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COMPLETENESS IN THE THEORY OF PROPERTIES, RELATIONS, AND PROPOSITIONS

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Higher-order theories of properties, relations, and propositions (PRPs) are known to be essentially incomplete relative to their standard notions of validity.[2] There is, however, a first-order theory of PRPs that results when standard first-order logic is supplemented with an operation of intensional abstraction. It turns out that this first-order theory of PRPs is provably complete with respect to its standard notions of validity. The construction involves the development of a new algebraic semantic method. Unlike most other methods used in contemporary intensional logic, this method does not appeal to possible worlds as a heuristic; the heuristic used is that of PRPs taken as primitive entities. This is important, for even though the possible-worlds approach is useful in treating modal logic, it seems to be of little help in treating the logic for psychological matters. The present approach, by contrast, appears to make a step in the direction of a satisfactory treatment of both modal and intentional logic. For, by taking PRPs as primitive entities, we remain free to tailor the statement of their identity conditions so that it agrees with the logical data—modal, psychological, etc. In this way, the present approach suggests a strategy for developing a comprehensive treatment of intensional logic.

In [1] and [2] I explore this prospect philosophically. The purpose of the present paper is to lay out the technical details of the approach and to present the completeness results.

§1. The first-order intensional language L_ω.

Primitive symbols:

Logical operators: &, $\neg$, $\exists$,

Predicate letters: F^1_1, F^1_2, $\cdots$, F^r_s, $\cdots$ (for $r, s \geq 1$),

Variables: $x, y, z, x_1, y_1, z_1, \ldots$,

Punctuation: (,), [,].

Simultaneous inductive definition of *term* and *formula* of L_ω:

(1) All variables are terms.

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[2]This follows from Gödel's first incompleteness theorem and the fact that the notions of first-order arithmetic are definable in higher-order theories of PRPs.

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(2) If $t_1, \ldots, t_j$ are terms, then $F_i^j(t_1, \ldots, t_j)$ is a formula.
(3) If A and B are formulas and v_k is a variable, then $(A \,\&\, B)$, $\neg A$, and $(\exists v_k)A$ are formulas.
(4) If A is a formula and $v_1, \ldots, v_m$, $0 \leq m$, are distinct variables, then $[A]_{v_1 \cdots v_m}$ is a term.

In the limiting case where $m = 0$, $[A]$ is a term. On the intended informal interpretation of L_ω, $[A]_{v_1 \cdots v_m}$ denotes a proposition if $m = 0$, a property if $m = 1$, and an m-ary relation-in-intension if $m \geq 2$.

The following are auxiliary syntactic notions. Formulas and terms are *well-formed expressions*. An occurrence of a variable v_i in a well-formed expression is *bound* (*free*) if and only if it lies (does not lie) within a formula of the form $(\exists v_i)A$ or a term of the form $[A]_{v_1 \cdots v_i \cdots v_m}$. A term t is said to be *free for v_i in A* if and only if, for all v_k, if v_k is free in t, then no free occurrence of v_i in A occurs either in a subcontext of the form $(\exists v_k)(\ldots)$ or in a subcontext of the form $[\ldots]_{\alpha v_k \beta}$, where α and β are sequences of variables. If v_i has a free occurrence in A and is not one of the variables in the sequence of variables α, then v_i is an *externally quantifiable variable* in the term $[A]_\alpha$. Let δ be the sequence of externally quantifiable variables in $[A]_\alpha$ displayed in order of their first free occurrence; $[A]_\alpha$ will sometimes be rewritten as $[A]_\alpha^\delta$. Let $A(v_1, \ldots, v_p)$ be any formula; $v_1, \ldots, v_p$ may or may not occur free in A. Then I write $A(t_1, \ldots, t_p)$ to indicate the formula that results when, for each k, $1 \leq k \leq p$, the term t_k replaces each free occurrence of v_k in A. Terms $[A(u_1, \ldots, u_p)]_{u_1 \cdots u_p}^\delta$ and $[A(v_1, \ldots, v_p)]_{v_1 \cdots v_p}^\delta$ are said to be *alphabetic variants* if and only if, for each k, $1 \leq k \leq p$, u_k is free for v_k in A and conversely. Terms of the form $[F_i^j(v_1, \ldots, v_j)]_{v_1 \cdots v_j}$ are called *elementary*. A term $[A]_\alpha$ is called *normalized* if and only if all variables in α occur free in A exactly once and α displays the order in which these variables occur free in A. The logical operators $\forall$, $\supset$, $\equiv$, $\equiv_{v_1 \cdots v_j}$ are defined in terms of $\exists$, $\&$, and $\neg$ in the usual way. Finally, F_1^2 is singled out as a distinguished logical predicate, and formulas of the form $F_1^2(t_1, t_2)$ are rewritten as $t_1 = t_2$.

§2. **Intensional semantics.** A *model structure* is any structure $\langle \mathscr{D}, \mathscr{P}, \mathscr{K}, \mathscr{G}, \mathrm{Id}, \mathscr{T}, \mathrm{Conj}, \mathrm{Neg}, \mathrm{Exist}, \mathrm{Pred}_0, \mathrm{Pred}_1, \mathrm{Pred}_2, \ldots, \mathrm{Pred}_k, \ldots \rangle$ whose elements satisfy the following conditions. $\mathscr{D}$ is a nonempty domain. $\mathscr{P}$ is a prelinear ordering on $\mathscr{D}$ that induces a partition of $\mathscr{D}$ into the subdomains $\mathscr{D}_{-1}, \mathscr{D}_0, \mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3, \ldots$. The elements of $\mathscr{D}_{-1}$ are to be thought of as particulars; the elements of $\mathscr{D}_0$ as propositions; the elements of $\mathscr{D}_1$ as properties, and the elements of $\mathscr{D}_i$, for $i \geq 2$, as i-ary relations-in-intension. Although $\mathscr{D}_i$, $i \geq 0$, must not be empty, we do permit $\mathscr{D}_{-1}$ to be empty. $\mathscr{K}$ is a set of functions on $\mathscr{D}$. For all $H \in \mathscr{K}$, if $x \in \mathscr{D}_{-1}$, then $H(x) = x$; if $x \in \mathscr{D}_0$, then $H(x) = T$ or $H(x) = F$; if $x \in \mathscr{D}_1$, then $H(x) \subseteq \mathscr{D}$; if, for $i > 1$, $x \in \mathscr{D}_i$, then $H(x) \subseteq \mathscr{D}^i$. These functions $H \in \mathscr{K}$ are to be thought of as telling us the alternate or possible extensions of the elements of $\mathscr{D}$. $\mathscr{G}$ is a distinguished element of $\mathscr{K}$ and is to be thought of as the function that determines the *actual* extensions of the elements of $\mathscr{D}$. Id is a distinguished element of $\mathscr{D}_2$ and is thought of as the fundamental logical relation-in-intension *identity*. Id must satisfy the following condition: $(\forall H \in \mathscr{K})(H(\mathrm{Id}) = \{xy \in \mathscr{D}: x = y\})$. In order to characterize the next element $\mathscr{T}$, consider the following partial functions on $\mathscr{D}$: Exp_i, defined on

$\mathscr{D}_i$, $i \geq 0$; Ref_i, defined on $\mathscr{D}_i$, $i \geq 2$; Conv_i, defined on $\mathscr{D}_i$, $i \geq 2$; Inv_i, defined on $\mathscr{D}_i$, $i \geq 3$.[3] For all $H \in \mathscr{K}$ and all $x_1, \ldots, x_{i+1} \in \mathscr{D}$, these functions satisfy the following conditions:

a. $x_1 \in H(\text{Exp}_1(u))$ *iff* $H(u) = T$ (for $u \in \mathscr{D}_0$).
$\langle x_1, \ldots, x_1, x_{i+1}\rangle \in H(\text{Exp}_i(u))$ *iff* $\langle x_1, \ldots, x_i\rangle \in H(u)$ (for $u \in \mathscr{D}_i$, $i \geq 1$).

b. $\langle x_1, \ldots, x_{i-2}, x_{i-1}\rangle \in H(\text{Ref}_i(u))$ *iff* $\langle x_1, \ldots, x_{i-2}, x_{i-1}, x_{i-1}\rangle \in H(u)$ (for $u \in \mathscr{D}_i$, $i \geq 2$).

c. $\langle x_i, x_1, \ldots, x_{i-1}\rangle \in H(\text{Conv}_i(u))$ *iff* $\langle x_1, \ldots, x_{i-1}, x_i\rangle \in H(u)$ (for $u \in \mathscr{D}_i$, $i \geq 2$).

d. $\langle x_1, \ldots, x_{i-2}, x_i, x_{i-1}\rangle \in H(\text{Inv}_i(u))$ *iff* $\langle x_1, \ldots, x_{i-2}, x_{i-1}, x_i\rangle \in H(u)$ (for $u \in \mathscr{D}_i$, $i \geq 3$).

A proto-transformation is defined to be a function that arises from composing a finite number of these functions in some order (repetitions permitted). A proto-transformation τ is said to be degenerate if and only if $\tau(x) = x$ for all $x \in \mathscr{D}$ for which τ is defined. A function τ is said to be *equivalent* to a proto-transformation τ' if and only if, for all $H \in \mathscr{K}$ and for all $x \in \mathscr{D}$ for which τ' is defined, $H(\tau(x)) = H(\tau'(x))$. Now $\mathscr{T}$ is a set of partial functions on $\mathscr{D}$: for every nondegenerate proto-transformation, there is exactly one equivalent function in $\mathscr{T}$, and nothing but such functions are in $\mathscr{T}$. The functions in $\mathscr{T}$ are called *transformations*. The remaining elements in a model structure are partial functions on $\mathscr{D}$. Conj is defined on each $\mathscr{D}_i \times \mathscr{D}_i$, $i \geq 0$; Neg, on each $\mathscr{D}_i$, $i \geq 0$; Exist, on each $\mathscr{D}_i$, $i \geq 1$; Pred_0, on each $\mathscr{D}_i \times \mathscr{D}$, $i \geq 1$; Pred_k, on each $\mathscr{D}_i \times \mathscr{D}_j$, $i \geq 1$ and $j \geq k \geq 1$. These functions satisfy the following, for all $H \in \mathscr{K}$ and all $x_1, \ldots, x_i, y_1, \ldots, y_k \in \mathscr{D}$:

1. $H(\text{Conj}(u, v)) = T$ *iff* $(H(u) = T \;\&\; H(v) = T)$ (for $u, v \in \mathscr{D}_0$).
$\langle x_1, \ldots, x_i\rangle \in H(\text{Conj}(u, v))$ *iff*
$(\langle x_1, \ldots, x_i\rangle \in H(u) \;\&\; \langle x_1, \ldots, x_i\rangle \in H(v))$ (for $u, v \in \mathscr{D}_i$, $i \geq 1$).

2. $H(\text{Neg}(u)) = T$ *iff* $H(u) = F$ (for $u \in \mathscr{D}_0$).
$\langle x_1, \ldots, x_i\rangle \in H(\text{Neg}(u))$ *iff* $\langle x_1, \ldots, x_i\rangle \notin H(u)$ (for $u \in \mathscr{D}_i$, $i \geq 1$).

3. $H(\text{Exist}(u)) = T$ *iff* $(\exists x_1)(x_1 \in H(u))$ (for $u \in \mathscr{D}_1$).
$\langle x_1, \ldots, x_{i-1}\rangle \in H(\text{Exist}(u))$ *iff*
$(\exists x_i)(\langle x_1, \ldots, x_{i-1}, x_i\rangle \in H(u))$ (for $u \in \mathscr{D}_i$, $i \geq 2$).

4.0 $H(\text{Pred}_0(u, y_1)) = T$ *iff* $y_1 \in H(u)$ (for $u \in \mathscr{D}_1$).
$\langle x_1, \ldots, x_{i-1}\rangle \in H(\text{Pred}_0(u, y_1))$ *iff*
$\langle x_1, \ldots, x_{i-1}, y_1\rangle \in H(u)$ (for $u \in \mathscr{D}_i$, $i \geq 2$).

[3]These functions—along with Conj, Neg, and Exist—are closely related to the operations Quine introduces in [7]. See also [8].

4.1. $\langle x_1, \ldots, x_{i-1}, y_1 \rangle \in H(\text{Pred}_1(u, v))$ *iff*
$\langle x_1, \ldots, x_{i-1}, \text{Pred}_0(v, y_1) \rangle \in H(u)$
(for $u \in \mathcal{D}_i$, $i \geq 1$, and $v \in \mathcal{D}_j$, $j \geq 1$).

4.2. $\langle x_1, \ldots, x_{i-1}, y_1, y_2) \rangle \in H(\text{Pred}_2(u, v))$ *iff*
$\langle x_1, \ldots, x_{i-1}, \text{Pred}_0(\text{Pred}_0(v, y_2), y_1) \rangle \in H(u)$
(for $u \in \mathcal{D}_i$, $i \geq 1$, and $v \in \mathcal{D}_j$, $j \geq 2$).

. . . .[4]

These functions, together with the transformations in $\mathcal{T}$, are to be thought of as fundamental logical operations on intensional entities. This completes the characterization of what a model structure is.

Now in the history of logic and philosophy there have been two competing conceptions of intensional entities, which I call conception 1 and conception 2. Conception 1 is suited to the logic for modal matters (necessity, possibility, etc.), and conception 2 appears to be relevant to the logic for psychological matters (belief, desire, decision, etc.).[5] According to conception 1, (i-ary) intensions are taken to be identical if they are *necessarily equivalent*. This leads to the following definition. A model structure is *type* 1 *iff*$_{\text{df}}$ it satisfies the following auxiliary

[4]In general,

$$\langle x_1, \ldots, x_{i-1}, y_1, \ldots, y_k \rangle \in H(\text{Pred}_k(u, v)) \textit{ iff}$$
$$\langle x_1, \ldots, x_{i-1}, \text{Pred}_0(\ldots \text{Pred}_0(\text{Pred}_0(v, y_k), y_{k-1}), \ldots, y_1) \rangle \in H(u)$$

where $u \in \mathcal{D}_i$, $i \geq 1$, and $v \in \mathcal{D}_j$, $j \geq k \geq 1$. The following examples help to explain the predication functions Pred_0, Pred_1, Pred_2, Pred_3, . . . :

$$\text{Pred}_0([Fxyz]_{xyz}, [Guvw]_{uvw}) = [Fxy[Guvw]_{uvw}]_{xy},$$
$$\text{Pred}_0([Fx]_x, [Guvw]_{uvw}) = [F[Guvw]_{uvw}],$$
$$\text{Pred}_1([Fx]_x, [Guvw]_{uvw}) = [F[Guvw]^{w}_{uv}]_w,$$
$$\text{Pred}_2([Fx]_x, [Guvw]_{uvw}) = [F[Guvw]^{vw}_{u}]_{vw},$$
$$\text{Pred}_3([Fx]_x, [Guvw]_{uvw}) = [F[Guvw]^{uvw}]_{uvw},$$
$$\text{Pred}_k([Fx]_x, [A]_{v_1 \cdots v_m u_1 \cdots u_k}) = [F[A]^{u_1 \cdots u_k}_{v_1 \cdots v_m}]_{u_1 \cdots u_k}.$$

(Note that I have just *used*, not mentioned, intensional abstracts from L_ω.) For further clarification of these predication functions Pred_0, . . . , see the definition of the associated syntactic operations given on page 35 f.

[5]On conception 1 PRPs are thought of as the actual qualities, connections, and conditions of things; on conception 2 PRPs are thought of as concepts and thoughts. (See §2 in [1] and §§40–41 in [2] for discussion of these distinctions.) Conception 1 and conception 2 correspond very closely to what Alonzo Church calls, respectively, Alternative 2 and Alternative 0 (pp. 4 ff. in [3] and pp. 143 ff. in [5]). Church states that he '. . . attaches greater importance to Alternative 0 because it would seem that it is in this direction that a satisfactory analysis is to be sought of statements regarding assertion and belief.' (p. 7 n. in [3]). A fuller defense of his approach to the logic for psychological matters is given by Church in [4], where he develops the criterion of strict synonymy upon which he bases Alternative 0. And I discuss at length the importance of conception 2 in [2] §§2, 4, 6–11, 18–20, 39.

For the present purposes, I advocate developing *both* conception 1 and conception 2 side by side without attaching greater importance to one over the other. An advantage of such a dual approach is that, once these two conceptions are well developed, it is relatively straightforward to adapt our methods to handle intermediate conceptions in the event that they condition: $(\forall x, y \in \mathscr{D}_i)((\forall H \in \mathscr{K})(H(x) = H(y)) \rightarrow x = y)$, for all $i \geq -1$. This auxiliary condition provides a precise characterization of conception 1. In contrast to conception 1, conception 2 places far stricter conditions on the identity of intensional entities. According to conception 2, when an intension is defined *completely*, it has a *unique, noncircular definition*. (The possibility that such complete definitions might in some or even all cases be infinite need not be ruled out.) This leads to the following definition. A model structure is *type* 2 *iff*$_{\text{df}}$ the transformations in $\mathscr{T}$ and the functions Conj, Neg, Exist, Pred_0, Pred_1, Pred_2, ... are all (i) one-one, (ii) disjoint in their ranges, and (iii) noncycling. Auxiliary conditions (i)-(iii) provide us with a precise formulation of conception 2.[6]

In order to state the semantics for L_ω, I must define some preliminary syntactic notions. First, I define certain syntactic operations on complex terms of L_ω. These operations have a natural correspondence to the logical operations Conj, Neg, Exist, Pred_0, ... in a model structure. If $[(A \;\&\; B)]_\alpha$ is normalized, it is the *conjunction* of $[A]_\alpha$ and $[B]_\alpha$. If $[\neg A]_\alpha$ is normalized, it is the *negation* of $[A]_\alpha$. If $[(\exists v_k)A]_\alpha$ is normalized, it is the *existential generalization* of $[A]_{\alpha v_k}$. Suppose that $[F_i^j(v_1, \ldots, v_{m-1}, t_m, t_{m+1}, \ldots, t_j)]_\alpha$ is normalized and that no variable occurring free in t_m occurs in α. Then this normalized term is the *predication*$_0$ of

$$[F_i^j(v_1, \ldots, v_{m-1}, v_m, t_{m+1}, \ldots, t_j)]_{\alpha v_m}$$

of t_m. (v_m is the alphabetically earliest variable not occurring in the normalized

should prove relevant. Consider two examples. First, according to construction of conception 2 presented in the text, the proposition $\text{Pred}_0(\text{Pred}_0([Lxy]_{xy}, a), b)$ is treated as distinct from the proposition $\text{Pred}_0(\text{Pred}_0([Lxy]_{yx}, b)a)$. If this distinction seems artificial, then along the lines of p. 54 [2] one can relax the identity conditions on PRPs within type 2 model structures so that these two propositions are treated as identical. Secondly, there are instances of the paradox of analysis involving analyses of logical operations themselves. (E.g., despite the usual definition of conditionalization in terms of negation and conjunction, someone might doubt that $(A \supset B) \equiv \neg(A \;\&\; \neg B)$ and yet not doubt that $(A \supset B) \equiv (A \supset B.)$ Such puzzles can be easily resolved along the lines of chapter 3 [2] once one enriches model structures with appropriate additional logical operations (including a primitive operation for conditionalization): e.g., for each nondegenerate finite composition of the present logical operations, one might add a primitive operation that is equivalent to it in H-values. The broader philosophical point is that, if there is artificiality in the construction given in the text, it appears not to be inherent in the general algebraic approach; evidently it can be removed by some combination of the above methods. It does not follow, of course, that these methods can be used to rid other approaches to intensional logic of their forms of artificiality. For example, the familiar approach that identifies PRPs with functions seems to have a form of artificiality that cannot be removed by any means (cf., §24[2]).

[6]Taken together, (i) and (ii) guarantee that the action of the inverses of the $\mathscr{T}$-transformations and Conj, Neg, ... in a type 2 model structure is to decompose each element of $\mathscr{D}$ into a unique (possibly infinite) complete tree. (A decomposition tree is *complete* if it contains no terminal node that could be decomposed further under the inverses of the $\mathscr{T}$-transformations and Conj, Neg, ...). Notice that without condition (iii) unwanted identities such as $[Fx]_x = [A \;\&\; Fx]_x$ could arise. For, as far as conditions (i) and (ii) are concerned, the property $[Fx]_x$ can have a unique complete decomposition tree in which $[Fx]_x$ occurs (denumerably many successive times) on a path descending from $[Fx]_x$. Condition (iii) rules out such a tree.

Examples of type 1 and 2 model structures are easily constructed. E.g., a type 1 model structure can be constructed relative to a model for first-order logic with identity and extensional abstraction, and a type 2 model structure can be constructed relative to a model for first-order logic with identity, extensional abstraction, and Quine's device of corner quotation.

term.) Finally, suppose that, for $k \geq 1$,

$$[F_i^j(v_1, \ldots, v_{m-1}, [B]_\gamma^\delta, t_{m+1}, \ldots, t_j)]_{v_1 \ldots v_{m-1} u_1 \ldots u_k \alpha}$$

is normalized, that $u_1, \ldots, u_k$ occur in δ, and that no variable in δ occurs in α. Then

$$[F_i^j(v_1, \ldots, v_{m-1}, [B]_\gamma^\delta, t_{m+1}, \ldots, t_j]_{v_1 \ldots v_{m-1} \ldots \alpha u_1 \ldots u_k}$$

is the *predication$_k$* of

$$[F_i^j(v_1, \ldots, v_{m-1}, u_1, t_{m+1}, \ldots, t_j)]_{v_1 \ldots v_{m-1} \alpha u_1}$$

of $[B]_{\gamma u_1 \ldots u_m}^{\delta'}$. ($\delta'$ is the result of deleting $u_1, \ldots, u_m$ from δ.)

Consider the following auxiliary operations on complex terms:

(a) $$\text{exp}_i([A]_{v_1 \ldots v_i}) =_{\text{df}} [A]_{v_1 \ldots v_i v_{i+1}}$$

(where $i \geq 0$ and v_{i+1} is the alphabetically earliest variable not occurring in $[A]_{v_1 \ldots v_i}$).

(b) $$\text{ref}_i([A(v_1, \ldots, v_{i-1}, v_i)]_{v_1 \ldots v_{i-1} v_i}) =_{\text{df}} [A(v_1, \ldots, v_{i-1}, v_{i-1})]_{v_1 \ldots v_{i-1}}$$

(where $i \geq 2$ and v_{i-1} is free for v_i in A).

(c) $$\text{conv}_i([A]_{v_1 \ldots v_{i-1} v_i}) =_{\text{df}} [A]_{v_i v_1 \ldots v_{i-1}}$$

(where $i \geq 2$).

(d) $$\text{inv}_i([A]_{v_1 \ldots v_{i-2} v_{i-1} v_i}) =_{\text{df}} [A]_{v_1 \ldots v_{i-2} v_i v_{i-1}}$$

(where $i \geq 3$).

Consider the operations σ that arise from composing a finite number of these operations in some order (repetitions permitted). A relation R_σ is a *term-transforming* relation if it is associated with one of these operations σ as follows: $R_\sigma(r, s)$ *iff* $\sigma(r') = s'$, where r' is an alphabetic variant of r, s' is an alphabetic variant of s, r is either an elementary complex term, a negation, a conjunction, an existential generalization, or a predication$_k$, $k \geq 0$, and s is none of these. Now for any model structure, a term-transforming relation R_σ is *associated* with a transformation τ in the set $\mathscr{T}$ in the model structure *iff*$_{\text{df}}$ (a) for some $\sigma_1, \ldots, \sigma_m$ selected from exp_i, ref_i, conv_i, inv_i, σ is the composition of $\sigma_1, \ldots, \sigma_m$; (b) for some $\tau_1, \ldots, \tau_m$ selected from Exp_i, Ref_i, Conv_i, Inv_i, τ is the transformation in $\mathscr{T}$ equivalent to the composition of $\tau_1, \ldots, \tau_m$; (c) for all k, $1 \leq k \leq m$, $\sigma_k = \text{exp}_i$ iff $\tau_k = \text{Exp}_i$; $\sigma_k = \text{ref}_i$ iff $\tau_k = \text{Ref}_i$; $\sigma_k = \text{conv}_i$ iff $\tau_k = \text{Conv}_i$; $\sigma_k = \text{inv}_i$ iff $\tau_k = \text{Inv}_i$. With these preliminary notions in hand I am finally ready to state the semantics for L_ω.

Denotation, truth, and validity. An *interpretation* $\mathscr{I}$ *for* L_ω *relative to model structure* $\mathscr{M}$ is a function that assigns to the predicate letter F_1^2 (i.e., $=$) the element $\text{Id} \in \mathscr{M}$ and, for each predicate letter F_i^j in L_ω, assigns to F_i^j some element of the subdomain $\mathscr{D}_j \subset \mathscr{D} \in \mathscr{M}$. An *assignment* $\mathscr{A}$ *for* L_ω *relative to model structure* $\mathscr{M}$ is a function that maps the variables of L_ω into the domain $\mathscr{D} \in \mathscr{M}$. Relative to interpretation $\mathscr{I}$, assignment $\mathscr{A}$, and model structure $\mathscr{M}$, the *denotation* relation for terms of L_ω is inductively defined as follows:

Variables. v_i denotes $\mathscr{A}(v_i)$.

Elementary complex terms. $[F_i^j(v_1, \ldots, v_j)]_{v_1 \ldots v_j}$ denotes $\mathscr{I}(F_i^j)$.

Nonelementary complex terms. If t is the conjunction—or predication$_k$—of r and s, and r denotes u, and s denotes v, then t denotes Conj(u, v)—or Pred$_k$(u, v). If t is the negation—or existential generalization—of r, and r denotes u, then t denotes Neg(u)—or Exist(u). If R_σ is a term-transforming relation associated with a transformation $\tau \in \mathscr{T}$ and $R_\sigma(r, t)$ and r denotes u, then t denotes $\tau(u)$.

The denotation relation is clearly a function. I henceforth represent it with $D_{\mathscr{I}\mathscr{A}\mathscr{M}}$. *Truth* is then defined in terms of $D_{\mathscr{I}\mathscr{A}\mathscr{M}}$ as follows: $T_{\mathscr{I}\mathscr{A}\mathscr{M}}(A)$ *iff*$_{\text{df}}$ $\mathscr{G}(D_{\mathscr{I}\mathscr{A}\mathscr{M}}([A])) = T$.[7] And finally two notions of validity are defined. A formula A is *valid*$_1$ *iff*$_{\text{df}}$ for every type 1 model structure $\mathscr{M}$ and for every interpretation $\mathscr{I}$ and every assignment $\mathscr{A}$ relative to $\mathscr{M}$, $T_{\mathscr{I}\mathscr{A}\mathscr{M}}(A)$. A formula A is *valid*$_2$ *iff*$_{\text{df}}$ for every type 2 model structure $\mathscr{M}$ and for every interpretation $\mathscr{I}$ and every assignment $\mathscr{A}$ relative to $\mathscr{M}$, $T_{\mathscr{I}\mathscr{A}\mathscr{M}}(A)$. This completes the semantics for L_ω.

§3. **The logic for PRPs on conception 1.** On conception 1 intensional entities are identical if and only if necessarily equivalent. Thus, on conception 1 the following abbreviation captures the properties usually attributed to the modal operator $\Box$: $\Box A$ *iff*$_{\text{df}}$ $[A] = [\,[A] = [A]\,]$. (That is, necessarily A *iff* the proposition that A is identical to any trivial necessary truth.) The modal operator $\Diamond$ is then defined in terms of $\Box$ in the usual way: $\Diamond A$ *iff*$_{\text{df}}$ $\neg\Box\neg A$.

The logic $T1$ for L_ω on conception 1 consists of the axiom schemas and rules for the modal logic S5 with quantifiers and identity and three additional axiom schemas for intensional abstracts.

Axiom schemas and rules of $T1$.

A1. Truth-functional tautologies.

A2. $(\forall v_i)A(v_i) \supset A(t)$ (where t is free for v_i in A).

A3. $(\forall v_i)(A \supset B) \supset (A \supset (\forall v_i)B)$ (where v_i is not free in A).

A4. $v_i = v_i$.

A5. $v_i = v_j \supset (A(v_i, v_i) \equiv A(v_i, v_j))$ (where $A(v_i, v_j)$ is a formula that arises from $A(v_i, v_i)$ by replacing some (but not necessarily all) free occurrences of v_i by v_j, and v_j is free for the occurrences of v_i that it replaces).

A6. $[A]_{u_1 \cdots u_p} \neq [B]_{v_i \cdots v_q}$ (where $p \neq q$).

A7. $[A(u_1, \ldots, u_p)]_{u_1 \cdots u_p} = [A(v_1, \ldots, v_p)]_{v_1 \cdots v_p}$ (where these terms are alphabetic variants).

A8. $[A]_\alpha = [B]_\alpha \equiv \Box(A \equiv_\alpha B)$.

A9. $\Box A \supset A$.

A10. $\Box(A \supset B) \supset (\Box A \supset \Box B)$.

A11. $\Box A \supset \Box\Diamond A$.

R1. If $\vdash A$ and $\vdash A \supset B$, then $\vdash B$.

R2. If $\vdash A$, then $\vdash (\forall v_i)A$.

R3. If $\vdash A$, then $\vdash \Box A$.

[7]Meaning may also be defined: $M_{\mathscr{I}\mathscr{A}\mathscr{M}}(A) =_{\text{df}} D_{\mathscr{I}\mathscr{A}\mathscr{M}}([A])$.

THEOREM (SOUNDNESS AND COMPLETENESS). *For all formulas A in L_ω, A is valid$_1$ if and only if A is a theorem of $T1$ (i.e., $\models_1 A$ iff $\vdash_{T1} A$).*[8]

PROOF (SOUNDNESS). First, the following lemmas are proved.

LEMMA 1. *$T1$ is equivalent to the theory that results when* A5, A8, and A11 *are replaced with the following simpler versions:*

A5*. $v_i = v_j \supset (A(v_i, v_i) \supset A(v_i, v_j))$ *(where $A(v_i, v_i)$ and $A(v_i, v_j)$ are as in* A5 *except that A is atomic).*

A8*(a). $\Box(A \equiv B) \equiv [A] = [B]$.

A8*(b). $(\forall v_i)([A(v_i)]_\alpha = [B(v_i)]_\alpha) \equiv [A(v_i)]_{\alpha v_i} = [B(v_i)]_{\alpha v_i}$.

A11*. $v_i \neq v_j \supset \Box v_i \neq v_j$.

LEMMA 2. *Let v_h be an externally quantifiable variable in $[B(v_h)]_\alpha$, and let t_k be free for v_h in $[B(v_h)]_\alpha$. Consider any model structure $\mathscr{M}$ and any interpretation $\mathscr{I}$ and assignment $\mathscr{A}$ relative to $\mathscr{M}$. Let $\mathscr{A}'$ be an assignment that is just like $\mathscr{A}$ except that $\mathscr{A}'(v_h) = \mathrm{D}_{\mathscr{I}\mathscr{A}\mathscr{M}}(t_k)$. Then,*

$$\mathrm{D}_{\mathscr{I}\mathscr{A}'\mathscr{M}}([B(v_h)]_\alpha = \mathrm{D}_{\mathscr{I}\mathscr{A}\mathscr{M}}([B(t_k)]_\alpha).$$

LEMMA 3. *For all $\mathscr{I}$, $\mathscr{A}$, $\mathscr{M}$ and for all $\mathscr{D}_k \subset \mathscr{D} \in \mathscr{M}$, $k \geq 0$,*

$$\mathrm{D}_{\mathscr{I}\mathscr{A}\mathscr{M}}([A]_{v_1 \ldots v_k}) \in \mathscr{D}_k.$$

LEMMA 4. *For all $\mathscr{I}$, $\mathscr{A}$, $\mathscr{M}$ and for all terms t and t', if $\mathscr{M}$ is type 1, then*

$$\mathrm{D}_{\mathscr{I}\mathscr{A}\mathscr{M}}([t = t]) = \mathrm{D}_{\mathscr{I}\mathscr{A}\mathscr{M}}([t' = t']).$$

LEMMA 5. *Let v_r be an externally quantifiable variable in $[A(v_r)]_\alpha$. Then, for all $\mathscr{I}$, $\mathscr{A}$, $\mathscr{M}$, if $\mathscr{M}$ is type 1,*

$$\mathrm{D}_{\mathscr{I}\mathscr{A}\mathscr{M}}([A(v_r)]_\alpha) = \mathrm{Pred}_0(\mathrm{D}_{\mathscr{I}\mathscr{A}\mathscr{M}}([A(v_r)]_{\alpha v_r}), \mathscr{A}(v_r)).$$

LEMMA 6. *For all $\mathscr{I}$, $\mathscr{A}$, $\mathscr{M}$:*

(a) $\mathrm{T}_{\mathscr{I}\mathscr{A}\mathscr{M}}(F_i^j(t_1, \ldots, t_j))$ *iff* $\langle \mathrm{D}_{\mathscr{I}\mathscr{A}\mathscr{M}}(t_1), \ldots, \mathrm{D}_{\mathscr{I}\mathscr{A}\mathscr{M}}(t_j)\rangle \in \mathscr{G}(\mathscr{I}(F_i^j))$.

(b) $\mathrm{T}_{\mathscr{I}\mathscr{A}\mathscr{M}}((A \mathbin{\&} B))$ *iff* $\mathrm{T}_{\mathscr{I}\mathscr{A}\mathscr{M}}(A)$ *and* $\mathrm{T}_{\mathscr{I}\mathscr{A}\mathscr{M}}(B)$.

(c) $\mathrm{T}_{\mathscr{I}\mathscr{A}\mathscr{M}}(\neg A)$ *iff it is not the case that* $\mathrm{T}_{\mathscr{I}\mathscr{A}\mathscr{M}}(A)$.

(d) $\mathrm{T}_{\mathscr{I}\mathscr{A}\mathscr{M}}((\exists v_k)A)$ *iff there is an assignment $\mathscr{A}'$ relative to $\mathscr{M}$ such that $\mathscr{A}'$ is just like $\mathscr{A}$ except perhaps in what it assigns to v_k and* $\mathrm{T}_{\mathscr{I}\mathscr{A}'\mathscr{M}}(A)$.

Then, given these lemmas, which are in most cases proofs by induction on the complexity of terms or formulas, the verification of the soundness of T1 is straightforward. (For example, the soundness of A6 follows directly from Lemma 3; the soundness of A8*(b), from Lemma 5; etc.)

PROOF (COMPLETENESS). The proof is Henkin style. Let L_ω^* be any extension of L_ω. A sentence A is said to be derivable in $T1$ from a set Γ of L_ω^*-sentences if, for some finite subset $\{B_1, \ldots, B_n\}$ of Γ, $\vdash_{T1} ((B_1 \mathbin{\&} \cdots \mathbin{\&} B_n) \supset A)$. A set $\mathscr{A}$ of sets of L_ω^*-sentences is said to be *perfect*$_1$ if (1) every set in $\mathscr{A}$ is maximal, consistent, and ω-complete; (2) for every identity sentence $t = t'$, if this sentence is in any set in $\mathscr{A}$, it is in all sets in $\mathscr{A}$; (3) for every sentence $[A]_{v_1 \cdots v_p} \neq [B]_{v_1 \cdots v_p}$ ($p \geq 0$),

[8] A corollary is that first-order logic with identity and extensional abstraction (i.e., class abstraction) is complete. Notice also that, in view of the definitions of $\Box$ and $\Diamond$ in terms of identity and intensional abstraction, modal logic may be thought of as a part of the identity theory for intensional abstracts.

if this sentence belongs to some $\Delta \in \mathscr{A}$, then there is some set $\Delta' \in \mathscr{A}$ (where possibly $\Delta = \Delta'$) such that the sentence $(\exists v_1) \cdots (\exists v_p) \neg (A \equiv B)$ belongs to Δ'; (4) for every closed term $[A]_{v_1 \cdots v_p}$, there is a primitive predicate letter F^p_q such that the sentence $[A]_{v_1 \cdots v_p} = [F^p_q(v_1, \ldots, v_p)]_{v_1 \cdots v_p} \in \Delta$, for some $\Delta \in \mathscr{A}$. The completeness of $T1$ follows from two lemmas:

LEMMA 1. *For every consistent set Γ of sentences in L_ω, there is a (denumerable) extension of L_ω relative to which there is a perfect$_1$ set $\mathscr{A}$ one of whose members Δ includes Γ.*

LEMMA 2. *For every extension of L_ω relative to which $\mathscr{A}$ is a perfect$_1$ set, every set Δ in $\mathscr{A}$ has a type* 1 *model (whose cardinality is that of Δ).*

To prove Lemma 1, we first form an extension L^*_ω of L_ω that has denumerably many primitive names and denumerably many new i-ary primitive predicates for each $i \geq 0$. The sentences of L^*_ω are then arranged into a sequence of consecutive sentences $A_1, A_2, A_3, \ldots$ having the following property: $A_1 = A_2$ and for every closed term $[B]_{v_1 \cdots v_p}$ in L^*_ω, there is at least one j such that A_j is the sentence $[B]_{v_1 \cdots v_p} = [F^p_q(v_1, \ldots, v_p)]_{v_1 \cdots v_p}$ where F^p_q is a primitive predicate letter that does not occur in B, Γ, or any A_h, $h < j$. Relative to this sequence, we use certain rules to construct an array of sets of L^*_ω-sentences:

$$
\begin{array}{cccccc}
\Delta_1 & \Delta_3 & \Delta_7 & \cdots & \Delta_{n^2+n+1} & \cdots \\
\Delta_2 & \Delta_4 & \Delta_8 & & \Delta_{n^2+n+2} & \\
\Delta_5 & \Delta_6 & \Delta_9 & & \Delta_{n^2+n+3} & \\
\vdots & & & \ddots & \vdots & \\
 & & & \Delta_{n^2} & \Delta_{n^2+2n} & \\
\Delta_{n^2+1} & \Delta_{n^2+2} & \Delta_{n^2+3} \cdots & \Delta_{n^2+n} & \Delta_{(n+1)^2} & \cdots \\
\cdots . & & & & &
\end{array}
$$

The rules are these. (1) $\Delta_1 = \Gamma$. (2) If A_n, $n \geq 1$, is $[A]_\alpha \neq [B]_\alpha$ and $A_n \in \Delta_{n^2}$, then $\Delta_{n^2+1} = \{(\exists\alpha)\neg(A \equiv B)\}$; otherwise, $\Delta_{n^2+1} = \Delta_{n^2}$. (3) Let Δ_m, $m > 1$, be in column $i > 1$ and row $k \geq 1$. Then if $m^+ \cup m^* \cup \{A_i\}$ is consistent, $\Delta_m = m^+ \cup m' \cup \{A_i\}$; otherwise, $\Delta_m = m^+ \cup m'$. The sets m^+, m^*, and m' are:

$m^+ =_{\text{df}}$ the set in row k and column $i-1$,

$m^* =_{\text{df}} \{[B]_\alpha = [C]_\beta : (\exists n < m)(\Delta_n \vdash_{T1} [B]_\alpha = [C]_\beta)\}$,

$m' =_{\text{df}} \{C_1(a_1), \ldots, C_s(a_s)\}$,

where the sentences $C_1(a_1), \ldots, C_s(a_s)$ are determined as follows: in the order in which they first occur in the sequence $A_1, A_2, \ldots, A_i, \ldots$, the sentences $(\exists v_1)C_1(v_1), \ldots, (\exists v_s)C_s(v_s)$ exhaust the existential sentences in m^+ that occur before A_i, and $C_1(a_1), \ldots, C_s(a_s)$ are the first substitution instances of $(\exists v_1)C_1(v_1), \ldots, (\exists v_s)\, C_s(v_s)$ occurring after A_i such that, for each r, $1 \leq r \leq s$, $C_r(a_r)$ contains the first occurrence of the primitive name a_r anywhere in the sequence $A_1, A_2, \ldots, A_i, \ldots$. Now the set Δ^j is defined to be the union of all sets in row j, $j \geq 1$. And the set $\mathscr{A}$ is defined to be the set of all sets Δ^j, $j \geq 1$.

Claim. $\mathscr{A}$ is perfect$_1$.

This claim, which entails Lemma 1, is easily proved once we have the following sublemma: for all $m \geq 1$, $\Delta_m \cup m^*$ is consistent. This sublemma, however, has a straightforward, though complex, proof by induction on m.

Lemma 2 is a proved as follows. Let L^*_ω be any extension of L_ω relative to which $\mathscr{A}$ is a perfect$_1$ set. For each $\Delta \in \mathscr{A}$ we construct a separate type 1 model $\langle \mathscr{M}_\Delta, \mathscr{I}_\Delta \rangle$ for Δ. Choose some well-ordering $<$ of the union of the class of individual constants and the class of primitive predicate letters in L^*_ω, where $=$ is the least primitive predicate letter in this well-ordering. The domain $\mathscr{D}_\Delta$ is then identified with the following union:

$$\{F^j_i \in L^*_\omega\text{: there is no } F^k_h \in L^*_\omega \text{ such that } F^k_h < F^j_i \text{ and the sentence } [F^k_h(v_1, \ldots, v_k)]_{v_1 \cdots v_k} = [F^j_i(u_1, \ldots, u_j)]_{u_1 \cdots u_j} \in \Delta\}$$
$$\cup \{a_j \in L^*_\omega\text{: there is no } F^k_h \in L^*_\omega \text{ such that the sentence } [F^k_h(v_1, \ldots, v_k)]_{v_1 \cdots v_k} = a_j \in \Delta, \text{ and there is no } a_i \in L^*_\omega \text{ such that } a_i < a_j \text{ such that the sentence } a_i = a_j \in \Delta\}.$$

The subdomain $\mathscr{D}_{-1}$ is the set of primitive names in $\mathscr{D}_\Delta$, and the subdomain $\mathscr{D}_i$, $i \geq 0$, is the set of primitive i-ary predicates in $\mathscr{D}_\Delta$. The prelinear ordering $\mathscr{P}$ is defined as follows: $\mathscr{P}(x, y)$ *iff*$_{\text{df}}$ for some i and j, $i < j$, $x \in \mathscr{D}_i$ and $y \in \mathscr{D}_j$. The set $\mathscr{K}$ of alternate extension functions $H_{\Delta'}$ is determined by the atomic sentences belonging to the various sets Δ' belonging to $\mathscr{A}$. The actual extension function $\mathscr{G} =_{\text{df}} H_\Delta$. The identity element Id $\in \mathscr{M}_\Delta$ is just the identity predicate $=$. And the transformations in $\mathscr{T}_\Delta$ and the logical operations Conj$_\Delta$, Neg$_\Delta$, ... are determined by the identity sentences in Δ. For example, Conj $(F^q_m, F^q_n) = F^q_p$ *iff*$_{\text{df}}$ $F^q_m, F^q_n, F^q_p \in \mathscr{D}_\Delta$ and, for some F^i_h, F^j_k, the following three identity sentences are in Δ:

$$[F^q_m(u_1, \ldots, u_i, v_1, \ldots, v_j)]_{u_1 \cdots u_i v_1 \cdots v_j} = [F^i_h(u_1, \ldots, u_i)]_{u_1 \cdots u_i v_1 \cdots v_j},$$
$$[F^q_n(u_1, \ldots, u_i, v_1, \ldots, v_j)]_{u_1 \cdots u_i v_1 \cdots v_j} = [F^j_k(v_1, \ldots, v_j)]_{u_1 \cdots u_i v_1 \cdots v_j},$$
$$[F^q_p(u_1, \ldots, u_i, v_1, \ldots, v_j)]_{u_1 \cdots u_i v_1 \cdots v_j} = [F^i_h(u_1, \ldots, u_i) \,\&\, F^j_k(v_1, \ldots, v_j)]_{u_1 \cdots u_i v_1 \cdots v_j}.$$

Finally, the interpretation $\mathscr{I}_\Delta$ may be defined as follows:
$\mathscr{I}_\Delta(a_i) =_{\text{df}}$ the individual constant $a_j \in \mathscr{D}_\Delta$ such that $a_i = a_j \in \Delta$,
$\mathscr{I}_\Delta(F^j_i) =_{\text{df}}$ the primitive predicate $F^j_k \in \mathscr{D}_\Delta$ such that

$$[F^j_i(v_1, \ldots, v_j)]_{v_1 \cdots v_j} = [F^j_k(v_1, \ldots, v_j)]_{v_1 \cdots v_j} \in \Delta.$$

With $\mathscr{M}_\Delta$ and $\mathscr{I}_\Delta$ so specified, it is then shown by induction on the complexity of formulas that, for all $\Delta \in \mathscr{A}$, $\langle \mathscr{M}_\Delta, \mathscr{I}_\Delta \rangle$ is a model of Δ.

§4. The logic for PRPs on conception 2. On conception 2 each definable intensional entity is such that, when it is defined completely, it has a unique, noncircular definition. The logic $T2$ for L_ω on conception 2 consists of (a) axioms A1-A7 and rules R1–R2 from $T1$, (b) five additional axiom schemas for intensional abstracts, and (c) one additional rule. In stating the additional principles, I will write $t(F^q_p)$ to indicate that t is a complex term of L_ω in which the primitive predicate F^q_p occurs.

Additional axiom schemas and rules for T2.

$\mathscr{A}8$. $[A]_\alpha = [B]_\alpha \supset (A \equiv B)$.

$\mathscr{A}9$. $t \neq r$ (where t and r are nonelementary complex terms of different syntactic kinds[9]).

$\mathscr{A}10$. $t = r \equiv t' = r'$ (where $R(t', t)$ and $R(r', r)$ for some term-transforming relation R, *or* t is the negation of t' and r is the negation of r', *or* t is the existential generalization of t' and r is the existential generalization of r').

$\mathscr{A}11$. $t = r \equiv (t' = r' \;\&\; t'' = r'')$ (where t is the conjunction of t' and t'' and r is the conjunction of r' and r'' *or* t is the predication$_k$ of t' of t'' and r is the predication$_k$ of r' of r'' for some $k \geq 0$).

$\mathscr{A}12$. $t(F^j_i) = r(F^k_h) \supset q(F^j_i) \neq s(F^k_h)$ (where t and s are elementary and r and q are not).

$\mathscr{R}3$. Let F^p_k be a nonlogical predicate that does not occur in $A(v_i)$; let $t(F^p_k)$ be an elementary complex term, and let t' be any complex term of degree p that is free for v_i in $A(v_i)$. If $\vdash A(t)$, then $\vdash A(t')$.[10]

THEOREM (SOUNDNESS AND COMPLETENESS). *For all formulas A in L_ω, A is valid$_2$ if and only if A is a theorem of T2 (i.e., $\models_2 A$ iff $\vdash_{T2} A$).*

PROOF. The proof of the soundness of $T2$ is quite straightforward. For example, the soundness of $\mathscr{A}8$ follows directly from Lemma 6 (stated ealier); $\mathscr{A}9$, from the fact that $\mathscr{T}$-transformations and the logical functions Conj, Neg, Exist, Pred$_0$, ... in a type 2 model structure all have disjoint ranges; $\mathscr{A}10$ and $\mathscr{A}11$, from the fact that all these functions are 1-1; $\mathscr{A}12$, from the fact that they are noncycling.

The soundness proofs for R1 and R2 are standard.

For the soundness of $\mathscr{R}3$, the induction hypothesis yields $\models_2 A(t(F^p_k))$. Hence, by the soundness of R2, A2, and A5 (Leibniz's law), we have $\models_2 t(F^p_k) = t' \supset A(t')$. But since F^p_k is a nonlogical predicate and does not occur in $A(t')$, $\models_2 A(t')$. The completeness proof is again Henkin style. A set of L^*_ω-sentences is said to be *perfect*$_2$ if (1) it is maximal, consistent, ω-complete and (2) for every closed term $[B]_{v_1 \cdots v_p}$ in L^*_ω, there is a primitive predicate letter F^p_k such that the sentence $[B]_{v_1 \cdots v_p} = [F^p_k(v_1, \ldots, v_p)]_{v_1 \cdots v_p} \in \Delta$. We show, first, that every consistent set of L_ω-sentences is included in some perfect$_2$ set of L^*_ω-sentences and, secondly, that every perfect$_2$ set has a type 2 model. The argument, while parallel to the argument used for $T1$, is much simpler.

§5. **The logic for PRPs and necessary equivalence on conception 2.** Let the 2-place logical predicate $\approx_N$ be adjoined to L_ω. $\approx_N$ is intended to express the logical relation of necessary equivalence.

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[9] That is, t and r are not in the range of the same term-transforming relation, nor are they in the range of the same syntactic operation—conjunction, negation, existential generalization, predication$_0$, ...

[10] $\mathscr{A}8$ affirms the equivalence of identical intensional entities. Schemas $\mathscr{A}9$–$\mathscr{A}11$ capture the principle that a complete definition of an intensional entity is unique. And schema $\mathscr{A}12$ captures the principle that a definition of an intensional entity must be noncircular. (The following instances of $\mathscr{A}12$ should help to explain what it says: $[Fxy]_{xy} = [Gxy]_{yx} \supset [Fxy]_{yx} \neq [Gxy]_{xy}$ and $[Fx]_x = [\neg Gx]_x \supset [\neg Fx]_x \neq [Gx]_x$.) $\mathscr{R}3$ says roughly that if $A(t)$ is valid$_2$ for an arbitrary elementary p-ary term t, then $A(t')$ is valid$_2$ for any p-ary term t.

A *type* 2′ model structure is defined to be just like a type 2 model structure except that it contains an additional constituent $\mathrm{Eq_N}$ which is a distinguished element of $\mathscr{D}_2$ satisfying the following condition:

$$(\forall H \in \mathscr{K})(H(\mathrm{Eq_N}) = \{xy: (\exists i \geq -1)(x, y \in \mathscr{D}_i) \,\&\, (\forall H' \in \mathscr{K})(H'(x) = H'(y))\}).$$

Thus, $\mathrm{Eq_N}$ is to be thought of as the distinguished logical relation-in-intension *necessary equivalence*. Now an interpretation $\mathscr{I}$ relative to a type 2′ model structure is just like an interpretation relative to a type 1 or type 2 model structure except that we require $\mathscr{I}(\approx_{\mathrm N}) = \mathrm{Eq_N}$. Then type 2′ denotation, truth, and validity are defined *mutatis mutandis* as before. The following abbreviations are introduced for notational convenience:

$$\Box A \text{ iff}_{\mathrm{df}} [A] \approx_{\mathrm N} [[A] \approx_{\mathrm N} [A]]$$
$$\Diamond A \text{ iff}_{\mathrm{df}} \neg\Box\neg A.$$

The intensional logic $T2'$ consists of the axioms and rules for $T2$ plus the following additional axioms and rules for $\approx_{\mathrm N}$:

$\mathscr{A}13$. $x \approx_{\mathrm N} x$.
$\mathscr{A}14$. $x \approx_{\mathrm N} y \supset y \approx_{\mathrm N} x$.
$\mathscr{A}15$. $x \approx_{\mathrm N} y \supset (y \approx_{\mathrm N} z \supset x \approx_{\mathrm N} z)$.
$\mathscr{A}16$. $x \approx_{\mathrm N} y \supset \Box x \approx_{\mathrm N} y$.
$\mathscr{A}17$. $\Box(A \equiv_\alpha B) \equiv [A]_\alpha \approx_{\mathrm N} [B]_\alpha$.
$\mathscr{A}18$. $\Box A \supset A$.
$\mathscr{A}19$. $\Box(A \supset B) \supset (\Box A \supset \Box B)$.
$\mathscr{A}20$. $\Box A \supset \Box\Diamond A$.
$R4$. If $\vdash A$, then $\vdash \Box A$.

Notice that these axioms and rules for $\approx_{\mathrm N}$ are just analogues of the special $T1$ axioms and rules for $=$. Finally, the soundness and completeness of $T2'$ can be shown by applying the methods of proof used for $T1$ and $T2$.

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