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Domain Extension and Ideal Elements in Mathematics†

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ABSTRACT

Domain extension in mathematics occurs whenever a given mathematical domain is augmented so as to include new elements. Manders argues that the advantages of important cases of domain extension are captured by the model-theoretic notions of existential closure and model completion. In the specific case of domain extension via ideal elements, I argue, Manders's proposed explanation does not suffice. I then develop and formalize a different approach to domain extension based on Dedekind's *Habilitationsrede*, to which Manders's account is compared. I conclude with an examination of three possible stances towards extensions via ideal elements.

1. INTRODUCTION

In field theory, algebraic number theory, and Galois theory, one often studies number domains of the form $\mathbb{Z}[\sqrt{2}]$, $\mathbb{Q}[i]$, $\mathbb{R}(i)$, *etc.* These are number domains which are obtained from $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, respectively, by *adjoining* new elements. This means the new elements are added to the old structure and then the mathematician works in the structure that results when the expanded domain

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is closed under the operations that were already defined on the old domain. A similar procedure can be carried out in geometry. There one can view the projective plane as obtained by adding points and lines at infinity to the standard Euclidean plane and then closing the structure under, *e.g.*, linear transformations. Historically, certain successful cases of such domain extensions have come to be referred to as extensions via *ideal elements* (see the discussion on ideal elements in the writings of Dedekind, but also Gauss, Veronese, and others, contained in [Cantù, 2013], summarised below in §2).

The philosophical significance of ideal elements and of the method of ideal elements has mainly been discussed in the context of Hilbert's philosophy of mathematics [cfr. Detlefsen, 1993; Hallett, 1990; Stillwell, 2014]. In his 1919 lecture 'The role of ideal entities' [Hilbert, 1992, pp. 90–101], Hilbert characterises the method of ideal elements as consisting in moving from a given 'system' in which the handling of certain questions is complicated to one where such questions become simple to handle (*op. cit.*, pp. 90–91). In addition, the new system contains a subsystem isomorphic to the old system. Thus, at least according to Hilbert, ideal elements are introduced to simplify certain mathematical problems, while preserving the old setting in which the problems arose.

Besides Hilbert, though, other mathematicians such as Poncelet (see for example [Chemla, 2016]), Kummer, and Dedekind [Cantù, 2013] talk of ideal elements; this suggests that domain extension via ideal elements was perhaps understood as a mathematical technique even before Hilbert. Other than the already cited treatments of ideal elements in the context of Hilbert's philosophy, a systematic investigation of what makes domain extensions successful, and domain extensions via ideal elements in particular can only be found in [Manders, 1989], where Manders sketches an account for domain extension. Manders argues that extended domains are productive to work with, because they are the existential closure of the original domain. In other words, for an extended domain to count as a good domain extension it is sufficient that it be the existential closure of the domain it extends.

In this paper, however, I will argue that if we understand ideal elements as heuristic tools affording the mathematician certain pragmatic or epistemic advantages, Manders's proposed explanation of the fruitfulness of domain extensions can only be a partial one, since it cannot explain some historically important cases of domain extension via ideal elements. I will then turn to a different approach to domain extension inspired by Dedekind [1854] and defend the view that, if interpreted correctly, it can provide a framework for the domain extensions motivated by closure under properties and operations. Given the historical context in which [Dedekind, 1854] was written, in §6 I explore the question of how this second criterion fares with respect to concurrent developments in number theory. I conclude (§§7, 8) that the comparison between Manders's framework and mine leaves us with three distinct options concerning the philosophical treatment of domain extension via ideal elements in mathematics.

2. IDEAL ELEMENTS

Cantù [2013] offers a historically informed reconstruction of the role ideal elements play in a mathematician's toolbox. She argues that 'ideal', 'imaginary' mathematical entities have been used by mathematicians in their proofs or theory building whenever the accepted mathematical domain would not warrant them in pursuing a certain simplification or generalisation of mathematics. Thus, the introduction of ideal elements is justified, in the eyes of the mathematician, on the basis of the following argument:

Premise (1) I, as a mathematician, have the goal (G'') of removing exceptions, allowing direct and inverse operations to satisfy closure properties, and dual transformations between models to be introduced, whenever possible.
Premise (2) The goal (G'') is supported by the set of values (V) and (V').
Premise (3) The method of introduction of ideal elements is a means for me, as a mathematician, to bring about (G'').
Conclusion (4) Therefore, I should (practically ought to) introduce ideal elements. [Cantù, 2013, pp. 86, 88]

The values Cantù recognises as supporting the mathematician's goal are the following:

(V) Value V. The generality of a theory, *i.e.*, its being without exceptions, is a desirable value in mathematics [Cantù, 2013, p. 83];

(V') Value V' as a warrant for value V. Generality is desirable because it increases simplicity [Cantù, 2013, p. 84].

Cantù reconstructs this argument on the basis of writings by Hilbert, Dedekind, Gauss, and Veronese. The new elements are *ideal*, or imaginary, *etc.* because they might enjoy an ontological, epistemic, or pragmatic status different from 'real' elements. In other words, they might exist in a different sense; they might be less epistemically secure; or they might be used differently than real elements [Cantù, 2013, pp. 79–80].

The argument above is supposed to offer a defence of the use of ideal elements in these mathematicians' work, based on their own writings on the matter. Cantù however is not arguing that this argument alone warrants the individual mathematician's use of ideal elements — she is noting though that several mathematicians use the above argument to justify the adoption of ideal elements in their practice. This argument cannot justify, for example, why a mathematician subscribes to (G''), or what happens when (G'') conflicts with another mathematical goal. Depending on the mathematician, these issues are fended off by different arguments.[1]

Having thus settled on a working notion of ideal elements as heuristic tools having epistemic and/or pragmatic advantages, I now introduce the first of the two accounts for domain extension via ideal elements that this paper considers.

3. MANDERS'S FRAMEWORK

Manders [1989] proposes to use the notions of existential closure and model completion from model theory to explain why certain historical cases of domain extensions, including some important cases of extension via ideal elements, are mathematically fruitful. Before sketching Manders's proposal, a few terminological clarifications are in order. For the remainder of the paper, a *structure* $\mathcal{A}$ is an ordered pair where the first element is a set of individuals, which is what we call a *domain* A, and the second element is the *interpretation* of all symbols of a given language $\mathcal{L}$ in $\mathcal{A}$. For each symbol l of $\mathcal{L}$, if l is a constant symbol then its interpretation is an element of A, if l is an n-ary relation symbol then its interpretation is a set of n-tuples of elements of A, and if l is an n-ary function symbol then its interpretation is an n-ary function on A, that is a function from A^n to A [see, *e.g.*, Tent and Ziegler, 2012, p. 2]. Now, let a theory T be a set of sentences in $\mathcal{L}$. If $\mathcal{A}$ makes those sentences true, we say that $\mathcal{A}$ is a *model of* T [Tent and Ziegler, 2012, p. 10]. We can now say what existential closure consists in. Roughly, existential closure is the property exhibited by a structure $\mathcal{A}$, considered as the model of a given theory T, or equivalently, as a member of a class K of structures (the class of all and only those structures which are models of T), whenever $\mathcal{A}$ contains in its domain all the solutions to equations and inequations which can be expressed in the language of $\mathcal{A}$. This language needs to be a first-order language with no relation symbols.

According to Manders, when performing domain extension via existential closure, the mathematician is trying to preserve three things: the original domain of objects, which we want to extend without modifying the objects we started with; conditions on said objects which we do not want to give up on, which he dubs 'invariant conditions' ('invariants' for short), indicated as $\varphi(), \psi(), \ldots$; and the properties these conditions give rise to, sentences of the form $\forall\overline{x}\varphi(\overline{x})$, where $\varphi()$ is itself an invariant. While the first one, namely the objects, are almost always preserved, invariants and the properties they give rise to sometimes have to be given up in order for the desirable extension to take place. Manders claims that this (informal) process has a formal counterpart in the notion of existential closure:

[1] For a more thorough treatment of objections to the argument (1)–(4), see [Cantù, 2013, pp. 89 ff.].

Definition (Existential Closure). Let $\mathcal{L}$ be a first-order language with no relation symbols (but possibly function symbols), and K be the class of $\mathcal{L}$-structures. Call a formula φ primitive if and only if $\varphi = \exists\, \overline{y} \bigwedge_{i \in I} \psi_i(\overline{y})$, where each ψ_i is either an atomic formula or a negated atomic formula.

Then a structure $\mathcal{A}$ from class K is existentially closed (e.c.) in K if and only if[2] for every primitive formula $\varphi(\overline{x})$ of $\mathcal{L}$, and every tuple $\overline{a}$ in A, whenever there is a structure $\mathcal{B}$ in K such that $A \subseteq B$ ($\mathcal{A}$ is a substructure of $\mathcal{B}$) and $\mathcal{B} \vDash \varphi(\overline{a})$ then already $\mathcal{A} \vDash \varphi(\overline{a})$.

Manders's goal is to convince his reader that by using existential closure (and model completion, where applicable) of contemporary model theory to conceptualise historical cases of domain extension in mathematics, one can achieve an analysis of what guides choices of fruitful theories in mathematics.

Manders's further claim is that, if we understand good domain extensions in terms of existential closure then we have accounted for the conceptual unification such extensions afford (*ibid.*, p. 554). This is how conceptual unification follows from existential closure. Once a given domain is existentially closed, the new structure, considered as a model of the old theory, will be such that for certain propositions, they will either hold universally or not at all (Manders calls this 'squeezing out the middle case'). Manders's example is that equations of second degree only have a solution in some cases over the real numbers, but once this is extended to the complex numbers, *every* second-degree equation has a solution in the extended domain.

The notion of existential closure is quite common in algebra: we can talk of an existentially closed (e.c.) lattice, an e.c. group, an e.c. field. One needs to use some caution, though, when talking about e.c. structures, for the notion itself is always relative to a class of structures. In the case of fields, for example, if K is the class of models of the theory of fields then the e.c. structures are exactly the algebraically closed fields (see, *e.g.*, [Hodges, 1993, p. 362]). If K on the other hand is the class of models for the theory of ordered fields, then the e.c. structures are the real closed fields — where algebraically closed and real closed fields are not extensionally the same class of structures.

If existential closure is, in a sense, quite common, what makes it noteworthy for the purposes of explaining the advantages of domain extensions? In short, existential closure can be a stepping stone towards an important model-theoretic feature of certain theories, *quantifier, elimination* (or properties which can approximate the advantages brought about by quantifier elimination proper). A theory T is said to have quantifier elimination whenever, for any formula φ in the language of T, T proves that φ is equivalent to a quantifier-free formula. Quantifier elimination is an important model-theoretic

[2] Existential closure in this formulation is due to [Hodges, 1993, p. 361].
As it turns out, one can strengthen the definition of existential closure so that the formula φ can be any existential formula, instead of just an existential quantifier followed by atomic or negated atomic formulas. This is just a consequence of the disjunctive normal form theorem for $\exists_1$ formulas [Hodges, 1993].

feature for algebraic theories, because it enables the proof of mathematically rich results such as the *Nullstellensatz*.[3]

4. DOMAIN EXTENSIONS AND IDEAL ELEMENTS

Manders's goal is to use cases of historical domain extensions which turn out to be existential closures of preexisting models[4] as evidence against the claim that fruitfulness of mathematical theories is an empirical, historical fact. Manders also suggests that existential closure is the formal model-theoretic notion that captures (Hilbert's) method of ideal elements. On the face of it, Manders sees existential closure as a sufficient condition for the success of certain domain extensions — in particular, successful domain extensions that Hilbert would consider as extensions via ideal elements. It is the scope of application of this explanation that I am interested in probing.

One of Hilbert's examples for ideal elements are lines and points at infinity. Manders [1984] shows how, under certain conditions, the models of projective geometry are existential closures of the Euclidean plane. So in that sense, Manders's account is correct in the case of ideal elements in geometry.

What about arithmetic and algebra? Let me start by the easiest case, namely the complex numbers. If we consider the field of complex numbers $\mathbb{C}$ as a structure in the class of models of the theory of fields, then, since it is an algebraically closed field, it is actually existentially closed (this follows almost immediately from the definitions). Moreover, the theory of algebraically closed fields is model-complete. So Manders's framework works well for this case — and indeed, if we look back at how he introduced the notion of existential closure, he generically spoke of all those cases in which one 'rounds off' a mathematical domain by adjoining roots. That is exactly one way of constructing the complex numbers, as $\mathbb{R}(i)$. Moreover, his historical discussion in [Manders, 1989] can be seen as a way of demonstrating how the extension of the reals into the complex number system is one of those instances of domain extension which does deliver conceptual unity; one can treat equations which used to be analysed separately as members of one and the same class of equations.

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[3] *Nullstellensatz* actually is the name given to several theorems of modern algebra, which however are generalisations of Hilbert's *Nullstellensatz*. One standard formulation of Hilbert's *Nullstellensatz* is as follows:

> Suppose A is an algebraically closed field, I is an ideal in the polynomial ring $A[x_0, \ldots, x_{n-1}]$ and $p(x_0, \ldots, x_{n-1})$ is a polynomial $\in A[x_0, \ldots, x_{n-1}]$ such that for all $\overline{a} \in A$ if $q(\overline{a}) = 0$ for all $q \in I$ then $p(\overline{a}) = 0$. Then for some positive integer $k, p^k \in I$. [Hodges, 1993, p. 366]

The *Nullstellensatz* is proved via quantifier elimination, and its generalisation called the Strong *Nullstellensatz* is used to establish certain results in duality theory. [nLab, 2019]

[4] The term is used in a non-technical sense, since these examples predate model theory by some time.

4.1. Infinitesimals As Ideal Elements

The next case one may want to consider is that of infinitesimals. Although infinitesimals are not explicitly listed by Hilbert as one of the canonical cases of ideal elements in his [1984], nor do they appear to be considered as such by the other authors Cantù considers,[5] I will briefly illustrate how modern authors such as Robinson [1996] and Goldblatt [1998] present the advantages of working in nonstandard analysis.

In his [1996, pp. 1–3], Robinson writes that the 'meaning' of a limit is more appealing if given in terms of infinitesimals — it is simpler.[6] Moreover,

> Leibniz's ideas [that is, infinitesimal calculus] can lead to a *fruitful* approach to classical Analysis and to many other branches of mathematics. [...] Infinitesimals have generalisations in topology which lead to fruitful applications. [Robinson, 1996, p. 2, emphasis added]

Thus, infinitesimals are fruitful; they lead to simplifications and generalisations in mathematics.

Similarly, one reads in the preface to [Goldblatt, 1998]:

> What does nonstandard analysis offer to our understanding of mathematics? [...] New definitions of familiar concepts, often simpler [...] New and insightful (often simpler) proofs of familiar theorems. [Goldblatt, 1998, p. vii]

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[5] With the exception perhaps of Veronese, *cf.* [Cantù, 2013, pp. 94–95].

[6] Here is the full quote:

> Underlying the fundamental notions of the branch of mathematics known as Analysis is the concept of a limit. Derivatives and integrals, the sum of an infinite series and the continuity of a function all are defined in terms of limits. For example, let $f(x)$ be a real-valued function which is defined for all x in the open interval $(0,1)$ and let x_0 be a number which belongs to that interval. Then the real number a is the *derivative of* $f(x)$ *at* x_0, in symbols 1.1.1 $f'(x_0) = (\frac{df}{dx})_{x=x_0} = a$ if 1.1.2 $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = a$. Suppose we ask a well-trained mathematician for the meaning of 1.1.2. Then we may rely on it that, except for inessential variations and terminological differences (such as the use of certain topological notions), his explanation will be thus:
>
> For any positive number ϵ there exists a positive number δ such that $|\frac{f(x)-f(x_0)}{x-x_0} - a| < \epsilon$ for all x in $(0,1)$ for which $0 < |x - x_0| < \delta$.
>
> Let us now ask our mathematician whether he would not accept the following more direct interpretation of 1.1.1 and 1.1.2.
>
> For any x in the interval of definition of $f(x)$ such that $dx = x - x_0$ is *infinitely close to* 0 but not equal to 0, the ratio $\frac{df}{dx}$, where $df = f(x) - f(x_0)$, is *infinitely close* to a. To this question we may expect the answer that our definition might be *simpler* [emphasis added] in appearance, but totally wrong. [...] [Robinson, 1996, pp. 1–2]

Thus, at least some mathematicians seem to argue in favour of infinitesimals because they allow for more perspicuous proofs, clearer expression of foundational concepts, and novel results. Working with infinitesimals, they claim, presents some epistemic advantages. Although the quotes above do not constitute conclusive evidence in that respect, it seems reasonable to allow infinitesimals under the umbrella of ideal elements as understood by Cantù.[7]

Let me now turn to the question of whether the reals, augmented by infinitesimals, constitute a good domain extension for Manders (and therefore whether infinitesimals count as ideal elements for him). If one considers the reals extended by infinitesimals (from now on denoted by $^*\mathbb{R}$), then the model one obtains is not the existential closure of $\mathbb{R}$ over the theory of the reals. Adjunctions that are conservative over the theory one is considering are not going to be existential closures, hence they cannot be good cases of domain extension according to Manders's framework. In the specific case of the real numbers, any nonstandard model for the theory is going to be conservative over the theory of the reals. Hence, the theory of the original model, namely, $\mathbb{R}$, does not undergo the simplification that Manders is after — *i.e.*, there is no 'squeezing out the middle case', nor any quantifier-elimination kind of simplification occurring.

Thus Manders's proposal seems to work well in several cases of adjunction of ideal elements, but not all.[8] While this does not undermine his proposal of existential closure as one sufficient condition for deeming a domain extension good or successful, it does seem to suggest that his explication of traditional theoretical virtues via model-theoretical ones is more limited than it might

[7] Even though I believe there is little doubt that infinitesimals are ideal elements at least in the epistemic sense, I should note that one would have to be pretty liberal with what counts as 'removing exceptions', if one wanted to argue that Cantù's argument (1)–(4) can be effortlessly read off of Robinson's and Goldblatt's quotes. Here is one possible modification of Cantù's argument: We replace goal (G'') with goal (G''') of making formal mathematics easier to understand and as close as possible to naïve intuitions, and the supporting values (V) and (V') with values (V''): Ease of comprehension of a mathematical theory is a desirable value in mathematics, and (V'''): Ease of comprehension is desirable because it increases fruitfulness. Historical proponents of infinitesimal calculus however may have appealed to the argument precisely as it is in [Cantù, 2013] though.

[8] Here the reader might wonder what happens if instead of considering $\mathbb{R}$ as the starting point of an extension, as I just did, we consider cases where $\mathbb{R}$ is the extended domain — for example, with respect to $\mathbb{Q}$. It is indeed true that there is a tradition regarding irrational numbers as ideal elements with respect to $\mathbb{Q}$, and the case of $\mathbb{R}$ as an extension of $\mathbb{Q}$ could potentially be problematic for Manders's account; $\mathbb{R}$ is not the existential closure of $\mathbb{Q}$ as a field. Since algebraic closure and existential closure collapse into the same notion for fields, this means that $\mathbb{R}$ is not the existential closure of $\mathbb{Q}$; so it is problematic to accommodate on Manders's framework. To this, the adopter of Manders's framework for extensions via ideal elements could give two replies. One is that indeed, $\mathbb{R}$ can be regarded as an extension of $\mathbb{Q}$ via ideal elements, but only if one departs from the classical mathematician's viewpoint. The second is that this is only to be expected, since what makes the real numbers worthy of the mathematician's attention is their completeness, and completeness is not expressible as a first-order formula, while existential closure only deals with preservation of first-order formulas. One may accept these two replies as satisfactory, but note that they seem to have the consequence of making Manders's account of domain extension more restrictive.

seem at first sight. If the adjunction of infinitesimals is not a case of existential closure, the 'fruitfulness' and 'simplification' afforded by infinitesimals remains unexplained on Manders's framework.

In the next section, I introduce an alternative conceptualisation of domain extensions and consider whether it can account for the status of infinitesimals as ideal elements.

5. DOMAIN EXTENSION ACCORDING TO DEDEKIND

In a footnote in his paper, Manders refers in passing to an alternative way of conceiving of domain extensions for number domains called the Law of Permanence of Forms [Manders, 1989, p. 555]. There he summarises the content of the law of permanence as requiring that certain universal properties about basic arithmetical operations be preserved in an extension of a mathematical domain. Manders seems to dismiss quickly the law of permanence as not being specific enough in determining what properties are worth preserving in a domain extension. In order to assess the limits of the law of permanence as an alternative to Manders's notion of successful domain extension, in this section I will (i) briefly discuss the origin of this law, and then (ii) introduce what seems to be Dedekind's take on the law of permanence. This will then form the basis for an alternative (semi-)formal criterion for good domain extension, against which I will compare Manders's own.

5.1. The Law of Permanence of Forms

The law of permanence, first introduced by British algebraist George Peacock (1791–1858), states that the only algebraic laws the mathematician should accept are those that — in modern terms — are conservative over certain[9] results of elementary arithmetic. Peacock introduces said 'law' or 'principle' in the context of justifying formal algebra as a generalisation of arithmetic, where 'formal' algebra stands for the part of algebra that studies forms (of equations). A much more detailed discussion of Peacock's views on mathematics and the precise role the principle was meant to fulfil in his philosophy of mathematics can be found in Detlefsen [2005, pp. 271–277]. Here I merely explain the principle to the extent that is needed to give some context to Dedekind's views (to be examined in the next subsection).

First let us consider one of Peacock's own formulations of the law of permanence:

> Let us again recur to this principle or law of the permanence of equivalent forms [...]. "Whatever form is Algebraically equivalent to another, when expressed in general symbols, must be true, whatever those symbols denote." Conversely, if we discover an equivalent form in Arithmetical

[9] As we will also see for Dedekind, this restriction of which laws of arithmetic are the ones to preserve under extensions is doing some rather non-trivial work in these criteria for extensions.

> Algebra or any other subordinate science, when the symbols are general in form though specific in their nature, the same must be an equivalent form, when the symbols are general in their nature as well as in their form. [Peacock, 1830, §132, p. 104]

'Arithmetical Algebra' in the passage above just is arithmetic, and 'Symbolic Algebra' is algebra. Peacock's claim is that expressions of elementary arithmetic such as $5 = 5$, or $5 + 5 = 2 \cdot 5$, which are valid only for arithmetical quantities, become laws of symbolic algebra when expressed via symbols that are 'general in their form' (*i.e.*, variables) and 'in nature' (*i.e.*, they are allowed to range over any kind of quantity, not just arithmetical quantities). As the quote below will clarify, Peacock sees arithmetic and algebra as being connected as a more specific and a more general formulation of the same science, the difference being in the semantic value of the symbols deployed by each in the statement of its propositions:

> But though the science of arithmetic, or of arithmetical algebra, does not furnish an adequate foundation for the science of symbolical algebra, it necessarily *suggests* its principles, or rather its law of combination; for in as much as symbolical algebra, though arbitrary in the authority of its principles, is not arbitrary in their application, being required to include arithmetical algebra as well as other sciences, it is evident that their rules must be identical with each other, as far as those sciences proceed together in common: the real distinction between them will arise from the *supposition or assumption that the symbols in symbolical algebra are perfectly general and unlimited both in value and representation, and that the operations to which they are subject are equally general likewise.* [Peacock, 1834, p. 195, emphasis original]

The principle roughly prescribes that 'symbolic algebra' is, for the most part, a recasting in variables of the already well-known truths of 'arithmetical algebra'. Thus, for example, if in arithmetic(al algebra) one finds that $+1-1 = 0, +2-2 = 0, +3-3 = 0, \cdots$, in symbolic algebra one can simply assert the general symbolic principle $+a - a = 0$.

Peacock however recognises that some of the laws (statements) of his symbolic algebra may not be so 'derived' (or to use Peacock's own terminology, 'suggested') from arithmetic. It is therefore necessary to offer a principled way of guiding formation of new principles in symbolic algebra, and what Peacock offers is more or less a conservativity criterion. If a certain statement is true in arithmetic, then one cannot accept into symbolic algebra another statement that would contradict the arithmetical one.

Peacock's law, as Detlefsen [2005, p. 272] also points out, was already somehow foreshadowed by other writers, and it is also quoted almost verbatim in the German-speaking context by Hankel [1867, pp. 11, 15].[10] Thus, even though I could not find direct evidence of Dedekind's having read Peacock's writings, there does seem to be a similarity in the mathematicians' ideas about generalisation of arithmetic via algebra, and extension of functions and domains in mathematics, respectively. Dedekind [1854] can be read as offering a criterion for fruitful domain extension which is strongly reminiscent of Peacock's principle. This is also noted by Ferreirós [2007, p. 219], who writes:

> This principle [Dedekind's, *author's note*] is analogous to Ohm's ideas on how to generalize arithmetical operations, and to the famous 'principle of permanence' formulated by Peacock around 1830 (still found in [Hankel, 1867]).

In the next subsection I thus present Dedekind's analogous ideas on domain extension as expressed in [Dedekind, 1854].

5.2. Early Dedekind on Domain Extension

Dedekind's *Habilitationsrede* [1854] main claim is that, just as in the other sciences,

> In mathematics too, the definitions necessarily appear at the outset in a restricted form, and their generalisation emerges only in the course of further development.

He then follows immediately with a remark that is both puzzling to the modern reader, and familiar to someone acquainted with Peacock's principle:

> But [...] these extensions of definitions no longer allow scope for arbitrariness; on the contrary, they follow with compelling necessity from the earlier restricted definitions, provided one applies the following principle: Laws which emerge from the initial definitions and which are characteristic for the concepts that they designate are to be considered as *of general validity*.

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[10] On p. 11 one reads:

> Der hierin enthaltene hodegetische Grundsatz kann als das Prinzip der Permanenz der formalen Gesetzen bezeichnet werden und besteht darin: Wenn zwei in allgemeinen Zeichen der arithmetica universalis ausgedrückte Formen einander gleich sind, so sollen sie einander auch gleich bleiben, wenn die Zeichen aufhören, einfachen Größen zu bezeichnen, und daher auch die Operationen einen irgend welchen anderen Inhalt bekommen.

(Author's translation: The introductory base principle herein contained can be dubbed as the Principle of Permanence of formal Laws and consists in the following: whenever two forms expressed in general signs of *arithmetica universalis* are equal to one another, they should also remain equal to one another when the signs cease to designate simple quantities and therefore also the operations acquire some other content [*i.e.*, meaning].)

Note how, just as Peacock rushes to defend algebra as a non-arbitrary generalisation of arithmetic, so does Dedekind not just for algebra, but for any extended mathematical definition (or function). How the extension happens is however somewhat different: for Peacock, the extension concerns the range of validity of certain algebraic propositions; for Dedekind, the extension seems to consist in augmenting the domain of objects that fall under a certain concept (for example, number). Dedekind's understanding of extension however can be seen as equivalent to Peacock's; for concepts are determined by 'characteristic' laws which 'emerge from the initial definitions' of said concepts. So in the end to expand a concept in Dedekind's sense (at least in arithmetic) is the same as interpreting certain special arithmetic statements as being not just about a restricted domain, but a wider, richer one. This is Peacock's law for the permanence of forms — the law guiding the generalisation of arithmetical results to algebra.

There is also a difference in scope between Dedekind's criterion and Peacock's law; for Dedekind seems to be offering a (prescriptive, as well as descriptive) criterion for *all* conceptual expansions in mathematics, while Peacock seems to be focused on the generalisation (where generalisation consists in expanding the domain of application of a statement) of arithmetic only. The claim of similarity between Dedekind and Peacock substantiated, there is still another aspect of Dedekind's reflections that is worth mentioning, namely, his focus on functions, *i.e.*, operations. That is, Dedekind's criterion seemingly applies to more than just the extended domain and codomain of functions. His interest is particularly clear in the following passage concerning numbers and basic arithmetical operations:

> [7] Elementary arithmetic is based upon the formation of ordinal and cardinal numbers; the successive progress from one member of the sequence of positive integers to the next is the first and simplest operation of arithmetic; all other operations rest on it. If one collects into a single act the multiply repeated performance of this elementary operation, one arrives at the concept of addition. From this concept that of multiplication is formed in a similar manner, and from multiplication that of exponentiation. But the definitions we thereby obtain for these fundamental operations no longer suffice for the further development of arithmetic, and that is because it assumes that the numbers with which it teaches us to operate are restricted to a very narrow domain. That is, arithmetic requires us, upon the introduction of each of these operations, to create the entire existing domain of numbers anew; or, more precisely, it demands that the indirect, inverse operations of subtraction, division, and the like be unconditionally applicable. And this requirement makes it necessary to create new classes of numbers, since with the original sequence of positive integers the requirement cannot be satisfied. Thus one obtains the negative, rational, irrational, and finally also the so-called imaginary

> numbers. Now, after the number domain has been extended in this manner it becomes necessary to define the operations anew [...]. [Dedekind, 1854, §7]

This passage lays bare how domain extension and operation expansion relate for Dedekind, at least in the case of numbers: the given domain is that of the natural numbers, and the given operation just successor. From the successor function one obtains the other direct operations of addition and multiplication, each defined as iterations of the previously defined function. Once all the 'direct' operations are defined, one may want to introduce the inverses. For addition, this is subtraction. However for subtraction to be defined between two arbitrary elements of the domain, the domain has to be extended (*i.e.*, the concept of number is expanded) to include also negative numbers. Similarly, introducing the inverse operation for multiplication, namely, division, together with a closure requirement for the domain under the new operation, leads to the introduction of rational numbers. This iterative construction (introduce a new operation, then new numbers so that the domain is closed under said operation) goes all the way up to the imaginary numbers. But with each round of extension of the number domain, old operations also need to be defined anew.[11] Dedekind is not explicit about this, but it seems that what allows the process to stop is the achievement of a sufficiently rich (number) domain that is also closed under all the defined operations, taken in their most general form. To see how one can adapt the 'definition' of an operation to an extended domain, consider Dedekind's example of multiplication:

> We already have a definite example in multiplication. This operation arose from the requirement that a multiply-repeated performance of an operation of the next lower rank [Ordnung] — namely the addition of a fixed positive or negative addend (the so-called multiplicand) — be collected together into a single act. The multiplier — that is, the number which states how often the addition of the multiplicand is to be thought of as repeated — is therefore at the outset necessarily a positive integer; a negative multiplier would, under this first definition of multiplication, make absolutely no sense. A special definition is therefore needed in order to

[11] Note that Dedekind seems to say that at each round of extension, strictly speaking one is not simply adding new elements to the number domain or redefining operations, but the whole number domain is 'creat[ed] [...] anew'. I find it plausible that here Dedekind is merely recognising that adding numbers to the old domain is effectively a change in the concept of number. Consequently, adding new numbers yields a rewriting of the definition of the concept of number altogether, and in that sense, the previously existing numbers are also recreated once the new numbers are in place. This reading is admittedly weaker than other readings of Dedekind's 'creationism' about numbers especially in *Was sind und was sollen die Zahlen* and *Stetigkeit und irrationale Zahlen* ([1888; 1872] translated in [Dedekind, 1963]) as presented, *e.g.*, in [Tait, 1996; Hallett, 2019]. A careful discussion of the relationship between definitions and creation in Dedekind's writings goes beyond the scope of the present paper.

> admit negative multipliers as well, and thereby to liberate the operation from the initial constraint; but such a definition involves *a priori* complete arbitrariness, and it would only later be decided whether then this arbitrarily chosen definition would bring any real use to arithmetic; and even if the definition succeeded, one could only call it a lucky guess, a happy coincidence — the sort of thing a scientific method ought to avoid. So let us instead apply our general principle. We must investigate which laws govern the product if the multiplier undergoes in succession the same general alterations which led to the creation of the sequence of negative integers out of the sequence of positive integers. For this it suffices if we determine the alteration which the product undergoes if one makes the simplest numerical operation with the multiplier, namely, allowing it to go over into the next-following number. By successive repetition of this operation we obtain the familiar addition theorem for the multiplier: in order to multiply a number by a sum, one multiplies it by each summand and then adds these partial products together. From this theorem a subtraction theorem immediately follows for the case where the minuend is greater than the subtrahend. If one now declares this law to be valid in general (that is, to hold also when the difference which the multiplier represents is negative) then one obtains the definition of multiplication with negative multipliers; and it is then of course no accident that the general law which multiplication obeys is exactly the same for both cases. [Dedekind, 1854, §8]

The 'original definition' of multiplication as iterated addition has to be amended so that it may also be defined for negative factors, because one cannot repeat an action a negative number of times. Instead, left distributivity is considered as the 'general law' that is to be preserved even in the extended domain.

At this point it is important to notice an element of imprecision in Dedekind's discussion, namely that he seems to be considering simultaneously two types of what one may call conceptual extensions in mathematics. On the one hand there is the introduction of new operations (or functions, as per his lecture title) alongside 'the chain of previous ones'. This is akin to an expansion of the language which one uses to 'talk about' the domain, and here is an example of how that is supposed to work. If we keep the domain of a structure $\mathcal{A}$ fixed, we can add, say, relation symbols to the language so as to obtain a new structure $\mathcal{A}'$ that also interprets these new symbols as well as the old ones. If we let N be the set of all natural numbers, then we can consider both the structure $\mathbb{N}$ of the natural numbers in the language $L = \{0, +\}$ and the structure $\mathbb{N}'$ of the natural numbers in the language $L' = \{0, 1, +, \cdot\}$. The domain underlying both $\mathbb{N}$ and $\mathbb{N}'$ is the same; no new elements have been added to N. Yet there is an expansion occurring between the two, which involves operations and constants only. The second type of extension consists in adding elements to the domain of functions, or the introduction of new objects under an old concept. For example, while multiplication as originally defined can only be performed between two positive integers, it can be later redefined so as to allow also negative integers among its domain (and range). This extension can be exemplified as follows. The $\mathcal{L}$-structure one starts with is $\mathbb{N}$, where the domain is just N, and the language $\mathcal{L}$ comprises a symbol for addition, '+', that $\mathbb{N}$ interprets as the function $\{((m,n), m+n) : m,n \in N\}$. One then adds the negative integers to N, thus using Z as the domain of the new $\mathcal{L}$-structure $\mathbb{Z}$, and moreover '+' is now interpreted as $\{((m,n), m+n) : m,n \in Z\} \supseteq \{((m,n), m+n) : m,n \in N\}$. This second change is more straightforwardly a case of adding elements to the total domain of the model as well as to the domains of the individual functions. These two (expansion of the language versus extension of the domain) are, in principle, two distinct kinds of extension, yet Dedekind does not seem to note this. I believe the reason why Dedekind does not examine the two cases of extension separately is because he does not believe one can occur in the absence of the other: if the mathematician introduces new elements to the domain in question, then she needs to be able to determine how the old operations or functions apply to the new objects.

5.3. Formalizing Dedekind's Proposal

As the quote illustrates, there is a lot happening in Dedekind's text. Hence, in order to bring out the points of comparison with Manders's notion of domain extension, it might be helpful to give a model-theoretic characterisation of Dedekind's views. I propose the following:

Definition (Dedekind-extension). Let $\mathcal{L}$, $\mathcal{L}'$ be two first-order languages without relation symbols such that $\mathcal{L} \subseteq \mathcal{L}'$. Let $\mathcal{A}$ be an $\mathcal{L}$-structure. Then a Dedekind-extension of $\mathcal{A}$ consists in finding a class of $\mathcal{L}'$-structures $\mathcal{K}$ such that for all $\mathcal{B} \in \mathcal{K}$:

(i) $A \subseteq B \restriction \mathcal{L}$;
(ii) $B \vDash \forall \overline{x} \varphi(\overline{x})$ whenever $A \vDash \forall \overline{x} \varphi(\overline{x})$, φ a quantifier-free, positive formula in $\mathcal{L}'$.

Condition (i) of the definition asks that $\mathcal{A}$ be embedded in $\mathcal{B}$. This ensures the preservation of functions among the individuals of the original model A, if the languages $\mathcal{L}$, $\mathcal{L}'$ include function symbols interpreted in $\mathcal{A}$ and $\mathcal{B}$.

Condition (ii) aims to capture Dedekind's rule about certain laws that are to be considered 'as of general validity'. The positivity restriction on φ is motivated by technical issues one would otherwise encounter,[12] and also by the fact that equations seem to have a privileged status over inequations. Preservation

[12] Consider for example $\mathbb{Z}$ as $\mathcal{A}$, $\mathbb{Q}$ as $\mathcal{B}$. If the formalization is to capture Dedekind's notion of good domain extension, then $\mathbb{Q}$ should turn out to be one such for $\mathbb{Z}$. In order to do that, my definition needs to rule out, *e.g.*, $\forall x\, 2x \neq 1$ from the class of sentences which one wants to preserve between $\mathbb{Z}$ and $\mathbb{Q}$, and one way of doing that is by excluding order relation(s) from appearing in φ.

of equations is an important theme of results in universal algebra, as witnessed by the stream of research in universal algebra consisting in generalisations and applications of Birkhoff's theorem.[13] Moreover, other nineteenth-century mathematicians such as Hankel [1867, pp. 26, 40–41], and Peacock [1834] implicitly recognise the importance of preserving equations when extending the number domain and arithmetical operations. Thus the restrictions on φ in the formalization do not have to be seen as arbitrary.

To be clear, the definition of Dedekind-extension alone does not answer the question of whether a given mathematical domain is or is not a good domain extension of another domain, according to the view I ascribe to Dedekind. The choice of languages $\mathcal{L}$, $\mathcal{L}'$ also plays a non-trivial role in that sense. Consider for instance the following example: let $\mathbb{N}$, $\mathbb{Z}$ be the models under considerations, with $< \in \mathcal{L}$. Then $\mathbb{Z}$ cannot be a Dedekind-extension of $\mathbb{N}$, because $\mathbb{N} \vDash \forall x(x > 0)$, which is false in $\mathbb{Z}$. If we exclude $<$ from our language, however, the problem does not arise and $\mathbb{Z}$ can be considered a Dedekind-extension of $\mathbb{N}$. This means that the notion of Dedekind-extension is still, to some extent, context dependent. This however is also true of Manders's notion, for existential closure and model completion are also language sensitive.

6. EXTENDING THE CONCEPT OF NUMBER

In the previous section, I presented Dedekind's 1854 reflections as suggesting a conception of domain extension akin to that underlying the principle of permanence of equivalent forms, and I proposed a model-theoretic semi-formalization of the criterion. This was done in an attempt to make progress on the *normative* question of what makes certain domain extensions 'good'. At the same time, work in the previous section might leave the reader wondering about the *historical* question of whether the criterion thus formalized truly does justice to Dedekind's attitude towards new number systems being developed in the mid-1850s. To answer this question, I briefly recall in this section two such number-theoretic developments and argue that in both cases it appears unlikely that Dedekind would regard them as extensions in the sense of his [1854].

6.1. Quaternions, Octonions and Other Hypercomplex Numbers

The first case in consideration is that of quaternions, or so-called hypercomplex numbers more generally.

A hypercomplex number is traditionally any number belonging to a (unital) algebra constructed on top of the real numbers. There are several distinct hypercomplex number systems that can be defined as vector spaces over the real numbers; quaternions and octonions are the number systems of dimensions 4 and 8, respectively as their names suggest. This means that each quaternion

[13] The theorem states that a class of algebras $\mathcal{K}$ is equationally axiomatisable if and only if it is a variety — *i.e.*, if and only if it satisfies certain closure properties. Establishing that a class of algebras is a variety is easier than having to give an axiomatisation of a class of algebras explicitly, and knowing that a certain structure is equationally definable is extremely valuable.

can be represented by a quadruplet of real numbers, while each octonion can be represented by an octuplet (or 8-tuple). The problem is that multiplication in quaternions is not commutative, and in the octonions it even fails to be associative. But commutativity and associativity of multiplication are expressible as universal positive statements of the kind a Dedekind-extension should preserve, by definition. This makes clear that the proposal at hand, although inspired by Dedekind, cannot account for these extensions as good extensions.

Dedekind writes on the hypercomplex numbers in two papers [1885; 1887]), and in both papers his presentation consists in letting the hypercomplex numbers be expressible as finite sums of the form $\Sigma\, \xi_\iota e_\iota$, where e_ι is a 'principal unit' (*Haupteinheit*) of the hypercomplex numbers (think i, j, k for quaternions). Then operations between any two hypercomplex numbers can be defined as operations on the units, which taken together form a *basis*. These operations can be expressed as linear transformations, that is, matrices, and some of their crucial properties are therefore determined by the value of the determinant of the corresponding matrix. This is what Dedekind is concerned with in these writings.

By way of conclusion in the 1885 paper, Dedekind writes (author's translation):

> [...] every system of n principal units, as it happens in Mr Weierstrass's investigation, may always be understood as an n-valued system from n ordinary numbers, in this way, that each rational equality between the n principal units is true if and only if it holds for each of the special systems $e_1^{(s)}, e_2^{(s)}, \ldots, e_n^{(s)}$ derived by us. So if we want to speak of such complex quantities as new numbers (which to me is inexpedient, because in our higher algebra there always emerge multi-value quantity systems in the manner here described), this can only be done though in a completely different, and indeed infinitely weaker sense, than in the introduction of imaginary numbers by hefty enrichment of the real-number field, or also in the introduction of Hamilton's quaternions, which although their usefulness seems to be limited to a very small field, make an unconditional claim to the character of novelty against the other numbers. ([Dedekind, 1885], in [Dedekind, 1931, p. 16])

If on the one hand this supports my interpretation that hypercomplex numbers are not genuine new numbers, it also undermines the idea that quaternions count as a special case of hypercomplex numbers in that respect. Dedekind considers them 'new enough' to count as genuine new numbers. So we are left with the following: the Dedekind-inspired account correctly aligns with a distinction between two cases of extension — namely, a domain extension due to expanding the very concept of number (such would be the quaternions) — and number domains obtained as unique extensions (up to isomorphism) of the natural numbers. My definition of Dedekind-extension adequately captures the latter kind of domain extension as good, but it leaves out hypercomplex numbers, quaternions included, despite what Dedekind writes in the excerpt above.

So, when it comes to numbers obtained by adjoining new imaginary units, it seems that Dedekind draws the line at quaternions in terms of what counts as genuine new numbers. For, if on the one hand it already seemed suggested in his [1854] that both imaginaries and quaternions are numbers, only not yet equipped with a satisfactory account of how they are obtained, on the other hand these number systems can be obtained in roughly the same way as hypercomplex numbers; so one would expect Dedekind to regard all these as either uniformly in or uniformly out of the category 'genuine domain extensions' (something needs to be a genuine domain extension to be a good one, needless to say). By looking closely at [Dedekind, 1854] one plausible suggestion is that, although both complex numbers and quaternions are genuine domain extensions (because in both cases genuinely new numbers are introduced), only the complex numbers are obtained as a closure of an already accepted number domain (the real numbers) under a certain inverse operation, namely, the inverse of exponentiation. Since the definition of Dedekind-extension strives to capture the idea of good domain extension expressed in [Dedekind, 1854], and that idea is that one extends domains to close them under operations, it is a positive feature of the definition of Dedekind-extension that it is satisfied by the complex numbers as an extension of the real numbers, but not by quaternions, since it is only complex numbers that are introduced as closure of the real numbers under square roots.

6.2. Dedekind's Ideals and Ideal Elements

A second case of what one might want and expect to turn out a case of 'good extension' for Dedekind is Dedekind's own ideals, or ideal numbers more generally. Ideal numbers were first introduced by Kummer in 1846 [Bordogna, 1996, p. 6; Edwards, 1980, p. 322] to solve the specific problem of uniqueness of factorisation for certain number domains. Unique factorisation in the case of natural (or even integer) numbers is pretty straightforward: for any non-prime natural number n, there is a unique decomposition of n into its prime factors, that is, into numbers that themselves cannot be written as the product of anything but themselves and the unit. While Kummer first introduced talk of 'ideal numbers' or 'ideal divisors' as numbers that exist beyond (outside) the domain of real (*i.e.*, existing in reality) numbers, and seemed to consider these as additional numbers to be added to the already existing ones, Dedekind's position on the status of his ideals (and the corresponding ideal numbers) is not as clear. In the upcoming subsection, I will sketch Dedekind's second version of the theory of ideals. On the basis of this sketch I will then be able to address the question of what kind of domain extension it is, if it is one at all. For interested readers [White, 2004] contains a detailed discussion of the differences between these two versions.

6.2.1. *The Class of Ideals and Dedekind's 'Rigorous Definition of Ideal Numbers'*

Dedekind presented his theory of ideals first in his supplements to Dirichlet's *Lectures on Number Theory* [1877; 1999], and then in a series of papers in the *Bulletin des Sciences Mathématiques et Astronomiques* [Dedekind, 1876; 1877]. In his *Bulletin* formulation of the theory, Dedekind comes to a 'precise' definition of *ideal number* in the following way.

The starting point is a finite degree extension (in the technical sense of a field, which can be seen as a one- or two-dimensional vector space over $\mathbb{Q}$) Ω of $\mathbb{Q}$. In this field one identifies a subring of elements that for the purposes of divisibility behave similarly to the integers. This is the ring of integers $\mathfrak{o}$ of the field Ω. The problem is that, in general, unique factorisation will fail in $\mathfrak{o}$. The point of introducing ideal divisors is to retrieve partially some of the advantages of unique factorisation even in the cases where it strictly speaking fails. Viewed as a set, $\mathfrak{o}$ is not just a subring of Ω; it is also an *ideal*, where an ideal I is a set of elements that is also an additive subring of the original ring R, and for any element $r \in R$ and $p \in I$, $rp \in I$.

Throughout, Dedekind is actually considering ideals of the ring $\mathfrak{o}$. I can now sketch Dedekind's definition of an ideal *number* (or *divisor*). Dedekind shows that for any ideal $\mathfrak{a}$, there is a positive integer h such that $\mathfrak{a}^h = \{\alpha^h; \alpha \in \mathfrak{a}\}$ is a principal ideal, *i.e.*, $\mathfrak{a}^h = \{b\alpha_1; b \in \mathfrak{o}\}$ for some α_1 in $\mathfrak{o}$. From this it immediately follows that any α^h in $\mathfrak{a}$ is of the form $b\alpha_1$ for some b, and thus that α itself is divisible by $\mu = \sqrt[h]{\alpha_1}$, and μ is an 'algebraic integer' that does not belong to the field one started with, Ω. Dedekind thus writes:

> Thus the ideal $\mathfrak{a}$ is composed of all the integer numbers contained in Ω and divisible by the integer μ; for this reason we will say that the number μ, although not contained in Ω, is *an ideal number of the field* Ω, and that it corresponds to the ideal $\mathfrak{a}$.[14] [1877, p. 246]

Dedekind stops short of identifying an ideal containing all the numbers divided by a certain ideal divisor with the ideal divisor itself. This distinction might seem analogous to that which Dedekind draws in the case of real numbers and cuts, where Dedekind says that to each cut that is not generated by a rational number there corresponds an irrational number, without saying that the cut and the number are *one and the same*.[15] One might then consider it a shallow distinction that should not be taken at face value. However there is a substantial difference between the way Dedekind then handles the real numbers versus the number domain he defines ideals over, once the ideals have been defined. In the first case, Dedekind tries to establish that the cuts *taken*

[14] Donc l'ideal $\mathfrak{a}$ est composé de tous les nombres entiers contenus dans Ω et divisibles par le nombre entier μ; pour cette raison nous dirons que le nombre μ, lors même qu'il n'est pas contenu dans Ω, est *un nombre idéal du corps* Ω, et qu'il correspond à l'ideal $\mathfrak{a}$.

[15] For a discussion of Dedekind's views on cuts and real numbers, and attending difficulties, see [Reck, 2020].

collectively as one domain, satisfy certain arithmetical and order properties. He thus establishes some continuity between the cuts and the irrational numbers they determine, and the arithmetic of natural numbers. If one looks at how Dedekind treats the integers and the rational numbers[16] one sees these steps have in common that the new numbers are defined as (ordered) pairs of the old numbers, and the arithmetic operations on the new numbers are defined in terms of the operations on the old numbers. Moreover, Dedekind seems to have a sense of the newly defined numbers as forming a new whole, a new structure (system), having certain arithmetical properties (for example, in the case of the integers, commutativity of addition and distributivity laws for addition and multiplication) that also hold for the natural numbers. It seems to me that no analogous interest can be detected in Dedekind's work on ideals. He is not trying to show that there is some deep continuity between the arithmetical properties of the natural numbers and the arithmetical properties of these putative new numbers (even though one might say that they are generated because of an investigation of divisibility and the fundamental theorem of arithmetic, and are therefore what one obtains when trying to define 'divisibility' in its most general form, *i.e.*, extend divisibility, in some sense).

6.3. Are Ideals and Hypercomplexes Good Dedekind-Extensions?

In the presentation of the ideals at hand, Dedekind explicitly distances himself from Kummer's approach to the ideals, which renders them as non-existing numbers which are only individuated by divisibility rules given through cumbersome equations [Bordogna, 1996; Edwards, 1983].

Dedekind by contrast defines ideals as classes of (complex) numbers. He claims that these equivalence classes are *not* to be seen as additions to the number domain. In other words, he does not see himself as having expanded the domain.

This is in accordance with the way extensions of the number concept are presented in the *Habilitation*. For Dedekind's ideals to count as new numbers, one should expect Dedekind to try and prove that the number domain extended to include the ideals still preserves certain 'laws' that were true of the same domain without the ideals. But Dedekind does not do that. Specifically, he does not try to prove that the fundamental arithmetical operations are preserved in the extension.

Similarly, such a concern seems to be absent from his treatment of hypercomplex numbers. Given that still in his [1872] and [1888] Dedekind proudly refers to [Dedekind, 1854] as a script the aim of which Gauss himself approved, it seems unlikely that he would not take notice of an extension of the concept

[16] Dedekind offers a construction for each of these in the *Nachlass*. I was able to gain access to his notes on the integers thanks to Emmylou Haffner, but not to those on the 'analogous construction' for the rationals, and am therefore relying on Sieg's and Schlimm's [2005] account on the matter.

of number, namely, the hypercomplex numbers, which does not tally with his description of what constrains such extensions.

I therefore favour a different position when it comes to ideals and hypercomplex numbers, namely, these are not meant to be extensions in the sense of [Dedekind, 1854]. There are two reasons for this position. First, as already mentioned, both cases present us with a conundrum: a (putative) case of extension that does not seem to satisfy Dedekind's criterion for a good extension. Second, I believe there is enough textual evidence to suggest that Dedekind treats these two cases differently from the way he treats the integers (as extensions of $\mathbb{N}$), the rationals, and the reals. (The case of complex numbers is not an issue, for it comes out as a good domain extension on my semi-formal rendition of Dedekind's 1854 criterion, and Dedekind himself does consider those as extending the real numbers). I believe that the latter (integers, rationals, reals, complexes) are genuine extensions of the number concept for Dedekind in a way that ideals and hypercomplex numbers are not, and this much is also what my semi-formalization of Dedekind's criterion suggests.

7. COMPARISON

If one considers Dedekind's criterion for extension, then the number-domain cases which Dedekind is interested in (extensions from $\mathbb{N}$ to $\mathbb{Z}$ all the way up to $\mathbb{C}$) come out as good cases of domain extension — if one limits the signatures so as to exclude order; otherwise, already $\mathbb{Z}$ as an extension of $\mathbb{N}$ would not satisfy condition (ii) in Dedekind's definition.

$\mathbb{R}(i)$, for example, would be a structure obtained as completion of another one, namely $\mathbb{R}$. One starts with domain $\mathbb{R}$, adds one new element, i, and then adds also all the appropriate algebraic combinations of i with all the elements of $\mathbb{R}$. It also seems that in the process we have been conservative over $\mathbb{R}$ as a field (not as an ordered field though, given that $\mathbb{R}$ is linearly ordered whereas $\mathbb{R}(i)$ is not). Thus this particular example is a 'good case' extension both for Manders and for Dedekind.

The extension from $\mathbb{R}$ to $^*\mathbb{R}$ also counts as a good domain extension under Dedekind's framework, unlike under Manders's. This is a significant difference which can be explained in terms of what the two different frameworks are trying to capture. Manders's use of existential closure is meant to capture cases of domain extension that aimed at gaining simplification in terms of reduced quantifier complexity of the theory. Dedekind's extension, on the other hand, is meant to capture cases of domain extension that aim at extending a given concept (*e.g.*, that of addition or of number) as much as the *essence* of the concept allows.[17]

[17] This remark might spur some readers to think that the case of forcing extensions in set theory are the kind of extensions that Dedekind should be able to account for. I have two replies to this issue. First, I am interested in Dedekind's notion of domain extension primarily in the instance where the elements added are 'ideal elements'. To the best of my knowledge, forcing extensions are not discussed in those terms in the literature. Secondly, a more appropriate condition (ii) for a notion of extension trying to capture good extensions

In the previous section I touched upon two prominent cases of putative domain extension and concluded that they seem not to qualify as such under my interpretation of Dedekind. We now turn briefly to the question of whether quaternions and ideals are well handled by Manders's notion. Quaternions ($\mathbb{H}$) are not obtained as an existential closure of $\mathbb{C}$, given that $\mathbb{C}$ is already existentially closed and not isomorphic to $\mathbb{H}$. At the same time, it is not straightforward that there should be some class of structures K over which $\mathbb{H}$ is existentially closed. Similarly for ideals defined over some field. Without such results then, one cannot definitively rule whether quaternions and ideals count as good domain extensions for Manders.[18]

In Section 3 I explained how Manders argues that for any theory such that each solvability condition has one weak complement, existential closure yields simplification and conceptual unification [Manders, 1989, pp. 554–556]. Manders spells out simplification and unification in terms of formal properties of the theories of the existentially closed models one obtains. In other words, Manders suggests that existential closure is a sufficient condition for considering a domain extension as a good, fruitful one,[19] and he points out that a few historically important cases (complex numbers, points, and lines at infinity) are indeed cases of existential closure. As such, they really are a means of partially pursuing goal (G''): via results like the *Nullstellensatz*, they allow 'dual transformations between models to be introduced', and in virtue of what Manders calls the 'squeezing out the middle case' property, they remove exceptions.

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of theories in the language of set theory would be one requiring preservation of *absolute*, that is, Δ_0 notions between the original structure and the extension.

[18] Given that on the face of it hypercomplex numbers and ideals do not seem to fall straightforwardly in the category of good domain extensions under either of the frameworks considered here, one might wonder whether a satisfactory account of domain extension is one that validates hypercomplex numbers and ideals as good domain extensions.

My semi-formalization of Dedekind's proposal allows considering commutativity of addition and multiplication as some of the laws any extension of the number concept (or of a number domain) should preserve. This has as a straightforward consequence that the quaternions therefore cannot count as a case of Dedekind-extension. Moreover, it is consistent with my explanation of what Manders's criterion is supposed to capture and what Dedekind's is supposed to capture that neither would then consider ideals and hypercomplex numbers as good extensions. For Manders's sufficient criterion, I believe, captures the cases of domain extension that are motivated by adjoining solutions to equations that are expressible, though unsolvable, in the original domain. Clearly, hypercomplex numbers and ideals are no such things. Dedekind's criterion on the other hand ought to capture the cases of domain extension that originate from expanding the domain of well-definedness of algebraic operations *as much as possible*. I want to also argue that introduction of ideals is not brought about by wanting to 'close' a domain under some operations — that is, functions — on the original, restricted, domain, which is the kind of domain extension I take Dedekind's notion to capture. The problem of course is that this makes both accounts (Manders's and mine, based on Dedekind) somewhat normative, instead of merely descriptive.

[19] More precisely, Manders suggests that for theories satisfying certain properties, existential closure is a sufficient condition for a good domain extension.

Dedekind's notion, meanwhile, focuses on the preservation of certain features (theorems) of a theory which are considered to be essential to the concepts involved (of addition, for example). Because of this, a Dedekind-extension pursues the goal (G'') by allowing direct and inverse operations to satisfy closure properties. This splitting of goal (G'') suggests the possibility of using both Dedekind's and Manders's proposal to develop a disjunctive characterisation of historical cases of ideal elements.

Nevertheless, there is a fundamental conceptual difference between Dedekind on the one hand, and Manders on the other. Manders insists that, after the fact of the extension, we might find ourselves in a position to reject properties or facts which, before the extension, had been considered essential to the concept that the structure in question was meant to represent or model (in a loose sense of the terms). As already noted in Section 5, he even refers to Peacock's principle of permanence of equivalent forms while remarking that, despite its *prima facie* plausibility, it cannot be held true at all times. This seems to be an irreconcilable difference in the way the two opposing camps — Manders on one hand, Dedekind on the other — conceive of the goals and benefits of domain extensions. Preservation of the essence of a function is the criterion, for Dedekind, that guides the mathematician to extend her functions and concepts in one way rather than another.

8. CONCLUSION

In this paper I started by giving an overview of ideal elements in mathematics, seen as an example of good extensions of mathematical domains. I considered [Manders, 1989] as a candidate for a model-theoretic explication of Hilbert's method of ideal elements and its role in the advancement of pure mathematics.

Manders's conception of domain extension however seemed to be ill-equipped to explain the 'ideality' of domain extensions which occur when the mathematician pursues closure under operations, or simplification and fruitfulness of a different sort than that granted by quantifier elimination. While it is true that Manders only aims at offering a sufficient condition for successful or good domain extensions, the number and kind of cases which do not exhibit the model-theoretic characteristics he focuses on suggest that Manders's explanation of the fruitfulness of domain extensions is, at best, partial.

In an attempt to shed light on the related questions of how one should understand attributions of theoretical virtues like simplicity and fruitfulness to extended mathematical domains (or attending theories), and of whether such virtues can be reduced to model-theoretic traits of the structures or theories in question the way Manders suggests we should do, I used [Dedekind, 1854] as a basis for an alternative model-theoretic criterion of good domain extensions. The upshot of the comparison between Dedekind and Manders is that they both consider the complex numbers as a fruitful case of domain extension, but then seem to disagree on most other cases. Quaternions and ideals are not straightforward to adjudicate on Manders's framework, but also they do not seem like the kind of extension his criterion is intended to capture as a good case of extension; they also do not satisfy the definition of Dedekind-extension.

The case of the reals with infinitesimals, on the other hand, constitutes a good extension for Dedekind, although in a way that does not reveal the (epistemological) advantages of working with infinitesimals. It cannot for Manders. For infinitesimals then one is left with the following two options: either the understanding of ideal elements offered by Manders's formalization is too restrictive, because it does not account for the role of ideal elements as 'proof simplifiers'; or, if one takes Manders's proposal as normative, infinitesimals are not ideal elements after all. However, there might be a third option if one looks more carefully at the discussion of ideal elements and extensions at the beginning of the paper, namely one might want to distinguish between ideal elements which are introduced to round off a domain in Manders's sense or to simplify the mathematics in Manders's way, and ideal elements which are introduced to satisfy closure under certain operations. Under this suggestion, Dedekind-extensions are the extensions that involve a genuine enlargement of the domain of objects in the domain, and an enlargement that achieves closure under certain operations. This solution would do justice to the historical discussion brought forward by Cantù, while highlighting both the strengths and potential limitations of the use of model-theoretic concepts to understand domain extension in mathematics.

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