Review of Tim Button's *Level Theory Parts 1–3*

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Tim Button, Level Theory, Part 1: Axiomatizing the Bare Idea of a Cumulative Hierarchy of Sets, *The Bulletin of Symbolic Logic*, vol. 27 no. 4 (2021), pp. 436-460.

Tim Button, Level Theory, Part 2: Axiomatizing the Bare Idea of a Potential Hierarchy, *The Bulletin of Symbolic Logic* vol. 27 no. 4 (2021), pp. 461–484.

Tim Button, Level Theory, Part 3: A Boolean Algebra of Sets Arranged in Well-Ordered Levels, *The Bulletin of Symbolic Logic* vol. 28 no. 1 (2022), pp. 1–26.

Introduction

Tim Button's Level Theory papers concern what has been come to be known as the *iterative conception of set*; the idea that new sets are successively formed from old. Primarily emerging as a way of responding to the set-theoretic paradoxes, the iterative conception has gained much traction in philosophy and seems well-understood mathematically (via the V_α hierarchy). It would thus be easy to dismiss these papers as revisiting an already understood and well-worn subject matter. This would be a mistake. By weaving together mathematical results and philosophical observations

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with a careful examination of the history of these ideas, Button has produced a series of highly informative and beautifully presented papers that shed new light on a topic of central foundational significance.

1 The main ideas

1.1 Part 1: Axiomatizing the bare idea of a cumulative hierarchy of sets

We start Part 1 with the following story:

> **The Basic Story.** Sets are arranged in stages. Every set is found at some stage. At any stage s: for any sets found before s, we find a set whose members are exactly those sets. We find nothing else at s.

We formalise this by working against the background of second-order logic, and taking two kinds of first-order variables; x, y, z etc. range over *sets*, and **q**, **r**, s range over *stages*. We also, in addition to set-membership, have three primitive binary predicates (i) '$\in$' for *set-membership*, (ii) '$<$' for one stage being *before* another, and (ii) '$\preceq$' is a relation for a set x being 'found' at a stage s.

Since the list of axioms needed is rather short, it's perhaps worth just stating the axioms of his stage theory or ST:

Extensionality As usual, the axiom that sets with the same members are identical.

Order $<$ is a transitive relation on the stages.

Staging Every set is a member of some stage.

Priority For any set, its members are found before it is.

Specification For any property F, if all Fs are found before a stage s, then the set of Fs is found at s.

Button then lays down a *set* theory (i.e. a theory just with '$\in$') called *Level Theory* (or LT). This consists of the familiar axioms of **Extensionality** and **Separation**, with a **Stratification** principle that every set exists at some *level*, where a level is effectively a set-theoretic proxy of a stage.

On to some important theorems. Let's start with Theorem 4.1:

$$\mathsf{ST} \vdash \phi \text{ iff } \mathsf{LT} \vdash \phi \text{ for any } \mathsf{LT} \text{ sentence } \phi$$

This vindicates the idea that LT is the 'right' set-theoretic correlate of the multiple-primitive ST; they prove exactly the same things about the sets.

An interesting payoff of this theorem is that it immediately implies that any cumulative hierarchy is well-ordered. The Basic Story might seem to leave open room to have different kinds ordering on the stages, beyond their length (e.g. partial, dense, or ill-founded orderings). No such orderings are possible.

A final theme of Part 1 concerns different kinds of categoricity. A theory is *categorical* when it has exactly one model up to isomorphism, and readers may be familiar with the *quasi*-categoricity result that any two models of second-order ZFC are either isomorphic or one is isomorphic to an initial segment of the other. Button shows that any set-sized model of Level Theory is isomorphic to a V_α and vice versa (Theorem 6.1). Moreover, he shows an *internal* deductive *full* categoricity result. Those wishing to learn about these different kinds of categoricity might consider reading Part B of his *Philosophy and Model Theory* (co-authored with Sean Walsh).

The axioms of ST and LT are rather *minimalist*; since (almost) *any* V_α provides a model for LT, both theories are compatible with the existence of there only being finitely many sets. As it turns out, there are various level-theoretic bolt-ons that one can consider — roughly corresponding to Infinity, Powerset, and Replacement — which yield the expected interpretability results with subtheories of ZF.

The conclusion of the paper provides a beautifully concise history of the subject, tracing how the results given emerged from previous work. In particular, the discussion of the Scott and Montague approach (contained within a manuscript that was sadly never published due to Montague's untimely death) is a historical gem that might be missed on a casual read through. This is an area that merits further study, and Button has done the community a service in drawing attention to it.

1.2 Part 2:Axiomatizing the bare idea of a potential hierarchy

The next two parts can be, in some respects, viewed as variations on the theme of the first. Each takes a slight tweak of the Basic Story, and then shows equivalences with Level Theory and categoricity results.

The second story is *tensed*:

> **The Tensed Story.** Always: for any sets that existed, there is a set whose members are exactly those sets; there are no other sets.

The setting for formalising the Tensed Story is negative-free second-order logic with an axiom assuming that time is past-directed. There are then three modal operators: ⬘ϕ asserts that it *was* the case that ϕ, ⬙ϕ asserts that it *will* be the case that ϕ, and there is an unlimited operator $\Diamond$ that may be glossed as 'sometimes ϕ'.

Button then provides a *Potentialist Set Theory* (PST) using these resources. I won't state the axioms here; they are simply modal reworkings of the axioms of the Stage Theory of Part 1. Interestingly in this context it is only well-*foundedness* that can be proved, linearity (and thus well-ordering) must be axiomatically thrown in.

One can then, similarly to Part 1, consider 'equivalence' theses between versions of LT and PST. One important distinction here concerns mutual interpretability and definitional equivalence. Mutual interpretability occurs when two theories can each provide an interpretation of another, whereas definitional equivalence (roughly speaking) adds the additional requirement that composing interpretations gets you back where you started. It may be helpful to consider some more mathematically familiar examples; many extensions of ZFC are mutually interpretable (ZFC and ZFC + $V = L$ for example). However we know from the work of Albert Visser and Ali Enayat (among others) that two non-identical extensions of ZFC are *never* definitionally equivalent (ZFC is a so-called 'tight' theory).

The equivalence results Button provides do not quite achieve definitional equivalence, but they do come desperately close. Button terms his kind of equivalences '*near-synonymies*'. He provides a helpful classification of the different synonymies available. Importantly, he shows that whether one is a necessitist or contingentist about second-order entities (i.e. whether or not one takes the second-order entities to exist out of necessity or not) has a significant role to play — a deductive near-synonymy is not available for full LT and the second-order contingentist, and a semantic version is only available when the full semantics are employed.

Button then goes on to assess the significance of the near-synonymies. In the case of definitional equivalence, many are happy with the idea that two definitionally equivalent theories 'say the same thing' or 'carve up the same facts' in different languages. This suggests the **Equivalence Thesis** that Actualism and Potentialism are different but equivalent ways of expressing the same facts. Button avoids overstating his case, and ultimately concludes that whether or not the Equivalence Thesis is plausible will depend on whether the potentialist can supply a sufficiently clear conception of the the mathematical modality in play. Whilst I would have liked Button to stick his neck out a *little* more on whether he thinks the Equivalence Thesis hold, I think it was an appropriate choice given the context of an article in the *Bulletin*.

Similar to Part 1, comparison with other modal approaches — in particular those of Parsons, Linnebo, and Studd — is handled in the conclusion. Whilst not as gripping as the conclusion to Part 1, it is nonetheless instructive for those wishing to learn about some subtleties of setting up modal set theories.

1.3 Part 3: A boolean algebra of sets arranged in well-ordered levels

Part 3 aims to show just how far the Basic Story can be modified whilst retaining categoricity and synonymy results. We are presented with:

The Complemented Story. Sets are arranged in stages. Every set is found at some stage. At any stage s: for any sets found before s we find both.

(Lo) a set whose *members* are exactly those sets, and

(Hi) a set whose *non-members* are exactly those sets.

We find nothing else at s.

To axiomatise this idea, we have the same primitives as in Part 1 (i.e. $\in$, $<$, and $\prec$). **Extensionality**, **Order**, and **Staging** are the same as in the Basic Story. Some small tweaks are required to handle complements, however. Call a set 'low' if it found using clause Lo and 'high' if it found using clause Hi. **Priority** and **Specification** then receive relativised versions for the high and low sets. An axiom **Cases** is also required to ensure that every set is either high or low.

As in Part 1, we define a *Boolean* Level Theory (BLT) with just the primitive $\in$. There's a couple of interesting differences to LT, notably (1) there is an axiom asserting the existence of absolute complements, and (2) we need a new notion of level (called a *Boolean level*, or just *bevel*). Again we have a result (Theorem 3.3) that BST proves ϕ iff BLT proves ϕ, and some examination of what can be proved with extensions of BLT (e.g. using BLT analogues of the level-theoretic bolt-ons of Part 1).

We then see that Boolean Level Theory admits of categoricity theorems (Theorem 5.1, 5.2, and 5.3). Moreover it turns out that ZF and and extension of BLT are *definitionally equivalent* (Theorem 7.1). This is *despite* the fact that BLT proves that *every set has a complement*.

The paper closes with some interesting remarks about Conway games and the surreals, showing that one can prove that they form a totally ordered field within BLT (Theorem 8.5). The appendices contain routine checking of the details, a choice which has rendered the main body of the text very readable.

2 Connection to literature and an open question

Button begins the Level Theory papers with the claim that they form part of a triptych, but this belies the fact that the papers currently form the first three parts of a pentaptych. 'The iterative conception of function and the iterative conception of set' (forthcoming in Antos, C, Barton, N. & Venturi, G. (eds.), *The Palgrave Companion to the Philosophy of Set Theory*) cashes out the iterative story in terms of *functions* and associated primitives. Again we have a categoricity result, and a synonymy result with ZF. Beyond this, one finds some more philosophical discussion of the Equivalence Thesis.

The second paper, 'Wand/Set Theories: A realization of Conway's mathematicians' liberation movement, with an application to Church's set theory with a universal set' (forthcoming in *The Journal of Symbolic Logic*) takes the general theme of these papers and pushes it to the extreme. There Button takes what he calls the **Wand/Set Template**: we build the low sets exactly as in the Basic Story, but also at any stage we find the result of tapping any set previously found with a 'wand' (something that constructs new objects from the usual sets). Button shows how this template can be extended to cover Conway games, partial functions, multisets, accessible pointed graphs, and Church's universal set theory. And, importantly, he provides a general theory for such wand/set theories, and shows a synonymy result with a ZF-like theory.

There is one open question that looms large. Whilst Button's papers explore different possible options, they are all set against the background of the iterative conception where we take the *full* powerset at each stage. There are however many kinds of set forming construction which need not be construed in this way, some of which are of philosophical and mathematical interest (e.g. constructibility, predicativism, forcing). The existence of such options presents a possibility for those who want to go a different way. Still, Button has elegantly and precisely laid out exactly what can be done with the 'full' iterative conception, and these papers are essential reading for those who wish to get to grips with this important foundational idea.

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