A GENERALIZED MODEL OF JUDGMENT AND PREFERENCE AGGREGATION

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Some basic notions of fuzzy logic for trapezoidal fuzzy numbers are extended. Purpose of these extensions is to give a generalized model of judgment and preference aggregation by using trapezoidal fuzzy numbers. Examples are given to illustrate the proposed technique to optimize the problem by avoiding paradoxical outcomes without the fear of indecision.

***Keywords*:** *judgment aggregation; preference aggregation; Condorcet paradox; doctrinal paradox; trapezoidal fuzzy number*

JEL Classification:

1. INTRODUCTION

In 18th century, social theorists[1] proposed the idea that group decision making is better than individuals decision. Voting theory studies the individual preferences aggregation method. Preference aggregation defined in social choice theory as forming collective preferences in given set of alternatives. Likewise, judgment aggregation pertains to forming collective judgments on a given set of logically related classical propositions. Arrow[2] and Sen[3] proved the first social choice theory (im)possibility result. Judgment aggregation to solve the problems with axiomatic method was initiated by List and Pettit[4,5]. List and Pettit [6] formalized judgment aggregation in social choice theoretic setting for impossibility theorems. Judgment aggregation is also an emerging research area in formal epistemology and economics.

[1] M.J.A.N de C. Marquis de Condorcet, 1785.
[2] K.J. Arrow, 1963.
[3] A. Sen, 1970.
[4] C. List and P. Pettit, 2002, 89-110.
[5] C. List, 2005, 25-38.
[6] C. List and P. Pettit, 2011.

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If a set of propositions $\{p,q,p\cap q\}$, expressed in propositional calculus then set $L=\{(0,0,0),(0,1,0),(1,0,0),(1,1,1)\}$ consists of all possible assignments of 0 or 1 as a truth values of propositions in $\{p,q,p\cap q\}$, that are logically consistent. Judges can make decision on the truth value of each proposition by an aggregator that is $L^n \to L.$ Proposition wise majority rule leads to inconsistent collective decisions called "Doctrinal Paradox". Area of judgment aggregation grows for the solution of this paradox. Most of the discussions on this paradox have been in the domain of social choice theory. An example of doctrinal paradox was given by Kornhauser and Sager[7]. In this example they mentioned the problem is that court's decision depends on the method adopted. Under conclusion based method, the defendant will be declared not liable, but under premise based procedure, the defendant would be declared liable. Kornhauser and Sager[8] stated: "We have no clear understanding of how a court should proceed in cases where the doctrinal paradox arises. Worse, we have no systematic account of the collective nature of appellate adjudication to turn to in the effort to generate such an understanding."

List and Pettit[9,10,11] recognized that the doctrinal paradox illustrates a more general problem than a court decision. They further introduced the term discursive dilemma to indicate a group decision in which proposition wise majority voting on related propositions may yield an inconsistent collective judgment. Here we gave an example to illustrate doctrinal paradox and discursive dilemma jointly. In an academic department, there is a three member graduate studies committee (GSC). These members are from experienced faculty of the university. GSC has to award doctorate degree and the decision rule is such that a candidate C will be awarded a degree only if the candidate is good at thesis defence presentation and good at research. We will say that degree awarded to C is the conclusion while good at thesis defence presentation and good at research are the premises. How shall we derive a group decision given the individuals' opinions on premises and conclusion? Assumed that each individual expresses opinions in the form of binary decision (Yes/No) on two logically inter connected propositions. If we formulate the decision on the basis of majority voting then we may reach to inconsistent decision. This shows that Graduate Studies Committee (GSC) may have to face a situation in which a majority does not deem C a degree awarded candidate. According to the GSC members, if the student is good at thesis defence presentation (proposition P) and student is good in research (proposition Q) then the student is eligible to get doctorate degree (proposition R). Now assume that each member of GSC makes a consistent judgment over these

[7] L.A. Kornhauser and L.G. Sager, 1993, 1-51.
[8] L.A. Kornhauser and L.G. Sager, 1993, 1-51.
[9] C. List and P. Pettit, 2002, 89-110.
[10] C. List and P. Pettit, 2004, 207-235.
[11] C. List and P. Pettit, 2011.

propositions P, Q and R as shown in Table 1, which is a form of doctrinal paradox and discursive dilemma.

Each GSC member assigns a binary truth value (Yes/No) to the propositions P, Q and R which gives rise to the doctrinal paradox. The paradox lies precisely in the fact that the two procedures may lead to contradictory results depending on whether the majority is taken on the individual judgments of P and Q or whether the majority is calculated on the individual judgments of R (discursive dilemma).

Table 1

	P	Q	$(P \wedge Q) \to R$	R
GSC member 1	Yes	Yes	Yes	Yes
GSC member 2	Yes	No	Yes	No
GSC member 3	No	Yes	Yes	No
Majority	Yes	Yes	Yes	?

Is the shift from doctrinal paradox to discursive dilemma is an innocent one? Investigation of Mongin[12] about this shift observed that: "The discursive dilemma shifts the stress away from the conflict of methods to the logical contradiction within the total set of propositions that the group accepts. Trivial as this shift seems, it has far reaching consequences, because all propositions are now being treated alike; indeed, the very distinction between premises and conclusions vanishes. This may be a questionable simplification to make in the legal context, but if one is concerned with developing a general theory, the move has clear analytical advantages."

How individual preferences can be aggregated into collectively preferred alternative is studied in social choice theory[13,14], whose origin is the work by Condorcet[15]. Condorcet made a result that majority voting was a good truth tracking method. He also found a problem of majority voting method. Condorcet suggested a method which is consisted in the comparison of each of the alternatives in pairs. Majority voting decides the winner in each pair and the collective result obtained by combination of all partial results. But, this method leads to cycles in the collective result: the Condorcet paradox.

Here, we gave an example for the clear illustration of Condorcet paradox. Let there are three alternatives x, y and z and three voters V^1, V^2 and V^3. Let $\succ i$ denote voter V_i's preference over X and $\succ$ denote the collective preference result over X. $V_1 = \{x \succ_1 y, y \succ_1 z\}, V_2 = \{y \succ_2 z, z \succ_2 x\}$ and $V_3 = \{z \succ_3 x, x \succ_3 y\}$

[12] P. Mongin, 2011.
[13] K.J. Arrow, 1963.
[14] A. Sen, 1970.
[15] M.J.A.N de C. Marquis de Condorcet, 1785.

are three voter's preferences over X (see Table 2). Each voter's preference is transitive; unfortunately transitivity fails to be mirrored at collective level 'V_s'. This situation is called Condorcet paradox.

Table 2

	$x \succ y$	$y \succ x$	$y \succ z$	$z \succ y$	$x \succ z$	$z \succ x$
V_1	1	0	1	0	1	0
V_2	0	1	1	0	0	1
V_3	1	0	0	1	0	1
V_S	1	0	1	0	0	1

When we combine individual opinions into a collective decision, we may lose at collective level that held at individual level, like logical consistency (in judgment aggregation) and transitivity (in preference aggregation). In doctrinal paradox individuals are consistent in terms of propositional logic and in Condorcet paradox consistency in terms of individual preferences. But these two paradoxes are not equivalent as stated by List and Pettit[16]: "When transcribed into the framework of preferences instances of the discursive dilemma do not always constitute of the Condorcet paradox; and equally instances of the Condorcet paradox do not always constitute instances of the discursive dilemma."

Van Hees[17] used multi-valued logic for some generalization of doctrinal paradox. Handling of realistic collective decision problems with some extension beyond classical propositional logic into the realm of general multi-valued logic (see Dietrich[18], Beg *et al.*[19]). In Duddy and Piggins[20] truth value of propositions is in degrees not in binary form. Scientists proved number of '(im)possibility theorems'[21,22,23]. Beg and Butt[24] proved the (im)possibility theorem in fuzzy framework, similar to those of Arrow[25] and Sen[26]. In fact, these theorems show that there cannot exist any judgment aggregation procedure that simultaneously satisfies certain minimal consistency requirements[27].
Often, human judgment and preference are ambiguous, vague and cannot be estimated with exact numeric value under many conditions, so the crisp values

[16] C. List and P. Pettit, 2004, 207-235.
[17] M. Van Hees, 2007, 649-666.
[18] F. Dietrich, 2007, 529-565.
[19] I. Beg and N. Butt, 2010, 1-11; I. Beg and N. Butt, 2012, article ID: 635043, 5 pages; I. Beg and A. Khalid, 2012, 911-924.
[20] C. Duddy and A. Piggins, 2009.
[21] K.J. Arrow, 1963.
[22] M.J.A.N de C. Marquis de Condorcet, 1785.
[23] A. Sen, 1970.
[24] I. Beg and N. Butt, (2010), 1-11.
[25] K.J. Arrow, 1963.
[26] A. Sen, 1970.
[27] F. Dietrich, 2006, 286-298.

are not suitable to model real world situations. Zadeh proposed the concept of fuzzy set theory[28] and successfully used it to handle imprecision (or uncertainty) in decision making problems[29], to solve the ambiguity and vagueness in information from human judgment and preference. In[30,31], using fuzzy logic they tried to solve doctrinal paradox to illustrate optimal judgment aggregation. Pigozzi[32] also tried to remove paradox in binary logic on the basis of least distance approach from profile and she get the result as dictatorship.

Aggregation procedures in fuzzy logic can help us make the collective judgment set more democratic in nature. Trapezoidal fuzzy numbers are the best way to show model judgment and preference. In this paper, we use trapezoidal fuzzy numbers as a truth value of the proposition. Rest of the paper is organized as follows: In Section 2 we proposed some basic relations and operations on trapezoidal fuzzy numbers and reformulation of problems in setting of trapezoidal fuzzy numbers. Section 3 illustrates how the paradoxes are resolved in trapezoidal fuzzy numbers to find optimal fuzzy aggregation functions. Some concluding remarks are given in the last section.

2. JUDGMENT AND PREFERENCE AGGREGATION BY USING TRAPEZOIDAL FUZZY NUMBERS

In some decision problems propositions are vague and thus can have truth values between true and false. This might be so if the economy is in a good shape. In a good shape is not precisely defined. To account for vagueness, Beg and Butt[33] tried to use fuzzy logic framework for the solution of doctrinal paradox. Trapezoidal fuzzy numbers are the best way to model the vagueness of human knowledge in decision problems.

A function ‘ A ’ given by

$$A(x) = \begin{cases} 0 & \text{if} \quad x < x_1 \cdot \text{or} \cdot x > x_4, \\ \dfrac{x - x_1}{x_2 - x_1} & \text{if} \quad x_1 \le x < x_2, \\ 1 & \text{if} \quad x_2 \le x < x_3, \\ \dfrac{x - x_4}{x_3 - x_4} & \text{if} \quad x_3 \le x < x_4, \end{cases}$$

[28] L. A. Zadeh, 1965, 338-356.
[29] R.E. Bellman and L.A. Zadeh, 1970, 141-164.
[30] I. Beg and N. Butt, 2012, article ID: 635043, 5 pages.
[31] I. Beg and A. Khalid, 2012, 911-924.
[32] G. Pigozzi, 2006, 285-298.
[33] I. Beg and N. Butt, 2012, article ID: 635043, 5 pages.

where $x_1 < x_2 < x_3 < x_4$, is called trapezoidal fuzzy number. Symbolically, A is denoted by (x_1, x_2, x_3, x_4) (see[34]). Let $D[0,1]$ denotes the set of all trapezoidal fuzzy numbers such that $0 \leq x_1$ and $x_4 \leq 1$ We assume that individual judgments take values on the $D[0,1]$. We define an operator Δ on trapezoidal fuzzy numbers, which is given by:

$$A\Delta B = \left(0 \vee (x^1 + y^1 - 1), 0 \vee (x^2 + y^2 - 1), 0 \vee (x^3 + y^3 - 1), 0 \vee (x^4 + y^4 - 1)\right)$$

where

$$A = (x_1, x_2, x_3, x_4),\ B = (y_1, y_2, y_3, y_4) \text{ and } A,\ B \in D[0,1].$$

Also

$$A = B \text{ if } x_1 = y_1, x_2 = y_2, x_3 = y_3 \text{ and } x_4 = y_4$$

The relation $\prec$ in $D[0,1]$ is introduced as follows:

Let $A = (x_1, x_2, x_3, x_4)$, $B = (y_1, y_2, y_3, y_4)$ and $A, B \in D[0,1]$

If $x_4 < y_4$ then $A \prec B$

If $x_4 = y_4$ and

(i) $x_3 < y_3$ then $A \prec B$;
(ii) $x_3 = y_3$ and
(a) $x_2 < y_2$ then $A \prec B$;
(b). $x_2 = y_2$ and $x_1 < y_1$ then $A \prec B$

Obviously, $D[0,1]$ is an ordered set. Also define

$$\min(A,B) = \left(\min(x^1, y^1), \min(x^2, y^2), \min(x^3, y^3), \min(x^4, y^4)\right)$$
$$\max(A,B) = \left(\max(x^1, y^1), \max(x^2, y^2), \max(x^3, y^3), \max(x^4, y^4)\right)$$

A fuzzy implication $\Rightarrow$ for trapezoidal fuzzy numbers is a map

$$\Rightarrow: D[0,1] \times D[0,1] \to D[0,1]$$

[34] H.T. Nguyen and E. Walker, 2006.

Satisfying

$\Rightarrow$	(0,0,0,0)	(1,1,1,1)
(0,0,0,0)	(1,1,1,1)	(1,1,1,1)
(1,1,1,1)	(0,0,0,0)	(1,1,1,1)

An example of a fuzzy implication for trapezoidal fuzzy numbers A,B is:

$$A \Rightarrow B = \begin{cases} 1-\max\left(\left|x_1-y_1\right|,\left|x_2,-y_2\right|,\left|x_3,-y_3\right|,\left|x_4,-y_4\right|\right) & \\ 1-\max\left(\left|x_1-y_1\right|,\left|x_2,-y_2\right|,\left|x_3,-y_3\right|,\left|x_4,-y_4\right|\right) & \\ 1-\min\left(\left|x_1-y_1\right|,\left|x_2,-y_2\right|,\left|x_3,-y_3\right|,\left|x_4,-y_4\right|\right) & ;\text{if} \quad A \succ B \\ 1-\min\left(\left|x_1-y_1\right|,\left|x_2,-y_2\right|,\left|x_3,-y_3\right|,\left|x_4,-y_4\right|\right) & \\ (1,1,1,1) & \text{if} \quad A \preccurlyeq B \end{cases}$$

Example 1. Here we formulate our problem in trapezoid al fuzzy number form which is illustrated and summarize in Table 1. Table 3 is clear illustration of Table 1 for trapezoidal fuzzy numbers. In this case we get at least one solution of doctrinal paradox for trapezoidal fuzzy numbers.

Table 3

	P	Q	R	$(P \wedge Q) \Rightarrow R$
GSC member 1	(0.45, 0.5, 0.6, 0.65)	(0.55, 0.6, 0.7, 0.75)	(0.45, 0.5, 0.6, 0.65)	(1, 1, 1, 1)
GSC member 2	(0.15, 0.2, 0.3, 0.35)	(0.65, 0.7, 0.8, 0.85)	(0.15, 0.2, 0.3, 0.35)	(1, 1, 1, 1)
GSC member 3	(0.75, 0.8, 0.9, 0.95)	(0.05, 0.1, 0.2, 0.25)	(0.05, 0.1, 0.2, 0.25)	(1, 1, 1, 1)
majority	$(\theta^1(x^1)$, $\theta^1(x^2)$, $\theta^1(x^3)$, , $\theta^1(x^4))$	$(\theta^2(x^1)$, $\theta^2(x^2)$, $\theta^2(x^3)$, $\theta^2(x^4))$	$(\theta^3(x^1)$, $\theta^3(x^2)$, $\theta^3(x^3)$, $\theta^3(x^4))$	(1, 1, 1, 1)

One important property that majority selection is:

$$\min\left((0.45,0.5,0.6,0.65),(0.15,0.2,0.3,0.35),(0.75,0.8,0.9,0.95)\right) \preccurlyeq$$

$$(\theta^1(x^1),\theta^1(x^2),\theta^1(x^3),\theta^1(x^4)) \preccurlyeq$$

$$\max\left((0.45,0.5,0.6,0.65),(0.15,0.2,0.3,0.35),(0.75,0.8,0.9,0.95)\right),$$

$$\min\left((0.55,0.6,0.7,0.75),(0.65,0.7,0.8,0.85),(0.05,0.1,0.2,0.25)\right) \preccurlyeq$$

$$(\theta^2(x^1),\theta^2(x^2),\theta^2(x^3),\theta^2(x^4)) \preccurlyeq$$

$$\max\left((0.55,0.6,0.7,0.75),(0.65,0.7,0.8,0.85),(0.05,0.1,0.2,0.25)\right)$$

and

$$\min\left((0.45,0.5,0.6,0.65),(0.15,0.2,0.3,0.35),(0.05,0.1,0.2,0.25)\right) \preccurlyeq$$

$$(\theta^3(x^1),\theta^3(x^2),\theta^3(x^3),\theta^3(x^4)) \preccurlyeq$$

$$\max\left((0.45,0.5,0.6,0.65),(0.15,0.2,0.3,0.35),(0.05,0.1,0.2,0.25)\right)$$

in Table 3.

Simply

$$(0.15,0.2,0.3,0.35) \preccurlyeq$$

$$(\theta^1(x^1),\theta^1(x^2),\theta^1(x^3),\theta^1(x^4)) \preccurlyeq$$

$$(0.75,0.8,0.9,0.95),\ \ (0.05,0.1,0.2,0.25) \preccurlyeq$$

$$(\theta^2(x^1),\theta^2(x^2),\theta^2(x^3),\theta^2(x^4)) \preccurlyeq$$

$$(0.65,0.7,0.8,0.85)$$

and

$$(0.05,0.1,0.2,0.25) \preccurlyeq$$

$$(\theta^3(x^1),\theta^3(x^2),\theta^3(x^3),\theta^3(x^4)) \preccurlyeq$$

$$(0.45,0.5,0.6,0.65)$$

such that

$$0.15 \le \theta^1(x^1) \le 0.75,\ 0.2 \le \theta^1(x^2) \le 0.8,\ 0.3 \le \theta^1(x^3) \le 0.9,$$

$$0.35 \le \theta^1(x^4) \le 0.95,\ 0.05 \le \theta^2(x^1) \le 0.65,\ 0.1 \le \theta^2(x^2) \le 0.7,$$

$$0.2 \le \theta^2(x^3) \le 0.8,\ 0.25 \le \theta^2(x^4) \le 0.85,\ 0.05 \le \theta^3(x^1) \le 0.45,$$

$$0.1 \le \theta^3(x^2) \le 0.5,\ \ 0.2 \le \theta^3(x^3) \le 0.6,$$

and

$$0.25 \le \theta^3(x^4) \le 0.65,$$

but the values of $\theta_i(x_j)$ will satisfy the condition for trapezoidal fuzzy numbers of $D[0,1]$.

At the same time, let $\pi(P)$ denote the degree of truth of the proposition P, and the fuzzy integrity constraint is $\{\pi(P)\Delta\pi(Q) \Rightarrow \pi(R)\}$. Suppose that the committee members are 'rational' they never violate the fuzzy integrity constraints (IC) (see List[35]). Now by using the above given Δ operator and fuzzy implication $\Rightarrow$ the IC can be translated as $\pi(R) \ge \max(0, \pi(P) + \pi(Q) - 1)$: Here $\pi(R) = f_i(\pi(P), \pi(Q))$ is a particular rule of inference for individual i (see Claussen and Roisland[36]).

We assume that decision makers are rational and they have the freedom to express their opinions on a proposition with which they do not agree or disagree fully. So any number from $D[0,1]$ that best represents their opinions can be opted. A finite set of n individuals and a finite set X of propositions over which individuals have to make their judgments is called an agenda. A judgment set A_i for an individual i is an n-tuple containing degree of truth for each proposition. Let $A_i = (h_{i1}, h_{i2}, ..., h_{i|X|})$, where $|X|$ denotes the cardinality of X and $h_{ij} \in D[0,1]$ for $j \in (1, 2, ..., |X|)$. A profile is an n-tuple $(A^1, A^2, ..., A_n)$ of individual judgment sets. Function f is an aggregation that assigns to each profile $(A^1, A^2, ..., A_n)$, a collective judgment set $(h_1^*, h_2^*, ... h_{|X|}^*) = f(A^1, A^2, ..., A_n)$. Here $h_j^* \in D[0,1]$ for $j \in (1, 2, ..., |X|)$.

[35] C. List, 2005, 25-38.
[36] C.A. Claussen and O. Roisland, 2005.

Truth value of some proposition $\phi \in X$ for the collective judgment set $f(A^1, A^2, ..., A_n)$ is $f(A^1, A^2, ..., A_n)(\phi)$. Similarly, truth value of some proposition $\phi \in X$ for the judgment set A_i is $A_i(\phi)$.

If $f(A^1, A^2, ..., A_n) = A_i$ for some $i \in (1,2,...,n)$ and every $(A^1, A^2, ..., A_n)$, then f is "dictatorship".

Function f is "manipulable" if and only if there exists some voter i, proposition ϕ and profile $(A^1, A^2, ..., A_n)$ such that $A_i(\phi) \neq f(A^1, A^2, ..., A_i, ..., A_n)(\phi)$ but $A_i(\phi) = f(A^1, A^2, ..., A_i^*, ..., A_n)(\phi)$ for some alternate judgment set A_i^*.

Function f is "independent" if and only if for all propositions $\phi \in X$ there is a function $g_\phi : D[0,1]^n \to D[0,1]$ such that for all $(A^1, A^2, ..., A_n)$, we have $f(A^1, A^2, ..., A_n)(\phi) = g_\phi(A^1(\phi), A^2(\phi), ..., A_n(\phi))$.

Belief aggregation formally investigates how to aggregate a finite number of belief bases into a collective one. This formal framework consists of a propositional language $\aleph$ which is built up from a finite set P of propositional letters standing for atomic propositions. Let the belief base K_i for the agent i be the following set $(\pi_i(p^1), \pi_i(p^2), ..., \pi_i(p_{|P|}))$, where $|P|$ denotes the cardinality of P. Here $\pi(\cdot)$ represents the truth function that maps elements in set P to $D[0,1]$. A belief set is the set $E = \{K^1, K^2, ..., K_n\}$. Given a set of integrity constraints *IC* in a fuzzy settings, ψ maps E and IC into a new (collective) belief base $\psi_{IC}(E)$, this process is called fuzzy aggregation for trapezoidal fuzzy numbers.

An interpretation is a function from P to $D[0,1]$. Let W denote the set of all interpretations and distance between interpretations is a real valued function

$$d : W \times W \to \Re$$

such that for all $w, w', w'' \in W$:

1. $d(w, w') \geq 0$.
2. $d(w, w') = 0$ if and only if $w = w'$.
3. $d(w, w') = d(w', w)$.
4. $d(w, w'') \leq d(w, w') + d(w', w'')$.

One possible choice for distance function is the Hamming distance:

$$d^*(w,w') = \frac{1}{4}\sum_{\forall x \in P} |w(x) - w'(x)|$$

$$d^*(w,w') = \frac{1}{4}\sum_{\forall x \in P} (|w(x^1) - w'(x^1)| + |w(x^2) - w'(x^2)| + |w(x^3) - w'(x^3)|$$
$$+ |w(x^4) - w'(x^4)|)$$

Another possible choice for distance function is the square distance:

$$d^{**}(w,w') = \sum_{\forall x \in P} ((w(x_1) - w'(x_1))^2 + (w(x_2) - w'(x_2))^2 + (w(x_3) - w'(x_3))^2$$
$$+ (w(x_4) - w'(x_4))^2)$$

Now let us define a belief aggregation operator in this framework. For any interpretation $w \in W$ and any profile of belief basis $K \in K^n$, the distance between an interpretation and a profile can now be defined as:

$$D^d(w,K) = \sum_{\forall i} d^*(w,K_i)$$

Our objective is to choose w which minimizes this distance and does not violate any IC in the fuzzy setting. To minimize the distance is same as to minimize a measure of disagreement in the society by bringing the collective judgment set close to the individual judgment sets as possible. Individual disagreement brings about individual disutility. Accordingly; we seek to minimize the societal disutility which is assumed to be the sum of individual disutilities. We have many distance and dissimilarity measures in literature like Hathaway *et al.*[37] distance 'd_h' Yang *et al.*[38,39] distances 'd_{LR}' and 'd_f' Hung *et al.*[40] distance 'd_{MLR}' etc. We have chosen Hamming distance d^* and Square distance d^{**} only for the sake of illustration in this paper. Choosing any distance or dissimilarity measure is solely at our discretion provided it satisfies certain normative principles.

Let *w* be any arbitrary interpretation. In this case $w(P) = (\theta_1, \theta_2, ..., \theta_{|P|})$ where we have $\theta_i \in D[0,1]$ for all $1 \le i \le |P|$ and $|P|$ denotes the cardinality of P Now d^* is a generalization of Hamming distance and we can use it in trapezoidal fuzzy numbers to avoid the doctrinal paradox. We can formulate the fuzzy aggregation as an optimization problem which can be stated as:

37 R.J. Hathaway, J.C. Bezdek and W. Pedrycz, 1996, 270-281.
38 M.S. Yang and C.H. Ko, 1996, 49-60.
39 M.S. Yang, P.Y. Hwang and D.H. Chen, 2004, 301-317.
40 W.L. Hung, M.S.Yang and E.S. Lee, 2011, 1776-1787.

Minimize

$$D^d(w,K) = \sum_{\forall i} d^*(w,K_i),$$

subject to the fuzzy integrity constraints IC.

Here

$$w(P) = (\theta^1, \theta^2, ..., \theta_{|P|}),$$

$$\min\{K_1(j), K_2(j), ..., K_n(j)\} \le \theta_j \le \max\{K_1(j), K_2(j), ..., K_n(j)\}$$

and

$$\theta_j \in D[0,1]$$

for

$K_1(j), K_2(j), ..., K_n(j$ (Where $K_i(j)$ denotes the j^{th} element of the belief base K^i).

The above optimization problem helps us to avoid doctrinal paradox and we can find an optimal fuzzy aggregation function. We say that an aggregation function is optimal if the collective judgment set is close to the individual judgment set as possible. Finding collective social choice function in Table 3 now becomes an optimization problem which can have multiple optimal solutions. The problem in Table 3 is framed in MATLAB (The language of technical computing). The optimal fuzzy aggregation function gives the solution for Table 3 as $(\theta_1, \theta_2, ..., \theta_3) = ((0.45, 0.5, 0.6, 0.601), (0.55, 0.6, 0.6, 0.75), (0.15, 0.2, 0.3, 0.351))$ with minimum $D^d = 1.7125$ (for details see Appendix 1). The optimal fuzzy aggregation function gives the solution for Table 3 as $(\theta_1, \theta_2, ..., \theta_3) = ((0.45, 0.5, 0.6, 0.65), (0.417, 0.467, 0.475, 0.671), (0.217, 0.267, 0.45, 0.45))$ with minimum $D^{d^{**}} = 1.9558$ (for details see Appendix 1). The fact that there is at least one solution to the problem shows that doctrinal paradox cannot occur in this case.

We like our aggregation procedure to be strategy-proof. Impossibility theorems proved by Dietrich and List[41], similar to Gibbard-Satterthwaite theorem on strategy proof aggregation rules. Given these theorems we do not claim that our distance based aggregation method is strategy proof. In fact, Dietrich[42] has proved that independence and monotonicity are properties of an aggregator that result in strategy proofness. Since, we do not claim that our distance based

[41] F. Dietrich and C. List, 2007, 269-300.
[42] F. Dietrich, 2006, 286-298.

aggregator is independent and monotone simultaneously, the strategy proofness of our aggregator is not clear. However, the nature of our objective function in the optimization problem is such that if an individual was to submit an insincere judgment (in an attempt to manipulate the collective judgment), any deviation of the collective judgment set from this insincere judgment, has a penalty in the objective function. In this situation there appears to be a partial corrective mechanism whereby our aggregation method is not easily prone to manipulation.

Example 2. Here we formulate our problem in trapezoidal fuzzy number form, which is illustrated and summarize in Table 2. Table 4 is clear illustration of Table 2 for trapezoidal fuzzy numbers. Now we get at least one solution of Condorcet paradox for trapezoidal fuzzy numbers.

Suppose the individuals have the following preference structure:

Table 4

	$x \succ y$	$y \succ x$	$y \succ z$	$z \succ y$	$x \succ z$	$z \succ x$
P_1	(0.45,0.5, 0.6,0.65)	(0.15,0.2, 0.3,0.35)	(0.45,0.5, 0.6,0.65)	(0.05,0.1, 0.2,0.25)	(0.35,0.4, 0.5,0.55)	(0.15,0.2, 0.3,0.35)
P_2	(0.75,0.8, 0.9,0.95)	(0.65,0.7, 0.8,0.85)	(0.15,0.2, 0.3,0.35)	(0.05,0.1, 0.2,0.25)	(0.45,0.5, 0.6,0.65)	(0.25,0.3, 0.4,0.45)
P_3	(0.75,0.8, 0.9,0.95)	(0.05,0.1, 0.2,0.25)	(0.25,0.3, 0.4,0.45)	(0.15,0.2, 0.3,0.35)	(0.65,0.7, 0.8,0.85)	(0.15,0.2, 0.3,0.35)
P_s	θ^1	θ^2	θ^3	θ^4	θ^5	θ^6

Now consider Table 4. Assume that there is a small economy with three individuals and three goods x, y, z. Individual binary relations over $X = \{x, y, z\}$ namely p_1, p_2 and p_3 are linear orders. Any optimal fuzzy social preference aggregation function map individual preference set into social preference set that must be a linear order. Accordingly, this becomes an optimization problem in which minimize the sum of the distances of social preference from the individual preferences using d^* (subject to fuzzy IC of linear order). Here preference aggregation is molded in to judgment aggregation by representing preference ordering as truth values over trapezoidal fuzzy numbers.

The problem in Table 4 is framed in MATLAB (The language of technical computing) in d^*. The optimal fuzzy preference function gives the solution for Table 4 as:

$$(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) = ((0.75, 0.8, 0.9, 0.95), (0.15, 0.2, 0.3, 0.35),$$

$(0.25, 0.3, 0.4, 0.45), (0.05, 0.1, 0.2, 0.25), (0.45, 0.5, 0.6, 0.65),$

$(0.15, 0.2, 0.3, 0.35))$

with minimum distance $D^{d^*} = 1.7$ (for details see Appendix 2). The optimal fuzzy preference function gives the solution for Table 4 as

$(0.28, 0.33, 0.43, 0.48), (0.08, 0.13, 0.23, 0.28), (0.48, 0.53, 0.63, 0.68),$

$(0.18, 0.23, 0.33, 0.38))$

with minimum distance $D^{d^*} = 1.494$ (for details see Appendix 2). Note: uniqueness of optimal solution is not guaranted.

3. DISCUSSION AND SUMMARY

Our aggregation procedure suffers from non-uniqueness problem (in other words indecision) whereby the aggregation function could become set-valued. It is an improvement upon Pigozzi's[43] aggregation method (in context of binary logic) because cases of dictatorship are highly unlikely in our method. We could devise a tie-breaking method in case of our optimal solution is not unique. Suppose we have a judgment set $C = (\text{average}(\pi_1(P_1)), \text{average}(\pi_2(P_2)), \ldots,$ $\text{average}(\pi_i(P_i)))$. Such a set C might violate the fuzzy integrity constraints. A tie-breaking method would narrow down solutions by picking solutions which are at a minimal distance from C. Aggregation function from the optimal solution would be another useful method in which minimal disagreement with other aggregation functions by using distance or dissimilarity measure. However, such a procedure does not ensure non-uniqueness of our final solution. An appropriate social welfare function can be useful in such cases. But important question is, how well behaved is our aggregation operator. We want the collective judgment set to be responsive to the individuals judgment set. On the other hand, we want the collective judgment set to obey rationality constraints. We note that our fuzzy aggregation method satisfies social axioms like unanimity (Pareto conditions), anonymity, compensativeness, non-dictatorship, collective rationality and universal domain. However, monotonicity and citizen sovereignty properties are unclear. It is worthy to compare our aggregation operator to other fuzzy aggregation operators; for example, fuzzy LAMA operator has anonymity, unanimity, unrestricted domain, monotonicity and citizen sovereignty (See Peláez and Doña[44]). But it does not ensure collective rationality. Yager[45] has proposed order weighted averaging (OWA) operators which are neutral, idempotent, monotone and compensative yet again do not ensure collective rationality.

[43] G. Pigozzi, 2006, 285-298.
[44] J.I Peláez and J.M. Doña, 2003, 809-820.
[45] R. Yager, 1988, 183-190.

Some time majority voting in classical propositional logic leads to doctrinal and Condorcet paradoxes. Distance based operator are used to attain collective rationality results in a situation of indecision or a tie. Fuzzy aggregation methods can help us to construct optimal fuzzy social judgment and preference aggregation functions as already given in the previous discussions. Finding such optimal fuzzy preference structures could have great applications in social choice theory by bringing it closer to reality with the use of trapezoidal fuzzy numbers for the clear illustration of vagueness. In this paper, distance based approach for trapezoidal fuzzy numbers is used to find an interpretation having the least distance with the profile of individual judgment and preference sets. The real challenge being to construct aggregation methods that satisfy desirable social properties and do not violate collective rationality. We believe that in the field of judgment and preference aggregation modeling for trapezoidal fuzzy numbers will have tremendous useful applications.

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APPENDICES

Both the appendices are written in MATLAB.

Appendix 1:

distance={ };

FOR

$$(\theta_1(x_1) = 0.15 : 0.001 : 0.75, \theta_2(x_1) = 0.05 : 0.001 : 0.65,$$

$$(\theta_3(x_1) = 0.15 : 0.001 : 0.75, \theta_1(x_2) = \theta_1(x_1) : 0.001 : 0.8,$$

$$(\theta_2(x_2) = \theta_2(x_1) : 0.001 : 0.7, \theta_3(x_2) = \theta_3(x_1) : 0.001 : 0.5,$$

$$(\theta_1(x_3) = \theta_1(x_2) : 0.001 : 0.9, \theta_2(x_3) = \theta_2(x_2) : 0.001 : 0.8,$$

$$(\theta_3(x_3) = \theta_3(x_2) : 0.001 : 0.6, \theta_1(x_4) = \theta_1(x_3) : 0.001 : 0.95,$$

$$(\theta_2(x_4) = \theta_2(x_3) : 0.001 : 0.85, \theta_3(x_4) = \theta_3(x_3) : 0.001 : 0.65,$$

if

$$(\theta_3(x_1) \geq \max(0, \theta_1(x_1) + \theta_2(x_2) - 1) \,\&\&$$

$$(\theta_3(x_2) \geq \max(0, \theta_1(x_2) + \theta_2(x_2) - 1) \,\&\&$$

$$(\theta_3(x_3) \geq \max(0, \theta_1(x_3) + \theta_2(x_3) - 1) \,\&\&$$

$(\theta_3(x_4) \geq \max(0, \theta_1(x_4) + \theta_2(x_4) - 1) \& \&$

D^d \\(here we use $d = d^*$ and $d = d^{**}$)

distance=distance <union> D^d

min(distance)

end

END

Appendix 2:

distance={ };

FOR

$(\theta_1(x_1) = 0.45 : 0.001 : 0.75, \theta_2(x_1) = 0.05 : 0.001 : 0.65,$

$(\theta_3(x_1) = 0.15 : 0.001 : 0.45, \theta_4(x_1) = 0.05 : 0.001 : 0.15,$

$(\theta_5(x_1) = 0.35 : 0.001 : 0.65, \theta_6(x_1) = 0.15 : 0.001 : 0.25,$

$(\theta_1(x_2) = \theta_1(x_1) : 0.001 : 0.8, \theta_2(x_2) = \theta_2(x_1) : 0.001 : 0.7,$

$(\theta_3(x_2) = \theta_3(x_1) : 0.001 : 0.5, \theta_4(x_2) = \theta_4(x_1) : 0.001 : 0.2,$

$(\theta_5(x_2) = \theta_5(x_1) : 0.001 : 0.7, \theta_6(x_2) = \theta_6(x_1) : 0.001 : 0.3,$

$(\theta_1(x_3) = \theta_1(x_2) : 0.001 : 0.9, \theta_2(x_3) = \theta_2(x_2) : 0.001 : 0.8,$

$(\theta_3(x_3) = \theta_3(x_2) : 0.001 : 0.6, \theta_4(x_3) = \theta_4(x_2) : 0.001 : 0.3,$

$(\theta_5(x_3) = \theta_5(x_2) : 0.001 : 0.8, \theta_6(x_3) = \theta_6(x_2) : 0.001 : 0.4,$

$(\theta_1(x_4) = \theta_1(x_3) : 0.001 : 0.95, \theta_2(x_4) = \theta_2(x_3) : 0.001 : 0.85,$

$(\theta_3(x_4) = \theta_3(x_3) : 0.001 : 0.65, \theta_4(x_4) = \theta_4(x_3) : 0.001 : 0.35,$

$(\theta_5(x_4) = \theta_5(x_3) : 0.001 : 0.85, \theta_6(x_4) = \theta_6(x_3) : 0.001 : 0.45)$

If

$$(((1-(\theta_1(x_1)\geq\theta_2(x_1))*(\theta_3(x_1)\geq\theta_4(x_1)))+(\theta_5(x_1)\geq\theta_6(x_1)))\geq 1\&\&$$

$$(((1-(\theta_2(x_1)\geq\theta_1(x_1))*(\theta_5(x_1)\geq\theta_6(x_1)))+(\theta_3(x_1)\geq\theta_4(x_1)))\geq 1\&\&$$

$$(((1-(\theta_5(x_1)\geq\theta_6(x_1))*(\theta_4(x_1)\geq\theta_3(x_1)))+(\theta_1(x_1)\geq\theta_2(x_1)))\geq 1\&\&$$

$$(((1-(\theta_6(x_1)\geq\theta_5(x_1))*(\theta_1(x_1)\geq\theta_2(x_1)))+(\theta_4(x_1)\geq\theta_3(x_1)))\geq 1\&\&$$

$$(((1-(\theta_3(x_1)\geq\theta_4(x_1))*(\theta_6(x_1)\geq\theta_5(x_1)))+(\theta_2(x_1)\geq\theta_1(x_1)))\geq 1\&\&$$

$$(((1-(\theta_4(x_1)\geq\theta_3(x_1))*(\theta_2(x_1)\geq\theta_1(x_1)))+(\theta_6(x_1)\geq\theta_5(x_1)))\geq 1\&\&$$

$$(((1-(\theta_1(x_2)\geq\theta_2(x_2))*(\theta_3(x_2)\geq\theta_4(x_2)))+(\theta_5(x_2)\geq\theta_6(x_2)))\geq 1\&\&$$

$$(((1-(\theta_2(x_2)\geq\theta_1(x_2))*(\theta_5(x_2)\geq\theta_6(x_2)))+(\theta_3(x_2)\geq\theta_4(x_2)))\geq 1\&\&$$

$$(((1-(\theta_5(x_2)\geq\theta_6(x_2))*(\theta_4(x_2)\geq\theta_3(x_2)))+(\theta_1(x_2)\geq\theta_2(x_2)))\geq 1\&\&$$

$$(((1-(\theta_6(x_2)\geq\theta_5(x_2))*(\theta_1(x_2)\geq\theta_2(x_2)))+(\theta_4(x_2)\geq\theta_3(x_2)))\geq 1\&\&$$

$$(((1-(\theta_3(x_2)\geq\theta_4(x_2))*(\theta_6(x_2)\geq\theta_5(x_2)))+(\theta_2(x_2)\geq\theta_1(x_2)))\geq 1\&\&$$

$$(((1-(\theta_4(x_2)\geq\theta_3(x_2))*(\theta_2(x_2)\geq\theta_1(x_2)))+(\theta_6(x_2)\geq\theta_5(x_2)))\geq 1\&\&$$

$$(((1-(\theta_1(x_3)\geq\theta_2(x_3))*(\theta_3(x_3)\geq\theta_4(x_3)))+(\theta_5(x_3)\geq\theta_6(x_3)))\geq 1\&\&$$

$$(((1-(\theta_2(x_3)\geq\theta_1(x_3))*(\theta_5(x_3)\geq\theta_6(x_3)))+(\theta_3(x_3)\geq\theta_4(x_3)))\geq 1\&\&$$

$$(((1-(\theta_5(x_3)\geq\theta_6(x_3))*(\theta_4(x_3)\geq\theta_3(x_3)))+(\theta_1(x_3)\geq\theta_2(x_3)))\geq 1\&\&$$

$$(((1-(\theta_6(x_3)\geq\theta_5(x_3))*(\theta_1(x_3)\geq\theta_2(x_3)))+(\theta_4(x_3)\geq\theta_3(x_3)))\geq 1\&\&$$

$$(((1-(\theta_3(x_3)\geq\theta_4(x_3))*(\theta_6(x_3)\geq\theta_5(x_3)))+(\theta_2(x_3)\geq\theta_1(x_3)))\geq 1\&\&$$

$$(((1-(\theta_4(x_3)\geq\theta_3(x_3))*(\theta_2(x_3)\geq\theta_1(x_3)))+(\theta_6(x_3)\geq\theta_5(x_3)))\geq 1\&\&$$

$$(((1-(\theta_1(x_4)\geq\theta_2(x_4))*(\theta_3(x_4)\geq\theta_4(x_4)))+(\theta_5(x_4)\geq\theta_6(x_4)))\geq 1\&\&$$

$$(((1-(\theta_2(x_4)\geq\theta_1(x_4))*(\theta_5(x_4)\geq\theta_6(x_4)))+(\theta_3(x_4)\geq\theta_4(x_4)))\geq 1\&\&$$
$$(((1-(\theta_5(x_4)\geq\theta_6(x_4))*(\theta_4(x_4)\geq\theta_3(x_4)))+(\theta_1(x_4)\geq\theta_2(x_4)))\geq 1\&\&$$

$(((1-(\theta_6(x_4)\geq\theta_5(x_4))*(\theta_1(x_4)\geq\theta_2(x_4)))+(\theta_4(x_4)\geq\theta_3(x_4)))\geq 1\&\&$

$(((1-(\theta_3(x_4)\geq\theta_4(x_4))*(\theta_6(x_4)\geq\theta_5(x_4)))+(\theta_2(x_4)\geq\theta_1(x_4)))\geq 1\&\&$

$(((1-(\theta_4(x_4)\geq\theta_3(x_4))*(\theta_2(x_4)\geq\theta_1(x_4)))+(\theta_6(x_4)\geq\theta_5(x_4)))\geq 1\&\&$

D^d \(here we use $d=d^*$ and $d=d^{**}$)

distance=distance <union> D^d

min(distance)

end

END