objective" randomness as mathematical are theoretically failing as self-contradictory – they point to mathematical knowledge to correct issues *a* and *b* (which are actually the distortions and not the perception itself of randomness in a cognitive-sensorial sense), but no mathematical knowledge (in the sense of relational structures) includes randomness. Only extended epistemic-mathematical knowledge includes this concept and its relationships with other mathematical concepts and properties in probability theory and statistics. In pragmatic terms, sending gamblers "back to school" to learn the mathematics applied in gambling would be of no theoretical help in correcting the GF, just because no mathematical course will tell them about the nature of randomness; only philosophical courses do that. An inadequate perception of randomness would indeed prevent one from grasping further the relationships with the mathematical concepts involved in issues *a* and *b*, but perceiving it as mathematical would be ineffective as well.

This thesis is also marginally supported by studies associating the GF with neurophysiologic conditions rather than poor external cognitive achievements (for instance, Huang & al., 2019; Xue & al., 2012) and can co-habituate with their results; such studies can be fairly correlated with the hypothesis that the GF also affects educated gamblers and non-problem gamblers (Marmurek & al., 2015; Matarazzo & al., 2019). With respect to our thesis and this latter research, the hypothesis that mathematicians themselves can also be affected by the GF is not hazardous at all and would worth testing it by further empirical studies.

In a prospective cognitive-educational model of correcting the GF, the kinds of relations between the concepts involved are essential as acquired knowledge. As I argued, distinguishing between these kinds is possible only in a conceptual framework where randomness is acknowledged in its *epistemic* dimension. Inducing artificially the idea that randomness has a strong mathematical dimension – even for the sake of simplifying things or of any lexicographic convention – is detrimental to the goal of such a cognitive model. Putting forward the epistemic dimension of randomness and not the mathematical one is still consistent with the traditional description of the GF in terms of perception, as cognitive psychology and epistemology share diffuse borders in many zones,and perception is a concept shared by the two disciplines.

## 4. **Conclusions**

I have argued in terms of foundation and history of probability theory, in an epistemological framework, that general randomness is not a mathematical concept because it does not have a mathematical definition to describe it in its full complexity. In sciences and mathematics, randomness has a theoretical-methodological dimension and role, which submits to its more general epistemic dimension. I qualified such dimensions of randomness as strong and its mathematical dimension as weak, the latter per its indirect relationship as a primitive notion with probability theory.

In industrial gambling, randomness has interdependent functional and ethical dimensions, as a necessary prerequisite for the functionality and fairness of games of chance. I have shown that its mathematical dimension is not involved decisively in these roles, although the PRNGs are constituted on the basis of mathematical algorithms and as such provide an algorithmic randomness. The concerns for the fairness of the PRNGs actually pertain to the concrete application of these algorithms rather than their mathematical constitution, and I have claimed that the effort and resources allocated by players for overcoming such concerns are not justified when weighed against the much more certain harms produced by a problematic gambling behavior, in particular gambling-specific cognitive distortions.

In problem gambling, randomness has been conventionally or artificially granted a strong mathematical dimension or mathematical nature, as the review studies revealed. While one may see such qualification as just a lexicographic simplification or simplistic reference, I argue that it has implications for both the research and the educational-cognitive programs in problem gambling having as topics cognitive distortions and gamblers' education.

I took the GF as a theoretical case study for discussing this distortion relative to the nature of randomness and found that its constituent misconceptions about 'equally probable' and 'statistical independence,' both directly related to the concept of randomness by its epistemic dimension, cannot be supposed to be corrected in a merely mathematical cognitive-educational framework if randomness is assumed to be a mathematical concept. Such an assumption would elude the nature of the relationships between the concepts involved in the fallacy, and implicitly the distinction between the kinds of these relationships, which constitutes essentially the structural knowledge about them. In pragmatic terms of correcting the GF, directing the individuals affected by the GF to a program or counseling based on mathematical curricular content would not change their perception of randomness from subjective to objective or from inadequate to adequate, simply because randomness is not part of the formal mathematics of gambling, but of the whole epistemic context of it. This of course does not apply to every math-related cognitive distortion in gambling. For instance, the conjunction fallacy consists of an incorrect understanding of or lack of knowledge about a specific property of probability, although circumstantial factors (such as the descriptive text of the situation) are known to influence the correct belief (Costello & Watts, 2017); correction of the conjunction fallacy would have no essential elements left outside the mathematical context of the issue. Instead, for the near-miss effect, in another paper (Bărboianu, 2019) I have argued that by focusing equally on the mathematical description of the near-miss fallacy and its epistemology, we can identify more precisely the cognitive tools recommended as strategies to correct the distortion. As in our GF analysis, the epistemic dimension of the mathematical description of the near-miss phenomenon is decisively associated with the inadequate perception of the near-miss, and its role manifests before any hypothetical mathematical fallacy (when splitting the "near-missed" outcome in a matching and non-matching part).

It follows from our current and cited research that the epistemic dimension of the math-related concepts involved in the gambling cognitive distortions should not be given merely marginal attention, and conceptual distinctions should be made before proceeding to any theoretical approach of these distortions. In particular, randomness has to be employed in its non-mathematical dimensions (including the epistemic one) as well as in its mathematical one in problem-gambling research. And since educational programs dedicated to prevention and awareness are the result of applying research, these should adopt the same distinctions and distributed focus.

As it follows from our analysis, there are three main arguments for employing the distinctions between the dimensions of randomness in problem gambling research and associated educational-cognitive programs, and focusing on the non-mathematical dimensions:

First, whether we talk about studies on educational interventions for gamblers or programs delivered in the awareness/prevention zone, they all have a didactic component which is associated with a certain academic discipline from which the specific curricular content is imported. The beneficiaries of the interventions or programs as non-experts are referred tacitly or directly to a certain discipline by the simple reference to the attribute or dimension of the subject matter of study. Telling them about a mathematical or statistical randomness will direct them to mathematics; however, I have already argued that courses in this discipline will tell them nothing about randomness. Call this the disciplinary argument.

Second, the studies about educational interventions on gamblers for evaluating the changes in their gambling behavior (such as those cited in a previous section) consists of an interventional knowledge base (what is taught) and evaluation of the new condition after the intervention (by answers to questionnaires, reflecting intentions and acquired knowledge). The results of such studies not only assess the changes in behavior (as declared by subjects), but also make associations between the various elements of the two components (units of the learning content, the values of the variables describing the acquisition of the new knowledge, items in the questionnaire and answers). Any distinction or fine-graining in a concept or unit delivered (such as would be the distinctions between the mathematical and non-mathematical dimensions of randomness) leads to a change of the set of possible associations and may change some associations themselves; this means changing the conclusions of a study, including in what concerns the interpretation of the results. Call this the methodological argument.

Third, merging all dimensions of randomness into a mathematical dimension or referring only to the latter actually means lessening the complexity of the concept; however, complexity is what characterizes randomness, and adequate understanding of a complex concept is inconsistent with excessive simplification. Moreover, inducing to the non-expert gambler the idea of a mathematical randomness is supposed to incline their cognitive balance toward order rather than disorder as an attribute of the concept, since mathematics applied in gambling exhibits a kind of order (equal probabilities for similar events and the LLN or Law of Averages). Under this cognitive condition, there is no reason to believe that a gambler will be more convinced that a black is not "due" after a streak of ten reds than they would had randomness been explained to them as a "non-mathematical disordered order" – more so, in fact, as the individual also may not distinguish between math and applied math in gambling. The latter may even be tricky in terms of interpretation in empirical terms, since probability itself is a tricky concept for those unfamiliar with it). Call this the epistemic argument.

The disciplinary and epistemic arguments apply to any kind of research, program, or counseling scheme having as its object the adequate understanding or perception of the concept of randomness in gambling, especially relative to the gambling-specific cognitive distortions. Focusing on the philosophical aspects of randomness for cognitive-educational goals should not be viewed as something unusual, as turning to philosophy for enhancing understanding in an educational context is not a novelty. In mathematics education, the potential of the associated philosophical disciplines (epistemology, philosophy of mathematics and of science, foundations of mathematics) in this respect has already been established (Kitcher, 1983; Ernest 1989, 1994; Godino & Batanero, 1998; Skovsmose, 2013; Ernest & al., 2016) and the theoretical research has concluded that teaching mathematics for an enhanced conceptual understanding includes teaching *about* mathematics in its complex nature. A similar though differently motivated necessity was advanced for the more general case of science education (Hills, 1992; Matthews, 1994; Mellado & al., 2006; Höttecke & Silva, 2011).

Per all the above arguments, I advance the thesis that the cognitive-developmental model of educational programs, focused on correcting the gambling cognitive distortions, envisioned by Keen & al. (2019), should be designed by assimilating the distinctions between non-mathematical and mathematical dimensions of randomness and give the former the deserved attention. Such distinctions should be also adopted by research dealing with the mathematically-related gambling cognitive distortions, where approaching the involved mathematical concepts beyond their mathematical nature is worth pursuing.

Further theoretical research is needed to provide the adequate design of future studies incorporating the advanced epistemic approach of randomness and other gambling-specific mathematical concepts, and of those investigating or assessing the effectiveness of this approach. Theoretical research is also needed to establish the adequate conceptual framework of an interdisciplinary cognitive model of educational programs that incorporates the epistemic approach here discussed.

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