What makes a 'good' modal theory of sets?*

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Abstract

I provide an examination and comparison of modal theories for underwriting different non-modal theories of sets. I argue that there is a respect in which the 'standard' modal theory for set construction—on which sets are formed via the successive individuation of powersets—raises a significant challenge for some recently proposed 'countabilist' modal theories (i.e. ones that imply that every set is countable). I examine how the countabilist can respond to this issue via the use of regularity axioms and raise some questions about this approach. I argue that by comparing them with the 'standard' uncountabilist theory, a new approach that brings in arbitrariness rather than the strict controls of forcing is desirable.

Introduction

A widespread idea in philosophy is that sets are more than merely extensional entities, but are somehow 'formed' from available objects. This paper is concerned with articulating and comparing different accounts of such methods of formation. Perhaps the 'standard' view of set-formation is that, if we're given some objects, we turn all possible subpluralities of those objects into sets, and continue this operation forever. This has the

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effect of iterating the powerset operation and collecting together at limits. Normally this view is regarded as in tension with *countabilism* (the position that every set is countable) since Cantor's Theorem implies that the powerset of any set x is strictly larger than x.

This standard account has some attractive features that we'll discuss shortly. However some authors question the standard picture, arguing that there are notions of set-formation that can support countabilism whilst denying the powerset axiom.[1] This raises the question of *why* the standard picture is so attractive, and whether the countabilist can come up with a similarly pleasing story. In this article I'll do the following:

Main Aim. I'll contrast the standard account with some countabilist modal theories of set-construction, isolate some 'good-making' features, and argue that the uncountabilist *currently* retains an advantage over the countabilist.

I argue as follows: §1 sets up some of the basic non-modal axiom systems we'll consider in the paper. §2 provides a presentation of the idea of a *modal theory* for set-formation. §3 explains a modal theory for the standard view, under which sets are formed via iterating the powerset operation, and using union to collect together at limits. §4 examines a different modal set theory due to Chris Scambler that ensures that every set is countable. §5 provides a modal axiomatisation of the Steel-Maddy-Meadows multiverse and presents some results about this system (including that it can interpret ZFC - Powerset + 'Every set is countable', as well as a certain regularity axiom). §6 then presents a challenge for both the Scambler and Steel-Maddy-Meadows potentialist conceptions regarding how they ensure that every set is a member of some stage. I suggest that future developments of countabilist modal set theory should focus less on *forcing* and more on the idea of an *arbitrary enumeration*. §7 provides some concluding remarks and raises some open questions.

1 Non-modal set theory

Later, we'll use a *modal* theory of stages to motivate a *non-modal* theory of sets. In this section, I'll lay down some of these non-modal theories for discussion later.

Since we'll be interested in sets, we'll want to consider the language of

[1]See, for example, [Scambler, 2021], [Builes and Wilson, 2022], [Scambler, MS], and [Barton, Fb].

set theory $\mathscr{L}_\in$ that has a single non-logical predicate $\in$. Let's now specify:

Definition 1. We will consider the following base theories in $\mathscr{L}_\in$:

(i) ZFC is standard Zermelo-Fraenkel set theory with the Axiom of Choice rendered as the claim that every set can be well-ordered and the axiom scheme of Replacement (but the Collection scheme—see below—is *not* included in the axioms, though over the rest of ZFC is provable from Replacement).

(ii) ZFC$-$ is ZFC with the Powerset Axiom deleted.

(iii) ZFC$^-$ is ZFC$-$ with the following Collection scheme added (for any formula $\phi(x, y)$ in two free variables):

$$(\forall a)\big((\forall x \in a)(\exists y)\phi(x, y) \rightarrow (\exists b)(\forall x \in a)(\exists y \in b)\phi(x, y)\big)$$

(i.e. If $\phi(x, y)$ defines a relation, and for some set a and for every $x \in a$ there is always a y ϕ-related to x, then there is a set z 'collecting' together at least one 'ϕ-witness' for every $x \in a$.)

(iv) Count is the axiom stating that all sets are countable.

It's important to distinguish ZFC$-$ from ZFC$^-$ since ZFC$^-$ has more consequences (in the absence of Powerset, Collection and Separation have strictly more consequences than Replacement alone).[2] Generally speaking we will be working with ZFC$^-$, but we will here and there have cause to mention ZFC$-$. Later, we will also discuss the use of some regularity axioms for sets of reals, but since they are a little more involved we'll avoid stating them for now.

A central focus of this paper concerns justifications of Count on the basis of modal theories of set construction. For foundational purposes, it's helpful to note the relationship between ZFC$^-$ + Count and *second-order arithmetic*. This theory is couched in a language we shall denote $\mathscr{L}_2$. This has two sorts of variables (in addition to the symbol $\in$) the first of which range over natural numbers (denoted with lower case variables $n, m, n_0, n_1, ...$) and the second sort ranging over sets of natural numbers or, equivalently, reals (these are denoted by upper-case variables $X, Y, Z...$), as well as symbols for 0, 1, $+$, $\times$, $<$. *Second-order arithmetic* (or SOA) is a theory in $\mathscr{L}_2$ comprising the basic arithmetic axioms (e.g. the recursive

[2]See here [Zarach, 1996] and [Gitman et al., 2016].

axioms for $+$ and $\times$), the induction axiom, and a comprehension scheme.[3] A folklore result shows that ZFC^- + Count and SOA can each interpret the other.[4] We can thus move freely between SOA and ZFC^- + Count, and henceforth we'll largely drop consideration of SOA, working solely in $\mathscr{L}_\in$. However, it bears mentioning that so long as one has ZFC^- + Count, one could be working in SOA, if one so desired.

2 Modal set theory

Since we're focused here on how sets get formed, it's very natural to provide *modal* descriptions of set construction. Some suitably idealised agent can be thought of building sets by:

(i) taking some starting sets, and

(ii) repeatedly applying some set-theoretic construction methods.

There are close links between the idea of forming sets using methods of set-construction, and so called 'iterative' conceptions of set.[5] These are often given a stage-theoretic treatment (for example by axiomatising the notion of stage directly), but I'll take a *modal* approach here and speak more generally about set-construction methods (I'll discuss relationships to iterative conceptions in due course).[6] We'll be examining theories that use various modal operators, intended to axiomatise what can be 'constructed' using a particular 'construction method' over a given domain. So, given some construction method m we'll introduce $[m]$ (the 'necessity operator for m') and $\langle m \rangle$ (the 'possibility operator for m'). $[m]\phi$ should be read as 'no matter how you do m, ϕ', and $\langle m \rangle\phi$ should be read as 'it is possible to do m such that ϕ'. As is standard in modal logic we'll take one (in this case $\langle m \rangle$) to be primitive, and define $[m]\phi$ by $[m]\phi =_{df} \neg\langle m \rangle\neg\phi$. We'll put aside how we interpret these modalities for now (in the end we'll handle them axiomatically and return to this philosophical question in §7).

[3]The details of SOA are available in [Simpson, 2009, p. 4, Def. I.2.4].

[4]Although the theorem is folklore, it is very nicely presented in §5.1 of Regula Krapf's PhD thesis [Krapf, 2017].

[5]See here [Barton, Fb].

[6]For stage-theoretic treatments, see [Boolos, 1971] and [Button, 2021a]. These have the advantage of expressive power, but for the sake of providing easy interpretations of set theory in $\mathscr{L}_\in$ (i.e. avoiding mentioning the stages), integrating better with the literature potentialism (the idea that the universe of sets can 'grow'), and keeping discussion more general, I'll take a *modal* approach. Thanks to Davide Sutto and Chris Scambler for some discussion of this distinction.

For now, let's see how one interprets our *non*-modal theories using a language including the modal operators. This can be done with the following device:

Definition 2. [Linnebo, 2013] The *potentialist translation* of a formula in $\mathscr{L}_\in$ into a language containing $\langle m \rangle$ is obtained by substituting every occurance of $\exists x$ by $\langle m \rangle \exists x \phi$ and every occurrence of $\forall x$ by $[m]\forall x \phi$.

In the context of set theory, we can think of the potentialist translation of an existential quantification as telling us that we can get a set (using m) such that ϕ, and a universal quantification as telling us that no matter what sets we go on to form sets using m, all sets will be ϕ. Clearly, this is a very natural translation for conceptions of set that depend on a notion of set formation.

Modal axiomatisations of set theory are often set up against the background of plural logic, which allows us to talk about how, given some sets, we can form new sets from old. This has new variables xx that range over things in the plural (e.g. the books on my table), a binary relation symbol $\prec$ (where $x \prec xx$ is to be read as 'x is one of the xx'), and has the expected definition of well-formed formula. We'll denote the language obtained by adding these resources to $\mathscr{L}_\in$ by '$\mathscr{L}_{\in,\prec}$'. We'll routinely abuse singularisation and speak of 'a plurality' (as is standard in this field).

For our plural axioms (here we're mostly following the presentation in [Scambler, 2021]) we'll take the following:

Definition 3. *Plural logic* (over set theory) has as axioms (we'll give these axioms informally, see [Linnebo, 2018, Ch. 12] for the formal details):

(i) A principle of extensionality for plurals (that if two pluralities xx and yy comprise the same things, then anything that holds of the xx also holds of the yy and vice versa).[7]

(ii) Additionally, *impredicative* plural logic has the following **Impredicative Comprehension Scheme**:

$$\exists xx \forall y \big(y \prec xx \leftrightarrow \phi(y)\big)$$

for any ϕ in $\mathscr{L}_{\in,\prec}$ not containing xx free.

[7] I'm suppressing some subtleties here about how one formulates the extensionality axiom, see [Roberts, 2022] for details.

(iii) *Predicative* plural logic does not contain the **Impredicative Comprehension Scheme** but rather has the following **Predicative Comprehension Scheme**:

$$\exists xx \forall y \big(y \prec xx \leftrightarrow \phi(y) \big)$$

for any ϕ in $\mathscr{L}_{\in,\prec}$ not containing xx free and not containing any plural quantifiers.

Plural logic has perhaps become the de facto standard in discussions of modal set theory. For the sake of integrating with the literature, I'll follow this convention, but really any extensional second-order entities would do. All we need is *some* device that lets us talk about extensional non-set-like entities of some world, and how they can (or can't) be formed into sets. This bears emphasising, since some authors (e.g. [Roberts, MS]) regard the use of plural resources as problematic for some of the views we'll discuss shortly. The reader who feels queasy is invited to paraphrase away all plural talk in favour of an interpretation of second-order variables more congenial to the present context.

As we'll see, by giving us a picture of how sets are formed, modal set theories allow us (via the potentialist translation) to motivate 'good' non-modal theories of sets, and do so in a 'good' way. But what are the desiderata on being such a 'good' modal theory? To see some, let's turn to an exemplary case.

3 The standard 'uncountabilist' picture

Later, we'll discuss modal set theories that are *forcing-based* and *countabilist* in nature. First though, it will be helpful to get the standard uncountabilist approach on the table, in order to see *why* it's such a *good* conception of set. I'll identify four desiderata (**Naturalness**, **Paradox Diagnosis**, **Interpretation**, and **Capture**) that I'll explain as we work through one axiomatisation of the standard picture and some of its features.

First, let's get a rough description of the standard picture on the table, using our now available plural and modal resources. This accounts of sets takes it that we can start with the empty set. Then given some sets xx, we can form every possible subplurality of xx into a set. Effectively, we are forming sets by iterating the *powerset* operation. We keep doing this, and whenever we hit a limit, we collect together all sets we've previously formed (effectively bundling all our previous applications of powerset together via *union*). So our starting sets are none, and our set-construction methods are powerset and union.

An axiomatisation corresponding to this idea has been provided by Øystein Linnebo. We'll quickly state the axioms, and then discuss them individually after. We introduce a modal operator $\Diamond$ into $\mathscr{L}_{\prec,\in}$ to form the language $\mathscr{L}^{\Diamond}_{\prec,\in}$. We then consider the following axioms:

Definition 4. [Linnebo, 2013], [Linnebo, 2018] (here we follow [Scambler, 2021]'s presentation) Lin is the following theory in $\mathscr{L}^{\Diamond}_{\in,\prec}$:

(i) Classical first-order predicate logic.

(ii) Impredicative plural logic.

(iii) Classical $\mathsf{S}4.2$ with the Converse Barcan Formula added.[8]

(iv) The Axiom of Foundation (rendered as normal using solely resources from $\mathscr{L}_{\in}$).[9]

(v) Extensionality (again using solely resources from $\mathscr{L}_{\in}$).

(vi) **Modal Collapse.** The principle that any things (at a stage) could form a set:

$$\Box\forall xx\Diamond\exists y\Box\forall x(z \in y \leftrightarrow z \prec xx)$$

(vii) Stability axioms for $\prec$ and $\in$ (these mirror the necessity of identity/distinctness):[10]

- $x \in y \rightarrow \Box(x \in y)$
- $x \notin y \rightarrow \Box(x \notin y)$
- $x \prec yy \rightarrow \Box(x \prec yy)$
- $x \not\prec yy \rightarrow \Box(x \not\prec yy)$

(viii) Two principles of plural definiteness:

[8] Normally the Converse Barcan Formula comes for free (one must take steps to block it), see [Linnebo, 2018] (p. 207). I've added it for the sake of explicitness and making the 'growing domains' conception of potentialism clear. I'll make no further mention of this complication.

[9] Another nice option here is to use $\in$-induction. Thanks to Øystein Linnebo for some discussion of this point.

[10] It should be noted that a choice point here concerns whether to use the quantified or free-variable, forms of these axioms, since the free-variable versions seem stronger. Thanks to Chris Scambler for some discussion of this point.

- **Plural Membership Definiteness** is given by the following scheme:

$$(\forall x \prec yy)\Box\phi(x) \rightarrow \Box(\forall x \prec yy)\phi(x)$$

- **Subplurality Definiteness**: Say that $xx \preceq yy$ holds just in case the xx are a subplurality of the yy, i.e. for every x such that $x \prec xx$ we have $x \prec yy$. Then the **Subplurality Definiteness** scheme states that:

$$(\forall xx \preceq yy)\Box\phi(xx) \rightarrow \Box(\forall xx \preceq yy)\phi(xx)$$

(ix) **Modal Infinity.** The axiom that there could be some things comprising all and only the possible natural numbers.

(x) **Modal Powerclass.** The axiom that there could be some things that are all and only the possible subsets of a given set.

(xi) **Modal Replacement.** Every potentialist translation of the Replacement Scheme of ZFC.

(xii) **Plural Choice.** A plural version of the Axiom of Choice 'For any pairwise-disjoint non-empty sets xx, there are some things yy that comprise exactly one element from each member of the xx'.[11]

The stability axioms capture the claim that a set or plurality cannot 'change their mind' about whether they contain a certain set. The plural definiteness axioms deserve some mention, since they will be disagreed upon by some of the axiomatisations we consider: **Plural Membership Definiteness** axiomatises the claim that pluralities cannot pick up members, and **Subplurality Definiteness** axiomatises the claim that a plurality cannot pick up subpluralities. **Modal Collapse**, in combination with **Modal Powerclass**, axiomatises the formation of sets via the powerset operation: Given that we are at some stage of the process, we can form *every possible subset* of a set into a set. **Modal Replacement** ensures that we can continue this transfinitely, collecting together at limits. I think that these observations serve to indicate that there's a reasonable conception of a process of set formation in play. This, I think, suggests the following desideratum on a modal set theory:

[11]Strictly speaking Linnebo does not include AC, but I'm happy to throw it in. Some other authors (e.g. [Studd, 2013]) do so. Nothing hangs on it for the results we have here, other than the fact that if Lin is run without a form of AC, the that gets interpreted will also not include AC (it will be ZF rather than ZFC).

Naturalness. The modal axioms provided should be reasonably 'natural' and not ad hoc in nature; there should be clear notions of set-construction method that motivate the axioms.

Of course the extent to which a modal set theory is natural and/or not ad hoc will likely be somewhat imprecise and a matter of degree.[12] Whilst a full defence of the naturalness of Lin will have to wait for another time, I'm happy to proceed with the assumption that Lin is reasonably natural for now (readers should consult [Linnebo, 2010] and [Linnebo, 2013] for a more thorough—though not uncontroversial—argument to this effect).

There is one small wrinkle regarding Lin and **Naturalness** that should be noted before we proceed. Though the modal set theory Lin axiomatises an *uncountabilist* conception of set—we *can* form the powerset of a given set—the axiomatisation does not *exactly* correspond to the formation of the universe via powersets and union. To see this, one can (with a little work) obtain a Kripke model for Lin from the L_α of a model of ZFC.[13] However, we don't get the full powerset when moving from L_α to $L_{\alpha+1}$ (since the cardinality of L_α and $L_{\alpha+1}$ are the same for any α). If one wishes to smooth over this wrinkle, one can use *bimodal* operators (as in [Studd, 2013] and [Button, 2021b]) to enforce the *immediate* collapse of *every* possible subplurality of a world into a set. For the sake of simplicity in the frames we'll consider, I'll stick with Lin, but we could run the same points using Button or Studd's systems too. As we'll see shortly, there's a very natural Kripke model for Lin corresponding to the formation of the sets via powerset and union, the reader who wishes for this to be enforced is welcome to read this article with Button or Studd's bimodal axiomatisations in Lin's stead, the main philosophical points of this article will remain unchanged if this route is taken.

A different good-making feature of Lin, is that it tells us *why* problematic classes don't exist as sets. New sets can always be formed over a world by collapsing pluralities that exist there. This includes the Russell plurality of all non-self-membered sets at a particular world which *never* has a set coextensional with it at a world. *Which* conditions do and do not form sets is handled on this basis—every formula whose elements are available

[12] Questions remain, for example, about how to motivate Foundation and its relationship to $\in$-induction. Thanks to Davide Sutto and Øystein Linnebo for some discussion of this point.

[13] *Sketch.* Work within a model $M \models$ ZFC. Let worlds be of the form $(L_\alpha, \mathcal{P}^L(L_\alpha))$, where L_α provides the first-order domain and $\mathcal{P}^L(L_\alpha)$ provides the second-order (plural) domain, and let $(L_\alpha, \mathcal{P}^L(L_\alpha))$ access $(L_\beta, \mathcal{P}^L(L_\beta))$ iff $\alpha \leq \beta$. It's now relatively easy to check that this Kripke model satisfies Lin.

at some world defines a set, but those that have satisfiers unbounded in the worlds cannot. We thus have another desideratum:

Paradox Diagnosis. A given modal set theory $\mathsf{T}^{\Diamond}$ should yield a criterion (on the basis of the construction methods allowed) of which conditions do and do not define sets, and in particular *why* the usual problematic conditions do not define sets.

So Lin is reasonably **Natural** and provides a **Paradox Diagnosis**. Let's now turn to **Interpretation**, which concerns how much *non-modal* mathematics we can squeeze out of our modal theory under the potentialist translation. And Lin performs well here:

Theorem 5. [Linnebo, 2010], [Linnebo, 2013] If $\phi^{\Diamond}$ is the potentialist translation of a sentence ϕ in $\mathscr{L}_{\in}$, then ZFC proves ϕ iff Lin proves $\phi^{\Diamond}$.

This means that from within Lin we can prove the potentialist translation of any theorem of ZFC. In this way, we could just work with the non-modal theory if we wanted, acknowledging that the modal theory and potentialist translation is available any time we wish to look at how the sets get formed. This suggests the following desideratum on modal set theories:

Interpretation. A modal set theory $\mathsf{T}^{\Diamond}$ should interpret a 'good' non-modal theory of sets T under the potentialist translation.

Again, **Interpretation** might not be as hard-and-fast a desideratum as we might like. Clearly, for example, the notion of a 'good' theory (as already remarked) might be something of a tricky notion to analyse. Moreover, it's not clear that it's necessary to adopt the potentialist translation for interpreting a non-modal theory of sets, and we might consider what theories can be interpreted under different translations (see [Brauer et al., 2021] for an alternative). Thankfully, all the theories considered in this paper will interpret reasonably nice non-modal theories under the usual potentialist translation, and so I'll set this complication aside (despite its interest).

We can now move on to the final desideratum that I'll consider, namely **Capture**. In short: Given a model of the non-modal theory (in this case ZFC) we can find a natural Kripke frame for our modal theory (in this case Lin) that contains all the sets. Thus, not only does Lin allow us to motivate a nice non-modal theory, but there is also a kind of 'reversal' from the non-modal theory to the modal one:

Theorem 6. [Linnebo, 2013] Let M be a transitive model of ZFC. We define the following Kripke-frame K^M_{Lin} for Lin:

- **Worlds** are pairs of the form $(V^M_\alpha, V^M_{\alpha+1})$, where V^M_α provides the first order domain and $V^M_{\alpha+1}$ provides the second-order (plural) domain (with $\prec$ interpreted by $\in$).
- **Accessibility** is given by $(V_\alpha, V_{\alpha+1}) \leq^M_{\mathsf{Lin}} (V_\beta, V_{\beta+1})$ iff $\alpha \leq \beta$

Then K^M_{Lin} is an M-proper-class-sized Kripke frame validating S4.3.

This shows us that not only can we get the axioms of our favoured non-modal theory from Lin, but we also get worlds for Lin from said non-modal theory. This point can be strengthened by the following observation:

Theorem 7. (ZF) For every set x there is an ordinal α such that $x \in V_\alpha$.

Thus, the model K^M_{Lin} isn't just *any* model of Lin (which would be enough to witness equal consistency strength with ZFC), but also M thinks that every set lives in a world of K^M_{Lin}.[14] Theorems 6 and 7 collaborate to ensure that given any transitive model M of the non-modal theory (in this case ZFC) there's a canonical way to recover a model of Lin such that every set in M is a member of (the first-order domain of) some world.

I think this point has been perhaps somewhat overlooked in the literature, but it's an important part of explaining why the standard view is so philosophically satisfying. Let's therefore identify the following additional desideratum:

Capture. Not only should our modal theory $\mathsf{T}^\Diamond$ motivate a good non-modal theory T, but given a transitive model $M \models \mathsf{T}$ we should have a general way of extracting a Kripke frame $K^M_{\mathsf{T}^\Diamond} \models \mathsf{T}^\Diamond$ from M such that for every $x \in M$ there is a world $W \in K^M_{\mathsf{T}^\Diamond}$ such that $x \in W$.[15]

Remark 8. Why restrict to transitive models? Since all the theories I consider take sets to be well-founded, and because it makes the metamathematical details cleaner, I'll focus my attention on *transitive* models. I'll leave it open whether the results below can be conducted in ill-founded and/or non-transitive settings. Clearly though, the **Capture** relationship also holds between Lin and ZFC in the non-transitive setting too.

[14] In other modal set theories (e.g. [Studd, 2013] and [Button, 2021b]) this is put in as an axiom, sometimes called **Stratification**.

[15] I'll routinely abuse notation and use $W \in K$ to say that W is a world of K.

Summing up: I think that Lin (and indeed other uncountabilist modal set theories) provides a good modal theory of set construction, and this is witnessed by **Naturalness**, **Paradox Diagnosis**, **Interpretation**, and **Capture**. In the rest of the paper, I'll examine how close *countabilists* can get to such a satisfying modal set theory.

4 Scambler's countabilist modal set theory

A countabilist theory has been provided recently by Chris Scambler (see [Scambler, 2021], [Scambler, MS]). He starts with the background of $\mathscr{L}^{\Diamond}_{\prec,\in}$ but adds two modal operators $\langle v\rangle$ and $\langle h\rangle$. $\langle v\rangle$ corresponds to 'vertical' modality—turning available pluralities into sets. In this respect is somewhat similar to Lin. However it also allows $\langle h\rangle$; a 'horizontal' modality and involves adding in forcing generics. Call this language $\mathscr{L}^{\Diamond,\langle h\rangle,\langle v\rangle}_{\in,\prec}$. In this context, the general $\Diamond$ can be thought of as 'possible in either $\langle v\rangle$ or $\langle h\rangle$'. He then provides the following axioms:[16]

Definition 9. Sca consists of the following axioms in $\mathscr{L}^{\Diamond,\langle h\rangle,\langle v\rangle}_{\in,\prec}$ (again I'll focus on giving more informal statements, the reader should go to [Scambler, 2021] for the formal details):[17]

(i) Classical first-order logic.

(ii) Impredicative plural logic.

(iii) Classical S4.2 with the Converse Barcan Formula for every modality.

(iv) **Plural Membership Definiteness**

(v) The necessity of distinctness and stability axioms for $\prec$ and $\in$.

(vi) **Foundation.** The Axiom of Foundation (the standard one from ZFC).

(vii) **Extensionality.** Extensionality for sets (again, no different from ZFC).

(viii) **Weakening Schemas.** $\langle h\rangle\phi \to \Diamond\phi$ and $\langle v\rangle\phi \to \Diamond\phi$, for every ϕ.

(ix) **Vertical collapse.** $\langle v\rangle\exists y\Box\forall z(z \in y \leftrightarrow z \prec xx)$.

[16] See [Scambler, 2021, p. 1091].

[17] Scambler uses the term 'M' (for **M**eadows) to denote Sca, as he takes inspiration for his view from [Meadows, 2015]. As we'll see below, Meadows' work (drawing on [Steel, 2014]) is slightly different (he considers proper class models), therefore I've chosen 'Sca'.

(x) **Modal Infinity.** The axiom that there could *vertically* be some things that necessarily comprise all and only the natural numbers: $\langle v \rangle \exists xx \Box \forall y(y \prec xx \leftrightarrow$ 'y is a natural number').

(xi) **Vertical Modal Powerclass.** The axiom that its *vertically* possible to have some things that are *vertically necessarily* all the subsets of a set: $\forall z \langle v \rangle \exists xx [v] \forall y(y \prec xx \leftrightarrow y \subseteq z)$.

(xii) **Possible Generics.** The axiom 'If $\mathbb{P}$ is a forcing partial order and dd is some dense sets of $\mathbb{P}$, then it's horizontally possible that there is a filter meeting each dense set that is one of the dd'.

(xiii) **Choice.** The plural version of the **Axiom of Choice** 'For any pairwise-disjoint non-empty sets xx, there are some things yy that comprise exactly one element from each member of the xx'.[18]

(xiv) **Modal Collection, Separation, and Replacement.** Potentialist translations of the axiom schemas of **Collection**, **Separation**, and **Replacement** under each modality.[19]

A full defence of the **Naturalness** of Sca and its underlying idea is available in [Scambler, 2021], but a few words will help to facilitate a contrast with the picture we'll examine in §5. The thought behind Scambler's axiomatisation is that we have two kinds of operation, in addition to the union operation that allows collecting together already constructed sets. One (that we might call **Reify!**) takes the pluralities of a given model and reifies them into sets. Another (call it **Generify!**) takes a given forcing partial order at a stage and throws in a generic filter for that partial order. Starting with no sets, we build up stages until we reach V_ω. Given different conceptions of classes for V_ω, we can start to form them into sets, but we could also throw in forcing generics.[20]

As well as providing us with a picture of how sets are generated, we also get **Paradox Diagnosis**. All the usual paradoxical conditions don't form sets, since we could always reify more pluralities (e.g. the Russell plurality) into sets. However, some other conditions are also shown to not

[18]Scambler throws this in with the plural logic, but we'll keep it separate.

[19]Strictly speaking, Replacement is redundant given Separation and Collection. The reason to separate these out is that Collection and Separation are strictly stronger than Replacement when Powerset is removed (see [Zarach, 1996] and [Gitman et al., 2016]). [Scambler, 2021] works only with the potentialist translations of Replacement, I'll discuss this fact in due course.

[20]Some readers may feel a little unease at the idea that forcing provides a legitimate process of set-formation. I'll address this in due course.

have corresponding sets. For instance, since any set can be made countable using forcing, the class of all possible hereditarily countable sets cannot form a set—in this context Cantor's Theorem shows that there *could* always be more subsets of a given set.

We should identify some points of difference between the Scambler's theory and Lin. Note that **Subplurality Definiteness** has to fail; since sets can pick up subsets at additional worlds (say if we introduce one by forcing), we cannot have all possible subpluralities at a world. Some (e.g. [Roberts, MS]) see this as an objection to the use of pluralities in this context. I'm willing to see it as a necessary consequence of Scambler's theory that some views about the nature of the second-order variables are off the table, rather than an objection per se.

Another point to be made is that the process of set formation is non-functorial. This arises as a result of **Generify!**—for several kinds of forcing, there are many generics one could add. Take the addition of a single Cohen real, for example. Given a structure M over which we can add such reals, there are always non-interdefinable Cohen reals G and H, and so there is a choice to be made about which to add. We can also get denseness in the ordering. Start by identifying that for any Cohen real G, there is a Cohen-generic real H from which G can be defined but not vice versa. Moreover, if G is definable from H but not vice versa, there is also a generic I that (i) G is definable from I, (ii) I is not definable from G, and (iii) I is definable from H. Thus, given any two single-Cohen-real forcing extensions $M[G]$ and $M[H]$ there is also a dense ordering of $M[I]$ between them.

One might be tempted to object to the **Naturalness** of this way of forming the sets via **Reify!** and **Generify!** on these grounds. This is especially so if one is attracted to modal theories as particular versions of the *iterative conception*—there is no sense of iterating a *determinate* operation to yield the universe here.[21] Moreover even if we allow an *indeterminate* construction methods (i.e. ones that are non-functorial), the natural accessibility relations we might come up with are non-well-founded. This contrasts sharply with the situation in which powerset and union are our only set-construction methods. There we can think of construction along the V_α as a perfectly determinate and well-founded.

I don't take this objection to be knock-down against the advocate of modal theories like Sca. I think that one can see this from the consideration of simpler modal construction cases that share many features with forcing. Let's suppose that we're given a finite line segment $l \subset \mathbb{R}$. I have a single construction method **Extend!** that allows me to extend l in

[21] [Brauer, MS] also makes some points along these lines.

a single direction. Now I could extend l left, or I could extend l right. Moreover, if I extend l left, I don't get what I get if I extend l right (I'm assuming an identity criterion on lines here where l_1 and l_2 have to comprise exactly the same points to be identical). And any time I extend l in one of the two directions to a line l', there's a dense ordering of smaller lines that I could have extended to (with length greater than l but smaller than l'). Clearly, however, this is a reasonable description of a modal line-construction method, just one that is non-functorial. So it is for the sets under Sca.

I thus think that when we use forcing, there's still a good notion of 'set-construction method' in play. In particular, the idea that we can always run through any family of dense sets, successively hitting each one by extending our previous choices, and get something in the end exactly corresponds to the production of a generic. I'll leave it open whether we can view these set-construction methods as feeding in to some form of the 'iterative conception', and I acknowledge that *something* is given up when we do so. It's an interesting question whether there are natural countabilist conceptions that are functorial (or at least well-founded), but not one I'll address here.

Some remarks are also in order regarding the potentialist translations of **Separation**, **Replacement**, and **Collection**. **Modal Separation** is very natural, capturing the idea that if we have some set at some stage, and some condition, we should be able to separate out the satisfiers of this condition. In this respect it is not different from the 'standard' ZFC context (though some extra care is needed, see [Roberts, MS]). **Modal Replacement**, however, is a little more controversial. Since we can assume without loss of generality that the domain of the relevant function is countable (since we may collapse it using forcing) we are effectively asking that any ω-sequence of possibilia can be brought together into a single world (where then non-modal **Replacement** within that world will yield our desired set for the potentialist translation). In this respect, **Modal Replacement** functions in this context a bit like a 'super-$.2$' axiom ($\omega.2$, if you will).[22] I think that it's reasonable to assert this axiom. In any case, when *comparing* Sca with Lin, we should note that Lin also requires a version of **Modal Replacement**. Moreover, in the Lin-context, **Modal Replacement** asserts that *very* many—i.e. lots more than countably-many—possibilia can be brought together.[23]

[22] I thank Øystein Linnebo for some discussion of this point.

[23] It should be noted at this point that one can motivate **Modal Replacement** on the basis of reflection-style ideas over the other axioms of Lin, but since the discussion of

Modal Collection is more controversial, though I think a case can be made for its naturalness. Where **Powerset** fails, **Collection** ceases to be equivalent to **Replacement** (one can prove more from **Collection** and **Separation**). Moreover, though not included in [Scambler, 2021] or [Scambler, MS], it is desirable to have **Modal Collection** for mathematical reasons. There's a long list of such reasons, but (to pick an especially salient one) you need **Collection** to get the Łoś Theorem for ultrapowers to work (see [Zarach, 1996] and [Gitman et al., 2016] for discussion). But what does it *say* in this context? Let's present a statement of the potentialist translation of **Collection**:

Definition 10. Let $\phi^\Diamond(x, y)$ be the potentialist translation of a formula $\phi(x, y)$ in $\mathscr{L}_\in$ defining a relation. The *modalised collection scheme* (or just **Modal Collection**) asserts (for each such ϕ) that:

$$\Box\forall a(\Box\forall x(x \in a \rightarrow \Diamond\exists y\phi^\Diamond(x, y)) \rightarrow \\ \Diamond\exists b\Box\forall x(x \in a \rightarrow \Diamond\exists y(y \in b \wedge \phi^\Diamond(x, y))$$

Note here that the relation $\phi^\Diamond$ might define a *modally unbounded proper class* over any particular given x (i.e. there might be an x for which there is no world containing all y such that $\phi^\Diamond(x, y)$ holds). In this way a given set a and $\phi^\Diamond$ are providing a *parameterised family of classes*, with each $x \in a$ providing the indices. **Modal Collection** is thus essentially *choice-like*; even in the case where for some xs the class of *possible* y such that $\phi^\Diamond(x, y)$ is 'modally smooshed' across all the worlds, it's still possible to get a set which *picks at least one* element from each 'member' of this parameterised class.

Again, I want to note a *degree* of parity with Lin here. There is substantial debate to be had about how AC might be justified on the basis of modal set theory, one that I won't enter into here. But, formally speaking, you get as much choice in the non-modal theory as you're willing to throw in, and AC *has* to be written in (either in the axioms of Lin or the plural logic). A similar situation holds here. Though we get *set* AC for free in Sca (after all, we have the modal translation of Count), stronger forms of choice that we want for many kinds of mathematical construction need to be added.

I thus think that whilst there are questions of justification to be addressed for Sca, it nonetheless performs fairly well with respect to **Naturalness**, and is at least not *clearly* worse than Lin. Certainly if it is worse off, it's not *substantially* so. Let's now move on to **Interpretation**. We've already discussed the fact that Sca interprets Count under the potentialist translation. We can, in fact, go much further:

reflection-style axioms for Sca would take us a little far afield, I'll suppress this detail.

Theorem 11. [Scambler, 2021] Sca interprets $\mathsf{ZFC}^- + \mathsf{Count}$ under the potentialist translation using $\Diamond$.

$\mathsf{ZFC}^- + \mathsf{Count}$ is a reasonably good theory for doing mathematics. However, as several authors have argued, the absence of **Powerset** makes it desirable to have contexts in which ZFC is true, even if strictly speaking there are no uncountable sets.[24] Since Sca includes all the axioms of Lin for $\langle v \rangle$, we can prove:

Fact 12. [Scambler, 2021] Sca interprets ZFC using the potentialist translation with the $\langle v \rangle$ modality.

So if we restrict to the $\langle v \rangle$ modality, we have models of ZFC. We should want more, however. The potentialist translation (standardly conceived) makes no mention of $\langle v \rangle$. There is no guarantee that we can 'see' the ZFC-contexts using the broader modality $\Diamond$. Can we do better than restricting?

Here we find a use for *regularity properties* for sets of reals, which will allow us to derive consequences for both **Interpretation** and **Capture**. Recently Scambler has mobilised such regularity properties (using work of Solovay and Taranovsky)[25] in satisfying **Interpretation**. Let's start with the following definition:

Definition 13. We say that a class of reals (possibly defined by a formula in the $\mathsf{ZFC}^- + \mathsf{Count}$ context) is *perfect* iff it closed and has no isolated points.[26]

We can further define:

Definition 14. We say that a (definable) class of reals *has the perfect set property* iff it is either countable or contains a perfect subclass.

We can then finally define:

Definition 15. The $\utilde{\Pi}^1_1$-*Perfect Set Property* ($\utilde{\Pi}^1_1$-PSP) is the schema asserting that every $\utilde{\Pi}^1_1$-definable (i.e. definable with parameters from a Π_1-formula of second-order arithmetic) class of reals has the perfect set property.

[24]See [Arrigoni and Friedman, 2013], [Builes and Wilson, 2022], [Scambler, 2021], [Scambler, MS], [Barton, Fa], and [Barton and Friedman, MS] (among several others).

[25]See [Solovay, 1974] and [Taranovsky, 2004].

[26]We should note that the notion of being perfect depends on the reals forming a Polish space. Note also that since the reals and the hereditarily countable can be thought of as coding one another, you can basically think of the reals as the universe of sets for the countabilist.

How do these regularity properties help the countabilist? As [Scambler, MS] notes, the $\underset{\sim}{\Pi}^1_1$-PSP has consequences for the existence of models of the form $L[x]$ satisfying ZFC.[27] In particular:

Theorem 16. [Solovay, 1974] Let $\mathbb{R}^{L[x]}$ denote the class of y such that $L[x]$ thinks that y is a real number. The $\underset{\sim}{\Pi}^1_1$-PSP is equivalent to the claim that 'For every real x, $\mathbb{R}^{L[x]}$ is countable'.

Solovay's proof is conducted against the background of ZFC, but it works in the ZFC$^-$ context too (see [Taranovsky, 2004]). We can now point to an observation made by Dmytro Taranovsky:

Theorem 17. [Taranovsky, 2004] Over ZFC$^-$ + Count, the $\underset{\sim}{\Pi}^1_1$-PSP implies (indeed is equivalent to) the schema asserting that $L[x]$ satisfies ZFC for every real x.

Thus, if we could prove the modal translations of the $\underset{\sim}{\Pi}^1_1$-PSP in Sca, we would be able to 'see' the fact that ZFC holds in inner models, just using the potentialist translation under the 'broad' $\Diamond$ (i.e. without having to restrict to $\langle v \rangle$). And, indeed, this is the case:

Theorem 18. [Scambler, MS] Sca proves the potentialist translations of the $\underset{\sim}{\Pi}^1_1$-PSP, and hence the potentialist translations of 'For every real x, $L[x] \models$ ZFC'.

Scambler takes this to show that Sca can do the work of the 'usual' ZFC-based foundations, but within inner models rather than the universe. If we accept that inner models are acceptable interpretations for mathematics, then Sca also satisfies **Interpretation**. I am sympathetic to this point of view, but acknowledge that there's more to be said about whether such inner model interpretations are satisfactory.[28] I do not have enough space to adjudicate this debate here, suffice to say that there are at least very nice models satisfying ZFC on the picture provided by Sca. What I do want to do is to identify that this has further consequences for Sca, in particular that Sca can provide a version of a **Capture** theorem.

This fact is in fact implicit in the proof that [Scambler, MS] provides for the consistency of Sca relative to ZFC$^-$ + $\underset{\sim}{\Pi}^1_1$-PSP:

[27] I am very grateful to Chris Scambler for several discussions regarding the issues around $\underset{\sim}{\Pi}^1_1$-PSP, [Solovay, 1974], and [Taranovsky, 2004], as well as his [Scambler, 2021] and [Scambler, MS].

[28] See [Barton, Fa], [Barton, Fb], and [Barton and Friedman, MS] for some further discussion of this point.

Theorem 19. [Scambler, MS] There is an interpretation (preserving theoremhood) from Sca to ZFC$^-$ + Count + $\utilde{\Pi}^1_1$-PSP.

Scambler employs these results in the service of showing (i) that Sca is consistent relative to ZFC, and (ii) there is no loss of interpretive power compared to the usual ZFC-picture. Indeed, because of the interpretation of Sca into ZFC$^-$ + Count + $\utilde{\Pi}^1_1$-PSP, we know that Sca has exactly the same consistency strength as ZFC.[29] However, it does more, in particular it shows that **Capture** is satisfiable under Sca.

Fact 20. Not only does Sca prove the potentialist translations of ZFC$^-$ + Count + $\utilde{\Pi}^1_1$-PSP (call this T) but given a transitive model M of T we can uniformly recover a Kripke model K^M_{Sca} from M such that for every $x \in M$, there is a $W \in K^M_{\mathsf{Sca}}$ such that $x \in W$.

Proof. The relevant Kripke model K^M_{Sca} is just given by Scambler's interpretation from [Scambler, MS]. We have:

- **Worlds** are pairs (t, r) where t is a transitive set and r is a real such that $t \in L[r]$.
- **Accessibility:**
  - (t_1, r_1) $\langle v \rangle$-accesses (t_2, r_2) iff $r_1 = r_2$ and $t_1 \subseteq t_2$.
  - (t_1, r_1) $\Diamond$-accesses (t_2, r_2) iff $t_1 \subseteq t_2$ and r_1 is constructible from r_2.
  - (t_1, r_1) $\langle h \rangle$-accessing (t_2, r_2) can be given the same clauses as $\Diamond$.

Remark 21. Before going through the proof, we pause to give the reader a feel for what's going on with the construction of this Kripke model.[30] Effectively the transitive set t is our first-order domain, and r (via $L[r]$) specifies how t will grow using the vertical modality—we iterate the $L[r]$-powerset operation (i.e. r provides a conception of **Reify!** forever over t). Note that for this reason, we may have (t, r_1) and (t, r_2) as different worlds, not because the first-order domain is different, or even the second-order domain is different, but rather because far in the vertical future $L[r_1]$ differs from $L[r_2]$. Formally speaking, we can handle horizontal modality using the same clauses as $\Diamond$, but if one wants a distinctive modality one could

[29]See [Taranovsky, 2004] for a proof that ZFC$^-$ + $\utilde{\Pi}^1_1$-PSP and ZFC have the same consistency strength. The reversal is obtained by noting that given a model of ZFC, ZFC$^-$ + Count + $\utilde{\Pi}^1_1$-PSP holds in the $Col(\omega, < Ord)$ class forcing extension.

[30]Thanks to Chris Scambler for some discussion of his construction.

easily specify that (t_1, r_1) $\langle h \rangle$-accesses (t_2, r_2) iff r_1 is constructible from r_2 and $t_1 \subseteq t_2$ and the ordinal height of t_1 is the same as that of t_2 (recalling that this latter notion is well-defined since both t_1 and t_2 are transitive).

On to the proof of Fact 20: [Scambler, MS] already shows that K^M_{Sca} satisfies Sca. We can thus proceed directly to showing that any set in M is a member of some world of K^M_{Sca}. Start by taking some arbitrary $y \in M$. Since M is transitive and models T, there is a transitive set x such that $y \in x \in M$ (e.g. the transitive closure of $\{y\}$). But we also know that (x, x) is a world of K^M_{Sca}, since $L[x] \models$ ZFC (by $\underset{\sim}{\Pi}^1_1$-PSP) and $x \in L[x]$ for any particular real x (a standard fact about constructibility). Thus there is a world of K^M_{Sca} containing y (namely (x, x)), and since the choice of y was arbitrary, every $y \in M$ is contained in some world of K^M_{Sca}. $\square$

We thus get **Capture** too—given a transitive model of the non-modal theory supported by the modal conception of the stages, we can always extract a Kripke model for Sca where every set lives.

So, to review, Sca (with it's idea of using **Reify!** and **Generify!**) provides a **Natural** modal theory of sets, that is able to give **Paradox Diagnosis**, **Interpret** mathematics, and provide a **Capture** theorem. Later (§6) I'll examine just how good this **Capture**-theorem is, contrasting the situation in Sca with that of Lin. For now, we'll examine a modal set theory that uses *solely* **Generify!** (though starting from a great many sets).

5 Modalising Steel's multiverse

In this section, I'll explain how to provide a modal set theory similar to Steel's multiverse. We'll show that this interprets ZFC$^-$ + Count + $\underset{\sim}{\Pi}^1_1$-PSP, and discuss it with respect to **Naturalness**, **Paradox Diagnosis**, **Interpretation**, and **Capture**.

In [Steel, 2014], John Steel proposes a two-sorted but *non-modal* theory with variables for sets $x_0, x_1, ...$ and variables for universes $W_0, W_1, ...$ with the following axioms:[31]

Definition 22. *Steel's Multiverse Axioms* are as follows:

(i) The axiom scheme stating that if W is a world, and ϕ is an axiom of ZFC, then ϕ holds at W.

(ii) Every world is a transitive proper class.

[31] Here I follow the presentation in [Maddy and Meadows, 2020].

(iii) If W is a world and $\mathbb{P}$ is a forcing partial order in W, then there is a universe W' containing a generic for W.

(iv) If U is a world, and U can be obtained by forcing over some world W, then W is also a world.

(v) If U and W are worlds then there are G and H that are generic over them such that $U[G] = W[H]$.

Steel wants to use his theory to isolate the determinate part of set theory, regarding some sentences (like CH) as indeterminate 'pseudo-questions' [Steel, 2014, p. 154]. Further analysis of Steel's project on its own terms is provided by [Maddy and Meadows, 2020]. However, we might instead use the multiverse axioms as inspiration for a modal theory of sets. Whilst this idea is anathema to the project proposed by Steel, Maddy, and Meadows, it's interesting that one can extract such a modal theory from their ideas. On this conception, we form sets by starting with proper-class models of ZFC, and then use forcing as our single set-forming operation. We provide the following modal formulation:

Definition 23. SteMMe (for **Ste**el-**M**addy-**Me**adows) comprises the following axioms in $\mathscr{L}^{\Diamond}_{\prec,\in}$:

(i) Classical first-order logic.

(ii) *Predicative* plural logic.

(iii) Classical S4.2 with the Converse Barcan Formula for $\Diamond$.

(iv) The necessity of distinctness and stability axioms for $\in$ and $\prec$.

(v) **Plural Membership Definiteness**.

(vi) **The Ordinal Definiteness Schema:** This is the schema of assertions of the form $\forall x(\text{'}x \text{ is an ordinal'} \rightarrow \Box\phi(x)) \rightarrow \Box\forall y(\text{'}y \text{ is an ordinal'} \rightarrow \phi(y))$

(vii) The necessitation of every axiom of first-order ZFC.

(viii) **Possible Set-Generics.** The axiom 'If $\mathbb{P}$ is a forcing partial order and $\mathcal{D}$ is a set of dense sets of $\mathbb{P}$, then it's possible that there is a filter meeting each dense set that is a member of $\mathcal{D}$'.

(ix) **Modal Separation, Replacement, and Collection.** The potentialist translations of every instance of the **Separation**, **Replacement**, and **Collection** schemas.[32]

Let's take each of the desiderata of **Naturalness**, **Paradox Diagnosis**, and **Capture** in turn.

Naturalness. The idea of the SteMMe is to take some proper-class-sized model of ZFC as our starting sets and **Generify!** as our sole way of forming new sets from old. There is no **Reify!** operation. Much of the choice of logic (e.g. (i)–(iii)) is the same as in the case of Sca, we want to talk about pluralities and sets, which ones can and can't form sets, and how a domain can grow (hence the Converse Barcan Formula, which in any case is provable in this context). I adopt *predicative* plural logic since we will only need to talk about definable classes and it will make some of the model-theoretic analysis easier later, I leave it open how one might modify the approach to make the underlying plural logic impredicative. .2, though it does not *guarantee* the directedness of the corresponding frame, as before axiomatises the idea that any two possibilities can be brought together, in line with Steel's **Amalgamation** axiom. Stability axioms and **Plural Membership Definiteness** are required again to ensure that neither $\in$ nor $\prec$ (nor subplurality-hood) can behave badly as new sets come into existence. The **Ordinal Definiteness Schema** essentially posits the Barcan Formula for the ordinals, axiomatising the principle that the ordinals can't get longer. This captures the idea that our stages are all proper-class-sized and we add the necessitation of first-order ZFC to capture the idea that ZFC holds in each of these proper class models. **Possible Set Generics** is motivated by the idea that our operation of set-formation is forcing. The points about **Modal Separation**, **Modal Replacement**, and **Modal Collection** are exactly the same as in Sca. **Modal Separation** is natural given worldly **Separation**. Since any set can be collapsed, we can assume without loss of generality that the domain of any function is countable, and so **Modal Replacement** functions like a super-.2 or ω.2 axiom Since any set can be made countable, the potentialist translation of AC is free, but we have to write in the 'choice-like' **Modal Collection** scheme.

In the service of providing a picture behind SteMMe to substantiate **Naturalness**, let's show some easy facts:

Lemma 24. SteMMe implies that the plurality of all ordinals cannot form a set.

[32]Again, there are redundancies here, but we separate them out in order to aid philosophical discussion.

Proof. Suppose otherwise. This would then be a transitive set well-ordered by the membership relation. Then use **Ordinal Definiteness** to get (per impossibile) that the ordinals form a set at the actual world. □

This then implies:

Corollary 25. SteMMe implies that **Modal Collapse** fails.

Proof. The plurality of all ordinals witnesses the failure of **Modal Collapse**. □

These indicate a sense in which SteMMe is different from both Lin and Sca; there are pluralities at every world who *cannot* be **Reified!** into sets. This vindicates further the idea that we're working with proper class models of ZFC.[33]

We can also note a fact about the way subsets (and corresponding subpluralities) can be added:

Fact 26. SteMMe implies that **Subplurality Definiteness** fails.

Proof. By **Possible Generics**, there could be a generic for the (currently existing) dense sets that doesn't currently exist. Note also that by **Predicative Comprehension** at any world, the following **Plurality Correspondence** principle holds:

$$\Box\forall x\exists yy\forall z(z \prec yy \leftrightarrow z \in x)$$

This asserts that necessarily there's a plurality co-extensional with any set x, and is a consequence of **Predicative Comprehension** by considering the formula (in the parameter $\bar{x}$ for x) '$y \in \bar{x}$'. Using **Plurality Correspondence**, we can then immediately infer that there could be a subplurality of a plurality that doesn't currently exist (namely the plurality corresponding to our generic G). □

We thus not only have a vindication of the idea that our worlds are proper classes containing very possible ordinal, but also that they can be expanded by forcing. SteMMe thus fills in the logical space in modal set theories. Lin is a solely height potentialist theory, but width actualist in

[33] This has the result that this conception runs counter to many 'height potentialist' modal set theories (e.g. all of [Linnebo, 2010], [Linnebo, 2013], [Studd, 2013], and [Scambler, 2021] do not support this idea). Many will take this to be an objection, but I think that it highlights an interesting sense in which you can 'calibrate' what's allowed along different dimensions. I leave it open whether this opens the advocate of SteMMe to revenge worries in the manner described by [Studd, 2013] and [Linnebo, 2010].

that we can get a set of all possible subsets of a given set. Sca is both height potentialist and width potentialist. And SteMMe is height actualist but width potentialist. I think therefore that there's a *reasonably* **Natural** picture behind SteMMe. Before we continue, I'd like to forestall a few objections.

First, one might object that neither Sca nor Lin depends on there being very many sets to start with. We could start with our initial domain just containing the empty set and go from there, whereas SteMMe contains the necessitation of ZFC. One might worry here that if we're explaining how the sets are 'built up', we shouldn't start with many sets. Every house so to speak, has to start from the first brick—you can't build a good house if you assume that the foundations are already laid.

There's a few responses to be made here. First, both Lin and Sca write in a modal existence assumption; both assert the possible existence of a set of all possible natural numbers (i.e. **Modal Infinity**). There is *always* going to be a jump in size at some point. And, we could make modifications to SteMMe to allow it to start with no sets, but will result in a more finnicky axiomatisation. For example, we might just assert that ZFC is *possible* (rather than necessary), and then conditionalise our other axioms on the existence of an infinite set (e.g. the necessitation of ZFC could be replaced by $\Box$('There is an infinite set' $\rightarrow \phi$) for every axiom ϕ of ZFC.). The core idea would then be the same—we might start with the empty set—but as soon as you've got an infinite set you jump to some proper class model of ZFC. I thus think that this difference is one of degree rather than kind, and doesn't particularly affect SteMMe's **Naturalness**.

A related worry concerns the *autonomy* of SteMMe.[34] For both Sca and Lin, beliefs about the vertical modality licence the truth of ZFC for that modality. But what recourse does the advocate of SteMMe have for believing the truth of the necessitation of ZFC?

In response, one might point to the experience that set theorists have of working within inner models and moving between them. This motivates the idea that there's a coherent conception or intuition to be captured, and this is SteMMe's role. Steel chooses to do so with a non-modal framework, but it is interesting that this intuition can be captured modally too.

This idea has some affinity with the strategy of biting the autonomy bullet. One might piggy back of the height potentialist motivation for ZFC given by Sca. In a way, SteMMe functions very like a theory on which the vertical modality of Sca has been 'factored out'—we look at all the ways that ZFC can be true under $\langle v \rangle$, and then, kicking away the modal

[34] I thank Ethan Brauer for pressing this point.

ladder, view the necessitation of ZFC as true. Some houses are built brick-by-brick, but other perfectly good houses can be built by slotting prefabricated structures together. Shortly, we'll see that this view can be somewhat mathematically vindicated; SteMMe shares several formal features with Sca (in particular interpreting ZFC$^-$ + Count + $\undertilde{\Pi}^1_1$-PSP under the potentialist translation).

Paradox Diagnosis is an interesting problem. SteMMe certainly tells us that many conditions do not define sets. The collection of all countable sets, for example will not be available at any world. However, there is a puzzle in that the plurality of all possible ordinals doesn't form a set, but does exist at every world. What's going on here?

The point is that we only have **Generify!** as our sole set-construction method. The explanation for why the ordinals do not form a set is that although they may be an 'available' plurality, we do not get enough set-forming pressure from **Generify!** to ossify them into a set. For a modal set theory what pluralities can be formed as sets is dependent upon a calibration between the pluralities available and the strength of our set-forming methods. SteMMe buys a great many available sets (indeed proper-class many!) at the price of a relatively weak set-forming operation. Some folks (e.g. friends of **Modal Collapse**) will no doubt find this unappealing, but I find it rather interesting that you can trade availability and set-construction methods against one another whilst only partially, if at all, compromising **Naturalness**.

Interpretation. Can we get a nice non-modal theory using SteMMe? The answer is yes:

Fact 27. SteMMe interprets ZFC$^-$ + Count under the potentialist translation.

Proof. (**Infinity**) We have to show that $\Diamond\exists x(\emptyset \in x \wedge \Box\forall y(y \in x \rightarrow \{y\} \in x)$. Since we have the axiom of infinity at each world, we know that $\Diamond\exists x(\emptyset \in x \wedge \forall y(y \in x \rightarrow \{y\} \in x))$. By the stability of membership, we can then infer $\Diamond\exists x(\emptyset \in x \wedge \Box\forall y(y \in x \rightarrow \{y\} \in x))$.

(Count) Is immediate given **Possible Generics**, necessarily any x can be made countable using forcing.

(**The Axiom of Choice**) Follows immediately (and cheaply) from Count.

(**Foundation**) Holds by using the necessitation of foundation at every world.

(**Extensionality**) We need to show $\Box\forall x\Box\forall y\Box\forall z((z \in y \leftrightarrow z \in x) \rightarrow x = y)$. Immediate using the T axiom and extensionality at any given world.

(**Pairing**) Here we need to show $\Box\forall x\Box\forall y\Diamond\exists z\Box\forall p(p \in z \leftrightarrow (p = x \vee p = y))$. Suppose that we have some arbitrary set x at some world w_1. Then, by Converse Barcan, necessarily if some y exists at a world w_2 accessible from w_1, x also exists at w_2 too. We can then use pairing at w_2 containing both x and y to get the possible existence of the pair set $\{x, y\}$.

(**Union**) Union requires $\Box\forall x\Diamond\exists y\Box\forall z(z \in y \leftrightarrow \Diamond\exists p(z \in p \wedge p \in x))$. Suppose we have some x. Then we can just use the standard axiom of union at that world (plus the fact that the set can't pick up members) to get the required possible p.

(**Separation** and **Collection**) Are just handled because we included the relevant potentialist translations in SteMMe, **Replacement** then follows for free. □

Can we hope for more? One such hope is that we might be able to *prove* the potentialist translations required for **Replacement** and **Collection** without assuming them (after all, we have the unbounded tower of ordinals to use at every world). Unfortunately I think that this is an unlikely possibility, and since this matter is somewhat orthogonal to matters at hand, I relegate these observations to a footnote.[35] However we can show:

Fact 28. SteMMe proves the potentialist translations of the $\underset{\sim}{\Pi}^1_1$-PSP.

Proof. The proof of this is not deep and we roughly just copy the Scambler-Solovay-Taronovsky ideas into the SteMMe context. Essentially the same technique works as in [Scambler, MS], but without any need to check some of the absoluteness facts related to the $\langle v \rangle$ modality. Using the Taranovsky and Solovay results, all we need to show is that SteMMe proves the potentialist translations of 'For every real x, $\mathbb{R}^{L[x]}$ is countable'. So, take any possible real x under SteMMe (in fact any possible set will do, since SteMMe satisfies the potentialist translation of Count any set can be thought of as a real). Since we know by assumption that ZFC holds at any world in which x exists, then it is a standard fact about constructibility that $L[x]$ satisfies ZFC and thus $\mathbb{R}^{L[x]}$ exists. We then use **Possible Generics** to collapse $\mathbb{R}^{L[x]}$ to be countable. Since the choice of x was arbitrary, this holds for any possible set x. The only thing to check is that $\mathbb{R}^{L[x]}$ is the same across worlds

[35] I lack formal proofs of independence, but here are my reasons for doubt. Start by fixing a model M of ZFC$^-$ + Count. **Replacement.** Suppose M also thinks that $0^\sharp_n$ (i.e. 0 with n-many sharps after it) exists for every $n \in \mathbb{N}$. Let worlds be of the form $L[0^\sharp_n, G]$ for each n and G either empty or generic over $L[0^\sharp_n]$ for some $\mathbb{P} \in L[0^\sharp_n]$ (with the second-order domain handled by the definable classes over a given world). Then the function mapping n to $0^\sharp_n$ is a legitimate (modally definable!) function whose domain is a set (namely $\mathbb{N}$) but whose range is not a set at any world. **Collection** seems dubious to me since there are models of SOA satisfying AC but not DC (see [Friedman et al., F]).

containing x, but this holds in virtue of the absoluteness of the construction of $L[x]$ (it is absolute between transitive models of ZFC with the same ordinals). $\square$

We thus have **Interpretation**, at least up to the same level as given by Sca. What now of **Capture**?

Theorem 29. Suppose $M \models \mathsf{ZFC}^- + \mathsf{Count} + \utilde{\Pi}^1_1\text{-}\mathsf{PSP}$. Then we can construct a Kripke model K^M_{SteMMe} of SteMMe such that for every $x \in M$ there is a $W \in K^M_{\mathsf{SteMMe}}$ such that $x \in W$,

Proof. For any structure X, let $Def(X)$ be the 'definable powerset' of X (i.e. the the collection of all $\{y|X \models \phi(y)\}$ for some formula ϕ in one free variable in the language of X, possibly with parameters from X). Our Kripke model $K^M_{\mathsf{SteMMe}} = (W_M, \leq_M)$ will be as follows:

- **Worlds:** $W_M = \{((L[x])^M, Def((L[x])^M))|x \in M\}$, i.e. worlds consist of pairs of the form $((L[x])^M, Def((L[x])^M))$. From now on, we will suppress the superscript relativising $L[x]$ to M.
- **Accessibility:** $\leq^M_{\mathsf{SteMMe}}$ can be defined as follows:
  $(L[x], Def(L[x])) \leq^M_{\mathsf{SteMMe}} (L[y], Def(y))$ iff $L[x] \subseteq L[y]$

Effectively, we let the first-order domain of the worlds be proper class models of the form $L[x]$ and the second-order domain over each world is composed of the first-order-definable subclasses of $L[x]$. We get predicative plural logic over any world since it is always satisfied by the definable subclasses of any world (with $\prec$ interpreted by $\in$). Note that this requires the use of higher-order resources over M (e.g. an ambient set-theoretic background) but this is standard when handling M-proper classes. The reader who doesn't like the use of higher-order resources can think of each class as coded by the formula (in some set-parameters) defining it. We'll discuss the possibility of getting impredicative plural logic below (in Remark 30). From now on, we'll suppress the consideration of the second-order part of the models (i.e. $Def(L[x])$ for each x).

S4 is trivial for any frame which is a preorder. To show that the .2 axiom holds, we'll show the frame is directed. If $L[x]$ is a world and $L[y]$ is a world, then $L[x, y]$ is also a world accessed by both $L[x]$ and $L[y]$. We just need to show that there's a single real r such that $L[r] = L[x, y]$. But this can be obtained by 'winding x into y' (i.e. put x onto all even bits of r and y onto all odd bits of r). The Converse Barcan formula is free given that domains only grow. The necessity of distinctness and stability axioms for $\prec$ and $\in$ are handled by the nature of set membership in M.

Plural Membership Definiteness holds since $Def(L[x])$ is extensional, so clearly can't pick up members.

The Ordinal Definiteness Schema is immediate, since the (set) ordinals of each world are the same.

Necessitation of the ZFC **axioms.** These follow from the fact that the $\underset{\sim}{\Pi}^1_1$-PSP in M implies that $L[x]$ is a model of ZFC for every real x.

Possible Set Generics. If $L[x]$ is a world of K^M_{SteMMe}, and $\mathbb{P}$ is a notion of forcing in $L[x]$, then the family $\mathcal{D}$ of all dense sets for $\mathbb{P}$ in $L[x]$ is countable in M (since M satisfies Count). By the usual Rasiowa–Sikorski Lemma in M, there is thus a $G \in M$ intersecting every member of $\mathcal{D}$. We then note that there is some $r \in M$ such that $L[r] = L[x, G]$ (using the previous technique of winding x into G) with $L[r] \models$ ZFC (by $\underset{\sim}{\Pi}^1_1$-PSP) and such that $G \in L[r]$, with $L[x] \subseteq L[r]$.

The **Modal Separation** and **Modal Collection** are handled by noting that any instance of $\Diamond\exists x\phi(x)$ and $\Box\forall x\phi(x)$, the claims that 'There is a real x such that $L[x] \models \phi$' and 'For every real x, $L[x]$ models ϕ' are each (schematically) definable in M. Therefore **Modal Separation** and **Modal Collection** hold in K^M_{SteMMe} in virtue of the the fact that there will be a set y for Collection/Separation in M using the relevant formulae, and then $L[y]$ will be a world of K^M_{SteMMe} containing y (using $\underset{\sim}{\Pi}^1_1$-PSP). □

Remark 30. A nearby theorem is available when we allow *impredicative* comprehension. If we work against the background of ZFC, and we consider some $M \models$ ZFC$^-$ + Count + $\underset{\sim}{\Pi}^1_1$-PSP, *and* $Ord(M)$ is inaccessible in $L[x]$ for every real $x \in M$, then one can get impredicative comprehension too by letting worlds be $((L[x])^M, \mathcal{P}^{L[x]}((L[x])^M))$. The problem with this move is that we depend on a strong higher-order background (ZFC) over M to interpret the impredicative comprehension, and so I prefer to work against the background of predicative plural logic (for that, we just need definable powersets over M, and thus the construction can work perfectly well in ZFC$^-$). It's an open question whether a theory like Kelley-Morse class theory minus powerset (KM$^-$) could produce the required models using impredicative plural logic in SteMMe.

So, we are now in a very similar situation with respect to SteMMe as we were with Sca. In fact, many of the results from Sca can be easily imported to the current context, avoiding the complications raised by the $\langle v \rangle$ operator. Still, the fact that we have a forcing potentialist but height actualist theory that fairs reasonably-well with respect to **Naturalness**, **Paradox Diagnosis**, **Interpretation**, and **Capture** is interesting.

I think that countabilist modal set theories like Sca and SteMMe deserve consideration as viable for providing mathematical foundations. Before I conclude, however, I want to linger on the issue of **Capture** a little longer. Though **Capture**-theorems can be provided for all the theories considered here, I think that the standard view and Lin are actually in slightly better shape in that their **Capture**-theorem is more satisfying. As we'll see, I think that this suggests some important points for countabilist modal set theories moving forward.

6 Capturing Capture?

Whilst we do have **Naturalness**, **Paradox Diagnosis**, **Interpretation**, and **Capture** for each of the theories considered, I do want to raise a worry for advocates of both Sca and SteMMe (and thus, in a sense, advocates of forcing potentialism more widely), regarding the 'satisfaction' of **Capture**. The important point is the following: If we look at how **Capture** is satisfied in each case, we can see that forcing had a very minimal role. Really, we just used it to to get Count and $\underset{\sim}{\Pi}^1_1$-PSP—forcing does not really appear in the specification of the relevant Kripke models. Count gets us the fact that any set can be coded by a real, and the $\underset{\sim}{\Pi}^1_1$-PSP guarantees that given a real x, $L[x]$ satisfies enough set theory to get us a world containing x for our Kripke frame for Sca/SteMMe.

An example might be instructive here; for this we'll consider the case of sharps (or just $\sharp$s). Suppose we have a model M that satisfies ZFC$^-$ + Count + $\underset{\sim}{\Pi}^1_1$-PSP. Suppressing the details, $0\sharp$ is an object that can be coded as a real, and allows us to define a non-trivial elementary embedding $j : L \to L$. This idea can be iterated, $0\sharp\sharp$ for instance, can also be coded as a real, and allows us to define an embedding $j : L[0\sharp] \to L[0\sharp]$, and we can consider $0\sharp\sharp\sharp$ and so on. Let's let $0^\sharp_n$ denote the real (if it exists) that results from adding n many sharps after 0 (so, $0^\sharp_0 = \emptyset$, $0^\sharp_1 = 0\sharp$, $0^\sharp_5 = 0\sharp\sharp\sharp\sharp\sharp$ etc.). An interesting fact is that $0^\sharp_{n+1}$ cannot be gotten from $0^\sharp_n$ using known forcing technology (this is a quick consequence of the fact that the standard forcing construction cannot result in a consistency-strength increase, and over ZFC$^-$ + Count + $\underset{\sim}{\Pi}^1_1$-PSP, '$0^\sharp_{n+1}$ exists' always has a higher consistency strength than '$0^\sharp_n$ exists'). Now suppose in fact that M also satisfies 'For every n, $0^\sharp_n$ exists'. Here we'll have $L[0^\sharp_{n+1}]$ providing enumerations of many $L[0^\sharp_n]$-uncountable sets (and indeed thus providing $L[0^\sharp_n]$-generics for many forcings $\mathbb{P} \in L[0^\sharp_n]$). But moving between $L[0^\sharp_n]$ and $L[0^\sharp_m]$ for $n < m$ has *nothing* really to do with forcing, and *everything* to do with the new (highly arbitrary) enumerations added by $0^\sharp_m$. Indeed, when one thinks about the forcing construction, it is quite a *limited* and *controlled* way of adding enumerations.

This has the consequence that *accessibility* of our Kripke frames does not *exactly* match up with the *informal construction methods* we started out with in motivating our modal theory. For instance, given a model M satisfying ZFC^- + Count + $\underset{\sim}{\Pi}^1_1$-PSP, we have in K^M_{SteMMe} that $(L^M, Def(L^M)) \leq^M_{\mathsf{SteMMe}} ((L[0\sharp])^M, Def((L[0\sharp])^M))$. But there's no way of getting from L^M to $(L[0\sharp])^M$ using forcing. Similarly, in K^M_{Sca} we have $(L_{\omega+1}, \emptyset) \leq^M_{\mathsf{Sca}} (L_{\omega+1}[0\sharp], 0\sharp)$, but no way from getting from $(L_{\omega+1}, \emptyset)$ to $(L_{\omega+1}[0\sharp], 0\sharp)$ by forcing, and if we grow $(L_{\omega+1}, \emptyset)$ vertically it will stay within L and never pick up $0\sharp$ (even if we subsequently force). Contrast this with the case of the standard view and its twin operations of powerset and union. Over any transitive model $M \models \mathsf{ZFC}$, in K^M_{Lin} if $V^M_\alpha \leq^M_{\mathsf{Lin}} V^M_\beta$, then there is always (according to M) some way of getting from V^M_α to V^M_β by iterating powerset and collecting together at limits.

So: Not only do we get *a* **Capture** theorem for Lin and the standard view, we get one that is very satisfying with accessibility exactly matching the informal set construction methods described. But neither Sca nor SteMMe (as I've presented them) has this feature. I contend that it is *exactly* the fact that the powerset encodes the idea of an *arbitrary* subset that lets *any* set by constructed by iterating powerset far enough.

What the countabilist needs is something that has the level of 'arbitrariness' that is enjoyed by the powerset operation. My suggestion is that in the specification of how sets are formed for the countabilist, the focus should be on arbitrary enumerations rather than forcing. And non-forcing-based enumerations (e.g. the various $0^\sharp_n$, or more generally $x^\sharp$ for any real x) *can* be thought of as arising out of arbitrary enumerations. But this is just to specify the germ of an idea for future study, rather than anything fully worked out.[36]

This isn't to say that forcing doesn't play an important philosophical role for the countabilist. The idea of forcing corresponds to a very natural set-construction idea—if you give me any family of dense sets for some forcing partial order, and I can run through them all, successively meeting each one extending my previous choices, and get something at the end, I'll have produced a generic for that family. The idea that this can be done for any family is *already* sufficient to conflict with the with the Powerset Axiom, putting aside whether fully *arbitrary* enumerations exist.

I therefore think that the forcing construction plays a similar philosophical role for the countabilist as the definable powerset operation (i.e. the operation of taking all sets definable over a structure) does for height-

[36]A possible line of inquiry would be to use the system of [Brauer, MS] (and in particular his use of free choice sequences) in making this idea precise.

potentialist modal set theories (which may be uncountabilist). Kanamori writes:

> As the importance of the construction of L was gradually digested, the sense it promoted of a cumulative hierarchy reverberated to become the basic picture of the universe of sets. [Kanamori, 2007, p. 173]

The point is that the definable powerset operation can be thought of as a way of iterating a *very controlled* and *concrete* height-potentialist set-construction method. Forcing plays a similar role for the countabilist—it can be used as a tool for adding enumerations in a very concrete and well-behaved way. This might *motivate* you to believe that every set can be made countable. But when we want to start considering the full power of set theory to generate *arbitrary* sets, this control is going to have to drop away.

7 Conclusions and open questions

In this article, I've compared and contrasted three modal set theories with respect to **Naturalness**, **Paradox Diagnosis**, **Interpretation**, and **Capture**. I've argued that whilst width potentialist theories like Sca and SteMMe are also able to fulfil these requirements, their **Capture**-theorem is a little unsatisfying as it stands. I think that this highlights the following important:

Challenge. Provide a modal set theory that implies both **Count** in the non-modal theory (or at least width potentialism) but satisfies **Capture** in a more pleasing fashion.

This is the most pressing challenge for the countabilist. However I think there are also some important other further questions.

First, we might ask how far we can go with **Interpretation**. Since we can use Lin to interpret any sentence of ZFC via the potentialist translation, it's easy to augment Lin with axioms that have a higher degree of interpretative power by adding the potentialist translations of your favourite first-orderisable large cardinal axioms. Whilst there are some questions for how to motivate these axioms on modal grounds—higher-order axioms like reflection principles seem difficult—so long as we stay within the domain of first-order axiomatisation and (probable) consistency with ZFC there's no obstacles here (e.g. we could add the potentialist translations of 'There is measurable cardinal', 'There is a proper class of Woodin cardinals' etc.). This strategy is not available to the countabilist since they think the Powerset Axiom is false. We therefore ask:

Question 31. Are there well-motivated countabilist modal-set-theoretic principles that result in a higher degree of interpretive power?

There is already some work in this direction. One promising strategy is to appeal to *axioms of definable determinacy*, many of which *can* be formulated in ZFC^- + Count, and imply the existence of large cardinals *in inner models*. For example $\undertilde{\Pi}^1_1$-Determinacy is enough to guarantee 'For every real x, $x^\sharp$ exists' and *Projective Determinacy* (PD) implies that there are Woodin cardinals in inner models. Moreover, if PD holds we would obtain a high degree of theoretical completeness for our axiomatisation—there are no known sentences independent of ZFC^- + Count + PD other than Gödel-style diagonal sentences (see [Welch, 2017]). In principle, we could just throw in the potentialist translations of determinacy axioms, but this strikes me as a philosophically unsatisfying solution. There are other possible directions, [Barton and Friedman, MS] for example propose an axiom (based on ideas of *absoluteness*) that guarantees the existence of $0^\sharp$ but conflicts with low levels of determinacy.

A different suggestion, especially germane to Sca and SteMMe, is to consider adding large cardinals in *inner models*. This is suggested, for instance, by Steel who writes (regarding his second-order axiomatisation):

> The central role of the theories axiomatized by large cardinal hypotheses argues for adding such hypotheses to our framework. The goal of our framework theory is to maximize interpretative power, to provide a language and theory in which all mathematics, of today and of the future so far as we can anticipate it today, can be developed. [Steel, 2014, p. 165]

Taking inspiration from Steel, we can define:

Definition 32. Let W be the claim that 'There is a proper class of Woodin cardinals' (or some other suitable first-order-isable statement) rendered in $\mathscr{L}_\in$. Let the theory SteMMe^+ be the result of adding the necessitation of W to SteMMe and let Sca^+ be the result of adding the $\langle v \rangle$-potentialist translation of W to Sca.

The idea for SteMMe^+ and Sca^+ is thus that we've got many (i.e. a proper class of) Woodin cardinals in inner models. I'll leave it open at this stage whether such an axiom can be easily motivated. But we might then ask:

Question 33. Is there a significant increase in what can be interpreted under the potentialist translation using either Sca^+ or SteMMe^+ as compared to Sca and SteMMe?[37]

Our next question concerns the relationship between 'iterative conceptions' and modal set theories. As we noted, both K^M_{Sca} and K^M_{SteMMe} have non-well-ordered (and indeed non-well-founded) accessibility relations. We therefore ask:

Question 34. Can we extract 'iterative conceptions' from the 'set-construction methods' axiomatised by Sca and SteMMe? What about countabilist theories more widely?

The final question I wish to pose links to the topic of this special issue. Though I have referred to height- and width-potentialist modal set theories, I have deliberately remained agnostic throughout whether we should think that there is, at the end of the day, a maximal and definite domain of set theory. We might, for instance, view these modal set theories as merely descriptive; they provide a modal description of a definite and maximal universe of set theory. However we might also view them as showing us that the universe of sets is indefinitely extensible inherently potential in nature. My agnosticism has not been shifted by consideration of these issues, and so I ask:

Question 35. Do these modal set theories suggest a potentialism about the subject matter of set theory? If yes, how *different* are height- and width-potentialism?

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[37]One might be tempted to try infer PD from the fact that PD holds at every world (the latter we know by the Martin-Steel theorem). The problem is that since the $\mathbb{R}$ of any world is countable, and we know determinacy holds for countable sets of reals, we already know PD for such sets without any large cardinals. It's unclear to me how to get extra juice from the large cardinals beyond their worldly consequences. A different hope would be to proceed via the equivalence with the existence of the relevant mice, but this strategy remains opaque to me.

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