Mathematics education

WHAT IS MATHEMATICS

School guide to conceptual understanding of mathematics

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If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.
John von Neumann

Do not worry about your difficulties in mathematics; I can assure you that mine are still greater.
Albert Einstein

I dedicate this book to my son, Alexandru, to whom I spoke about mathematics during a time in which he had not yet come into contact with the formalism and symbolism of mathematics. I am sure that the potential "shock" of this contact will thereby be at least ameliorated.

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