Problem 11589: Where are the zeros?

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next points are $A_3 = (5rs : -3sq : -3qr)$, $A_4 = (11rs : -5sq : -5qr)$, and in general $A_n = (a_n rs : -a_{n-1}sq : -a_{n-1}qr) = ((a_n/a_{n-1})rs : -sq : -qr)$, where $\{a_n\}$ is defined recursively by $a_0 = 1$, $a_n = 2a_{n-1} + (-1)^n$, the Jacobsthal sequence. Take the limit as $n \to \infty$ to find (since $a_n/a_{n-1} \to 2$) that $A_n \to A^* = (2rs : -sq : -qr)$, so A^* is the intersection of p with the line A_0P ($ry = sz$, same as A_0A_1), since its coordinates satisfy both equations.

Also solved by R. Chapman (U. K.), M. Goldenberg & M. Kaplan, J.-P. Grivaux (France), A. Habil (Syria), M. E. Kidwell & M. D. Meyerson, L. R. King, O. Kouba (Syria), J. C. Linders (Netherlands), O. P. Lossers (Netherlands), J. Minkus, R. Stong, GCHQ Problem Solving Group (U. K.), University of Louisiana at Lafayette Math Club, and the proposer.

Where Are the Zeros?

11589 [2011, 653]. *Proposed by Catalin Barboianu, Infarom Publishing, Craiova, Romania.* Let P be a polynomial over $\mathbb{R}$ given by $P(x) = x^3 + a_2x^2 + a_1x + a_0$, with $a_1 > 0$. Show that P has a least one zero between $-a_0/a_1$ and $-a_2$.

Solution by William J. Cowieson, Fullerton College, Fullerton, CA. Note that $P(-a_0/a_1) = (a_0^2/a_1^3)(a_1a_2 - a_0)$ and $P(-a_2) = a_0 - a_1a_0$, so that

$$P(-a_0/a_1) = -(a_0^2/a_1^3)P(-a_2). \tag{1}$$

There are three cases.

(1) If $a_0 - a_1a_2 = 0$, then the interval reduces to a single point, and that point is a zero of P.

(2) If $a_0 = 0$, then $P(x) = x(x^2 + a_2x + a_0)$ has zeros at 0 and at $(-a_2 \pm \sqrt{a_2^2 - 4a_1}\,)/2$. If $a_2^2 - 4a_1 < 0$, then 0 is the only real zero of P. Otherwise, $(-a_2 \pm \sqrt{a_2^2 - 4a_1}\,)/2$ are both strictly between $-a_0/a_1 = 0$ and $-a_2$, since $a_1 > 0$.

(3) Both $a_0 - a_1a_2 \neq 0$ and $a_0 \neq 0$. In this case, from (1) we see that $P(-a_0/a_1)$ and $P(-a_2)$ are nonzero and of opposite sign when $a_1 > 0$. Hence the Intermediate Value Theorem implies that there is a zero between $-a_0/a_1$ and $-a_2$.

Also solved by B. K. Agarwal (India), G. Apostolopoulos (Greece), S. J. Baek & D.-H. Kim (Korea), B. D. Beasley, M. W. Botsko, D. Brown & J. Zerger, V. Bucaj, P. Budney, H. Caerols (Chile), E. M. Campbell & D. T. Bailey, M. Can, M. A. Carlton, T. Castro, J. Montero & A. Murcia (Colombia), R. Chapman (U. K.), H. Chen, W. ChengYuan (Singapore), J. Christopher, D. Constales (Belgium), W. J. Cowieson, C. Curtis, P. P. Dályay (Hungary), C. Degenkolb, C. R. Diminnie, K. Farwell, J. Ferdinands, D. Fleischman, V. V. García (Spain), O. Geupel (Germany), W. R. Green & T. D. Lesaulnier, J.-P. Grivaux (France), M. Hajja (Jordan), E. A. Herman, G. A. Heuer, S. Kaczkowski, B. Kalantari, B. Karaivanov, T. Keller, L. Kennedy, J. C. Kieffer, O. Kouba (Syria), P. T. Krasopoulos (Greece), R. Lampe, K.-W. Lau (China), J. C. Linders (Netherlands), J. H. Lindsey II, O. López (Colombia), O.P. Lossers (Netherlands), J. Loverde, Y.-H. McDowell & F. Mawyer, F. B. Miles, S. Mosiman, K. Muthuvel, M. Omarjee (France), Á. Plaza & K. Sadarangani (Spain), P. Pongsriiam & T. Pongsriiam (U. S. A. & Thailand), V. Ponomarenko, C. R. Pranesachar (India), R. E. Prather, R. Pratt, D. Ritter, A. J. Rosenthal, U. Schneider (Switzerland), C. R. Selvaraj & S. Selvaraj, A. K. Shafie & S. Gholami (Iran), J. Simons (U. K.), N. C. Singer, E. A. Smith, N. Stanciu & T. Zvonaru (Romania), J. H. Steelman, A. Stenger, R. Stong, M. Tetiva (Romania), N. Thornber, V. Tuck & A. Stancu, D. B. Tyler, D. Vacaru (Romania), E. I. Verriest, J. Vinuesa (Spain), T. Viteam (Germany), Z. Vörös (Hungary), M. Vowe (Switzerland), T. Wiandt, H. Widmer (Switzerland), R. Wieler, S. V. Witt, N. Youngberg, J. Zacharias, Z. Zhang, Fejéntaláltuka Szeged Problem Solving Group (Hungary), GCHQ Problem Solving Group (U. K.), University of Louisiana at Lafayette Math Club, Missouri State University Problem Solving Group, NSA Problems Group, and the proposer.