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Authors' Response

Numbers, numerosities, and new directions

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Abstract

In our target article, we argued that the number sense represents natural and rational numbers. Here, we respond to the 26 commentaries we received, highlighting new directions for empirical and theoretical research. We discuss two background assumptions, arguments against the number sense, whether the approximate number system (ANS) represents numbers or numerosities, and why the ANS represents rational (but not irrational) numbers.

We are humbled to have received 26 commentaries from 62 researchers, among them many of our academic heroes. Unsurprisingly, these commentaries reflect a diversity of opinion. Some endorse and build upon the main conclusions of our paper; others highlight points of disagreement. Although we remain confident in our main theses, we learned a great deal from our commentators – about soft spots in our arguments, points that require development, where we could have been clearer, and avenues for future research. We're extremely grateful for their insights.

Our replies follow the order of our target article. We discuss two background assumptions, arguments against the number sense, whether the approximate number system (ANS) represents numbers or numerosities, and why the ANS represents rational (but not irrational) numbers.

R1. Background assumptions

Explanation needs to start somewhere, and our discussion presupposed that the ANS is representational and that it sometimes operates in perception, enabling numbers to enter perceptual contents. Some commentators challenged these background assumptions.

R1.1. Is the ANS representational?

While the idea that the ANS *represents* anything at all is relatively uncontroversial among ANS researchers (but see Beck [2015] for a defense), **Jones, Zahidi, and Hutto (Jones et al.)** suggest that our commitment to representations imports unnecessary "philosophical baggage." They recommend instead embracing an anti-representational *Radical Enactivism*.

In general, we're dubious when people tell us we can avoid philosophical baggage by embracing views with "radical" in the name. **Jones et al.**'s "radical" vision is that we acquire a perceptual sensitivity to numbers simply by virtue of our sensitivity to the affordances they enable: "The 'sevenness' is not a property of the apples, nor of the perceiver, but of what the perceiver can do with them." The trouble is: It's essentially open ended what you can do when you perceive there to be seven of something. So we don't see how the perception of number can be specified in these terms. Furthermore, representation is fundamental to explanations of the ANS's internal computations. For instance, when children use their ANS to add the number of blue dots and red dots in a sequence of events (e.g., Barth et al., 2005), it's not just that they're afforded with (say) the sevenness of the blue dots, the tenness of the red dots, and then magically afforded with the seventeenness of the red and blue dots; they engage in a computational transition, in which internal states of the organism interact in content respecting ways. This presupposes representation.

R1.2. Are numbers perceivable?

Aulet and Lourenco, **Marshall**, **Novaes and dos Santos**, and **Opfer, Samuels, Shapiro, and Snyder** (**Opfer et al.**) all questioned our assumption that numbers are perceivable.

Because numbers are higher-order, **Novaes and dos Santos** and **Marshall** think they cannot be perceived. Marshall proclaims that "second-order entities are no part of the sensible realm," while Novaes and dos Santos write that numbers "emerge only after an agent has adopted a given sortal" and thus are not "out there, inexact or otherwise, to be represented." But we find this puzzling. Surely, it's an objective fact that the apples on the kitchen counter total five in number. This fact is "out there" and does not require anyone's mind to "emerge." It's also the sort of thing one should expect perceptual systems to be capable of picking up on. We're not sure why anyone would think otherwise unless they were committed to an outdated view of perception according to which perception only represents properties for which we have dedicated sensory transducers. But perception is not sensation. At least since Helmholtz, we've known that perception is a constructive and ampliative process whose outputs go beyond its inputs. The visual system makes assumptions that are non-demonstrative but generally ecologically valid. Some of these assumptions concern default sortals that individuate the incoming sensory array into objects and are used to enumerate concrete pluralities. There is no mystery concerning how something higher-order could be perceivable.

Aulet and Lourenco object that our contention that we perceive numbers "conflates the percept with the concept." But studies show adaptation to number in external coordinates (Burr & Ross, 2008; DeSimone et al., 2020), including cross-modal adaptation (Arrighi et al., 2014), which naturally controls for most non-numerical confounds. Because adaptation is a mark of the perceptual (Block, 2014), these studies suggest that we really do *perceive* numbers.

Opfer et al.'s rich commentary questions whether numbers are perceivable on different grounds. Because we assume numbers enter perception through property attribution, they accuse us of assuming the *Identity Thesis*, according to which, "Natural numbers are identical to cardinality properties." They object that the Identity Thesis is linguistically problematic because it's felicitous to say things like (1) but not (2).

(1) The number of women at the party is four.
(2) ?? The number of women at the party is the number four.

We're wary of drawing strong metaphysical conclusions from facts about how people talk (does the language faculty have a hotline to the Forms?) but happily concede that the Identity Thesis might be false. Plausibly, being eight in number is a property, while the number eight is an (abstract) particular. But, contrary to what **Opfer et al.** suggest, we don't think we're committed to denying this or affirming the Identity Thesis. We simply maintain that perception attributes properties like being eight in number, and that the attributed properties are complex, and include reference to natural numbers. For example, the attribution *is eight in number* includes reference to the number eight even though it is not identical to the number eight. (Compare: The property of being as rich as Jeff Bezos is not identical to Jeff Bezos; but when you wish you were as rich as Jeff Bezos you refer to Jeff Bezos.) When we said that "the ANS refers to numbers… by enabling numbers to enter into contents via property attribution," this is what we meant.

Opfer et al. raise a further objection that may now seem pressing: if (as we suggested in our target article) there's a puzzle about how numbers *qua* abstracta could be referred to in perception, and we maintain that attributing cardinality properties involves referring to numbers, how do we avoid our original puzzle? The answer lies in the fact that the puzzle was not supposed to be that it's mysterious how perception could refer to abstract entities. Rather, our worry was that it's mysterious how perception could veridically refer to abstract entities on their own, without simultaneously referring to something concrete. That's why we said, you can't "perceive the number seven itself – on its own." To veridically refer to the number seven in perception, you need to simultaneously perceive a concrete plurality. When you perceive the apples as being seven in number, you refer to a concrete plurality of apples and attribute a property to it, with reference to the number seven occurring within that attribution. (Admittedly, we could have been clearer on this point.)

Now, even though *our* worry didn't concern how perception could refer to abstract objects full stop, this is a worry others might have. But note two things. First, the worry is not unique to perception. Others have worried about how we can *think* about numbers given that they are abstract objects (Benacerraf, 1973). There is, thus, a version of this puzzle that arises for everyone.

Second, with respect to the specific puzzle of *perceiving* numbers, we don't see it as fundamentally different from the puzzle of how one can be perceptually related to other abstracta, such as shapes and colors. Even **Opfer et al.**'s linguistic point doesn't distinguish between these.

(3) The {color/shape} of the ball is {orange/a sphere}.
(4) ?? The {color/shape} of the ball is the {color orange/shape a sphere}.

Admittedly, colors and shapes are often taken to be universals rather than abstract particulars, and if that's right, the two puzzles are not identical. To say exactly *how* they differ, however, would require staking out various controversial positions in metaphysics and the philosophy of mathematics. In an article addressed to an interdisciplinary audience, we tried to bracket such issues. But the two perceptual relations (to shapes/colors and numbers) strike us as sufficiently similar that we've been able to sleep at night. Still, this issue deserves further attention and we're grateful to **Opfer et al.** for highlighting it.

R2. Arguments against the orthodox view

In our target article, we cleared space for the orthodox view that the ANS represents numbers by noting a lack of compelling arguments to the contrary. Thus, our target article replied to three arguments which have been pressed against this orthodox view – the arguments from *congruency*, *confounds*, and *imprecision*.

Notably, few commentators came out in support of these arguments. Indeed, **Marinova, Fedele and Reynvoet (Marinova et al.)** suggest that we "somewhat misinterpreted" the "key message" of the congruency and numerical interference studies they have been involved with – studies which formed the backbone of the arguments from congruency and confounds. Their commentary is helpful because it serves to distinguish two closely related views these studies could be seen to support. A radical interpretation uses them to argue that the ANS fails to exist or represent numbers at all. This radical interpretation seems to be in play when, for instance, Gebuis et al. (2016) claim that congruency studies of this sort indicate that "the output" of the ANS "is not an abstract number" (p. 28). By contrast, a modest interpretation simply takes these results to support an indirect model of ANS processing, on which numerical quantity is derived from continuous percepts. Our aim was to tease these possibilities apart and rebut the radical interpretation. It's good to learn that researchers behind some of these studies also want to distance themselves from the radical interpretation.

Marinova et al. also helpfully observe – and we agree – that it may not be necessary to choose between direct and indirect models of the ANS. Rather, there could be multiple ways perception extracts number, some direct and some indirect.

Aulet and Lourenco were more dubious. They note (correctly) that on our view "whether elephants, mice, or apples" are being counted, the ANS can attribute a numerical value to these "irrespective of their physical differences." Against this, they claim that "number is not (perceptually) independent of other magnitudes," citing evidence that number and area are perceived as integral dimensions. Two dimensions are *integral* when they cannot

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be represented independently and *separable* otherwise. But representing A and B as integral dimensions doesn't preclude representing A. Loudness and pitch are integral dimensions, and each is perceptually represented. Two integral dimensions may even be represented by distinct vehicles, with their integrality (often measured by interference effects) deriving from causal or structural relations among vehicles (Lande, 2020).

Aulet and Lourenco criticize one reason we provided for thinking that number is represented by the ANS. In our target article, we emphasized work by Cicchini et al. (2016) and others, suggesting that when subjects estimate the area, density, and number of dots in a visual array, they are more sensitive to number than area or density, and thus do not simply represent area and density, but also number. Against this, Aulet and Lourenco cite evidence that when perceived area is distinguished from true area, we are less sensitive to number. **Barth and Shusterman** bolster the objection, citing a wider range of studies in support of this claim. Thus, both commentaries suggest that when perceived area is distinguished from mathematical area, it's not true that we're more sensitive to number than area or other non-numerical confounds.

These are fascinating issues. But it's vital to keep three things in mind. First, the correct interpretation of these studies remains hotly contested, so it's probably too early to draw strong conclusions. For instance, Park (in press) objects that studies which control for perceived area tend to introduce *massive* incongruencies between number and non-numerical magnitudes – so, given that incongruencies of this sort suppress numerical sensitivity (DeWind et al., 2015), counterevidence of this sort probably underestimates true numerical sensitivity. Second, our target article did acknowledge some of the counterevidence these authors cite. For instance, we discussed Yousif and Keil's study, suggesting that subjects are more sensitive to "additive area" than number. But, as we noted then, Yousif and Keil are clear that their results cannot be fully explained by non-numerical confounds and still require that numbers are represented. But third, even if none of this were so, the idea that number is uniquely salient was just one reason we gave for rejecting *the arguments from congruency and confounds* (the arguments in which these studies featured). The arguments also fail for independent reasons. For example, the argument from congruency overgeneralizes in absurd ways; and the argument from confounds relies on an *ad hoc* strategy of explaining away success in number tasks that struggles to explain key findings (e.g., cross-modal comparisons and dumbbell effects).

R3. Number versus numerosity

We next proposed that the ANS represents numbers rather than numerosities or other exotic entities. To this end, we observed that the ANS tracks the cardinal number of entities in concrete pluralities (albeit imprecisely), supports arithmetic computations, and exhibits a higher-order sensitivity that's characteristic of number representation. We also argued that the thesis that the ANS represents number admits of no plausible alternatives, promotes integration with other sciences, and avoids a curious double standard with respect to the treatment of non-numerical quantities. Our commentators pushed back on many of these claims.

R3.1. Higher-order sensitivity

One reason to think the ANS represents number is that the ANS is sensitive to the higher-order character of number. Numerical quantities are assigned relative to a sortal, and this distinguishes them from other kinds of quantity. To illustrate, note that the group entering the restaurant is one party of diners, four couples, and eight people, while the group's weight remains constant irrespective of which sortal we apply. That the ANS is sensitive to this higher-order feature of number is especially clear from the dumbbell studies reviewed in our target article (Franconeri et al., 2009; He et al., 2009). Judgments of the number of items are influenced by whether they are connected to one another (even though subjects are told to ignore the connecting lines), suggesting that the system takes a stand on how items are individuated.

Marshall complains that we "hang *an awful lot*" on these dumbbell studies in making this point, but he doesn't question our logic or criticize the studies themselves. By contrast, **Buijsman** objects that the dumbbell studies only contain "a relatively small number of connected dots/squares" and that, as such, performance might result from the object-tracking (or subitizing) system rather than the ANS. But this worry appears to be based on a misinterpretation of the original studies. Buijsman writes, "the fourth experiment of Franconeri et al. (2009) has four circles, and in the connected format these form two dumbbell shapes." While that accurately describes the *figure* accompanying Franconeri et al.'s fourth experiment, the text clarifies that the actual stimuli consisted of 12, 24, or 48 circles, of which 0, 25, 50, 75, or 100% were connected.

This is not a one-off finding. Fornaciai et al. (2016) report that numerical adaptation effects are influenced by whether the post-adaptation stimuli consist of 20 unconnected dots or 10 pairs of connected dots. Fornaciai and Park (2018) confirmed that displays of 16 or 32 dots were underestimated when they were connected (compared to displays containing unconnected dots). In fact, the stimuli needn't really be connected. Kirjakovski and Matsumoto (2016) found that pacman-like stimuli that only *appeared* to be connected via Kanisza-like illusory contours also caused subjects to underestimate their total.

Aulet and Lourenco object that the dumbbell studies do not reveal that the ANS has a higher-order character because "if number perception was genuinely second-order, then it should be just as easy to continue perceiving the number of dots, instead of being biased towards the number of dumbbells." But this worry conflates two things: whether the ANS is higher-order, and whether the sortals it uses are under voluntary control. Crucially, the ANS could be higher-order even if the sortals it uses aren't under voluntary control.

Consider that the visual system is biased toward individuating the world into what are sometimes called *Spelke objects* – bounded, coherent, three-dimensional, continuous wholes (Carey, 2009; Spelke, 1990; but see Green, 2018.) Consequently, when the ANS takes inputs from the visual system, it enumerates Spelke objects by default. That is what the dumbbell studies show because connecting two items turns them into a single Spelke object. These studies evince a higher-order character to the referents of ANS representations because they show that the ANS is applying a sortal – the sortal *Spelke object*. This default can be overridden to some extent (subjects do not treat 10 pairs of connected circles as numerically identical to 10 unconnected circles), but not completely. Moreover, it is only the default in certain circumstances. The ANS also spontaneously enumerates events such as rabbit jumps, heard tones, and, as **Burr, Anobile, Castaldi, and Arrighi** demonstrate, self-generated actions such as hand taps. Thus, the sortal used by the ANS *is* capable of varying, even if (like most of the mind) it isn't under full voluntary control.

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R3.2. Numerosity

In our target article, we objected to the idea that the ANS represents *numerosities* rather than *numbers* themselves. For one, we objected that, while the term "numerosity" is widely used, no one seems to know what a numerosity is. This prompted many commentators to tell us just *what* a numerosity is, although their disagreements are notable, suggesting that insofar as researchers associate a distinctive meaning with "numerosity," it's not universally shared.

In their commentaries, **Núñez, d'Errico, Gray, and Bender (Núñez et al.)** and **Novaes and dos Santos** trace the term "numerosity" to S. S. Stevens. As Núñez et al. report, Stevens (1939/2006, p. 23) defined numerosity as "a property defined by certain operations performed upon groups of objects." We'll confess, this doesn't exactly clear things up for us.

Some commentators had more to say. **Novaes and dos Santos**, along with **Bermúdez** and **Opfer et al.**, suggest that numerosities are *cardinalities*. By contrast, **Núñez et al.** double down on the idea that the ANS is *quantical* rather than numerical, while **Buijsman** and **Gross, Kowalsky, and Burge (Gross et al.)** defend the view that numerosities are *pure magnitudes*. We discuss each of these proposals in the following sub-sections. First, however, we want to reply to two further commentaries that defend the concept of numerosity without neatly fitting into these three proposals.

Barth and Shusterman wonder whether researchers share an "understanding of what 'numerosity' means." Given the variety of proposals made by our commentators, we think it's clear they do not. The common term masks a diversity of concepts. Still, Barth and Shusterman think that the term should be retained. According to them, "number" is ambiguous because it can refer to a number word, a numeral, a mathematical entity, or a property of a stimulus. They think it is useful to have a term that refers just to the last of these, and that "numerosity" is up to the task.

Our view is that words and numerals are clearly not numbers – no more than the word "square" has four equal sides or the dinner bell is fit to eat. Therefore, we don't think anyone should be concerned about confusion on that front. (If one is concerned, using "number word" and "numeral" for number words and numerals, respectively, should guard against mix-ups.) We do think there's a difference between mathematical entities and properties of stimuli, but that's not a distinction that's unique to numbers. When a mathematician says, "A square is a plane figure with four equal sides and right angles," she's talking about a mathematical entity, not a property of a stimulus. But the tiles on Rachael Ray's kitchen floor can have the property of being square just as surely as they can have the property of being 30 in number. Would **Barth and Shusterman** also want to introduce the term "squareosity" to capture the shape property that these stimuli can have? If not, why introduce "numerosity" to capture their numerical property? Barth and Shusterman don't say.

Gallistel also defends the term "numerosity," arguing that "coherent discussion" requires a three-way distinction between *numerons*, *numbers*, and *numerosities*. A numeron is "a symbol in a computing machine like the brain." This strikes us as a helpful concept. Just as we use *numerals* (e.g., in Arabic notation) in language, the brain uses numerons in its internal code. Numerons are thus vehicles of representation – symbols in the language of thought. But Gallistel also says that *numbers* are symbols. This leaves us confused. If numerons and numbers are both symbols, aren't they the same thing? And wouldn't numbers then be vehicles of representation too? This seems like a mistake, analogous to confusing a rose with the word "rose." Symbols refer to numbers, but they aren't identical to numbers. After all, different symbols can refer to the same number (e.g., "4," "four," and "IV"), and the same symbol can refer to different numbers in different notations (e.g., "100" refers to one hundred in decimal notation and to four in binary).

Finally, **Gallistel** claims that a *numerosity* is "the number you get when you correctly count" a collection. But, if a numerosity is just a number, Gallistel has one more distinction than he needs.

Gallistel provides one further reason to think that we need "numerosity" in addition to "number." Just as psychophysicists use "brightness" for the percept and "luminance" for the distal stimulus, they need "number" for the percept and "numerosity" for the distal stimulus. But, while some objective magnitudes have an associated term that naturally applies to the percept (e.g., luminance/brightness, sound wave amplitude/loudness, and sucrose concentration/sweetness), others do not (e.g., distance/?, duration/?, and area/?). And yet psychophysicists seem to get along just fine measuring these percepts. As such, we should ask what feature of the percept "numerosity" is supposed to capture. Is it the vehicle? Gallistel already gave us "numeron" for that. Perhaps, instead, it's the phenomenal character of the percept (as it might be with "brightness")? But, beyond the oddity of using "number" to refer to a phenomenal property, it's far from clear that there's a phenomenal character that's common to how number is represented in vision, audition, action, and so on. The distinction between number and numerosity serves no apparent purpose.

R3.3. Cardinalities

Novaes and dos Santos write that "in the contemporary literature one finds 'numerosity' defined as a synonym for cardinality." In defense of this claim, they cite Nieder (2016, p. 366), who writes, "Cardinality (also known as numerosity) corresponds to the empirical property of quantity, and is the number of countable elements in a given group (for example, five runners)."

On one interpretation of this passage, cardinalities are just a specific type of number: cardinal numbers. And to represent a cardinality is to represent a cardinal number. This interpretation is obviously consistent with the hypothesis that the ANS represents numbers.

On a second interpretation, cardinalities are properties of concrete pluralities rather than numbers themselves. This would put **Novaes and dos Santos** in line with **Barth and Shusterman** and **Opfer et al.** Here, a distinction is drawn between the number five (a mathematical entity) and being five in number (a property of a concrete pluralities). To represent the cardinality of the runners is to represent the runners as being five in number – that is, as having a particular property. As we noted in section R1.2, we agree that the ANS attributes cardinality properties in this sense; but we maintain that in so doing it refers to numbers. Therefore, this proposal is also compatible with our hypothesis.

Is there some other way to use cardinality as an alternative to number? The notion of a cardinality derives from set theory, and **Novaes and dos Santos** suggest appealing to the set-theoretic notion of one-to-one correspondence, such that two sets have the same cardinality if and only if their members can be put in one-to-one correspondence. **Bermúdez** develops this suggestion, showing how it predicts that the ANS can represent *comparative* properties, but not *absolute* properties. That's because a computation of one-to-one correspondence can tell you whether two collections are equinumerous or not, but not how many elements are in either one. Bermúdez argues this proposal is consistent with much of the data associated with the ANS, including the many studies that require subjects to determine which of two presented pluralities is greater, and Weber's Law itself.

One might worry that this proposal falsely predicts that numerical comparisons will be precise, because the operation of one-to-one correspondence is precise. (It grounds some definitions of the integers.) Carey and Barner (2019) reject the proposal on exactly these grounds, writing that "the ANS lacks a mechanism like one-to-one correspondence that can establish the exact equality of sets" (pp. 826–827). But we see no reason that noise couldn't corrupt a computation that places representations in one-to-one correspondence, thereby giving rise to the imprecision associated with the ANS.

There are, however, two difficulties with the proposal. First, while numerous studies show that ANS representations can be stored in working memory, working memory for seen objects degrades quickly after three or four objects (Alvarez & Cavanagh, 2004; Vogel et al., 2001). A large collection of objects is not represented in memory in full detail, but as an "ensemble" using summary statistics (Alvarez, 2011). For example, the mean area of a collection of dots might be recorded in memory, but not the individual areas (Ariely, 2001). And similarly for orientation, brightness, location, and other properties. But, if the collection of individual objects in a display aren't stored in memory, then the comparative cardinality view cannot explain how subjects perform numerical comparisons once that collection is no longer perceivable. Consider the studies by Barth et al. (2005) that **Bermúdez** cites. In the very first experiment, preschoolers see some dots on a screen, then see those dots being covered, and then see some new uncovered dots. They then have to say whether the covered dots are more or less numerous than the uncovered dots. To do this, they must maintain in memory either a representation of the covered dots themselves or a summary representation of the covered dots' number. If they maintained a representation of the covered dots themselves, then they could put those dots in one-to-one correspondence with the still-visible dots to determine their comparative cardinality. But the displays contained up to 58 dots, well above the limits of visual working memory. Therefore, memory must instead store a summary representation of their total number.

Second, when **Bermúdez** writes, "Clarke and Beck readily concede that there is no evidence that the ANS is sensitive to the successor function or to basic arithmetical operations," he's only half right. We did concede that the ANS isn't sensitive to the successor function. But we noted "that ANS representations enter into arithmetic computations such as greater-than and less-than comparisons, addition, subtraction, multiplication, and division." This matters because most arithmetic computations require more than one-to-one correspondence. While other set-theoretic operations might be appealed to (e.g., addition might be explained in terms of the union operation), this approach gets trickier when we consider that ANS representations are believed to enter into arithmetic computations with other magnitudes. For example, there is evidence that the mind takes representations of number and divides by its representations of duration to yield representations of rate (Gallistel, 1990). We find it hard to envision how comparative cardinalities can explain such computations. (**Núñez et al.** claim it's a "biological no-go" to suppose that the nervous system implements arithmetic operations such as division. But they don't explain why; nor do they provide an alternative explanation of the many studies we cited that are indicative of such operations.)

R3.4. The quantical

Núñez et al. accuse us of "biological misconceptions," "mathematical naïveté," "serious inconsistencies," having "only [one] novel claim," "erroneous" characterizations, "misrepresent[ing]" distinctions, "an unnecessary condescending tone," and torturing puppies for fun. (We're reading between the lines on that last one.)

In an earlier article, Núñez argued that the capacities associated with the ANS "are not about numbers, but are about quantity, and therefore should not qualify as numerical… I propose to refer to these biologically endowed capacities as *quantical*" (Núñez, 2017, p. 419; emphasis in original). We interpreted these claims as implying that the ANS does not represent number, and instead represents something "quantical." **Núñez et al.** stress that "quantical" is an adjective to describe non-numerical quantities, and not a noun as we sometimes used it in our article. Fair enough. But, if the ANS is about something "quantical" rather than something numerical, what *exactly* does it represent? Núñez (2017) tells us that "quantical" pertains to quantity. But as we stressed in section 5.3 of our target article, just saying that the ANS represents quantities doesn't capture its second-order sensitivity or distinguish it from systems that represent magnitudes such as distance or duration.

Núñez et al. offer some clarificatory remarks. For one, they say that the quantical–numerical distinction is not about (im)precision. This was one of Núñez's (2017) stated reasons for thinking that the ANS is non-numerical, when he wrote, "A basic competence involving, say, the number 'eight,' should require that the quantity is treated as being categorically different from 'seven,' and not merely treated as often – or highly likely to be – different from it" (p. 417). And again, when he wrote that quantifying "in an exact and discrete manner" is part of the "minimal criteria" for a capacity to be numerical (p. 418). In section 5.3 of our target article we argued that this is not a good reason to reject the hypothesis that the ANS represents numbers. Núñez et al. seem to agree.

Núñez et al. claim that the core difference between quantical and numerical cognition lies in the distinction between non-symbolic and symbolic reference. By a "symbol" they seem to mean public symbols from a spoken or written language, and not internal mental symbols such as **Gallistel**'s numerons. (Thus, they deny not only that the ANS is symbolic, but also that subitizing is symbolic even though subitizing has been argued to recruit demonstrative-like mental symbols [Pylyshyn, 2007].) The way they use the quantical–numerical distinction is open to two interpretations, however, one weak and one strong.

According to the weak interpretation, the distinction is merely supposed to emphasize that the capacities that come online with public numerical symbols are importantly different from the capacities associated with the ANS. We agree wholeheartedly and said as much in our target article. Mastering a public numerical system makes it possible to do things that one could not do before. According to the strong interpretation, not only are the capacities different, but also the capacities associated with the ANS *are not numerical*, and so the ANS does not represent numbers. For reasons glossed above, we interpret Núñez (2017) as endorsing this stronger interpretation. But, as we note in our

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target article, the stronger claim faces two problems **Núñez et al.** don't address. First, it struggles to explain the higher-order sensitivity of the ANS. And second, it owes an account of what the ANS represents, if not numbers. Saying that it is "quantical" is insufficient because that either reduces to the trivial claim that internal mental representations are not public symbols (if "quantical" is just taken to mean *not symbolic*) or else fails to distinguish the capacities associated with the ANS from the capacities associated with systems devoted to quantities like duration or distance (if "quantical" means *quantitative*).

R3.5. Pure magnitudes

Most researchers who claim that the ANS represents numerosities fail to adequately explain what a numerosity is. Burge (2010) is an exception. His proposal that the ANS represents Eudoxan pure magnitudes is substantive and specific. We view this as the leading competitor to our proposal that the ANS represents number. **Buijsman** and **Gross et al.** defend it.

Gross et al. argue that pure magnitudes are preferable to natural numbers for two reasons. First, the ANS isn't sensitive to the full structure of the natural numbers. For example, its capacities do not include "counting, one-to-one matching, or a successor operation." By contrast, they claim pure magnitudes have all the structure needed to explain the ANS, and no more. We disagree. Pure magnitudes are extremely fine grained. The ancient Greeks introduced them to capture ratios that we would now express using irrational numbers. But, as we argued, there is no evidence that the ANS is sensitive to irrational numbers. Pure magnitudes have *more* structure than is reflected in the ANS (Beck, 2015).

Second, **Gross et al.** argue that perception already represents pure magnitudes when it represents continuous magnitudes like length and weight. To motivate this claim, they appeal to Peacocke's (1986) thesis that perception is unit-free. (When you see the length of a piano, you don't represent that length in meters, yards, or any other units.) But, given a background realism about magnitudes like length, the view that perception represents these can also respect the unit-free character of perception. Veridical perception of length is unit-free because length itself is unit-free (Peacocke, 2020). Pure magnitudes aren't needed. Furthermore, even if pure magnitudes were needed to represent continuous magnitudes, it wouldn't follow that they are also needed for the ANS unless continuous magnitude representations and the ANS draw on the same representational elements. While this hypothesis has been defended (Feigenson, 2007; Walsh, 2003), some recent evidence speaks against it. For example, Odic (2018) found that the precision of continuous and numerical magnitude representations follows distinct developmental trajectories.

We argued that the ANS represents numbers rather than pure magnitudes because only numbers have a second-order character and the ANS exhibits sensitivity to a second-order property of collections. **Buijsman** thinks the pure magnitude hypothesis "cannot (yet) be dismissed" because he is skeptical that ANS representations are genuinely second order. But, as we explain above (section R3.1), these concerns are misplaced.

Gross et al. grant that the ANS exhibits second-order sensitivity, but claim that this is equally well captured by the pure magnitude hypothesis. On their view, perception represents a variety of magnitude types in terms of pure magnitudes, including distance, weight, duration, and "aggregate membership." (Whereas sets are abstract, an *aggregate* is roughly what we called a "concrete plurality," or a spatiotemporally located collection.) When pure magnitudes measure continuous magnitudes like distance, weight, and duration, sortals are not involved. But, when they measure aggregate membership, sortals must be involved. Thus, Gross et al. conclude that representations of pure magnitudes can also exhibit second-order sensitivity.

We're not so sure. To see why, it's helpful to distinguish the *genus* pure magnitude, which divides "into discrete and continuous subspecies" and "is not specific to any further type of magnitude – such as spatial extent or size, temporal duration, weight, and so forth" (Burge, 2010, p. 482), from various *species* of pure magnitude, such as duration and weight. We interpreted Burge (2010) as claiming that the ANS represents the genus pure magnitude. On that interpretation, we think our original criticism stands. The genus pure magnitude does not differentiate between being first order or second order; but the ANS is second order; so ANS representations are, in that respect, not well captured by the genus.

By contrast, **Gross et al.** seem to take the ANS to represent a particular species of pure magnitude – namely, the discrete species that measures "aggregate membership" (see also **Ball**). This evades our criticism because the discrete species is second order. But, as we understand it, this discrete species of pure magnitude just is natural number. What makes numbers a species of pure magnitude is that they can stand in ratios (analyzed in terms of equimultiples). But the ancient Greeks held that there is more to numbers than that. For example, they maintained that numbers are composed from discrete units. While it's true that they didn't attempt a reductive definition of number in terms of the successor relation or one-to-one correspondence (that would have to wait for the late nineteenth century), it doesn't follow that they were talking about something else. Therefore, if Gross et al. take the ANS to represent the discrete species of pure magnitude that measures aggregate measurement, that sounds to us like another way of saying that the ANS represents natural numbers.

R3.6. The scientific-ontology bias

We argued that entities that appear in our scientific ontology should be favored as contents of the ANS. We agree with **Gross et al.** that this consideration is only *prima facie*. It can be overruled by other considerations. While we take it to be an advantage of our hypothesis that it meets this consideration, we never meant to claim the advantage as unique. Other hypotheses may meet it too.

Brown objects that psychologists justifiably attribute contents that are not part of our scientific ontology. For example, developmental psychologists attribute representations that do not differentiate between heat and temperature. But, in that case, there is overwhelming evidence that children systematically conflate heat and temperature, so neither content on its own is appropriate. The consideration we adduced is thus overruled. But we argued at length that the ANS does not systematically conflate number with other magnitudes.

Brown also considers color vision. If we say that the ANS represents numbers, shouldn't we also say that color vision represents wavelengths? We think not. The way the ANS represents number (and, for that matter, the way perception represents distance, duration, weight, and a host of other magnitudes) is lawlike. By contrast, the relation between wavelength and color percepts is notoriously arbitrary.

If a bias toward scientific ontology can be overruled, is it needed? One reason to think so is that mental representation is always noisy and imprecise. The mind is an imperfect instrument

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with limited resources. Thus, one can always improve the fit of a content assignment by inventing a new concept that accommodates the noise and imprecision. But that would lead to overfitting, idiosyncratic contents, and a missed opportunity to capture genuine connections between the mind and world. The bias serves a useful purpose.

R3.7. Modes of presentation

In claiming that the ANS represents number, we did not mean to deny differences between ANS representations and mature number concepts. We simply argued that these could be captured by differences in their modes of presentation. Such differences are important, and we certainly didn't mean to treat them as "an afterthought" (**Barth & Shusterman**). On the contrary, we emphasized differences in mode of presentation.

Jones et al. think our appeal to modes of presentation is problematic. As they see it, "the distinction between the 'sense' and 'reference' of neural representations is an ad hoc construction without any independent justification." They grant that modes of presentation are legitimate when applied to person-level states, like experiences or beliefs, but deny that this is so when applied to sub-personal representations.

We disagree. For one, sub-personal states can differ in format (Marr, 1982), and this implies differences in mode of presentation because computational work is needed to translate between format types. Elsewhere, both of us have argued that ANS representations differ from conceptual thoughts in precisely this way (Beck, 2015; Clarke, forthcoming). But ANS representations are also not purely sub-personal. When you perceive a group of dots, they look to be a certain number to *you* and not just some component of your brain/mental machinery (see the demonstration in Burr and Ross, 2008).

Peacocke helpfully characterizes ANS representations as having the form *roughly that many Fs* and teases apart three aspects of their modes of presentation. ANS representations are *unspecific* (they don't refer to one specific number), *non-canonical* (they don't use a canonical system of representation to refer to numbers), and *non-recognitional* (the ANS doesn't enable subjects to reliably recognize the same number presented at two different times). Peacocke suggests that these three features should be distinguished conceptually because they co-occur only contingently. There could be a numerical perceptual system that was specific (unlike the ANS) but also non-canonical and non-recognitional. Its mode of presentation would have the form *that many Fs*. It's unclear to us, however, what would ground the specificity of this hypothetical system. If you say, "That many Fs," the specificity plausibly derives from your mature counting abilities, or at least an ability to place items in one-to-one correspondence. In communities lacking those abilities, an utterance of "That many Fs" would not be specific. By contrast, if we are meant to imagine that the specificity is grounded in the perceptual system's discriminative abilities (in the way that having perfect pitch grounds reference to a specific pitch in someone who says "that pitch"), then the system is plausibly recognitional too. While this doesn't show that it's *impossible* for being specific, recognitional, and canonical to come apart in the ways Peacocke suggests, there may be important and deep connections among them.

R4. What kind(s) of number?

The preceding discussion notwithstanding, many commentators sympathize with our suggestion that the ANS represents numbers. But our target article considered a further question: What *kinds* of number does it represent? We speculated that the system goes beyond representing natural numbers by representing rational numbers. At the same time, we expressed skepticism that the ANS goes so far as representing irrational numbers and, hence, the reals more generally. Various commentaries pick up on these claims.

R4.1. Rational numbers

Some commentators welcome our suggestion that the ANS represents rational numbers. **Libertus, Duong, Fox, Elliott, McGregor, Ribner, and Silver** highlight evidence that ANS acuity predicts math skills at school, and suggest that it may be fruitful to explore whether the ANS's involvement in rational number processing relates to children's later understanding of fractions and decimals. This could be an important application of the conjecture.

In a similarly constructive spirit, **Zhang** draws on measurement theory to offer a technical proposal for how rational numbers might be constructed from placing ANS representations in the numerator and denominator of a fraction. Meanwhile, **Yousif** notes that the ANS's (alleged) representation of rational numbers offers to reframe findings typically interpreted as congruency effects. A bias toward treating smaller objects as fewer may result not from congruency effects of area/volume on number, but from interpreting the smaller objects as partial objects. This possibility is certainly worth testing. We also agree with his suggestion that more attention should be paid to the concept of a visual/perceptual object (Green, 2018, 2019).

Extending our conjecture, **Pinhas, Zaks-Ohayon, and Tzelgov** review fascinating evidence that the ANS represents zero. If that's right, we should say that the ANS most basically represents non-negative integers (0, 1, 2, …) rather than natural numbers. But, if zero enters into rational number representations in the way positive integers might, an intriguing possibility arises: Zero might feature as the denominator of a fraction, enabling the ANS to represent infinity!

Ball, **Gómez**, **Lyons**, and **Peacocke** took a more skeptical tone. In our target article, we proposed that, while the ANS most basically represents natural numbers, those natural numbers can enter into ratios, indicating that the system represents rational numbers as well. But these commentators worry that representing a ratio of natural numbers is not the same as representing a rational number.

As **Peacocke** puts the concern: "Appreciation that 6:4, 12:8, 3:2 are all the same ratio is not yet encoding that ratio as a single number 1 ⅔." He then proceeds to claim that the evidence we cited in favor of our proposal only supports the conjecture that the ANS represents ratios, not rational numbers as we suggest. But what's required for representing something as a rational number as opposed to a *mere* ratio? At one point, Peacocke says that this would involve representing these "as having a certain position in a rational number line." This suggests that if the ANS could go beyond matching ratios (e.g., representing that 6:4 and 12:8 are equal) by ordering these into greater/lesser relations (e.g., representing that 6:4 < 7:4), then this would go some way toward showing that the ANS is capable of representing rational numbers. But, if this were all that's required, our proposal would be favored by the studies we described in which subjects use their ANS to gamble on the more favorable of two ratios (Matthews & Chesney, 2015; Szkudlarek & Brannon, 2021).

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Lyons thinks the evidence we cited fails to show that rational numbers are represented because it fails to show that the ANS represents non-natural rational numbers greater than 1. But there are non-natural rational numbers less than 1 (e.g., ½). So, even if Lyons were right, the ANS could still represent *some* non-natural rational numbers. Moreover, the fact that subjects can gamble on the more favorable of two ratios suggests that they can distinguish a ratio of (say) 2:3 from a ratio of 3:2, and thus, if they represent any rational numbers at all, they do not merely represent non-natural rational numbers less than 1.

Gómez notes that distance effects (a signature of the ANS) show up when subjects compare certain symbolic numerical representations (e.g., single Arabic digits), but appear less consistently when they compare symbolic fraction representations. This leads him to infer that the ANS may not represent rational numbers. But, whether the ANS represents rational numbers is one thing; whether it maps those representations onto symbolic fractions is another.

Ball offers a means of adjudicating whether the ANS represents rational numbers or just ratios. He notes that extensive magnitudes (numerical or otherwise) can be added to one another, while intensive magnitudes cannot. But rational numbers are extensive (½ and ¼ can be summed) while ratios are not (1:2 and 1:4 cannot be summed). To decide whether the ANS represents ratios or rational numbers, we should thus investigate whether the ANS can *add* rational numbers.

In short, we *love* this suggestion. While we don't know of existing evidence that speaks directly to **Ball**'s point, it nicely distinguishes ratios from rational numbers and is empirically testable. (See footnote 6 of our target article for a complementary suggestion about how to distinguish ratios from fractions.)

R4.2. Precision

Lyons takes the ANS's imprecision to imply that it represents "approximate number" (e.g., 13ish), suggests that this is at odds with our proposal, and claims that this is something which cannot be "easily squared" with our suggestion that the ANS represents rational numbers – a conjecture which attributed "greater precision [to the ANS]… when what was needed was less." But we suggested that the ANS might represent "numerical intervals (5–9, 1.25–1.75, etc.) (Ball, 2017), or probability distributions over numerical intervals." Either option would involve the ANS referencing numbers, and be compatible with the representation of rational numbers. If Lyons has something else in mind by "approximate number" and "13ish," it's not clear to us what it is.

Lyons also claims that ANS imprecision should be attributed to ANS content, and not ANS vehicles. This leads naturally to the view that the ANS represents a (probability distribution over) a range of values. When you see 10 dots flashed on a screen, you represent there being 8 to 12 dots (or a bell-shaped probability distribution that peaks at 10). But that can't fully capture the imprecision in the ANS. For if it did, then when queried as to the number of dots, you should be able to reliably report the midpoint of the range or the peak of the distribution (i.e., 10). But subjects cannot do that. Some of the imprecision associated with the ANS is exogenous to its content.

R4.3. Is the RPS part of the ANS?

Commentators such as **Dramkin and Odic**, **Hecht, Mills, Shin, and Phillips (Hecht et al.)**, and **Hubbard and Matthews** raise a quite different worry for our hypothesis. They concede that rational numbers are represented but deny that the ANS *itself* produces these representations. Rather, they think that there is a separate domain-general ratio processing system (RPS) that does all the computing over ratios (for numbers, durations, distances, etc.).

It's important to recognize that this algorithmic-level hypothesis is consistent with our computational-level hypothesis that the ANS represents ratios. The key to seeing this is noting that "ANS" is ambiguous. We use it to refer to a system that is individuated in terms of its function: representing and computing over numbers in accordance with Weber's Law. But these commentators use it to refer to a module that's individuated by its inputs, outputs, and algorithms. On their proposal, what we call the ANS is realized by (at least) two modules: a number-specific module (which confusingly is also sometimes called the ANS) and a domain-general module for processing ratios (the RPS). Of course, the RPS could be a component of non-numerical systems too. (The respiratory system is distinct from the circulatory system, but the lungs belong to both).

On our usage, the ANS is distinctive because it concerns numbers (rather than other magnitudes) *and* because it obeys Weber's Law (unlike other numerical or quasi-numerical systems, such as the subitizing system). Our conjecture was thus not wedded to any given account of the system's underlying architecture. By analogy, we noted that the visual system is often said to be unified by its computational level description, despite comprising myriad submodules (Clarke, 2021; Marr, 1982).

Henik, Salti, Avitan, Oz-Cohen, Shilat, and Sokolowski acknowledge the point about levels of description but reject the proposed unity of the ANS, claiming that neurophysiological evidence supports a multi-system architecture which involves at least one generalized magnitude system (cf. Walsh, 2003). But, even bracketing evidence that tells against a generalized magnitude system (Odic, 2018; Pitt et al., 2021), such possibilities are precisely what a computational level description of the system leaves open (Marr, 1982).

In emphasizing a computational level description of the ANS we didn't mean to suggest that an algorithmic or neurophysiological description of the system is unimportant. Indeed, our target article offered some brief speculations on this point. For instance, we tentatively suggested that the ANS's representation of rational numbers may derive from its first assigning natural numbers to concrete pluralities and only then deriving ratios or rational numbers from the relations between these. In so doing, our speculations went beyond a bare computational level description, suggesting possible stages of processing in the ANS's analysis of rational numbers. These speculations were put under pressure by **Hubbard and Matthews**. They noted evidence that one's capacity to discriminate ratios is not correlated with one's acuity discriminating natural numbers under relevant conditions, and that ANS training does not transfer to ratio tasks. Insofar, as these studies are successfully measuring *numerical* ratios, they are hard to square with our tentative proposal. Note, however, that they are also hard to square with the proposal by **Hecht et al.** and **Dramkin and Odic** that there is a domain-general RPS that takes inputs from a variety of magnitude-specific modules. For that proposal also predicts that ANS acuity and numerical ratio acuity should be correlated. In any case, we agree with Hubbard and Matthews that "More research is necessary for the final adjudication" and look forward to learning about future findings in this area.

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R4.4. Irrational numbers

In proposing that the ANS represents rational numbers, we stopped short of claiming that it represents irrational numbers and, hence, the reals more generally. **Gallistel** now agrees. But, while we take it to be a contingent matter that the ANS cannot represent irrational numbers, Gallistel thinks this nomologically necessary, claiming that no irrational number "can be represented exactly by any physically realized system."

We're reluctant to go this far. The symbols "π" and "$\sqrt{2}$" represent exact irrational numbers. Perhaps, **Gallistel** simply means that use of the symbol by a physical system will never be perfectly precise. But, short of assuming the sensitivity principle, which we were at pains to reject (and which no commentaries sought to defend), it's hard to see why this should rule out the representation of irrational numbers.

Our proposal was that extant behavioral evidence fails to support the suggestion that the ANS represents irrationals. In saying this, we acknowledged that future research could, potentially, uncover evidence in favor of this suggestion. For instance, if behavioral evidence were to suggest that the ANS is involved in calculating square roots, this might provide evidence that we had not gone far enough.

Dramkin and Odic object to our emphasis on behavioral studies. They point out that behavioral measures can struggle to disambiguate performance from competence and may, therefore, lead us to underestimate the full range of numbers the ANS represents. This is a genuine methodological worry. But, to overcome these limitations, Dramkin and Odic claim that emphasis should be diverted away from behavioral evidence and instead placed on psychophysical models of ANS performance which treat "perceptual signals as highly continuous and in the domain of the reals."

While we don't wish to downplay the importance of psychophysical models, we're not convinced. The potential problems are two-fold. First, models are always idealizations (Weisberg, 2013). They allow us to abstract away from details of the real world, and it's not always clear whether details of the model reflect simplifying assumptions or not. A good model answers not only to reality, but also to the convenience of the modeler. Thus, the "highly continuous" signals in models may not reflect psychological reality. Second, it's important to distinguish the question of whether a model posits internal signals that are continuous from the question of whether the model posits representational contents that are continuous. A continuous vehicle can represent discrete contents. Thus, even if the models to which **Dramkin and Odic** allude were committed to a continuous perceptual signal, it wouldn't follow that they were committed to continuous contents.

R5. Concluding remarks

Our defense of the view that the ANS represents number, and our attempts to clarify the kinds of number it represents, have divided opinion. While we remain optimistic about the main proposals in our target article, understanding what the ANS represents strikes us as an important and neglected issue regardless. Therefore, if our discussion has helped highlight what is (and isn't) at issue in these debates, and inspired the pursuit of further empirical and conceptual lines of inquiry, we'd take our efforts, and those of our commentators, to have been worthwhile.

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