Indicative Conditionals: Probabilities and Relevance

Francesco Berto[1] and Aybüke Özgün[2]

[1] Arché, University of St Andrews and ILLC, University of Amsterdam,
fb96@st-andrews.ac.uk

[2] ILLC, University of Amsterdam, a.ozgun@uva.nl

Abstract

We propose a new account of indicative conditionals, giving acceptability and logical closure conditions for them. We start from Adams' Thesis: the claim that the acceptability of a simple indicative equals the corresponding conditional probability. The Thesis is widely endorsed, but arguably false and refuted by empirical research. To fix it, we submit, we need a relevance constraint: we accept a simple conditional $\varphi \rightarrow \psi$ to the extent that (i) the conditional probability $p(\psi|\varphi)$ is high, provided that (ii) φ is relevant for ψ. How (i) should work is well-understood. It is (ii) that holds the key to improve our understanding of conditionals.

Our account has (i) a probabilistic component, using Popper functions; (ii) a relevance component, given via an algebraic structure of topics or subject matters. We present a probabilistic logic for simple indicatives, and argue that its (in)validities are both theoretically desirable and in line with empirical results on how people reason with conditionals.

Keywords: Conditionals, Conditional Probabilities, Relevance, Adams' Thesis, Subject Matter

> People do not consider the conditional probability to be the only basis for asserting a conditional, even if they judge the probability of the conditional to be the conditional probability.

Evans & Over, *If*

1 Adams' Thesis and the Problem of Relevance

Adams' Thesis (Adams, 1966, 1975) has it that the acceptability of a simple indicative conditional $\varphi \rightarrow \psi$ – an indicative with no indicatives in φ or in ψ – equals the corresponding conditional probability:

(AT) $Acc(\varphi \rightarrow \psi) = \mathrm{p}(\psi|\varphi)$

Stalnaker's Hypothesis (Stalnaker, 1975), also called the Equation, has it that the probability of an indicative $\varphi \rightarrow \psi$ equals its conditional probability:

(SH) $\mathrm{p}(\varphi \rightarrow \psi) = \mathrm{p}(\psi|\varphi)$[1]

AT is popular in philosophy among proponents of the non-propositional view of indicatives (Edgington, 1995; Adams, 1998; Bennett, 2003). SH is popular in psychology: it is in line with the New Paradigm Psychology of Reasoning (Over, 2009; Elqayam and Over, 2013), which puts probabilities at center stage in the study of reasoning, and handles conditionals probabilistically (Evans and Over, 2004; Oaksford and Chater, 2010). The Paradigm is becoming so dominant that even proponents of essentially non-probabilistic accounts of the conditional, like the mental models theory (Johnson-Laird and Byrne, 2002), feel the need to relate their view to probabilities (Girotto and Johnson-Laird, 2010).

However, Lewis' and others' notorious triviality results (Lewis, 1976; Hajek, 1989) are often taken as showing that SH can't be *quite* right. On the other hand, by endorsing AT rather than SH, non-propositionalists can insist that indicatives be handled probabilistically: they are safe from triviality for they don't express propositions and cannot generally be embedded (hence the limitation to simple conditionals in AT).[2] They generally lack truth values,[3] thus probabilities of truth properly so called, as Adams realized. But they can have acceptability conditions, as per AT. McGee (1986) claims that '[AT] describes what English speakers assert and accept with unfailing accuracy' (485). And Jackson:

> There is a great deal of evidence for [AT]. There is head-counting evidence. Very many philosophers of otherwise differing opinions have found [AT] highly intuitive. There is case-by-case evidence. Take a conditional which is highly assertible [...]; it will invariably be one whose consequent is highly probable given its antecedent. (Jackson, 1987, p. 12)

But AT is false. A conditional probability $\mathrm{p}(\psi|\varphi)$ for an unacceptable indicative can be high because ψ is already likely and has little to do with φ:

1. If Brexit causes a recession, then Jupiter is a planet.

One may claim that (1) is unacceptable for its consequent has probability 1. We'll come back to the issue of conditionals with extreme antecedent or consequent probabilities. Even granting the claim, sometimes conditionals with high but less than 1 probability of their consequent are unacceptable because this has little to do with their antecedent:

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[1]Recall that $\mathrm{p}(\psi|\varphi) = \frac{\mathrm{p}(\psi \wedge \varphi)}{\mathrm{p}(\varphi)}$ when $\mathrm{p}(\varphi) > 0$; and undefined otherwise. AT and SH are at times formulated with the proviso that $\mathrm{p}(\psi|\varphi) = 1$ when $\mathrm{p}(\varphi) = 0$ for the conditional probability to be defined for all φ (see, e.g., Adams, 1998, p. 150).

[2]For thorough discussions of the issues whether indicatives express propositions and can be embedded see, respectively, Rothschild (2013) and Kaufmann (2009).

[3]Or, they lack a complete truth table. One may take them as false when the antecedent is true and the consequent false; true when both are true: see e.g. the ersatz truth values of (Adams, 1998, pp. 121-3), (Bennett, 2003, Ch. 8).

2. If Brexit causes a recession, then there will be some heads in the first 100 tosses of this fair coin.

AT is empirically inadequate.[4] In the experiments reported in Douven and Verbrugge (2010), one group of subjects was given contexts C_i, $1 \leqslant i \leqslant 30$, and asked to rate the acceptability of conditionals $\varphi_i \to \psi_i$ in C_i. Another group was given the same contexts C_i and asked to judge the probability of ψ_i in C_i on the supposition that φ_i. People's patterns of (degrees of) acceptance for conditionals generally don't even approximate the corresponding conditional probabilities: this 'manifestly refute[s] Adams' Thesis, both in its strict form AT and in its approximate form' (Douven, 2016, p. 99).[5]

What's wrong with AT? Compare (2) above with the following, adapting (Douven, 2016, p. 104):

3. If there's some heads in the first 10 tosses, then there will be some heads in the first 100 tosses of this fair coin.

We accept (3), not (2), because (3)'s antecedent is relevant for the consequent, which is, instead, off-topic with respect to (2)'s antecedent. This suggests a fixing for AT: we accept a conditional to the extent that (i) the consequent is likely conditional on the antecedent, *provided* (ii) some relevance or topicality connection linking antecedent and consequent is satisfied. How (i) should work is well-understood already. It is (ii) that holds, we submit, the key to improve our understanding of conditionals.

What is relevance or topicality for indicatives? A venerable idea going back to Grice (1989), and sometimes invoked to save the material conditional analysis from apparent counterexamples, is that it's a pragmatic issue: some perfectly true or probabilistically all right conditionals are unassertable, lacking a connection between antecedent and consequent:[6] see e.g. (Johnson-Laird and Byrne, 1991, p. 8). But we shouldn't take for granted that relevance has to be handled merely as pragmatic coherence. Sophisticated approaches to the logic of conditionals, such as relevance logics (Dunn and Restall, 2002), make relevance amenable to a rigorous, compositional, and properly semantic treatment, and have also been developed for *ceteris paribus* conditionals (Mares and Fuhrmann, 1995; Mares, 2004). Besides, as Krzyżanowska et al. (2017) have shown, pragmatic discourse coherence is a weaker constraint than proper relevance of conditional antecedents for their consequents: the former only requires not to assert a conjunction when the two

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[4]The reader should not confuse the empirical support for SH with the empirical support for AT. As noted also in (Douven and Verbrugge, 2010, Section 4), there is significant experimental work that supports SH, finding high correlation between the probabilities that the participants assign to conditionals and the corresponding conditional probabilities. However, to the best of our knowledge, Douven and Verbrugge are the first to test AT by asking a group of participants to grade the *acceptability of conditionals* rather than their probability of truth. We refer to the aforementioned source for further references of empirical results supporting SH and a detailed discussion on how experiments on AT and SH differ.

[5]The conditionals that fare better are what Douven and Verbrugge call 'deductive inferential', i.e., such that the consequent follows deductively from the antecedent plus background, unstated assumptions; for these, at least a high correlation was found: see (Douven, 2016, p. 100).

[6]We take acceptance as a mental state, assertion as the linguistic act manifesting it. We speak of assertability in strictly pragmatic contexts, but we are after acceptability conditions: we follow Douven (2016, p. 94) in taking the latter as the core notion.

conjuncts have nothing to do with each other in the given conversational context ('Brexit will cause a recession and Jupiter is a planet' is an odd thing to say in one breath, in natural conversational contexts where what happens after Brexit, and what kind of thing Jupiter is, are completely disconnected topics). However, sometimes a conjunction is assertable in a given context because of some topic overlap between the two conjuncts, but we don't want to assert the corresponding conditional in the very same context. Picking Krzyżanowska et al.'s own example, one can easily think of contexts where this is clearly assertable:

4. Raccoons have no wings and they cannot breath under water.

What makes (4) pragmatically all right is that the two conjuncts overlap in topic – both are about *raccoons*, although they say disparate things about them. But we may not want to assert, in the very same contexts in which (4) is a fine thing to say, the corresponding conditional:

5. If Raccoons have no wings, then they cannot breath under water.

Or suppose this (now drawing on Priest (2008, p. 96)) is true:

6. A fortune-teller predicts that you'll win the lottery, and you do.

It's pragmatically all right to assert (6) in a number of contexts, for its conjuncts overlap in topic, both having to do with *your winning the lottery*. But we may not want to assert, in those same contexts, the corresponding superstitious conditional:

7. If a fortune-teller predicts that you'll win the lottery, then you do.

Krzyżanowska et al. (2017) exhibit experimental results showing that, even when people find it pragmatically appropriate to assert conjunctions like (4) or (6), they tend not to assert the corresponding irrelevant conditionals, (5) and (7).[7]

The moves from (4) to (5), or from (6) to (7), are instances of the And-to-If schema, licensing the inference from a conjunction to the corresponding conditional:

(And-to-If) $\varphi \wedge \psi \models \varphi \rightarrow \psi$[8]

And-to-If is sometimes called 'Centering', for it holds in the mainstream similarity-based possible worlds semantics for conditionals due to Stalnaker (1968) and Lewis (1973) (Lewis advertised it for subjunctives or counterfactuals, however, not indicatives), when one assumes that the world of evaluation is always the single world most similar to itself (it's the unique one at the center of the nested spheres of worlds arranged around it, more similar worlds inhabiting smaller spheres). It doesn't hold only there. A number of mainstream theories of indicatives validate And-to-If: the material conditional view (Jackson,

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[7]Krzyżanowska et al. understand relevance itself probabilistically, but we don't need to follow them down this route, as we will see.

[8]Logical consequence $\models$ here may be understood standardly, as truth preservation (in all models), or, if indicatives lack truth conditions, as preservation of degrees of probability, or of acceptability, or so, as e.g. in Adams (1998).

1987; Grice, 1989) and the probabilistic-suppositional view (Adams, 1975; Edgington, 1995; Evans and Over, 2004), for instance, have it. They shouldn't: a true, acceptable conjunction doesn't generally warrant the truth, or acceptability, of the corresponding conditional, and the mismatch between the two is not easily reducible to the pragmatics of discourse coherence.

Two theories of indicatives, however, naturally invalidate And-to-If. Their discussion in Section 2 will make a number of useful points emerge, in view of the presentation of our own account starting in Section 3.

2   Inferentialism and Evidential Support

Some *inferentialist* (Braine, 1978; Braine and O'Brien, 1991) approaches to conditionals have it that conditionals express enthymematic arguments.[9] The idea goes back to Mill's *System of Logic*, and was endorsed by Ramsey in the same work where he introduced (what we now call) his 'test', whereby we evaluate a conditional by supposing the antecedent and assessing the consequent under that supposition:[10]

> [W]e can say with Mill that 'If p, then q' means that q is inferrible from p, that is, of course, from p together with certain facts and laws not stated but in some way indicated by the context. (Ramsey, 1990, p. 156)

So $\varphi \to \psi$ says that there's some good inference from φ and background assumptions ('facts and laws') BA_φ to ψ. Besides plausibly depending on the antecedent (that's what the subscript is there for), background assumptions depend on context and their list can be open-ended. They capture the idea that everyday conditionals are for the most *ceteris paribus* and non-monotonic: 'If the weather is good, I'm going to play golf' doesn't imply 'If the weather is good and I break my ankle, I'm going to play golf'. (Different approaches capture *ceteris paribus* features in different ways, e.g., variable strictness does it in the Lewis-Stalnaker semantics.)

For lots of good conditionals, there is no way to deduce ψ from φ, no matter what BA_φ come to help. But we needn't assume that the valid inference at issue be deductive: ψ may follow from φ and BA_φ also inductively, abductively, or via a mixture of different ways of inferring. Krzyżanowska (2015, pp. 64-5) imposes constraints on the connection between premises and conclusion which ensure that ψ doesn't follow trivially from φ and BA_φ, thus capturing a kind of relevance. The view makes And-to-If fail in a most natural

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[9]The label 'inferentialism' is used more generally in the literature, to refer more or less to any account that emphasizes relevance as influencing the acceptability of conditionals. So used, the label would apply to the evidential support theory to be discussed below, to other approaches that handle relevance probabilistically, e.g., Skovgaard-Olsen et al. (2016), or causally e.g., van Rooij and Schulz (2019), or by resorting to non-classical logics, e.g., Dunn and Restall (2002), and to our own view as well. But the label, however popular, is a misnomer, just as 'counterfactual' is a misnomer for the subjunctive (Lewis, 1973; Williamson, 2007). As we will see, relevance needn't be understood as inferential, unless one stretches 'inferential' beyond usefulness.

[10]Ramsey's legendary footnote: 'If two people are arguing "If p will q" and are both in doubt as to p, they are adding p hypothetically to their stock of knowledge and arguing on that basis about q; so that in a sense "If p, q" and "If p, $\neg q$" are contradictories. We can say that they are fixing their degrees of belief in q given p.' (Ramsey, 1990, 155n).

way: the mere fact that φ and ψ are true together doesn't warrant there being a good argument from the former (and, BA_φ) to the latter.

Inferentialism (of this kind) has not been proposed, as far as we know, as a general account of indicatives. It can hardly be one. Linguists distinguish inferential from content conditionals (Declerck and Reed, 2001; Haegeman, 2003; Dancygier and Sweetser, 2005) expressing non-logical connections between states of affairs: 'If John passes the exam, we'll have a party'; 'She's such a disappointment if she thinks so highly of him'. As stressed by Douven, the connections between antecedent and consequent in relevant conditionals can be of the most diverse kinds:

> [C]onditionals have been said to require for their truth the presence of a 'connection' linking their antecedent and consequent. Proposals in this vein immediately raise the question of what the nature of the supposed connection could be. Candidate answers abound: it could be logical, statistical, causal, explanatory, metaphysical, epistemic; or the 'connector' could be a second-order functional property, notably, the property that there is some first-order property or other that links antecedent and consequent. (Douven, 2016, pp. 35-6)

It's dubious that all relevant conditionals express the existence of some argument from their antecedent and contextually determined background assumptions to their consequent. It is surely in agreement with the Ramsey test to say that their assessment always involves some form of mental simulation, whereby we assess the consequent under the supposition of the antecedent. To label the process 'inferential' in all cases just on this basis, however, would be to stretch the term beyond usefulness: surely any conditional trivially says that its consequent follows, in some sense or other, from its antecedent. A general account of indicatives calls for a general notion of relevance. The criterion of relevance proposed in our account below aims at giving a catch-all condition, covering relevance of *any* kind, whether inferential or not.

Next, whenever an argument condensed in $\varphi \to \psi$ is not purely deductive, it may be valid even when φ is true and ψ isn't: good arguments involving inductive or abductive steps may fail to be necessarily truth-preserving. Thus, inferentialism is bound to invalidate Modus Ponens (Krzyżanowska, 2015, pp. 70-1). But preserving X forwards (X being truth, or degrees of probability, of acceptability, or whatnot) has often been taken as a minimal requirement for an operator to count as a conditional. Putative exceptions are very controversial, and anyway involve peculiar sentences (paradoxes like the Liar, see Beall (2015)), or contexts like the famous McGee cases – which anyway don't affect *simple* conditionals, as they crucially involve right-nested ones. As McGee himself admitted, 'there is every reason to suppose that, restricted to [simple] conditionals, modus ponens is unexceptionable' (McGee, 1985, p. 468). Additionally, with over 97% endorsement across a range of empirical tests (Oaksford, 2005; Oaksford and Chater, 2010), Modus Ponens is by far the most popular inference involving conditionals and one of the most popular tout-court (Evans and Over, 2004, pp. 46–52). Admittedly (as suggested by an anonymous referee) the inferentialist can make sense of this by claiming that Modus Ponens is highly reliable, to the extent that the relevant non-deductive inferences are, and this is enough to explain both our intuitions about it and the high endorsement rate in various experiments.

The *evidential support thesis* (EST) championed by Douven (2016) proposes to fix AT by adding to it a relevance condition of evidential support. Evidence is understood probabilistically: φ is evidence for ψ by making it more likely.[11] The qualitative (non-graded) acceptability conditions for a simple indicative are:

(EST) $\varphi \rightarrow \psi$ is acceptable iff (i) $p(\psi|\varphi) > \theta$ and (ii) $p(\psi|\varphi) > p(\psi)$

(i) is a qualitative variant of AT, saying that the conditional probability passes a threshold (say, $\theta \in [0.5, 1)$). (ii) is the evidential constraint: ψ is more likely conditional on φ than it is unconditionally. And-to-If nicely fails: $\varphi \wedge \psi$ can be true and acceptable without φ raising one bit the probability of ψ. This seems to be what is going on in a number of cases where we don't accept a conditional with true antecedent and consequent.

One issue with the view is its inferential weakness. Douven defines a notion of logical consequence as acceptability-preservation: when all premises reach a threshold of acceptability θ, the conclusion does, too. An inference is *valid for t* when it's acceptability-preserving for $\theta = t$, *invalid for t* otherwise. An inference is *valid (invalid) simpliciter* when valid (invalid) for all $t \in [0.5, 1)$ (Douven, 2016, p. 130). When '$\models$' is such validity, we have the following failures for the EST conditional:[12]

(Modus Ponens) $\varphi \rightarrow \psi, \varphi \not\models \psi$

(CC) $\varphi \rightarrow \psi, \varphi \rightarrow \chi \not\models \varphi \rightarrow (\psi \wedge \chi)$ (Conjunction in the Consequent)

(CMon) $\varphi \rightarrow \psi, \varphi \rightarrow \chi \not\models (\varphi \wedge \psi) \rightarrow \chi$ (Cautious Monotonicity)

(CT) $\varphi \rightarrow \psi, (\varphi \wedge \psi) \rightarrow \chi \not\models \varphi \rightarrow \chi$ (Cautious Transitivity)[13]

We've already highlighted the badness of Modus Ponens failure. Segerberg (1989) claims that CC should hold in *any* reasonable system of conditional logic.[14] There is wide agreement on CMon and CT, too, being required in a good conditional logic. Both

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[11]Three other approaches broadly in this ballpark are the very recent van Rooij and Schulz (2019) and Rott (2019), and the influential Skovgaard-Olsen et al. (2016). We will not discuss them in detail but, we mention that the van Rooij-Schulz paper is based on the promising idea that relevance can be accounted for via condition of dependence between antecedent and consequent understood as causal correlation. Van Rooij and Schulz argue that this is compatible with a general probabilistic view, insofar as it reduces to conditional probability in natural cases. As for Skovgaard et al., it is based on the idea that the acceptability of a conditional (they actually phrase the result in terms of probability assignments) correlates well with the corresponding conditional probability precisely when their relevance condition is satisfied. Skovgaard-Olsen et al. (2016) propose what they call the *Default and Penalty Hypothesis* (DPH): by default, people evaluate $\varphi \rightarrow \psi$ expecting the consequent to be positively relevant for the antecedent. When the expectation is fulfilled, they go for $p(\psi|\varphi)$. When not, people add a 'penalty' to their estimate. This is in line with our proposal below, where, as we will see, acceptability equals conditional probability when our relevance condition is satisfied, and drops otherwise. We, however, do not understand relevance probabilistically.

[12]Like Douven, we use popular labels from the literature on conditional logics for these closure conditions: see e.g. Chellas (1975).

[13]Douven calls this Cumulative Transitivity.

[14] Admittedly, CC may be controversial in a probabilistic context for special cases, e.g., the Lottery Paradox (Kyburg, 1961): for each ticket i, $1 \leqslant i \leqslant n$, of a large enough fair lottery L, 'If L has exactly one winner then ticket i will lose' is acceptable, but 'If L has exactly one winner then ticket 1 will lose, and ticket 2 will lose, and..., and ticket n will lose' is not. We'll come back to this.

feature in Chellas (1975)'s basic conditional logic. The basic system of non-monotonic entailment **C** in the seminal Kraus et al. (1990) has CMon and CT, which according to Gabbay (1985), are two minimal inferental schemata any non-monotonic notion of entailment must comply with. The popular non-monotonic logic **P** of preferential models by Kraus et al. (1990) has them, too. It has been claimed that such principles are both theoretically and empirically desirable (Pfeifer and Kleiter, 2010): they are strong enough to do the job of the invalid unrestricted monotonicity (from $\varphi \to \psi$ to $\varphi \land \chi \to \psi$) and transitivity (from $\varphi \to \psi$ and $\psi \to \chi$ to $\varphi \to \chi$) in most cases, while helping to explain why people sometimes endorse the latter by over-generalizing – as argued, among many, by Adams (1975), Bennett (2003), Pfeifer and Kleiter (2010).

The inferential weakness of EST may be tied to the particular way in which Douven defines his probabilistic logic.[15] But one issue besets the view due to its understanding relevance as probabilistic evidence: EST does not fare well with extreme probabilities. If $p(\varphi) = 0$, φ can hardly be evidence for anything. If $p(\psi) = 1$, nothing can raise the probability of ψ. Then any conditional $\varphi \to \psi$ with 0 antecedent-probability or 1 consequent-probability is unacceptable. Many such conditionals, however (the relevant ones!), sound perfectly acceptable in a number of contexts. Douven (2016, p. 113) discusses one example of relevant conditional whose consequent has probability 1:

8. If Obama is president of the United States, his residence is in the White House.

While he finds (8) odd, we can think of a number of contexts in which it would be perfectly acceptable.[16] If this can be done for a conditional like (8), whose antecedent and consequent are both not only true (at the time of Douven's writing), but also widely shared knowledge, it should't be too difficult to find such contexts for a number of probability 1 consequents.

The case of 0 probability antecedents, anyway, is more telling. *Pace* Bennett (2003) and others, one can non-trivially assess, and reason with, indicatives whose antecedent one fully takes to be false. One is pretty sure that Oswald killed Kennedy but has no troubles assessing 'If Oswald did not kill Kennedy, then someone else did' (Gillies, 2004). As stressed by Joyce (1999), unpretentious thinkers can suppose *in the indicative mood* that φ also when they utterly disbelieve φ, and assess whether ψ is the case under that supposition:

> [I]t is often assumed that any form of probabilistic belief revision that involves 'raising the dead' by increasing the probabilities of certainly false propositions must involve counterfactual beliefs. This is not so. It is logically consistent both to be certain that some proposition is false and yet to speculate about what the world is like if one is in fact wrong. To be subjectively certain of something is, after all, not the same as regarding oneself infallible on the matter. (Joyce, 1999, p. 203)

This holds even for conditionals whose antecedents are taken as necessarily false:

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[15]See the recent Crupi and Iacona (2019) for a reworking of the evidential idea with an eye on this.

[16]In a plot to blackmail the president, the conspirators are pondering the best strategy. Suddenly one asserts: 'But if Obama is president, then his residence is in the White House; so we should infiltrate someone in the personnel working at the White House, who will manage to spy him; it's hard but not unfeasible.'.

9. If all even numbers are prime and 5 is even, then 5 is prime.

10. If all even numbers are prime and 5 is even, then one cannot square the circle.

(9) seems perfectly acceptable although its antecedent is a necessary falsity. Its same-antecedent (10) doesn't look acceptable – because a relevant connection with the consequent is missing. Douven mentions that these cases could be handled by resorting to a non-standard probabilistic account that doesn't assign probability 0 to all logical and mathematical falsehoods (Douven, 2016, p. 114). The issue with (9), though, is that it seems to be acceptable also for one who is certain that its antecedent is false.

Cases like (9) and (10) give some evidence for a point at times neglected in the literature: conditionals are *hyperintensional*, at least as far as their acceptability conditions go.[17] We sometimes have different attitudes towards conditionals whose antecedents and, respectively, consequents, are necessarily equivalent, having the same truth value across all possible worlds: we accept the relevant ones, not the irrelevant ones. The account we propose below makes conditionals hyperintensional precisely in this way.[18]

The so-called Ratio Formula, which defines a conditional probability $p(\psi|\varphi)$ as the ratio of two unconditional probabilities $p(\varphi \wedge \psi)/p(\varphi)$, makes a conditional probability undefined for $p(\varphi) = 0$. This should be taken as a problem for a treatment of conditionals that resorts to it, rather than for the claim that one can non-trivially reason with, or assess, indicatives with zero antecedent-probability. The use of Popper functions, whereby one doesn't define conditional probabilities via unconditional ones, is often recommended because they easily handle such cases. Several approaches to conditional belief and belief revision (e.g. Van Fraassen (1995); Arlo-Costa and Parikh (2005); Baltag and Smets (2008)), thus, endorse an extension of classical probability theory using Popper functions. We are doing the same for our account.

3 Topicology

We need a relevance constraint to fix AT: we accept $\varphi \to \psi$ to the extent that (i) $p(\psi|\varphi)$ is high, provided (ii) φ is relevant for ψ. Unlike EST, we understand relevance in (ii) as *topic-sensitivity*: a relevant conditional is one whose consequent is *about* the right *topic*, as contextually determined by its antecedent. We thus need to say what aboutness and topics are.

Aboutness, as Yablo has it, is 'the relation that meaningful items bear to whatever it is that they are *on* or *of* or that they *address* or *concern*' (Yablo, 2014, p. 1): this is their *topic*, or *subject matter*. Work on aboutness has been burgeoning among philosophers (Putnam, 1958; Goodman, 1961; Lewis, 1988; Plebani and Spolaore, 2020), linguists (Roberts, 2011), and logicians (Fine, 1986; Humberstone, 2008; Fine, 2017; Berto and Hawke, 2018; Berto, 2019; Özgün and Berto, 2020). Declarative sentences are used to

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[17]Not completely neglected, though: Jackson (1979), Bennett (2003, p. 23) remark that assertability is hyperintensional in that logically equivalent sentences can differ in assertability, and take this as a problem for Gricean pragmatics.

[18]As an anonymous referee pointed out to us, probabilities may be hyperintensional in reality as well, although standard treatments don't make them so. Thus, accounts that deal with relevance probabilistically may deal with this, if they are supplied with a notion of subjective probability taking hyperintensionality into account.

say *true* things *about* all kinds of *topics*. One says: 'John is a plumber'. One thereby communicates something about *John's profession* and, more generally, *John*. What one says is true just in case John's profession includes being a plumber. One addresses certain topics and says that things are such-and-so with respect to them.

Topics are naturally linked to questions or issues under discussion in a discourse context (Lewis, 1988; Roberts, 2012): 'Our topic today is whether Brexit will cause a recession' maps to 'Will Brexit cause a recession?'. Topics needn't be framed as questions ('Our topic is the number of stars'), but there will always be a question in the vicinity ('How many stars are there?'). Thus, Lewis (1988) took topics as partitions of the set of possible worlds: the topic *the number of stars* is the partition determined by the question, 'How many stars are there?'. Worlds end up in the same cell when they agree on the answer: all zero-star worlds end up in one cell, all one-star worlds in another, and so on. Others have understood topics as sets or fusions of a sentence's truthmakers or falsemakers (Fine, 2016, 2017; Fine and Jago, 2018), taken in their turn as states or situations à la Barwise and Perry (1983), which, unlike possible worlds, can fail to be maximal or consistent.

We don't need to take a stance on the exact nature of topics, however: for our purposes, we only need them to obey three constraints. These are widely agreed upon in subject matter semantics:

(1) Logically or necessarily equivalent sentences φ and ψ can differ in content when they are about different things. In Yablo (2014)'s version, for instance, the propositional content of a sentence (in context) is not specified just by the set of worlds in which it is true ('proposition' in the UCLA sense of Montague-Lewis-Stalnaker-etc.), but also by what it's about. Subject matter semantics is, thus, hyperintensional, making distinctions more fine-grained than what standard intensional (possible worlds) semantics allows: '$2 + 2 = 4$' and 'Either Jupiter is a planet, or not' differ in content in spite of being true at the same worlds (all of them), for they say different things: only one is about *the number 2*.

(2) The space of topics must have a mereological structure (Yablo, 2014; Fine, 2016): topics can have proper parts; distinct topics may have common parts; and one topic may be included in another in that every part of the former is also a part of the latter. *Mathematics* includes *arithmetic*. *Mathematics* and *philosophy* overlap, having (certain parts of) *logic* as a common part. This reflects on notions of *same-saying, saying more, saying less* for the contents of assertions. Plausibly the topic of $\varphi \wedge \psi$ includes that of φ as a (proper) part; so if Mary says 'Obama is tall and handsome' and John says 'Obama is tall', what John said has already been said by Mary – who also said more: John has addressed a topic, say *Obama's height*, which is a (proper) part of the larger topic addressed by Mary, *the height and looks of Obama's*. Plausibly, the topic of $\varphi \wedge \psi$ is the same as that of $\psi \wedge \varphi$ when '$\wedge$' represents commutative, order-insensitive Boolean conjunction. So if Mary says 'Obama is tall and handsome', and John says 'Obama is handsome and tall', they have said the same thing.[19]

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[19]The ordering difference boiling down, at most, to an extra-semantic, pragmatic implicature: perhaps John wanted to stress the importance of Obama's looks in a certain discourse context.

(3) A third point of agreement (Perry, 1989; Yablo, 2014; Fine, 2016) is that the Boolean logical operators should add no subject matter of their own: they are 'topic-transparent'. The topic of $\neg\varphi$ is the same as that of φ ('Obama is not tall' is exactly about what 'Obama is tall' is about – *Obama's height*; it certainly is not about *not*).[20] Conjunction and disjunction merge topics ('Obama is tall and handsome' and 'Obama is tall or handsome' are both about the same topic: *the height and looks of Obama's*). Transparency accommodates the venerable idea that the laws of logic are formal in the sense of being topic-neutral, or subject-matter-independent.

Here's what we are going to do next. In Section 4, we will introduce a propositional formal language including a simple indicative conditional, for which we are giving acceptability conditions in terms of probabilities and topics. In Section 5, we'll then define a notion of logical consequence in terms of preservation of degrees of (un)acceptability from premises to conclusion. We'll show that our conditional has logical closure properties which are not only intuitively plausible, but also mirror people's empirical judgments on a number of inferences involving indicatives.

We will focus on simple indicatives and give only graded acceptability conditions for them, not truth conditions, to accommodate non-propositionalist views. We want to be able to conditionalize on 0 probabilities in a non-trivial way. We will therefore use Popper functions, following Hawthorne (1996) and Leitgeb (2012), among others. We interpret the conditional probability measures subjectively-epistemically, not as objective frequencies, following the mainstream on indicatives (Adams, 1966, 1975, 1998; McGee, 1986; Douven, 2016).

4   A Topic-Sensitive, Probabilistic Semantics

Here is our core idea: the degree of acceptability of a simple indicative $\varphi \to \psi$ is given, (i) as per AT, by the corresponding conditional probability, $\mathcal{P}(\psi|\varphi)$, provided (ii) the conditional is *on-topic* – otherwise, $\varphi \to \psi$ has zero acceptability. A conditional is on-topic when the topic of its consequent is *fully included* in a topic contextually determined by its antecedent. Arguably, this latter is not just the topic of the antecedent φ. Rather, it is the topic of the relevant background assumptions BA_φ determined by φ and context (where, plausibly, $\varphi \in BA_\varphi$). That's because we sometimes accept $\varphi \to \psi$ without direct topic-inclusion between φ and ψ:

11. If we keep burning fossil fuels at this pace, the polar ice will melt.

12. If Brexit causes a recession, the Tories won't win the next election.

13. If you push the button, the engine will start.

In cases like (11)-(13), the antecedent is relevant for the consequent although it doesn't, on its own, address an issue with respect to which the consequent is fully on-topic. Rather, the supposition of the antecedent triggers, in context, background assumptions

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[20]That doesn't make negation ineffable, of course: 'Negation is a logical connective'.

with respect to which the consequent is fully on-topic (e.g., for (11), *fossil fuel burning* triggers topics such as *the emission of CO_2, raising global temperatures*, etc.). The topicality is between the background BA_φ and ψ.[21] The topic of BA_φ is determined, given that of φ, by a function f obeying plausible constraints.

Let $\mathcal{L}_{PL}$ denote the language of classical propositional logic defined on a countable set of propositional variables $\mathsf{Prop} = \{p, q, \dots\}$ with connectives $\neg$ and $\wedge$. The well-formed formulas are the elements of Prop, $\neg\varphi$, and $(\varphi \wedge \psi)$ whenever φ and ψ are formulas. We identify $\mathcal{L}_{PL}$ with the set of its well-formed formulas and employ the usual abbreviations for propositional connectives $\vee, \supset, \equiv$ as $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, $\varphi \supset \psi := (\neg\varphi \vee \psi)$, and $\varphi \equiv \psi := (\varphi \supset \psi) \wedge (\psi \supset \varphi)$. So, $\supset$ is the material conditional and $\equiv$ is the material biconditional. As for $\top$ and $\bot$, we set $\top := p \vee \neg p$ and $\bot := \neg\top$. We call the elements of $\mathcal{L}_{PL}$ *Boolean sentences*. For any $\varphi \in \mathcal{L}_{PL}$, P_φ denotes the set of propositional variables occurring in φ. We use the symbol $\vDash_{PL}$ for classical logical truth/consequence.

The full language $\mathcal{L}$ of simple indicative conditionals extends $\mathcal{L}_{PL}$ by an indicative conditional operator $\rightarrow$, which connects only the elements of $\mathcal{L}_{PL}$. The well-formed formulas in $\mathcal{L}$ are the elements of $\mathcal{L}_{PL}$ and $(\varphi \rightarrow \psi)$ whenever φ and ψ are in $\mathcal{L}_{PL}$. We again identify $\mathcal{L}$ with the set of its well-formed formulas.

(i) We resort to Popper functions for the probabilistic component of our acceptability conditions:

Definition 1 (Popper Functions). $\mathcal{P} : \mathcal{L}_{PL} \times \mathcal{L}_{PL} \rightarrow [0, 1]$ *is a* Popper function *on* $\mathcal{L}_{PL} \times \mathcal{L}_{PL}$ *iff*

1. *for some* $\alpha, \beta \in \mathcal{L}_{PL}$, $\mathcal{P}(\alpha|\beta) \neq 1$;

and for all $\varphi, \psi, \chi, \eta \in \mathcal{L}_{PL}$,

2. *if* $\vDash_{PL} \psi \equiv \chi$, *then* $\mathcal{P}(\varphi|\psi) = \mathcal{P}(\varphi|\chi)$,

3. *if* $\varphi \vDash_{PL} \psi$, *then* $\mathcal{P}(\psi|\varphi) = 1$,

4. *if* $\varphi \vDash_{PL} \neg(\psi \wedge \chi)$, *then* $\mathcal{P}(\psi \vee \chi|\varphi) = \mathcal{P}(\psi|\varphi) + \mathcal{P}(\chi|\varphi)$ *(i.e.,* $\mathcal{P}(\cdot|\varphi)$ *is a finitely additive probability measure) or* $\mathcal{P}(\eta|\varphi) = 1$;

5. $\mathcal{P}(\psi \wedge \chi|\varphi) = \mathcal{P}(\psi|\varphi)\mathcal{P}(\chi|\psi \wedge \varphi)$.

One could define Popper functions on $\mathcal{L}_{PL}$ without relying on the classical notion of logical truth/consequence (Hawthorne, 1996, Definition 3). We work with the above definition, however, because it makes the connection between Popper functions and unconditional probability measures clear. The latter can be recovered from Popper functions by conditionalization on $\top$. If $\mathcal{P}(\varphi|\top) > 0$, we have

$$\frac{\mathcal{P}(\varphi \wedge \psi|\top)}{\mathcal{P}(\varphi|\top)} = \mathcal{P}(\psi|\varphi \wedge \top) = \mathcal{P}(\psi|\varphi).$$

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[21]See Khoo (2016) for a recent view in the same ballpark. According to Khoo, what an indicative expresses is given by a contextually salient question under discussion determining a partition of modal space. We have seen that mainstream approaches to topics or subject matters (Lewis, 1988; Yablo, 2014) take these as given by questions determining partitions or divisions.

Popper functions allow for non-trivial conditionalization on 0 probabilities: we can have that $\mathcal{P}(\varphi|\top) = 0$ but $\mathcal{P}(\psi|\varphi) \in (0,1)$. We call an element φ of $\mathcal{L}_{PL}$ *abnormal with respect to $\mathcal{P}$* when $\mathcal{P}(\eta|\varphi) = 1$ for all $\eta \in \mathcal{L}_{PL}$; *normal* otherwise.

(ii) The second component of our acceptability conditions needs a *topicality filter*:

Definition 2 (Topic models with operators). *A topic model with operators (in short, topic model) $\mathcal{T}$ is a tuple $\langle T, \oplus, t, f \rangle$ where*

1. *T is a non-empty set of* possible topics. *We use variables a, b, c ($a_1, a_2, \dots$) ranging over possible topics.*

2. *$\oplus : T \times T \to T$ is a binary idempotent, commutative, associative operation:* topic fusion. *We assume unrestricted fusion, that is, $\oplus$ is always defined on T: $\forall a, b \in T \,\exists c \in T (c = a \oplus b)$. We define* topic parthood, *denoted by $\sqsubseteq$, in a standard way as*

$$\forall a, b (a \sqsubseteq b \text{ iff } a \oplus b = b).$$

Easily, $\sqsubseteq$ is a partial order on T.

3. *$t : \mathsf{Prop} \to T$ is a* topic function *assigning a topic to each element in Prop. t extends to the whole $\mathcal{L}_{PL}$ by taking the topic of a sentence φ as the fusion of the elements in P_φ:*

$$t(\varphi) = \oplus \mathsf{P}_\varphi = t(p_1) \oplus \cdots \oplus t(p_k)$$

where $\mathsf{P}_\varphi = \{p_1, \dots, p_k\}$. We abbreviate $t(\varphi)$ as t_φ.

4. *$f : T \to T$ is a function on T that satisfies for all $a, b \in T$:*

   (a) *$a \sqsubseteq f(a)$ (Inclusion);*

   (b) *$f(a) = f(f(a))$ (Idempotance);*

   (c) *$f(a \oplus b) = f(a) \oplus f(b)$ (Additivity);*

T provides the topics sentences of our language can be about. Fusion $\oplus$ gives our little mereology of topics, as per Constraint (2) from our topicology above: $a \oplus b$ is the merging of topics a and b ('Obama is tall and handsome' and 'Obama is tall or handsome' are about *Obama's height and looks*, the merging of the topics of 'Obama is tall' and of 'Obama is handsome'); and it makes sense to say that topic a is part of topic b, $a \sqsubseteq b$ (*Obama's height* is included in *Obama's height and looks* as a part). Topic function t assigns topics recursively, in such a way as to guarantee that the logical connectives in $\mathcal{L}_{PL}$ be topic-transparent, as per Constraint (3) of our topicology above: $t_{\neg\varphi} = t_\varphi$ and $t_{\varphi \wedge \psi} = t_\varphi \oplus t_\psi$. Finally, f is a Kuratowski closure operator on the poset $(T, \sqsubseteq)$, mapping the topic of a sentence φ to the topic of the relevant background assumptions BA_φ determined by φ and context.

Given this role of f, (4a-4c), the so-called Kuratowski axioms are well-motivated: Inclusion (4a) guarantees that the topic of the relevant background assumptions BA_φ possibly expands, but always includes, the topic of the antecedent φ that triggers the conditional supposition. This constraint fits with our assumption that $\varphi \in BA_\varphi$ and allows us to account for cases such as (11)-(13). Idempotence (4b) states that the set

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of background assumptions BA_φ determined by φ is *complete*: contemplating on the background assumptions triggered by φ does not lead to new background assumptions unless given additional inputs. Finally, Additivity (4c) ensures that the topic of the relevant background assumptions BA_φ determined by φ is the same as the fusion of the topics of the relevant background assumptions determined by its more primitive components.[22]

Given a topic model $\mathcal{T} = \langle T, \oplus, t, f \rangle$, we call a conditional of the form $\varphi \rightarrow \psi$ an *on-topic conditional with respect to $\mathcal{T}$* iff $t_\psi \sqsubseteq f(t_\varphi)$. We call $\varphi \rightarrow \psi$ an *on-topic conditional* (*simpliciter*) if it is an on-topic conditional with respect to every topic model. Being on-topic is what makes a conditional relevant: the topic of its consequent is included in that contextually determined by its antecedent, and given via f.

Lemma 1. *For any topic model $\mathcal{T} = \langle T, \oplus, t, f \rangle$ and $a, b \in T$, if $a \sqsubseteq b$ then $f(a) \sqsubseteq f(b)$.*

Proof. Let $\mathcal{T} = \langle T, \oplus, t, f \rangle$ be a topic model and $a, b \in T$ such that $a \sqsubseteq b$, i.e., $a \oplus b = b$. Then, since f is well-defined, we have $f(a \oplus b) = f(b)$. Therefore, Definition 2.4c guarantees that $f(a) \oplus f(b) = f(b)$, i.e., $f(a) \sqsubseteq f(b)$. $\square$

We can now define the graded (un)acceptability conditions for the formulas of $\mathcal{L}$, and in particular for our indicatives:

Definition 3 (Degrees of (Un)Acceptability). *For any Popper function $\mathcal{P}$ and topic model $\mathcal{T}$ defined on $\mathcal{L}_{PL}$, the degree of acceptability $\mathcal{A}_{\mathcal{P},\mathcal{T}} : \mathcal{L} \rightarrow [0,1]$ of an element in $\mathcal{L}$ is defined as:*

1. for all $\varphi \in \mathcal{L}_{PL}$, $\mathcal{A}_{\mathcal{P},\mathcal{T}}(\varphi) = \mathcal{P}(\varphi|\top)$; and

2. $\mathcal{A}_{\mathcal{P},\mathcal{T}}(\varphi \rightarrow \psi) = \begin{cases} \mathcal{P}(\psi|\varphi), & \text{if } t_\psi \sqsubseteq f(t_\varphi) \\ 0 & \text{otherwise.} \end{cases}$

For any $\varphi \in \mathcal{L}$, the degree of unacceptability $\mathcal{U}_{\mathcal{P},\mathcal{T}}(\varphi)$ is then given by $\mathcal{U}_{\mathcal{P},\mathcal{T}}(\varphi) = 1 - \mathcal{A}_{\mathcal{P},\mathcal{T}}(\varphi)$.[23]

When it is clear which Popper function and topic model are used, we omit the subscripts and simply write $\mathcal{A}$ and $\mathcal{U}$.

Definition 3 includes our main claim: the degree of acceptability of $\varphi \rightarrow \psi$ is (i) the probability of ψ conditional on φ, as per AT, as long as (ii) $\varphi \rightarrow \psi$ is an on-topic indicative; otherwise $\varphi \rightarrow \psi$ is plainly unacceptable.

The degree of acceptability of a Boolean sentence $\psi \in \mathcal{L}_{PL}$ goes by $\mathcal{P}(\psi|\top)$, that is, $\mathcal{A}(\psi) = \mathcal{P}(\psi|\top)$. Topic models play no role in stating the degree of acceptability of a Boolean sentence.

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[22]Additivity may seem more contentious than the other closure conditions. This is because it is keyed to the assumptions that (a) the Boolean connectives are topic-transparent (Constraint (3) in Section 3) and (b) the topic of BA_φ includes all those topics that the reasoner considers relevant for φ and all topics included in the topic of φ are trivially relevant for φ.

[23]It is easy to see that, given a Popper function $\mathcal{P}$ and a topic model $\mathcal{T}$, we have

1. for all $\varphi \in \mathcal{L}_{PL}$, $\mathcal{U}_{\mathcal{P},\mathcal{T}}(\varphi) = 1 - \mathcal{P}(\varphi|\top)$; and

2. $\mathcal{U}_{\mathcal{P},\mathcal{T}}(\varphi \rightarrow \psi) = \begin{cases} 1 - \mathcal{P}(\psi|\varphi), & \text{if } t_\psi \sqsubseteq f(t_\varphi) \\ 1 & \text{otherwise.} \end{cases}$

5 The Logic of On-Topic Indicatives

'Any complete theory of conditionals requires a theory of conditional inference' (Evans and Over, 2004, p. 168). Thus, we now investigate the logic of our on-topic indicatives. We present the closure principles of interest as premise-conclusion rules of the form '$\Gamma \vdash \Delta$' where $\Gamma, \Delta \subseteq \mathcal{L}$ with $\Gamma = \varnothing$ for zero-premise rules. For any $\varphi \in \mathcal{L}_{PL}$, '$\vdash_{PL} \varphi$' says that φ is theorem of classical propositional logic. Following Adams (1998), we define validity probabilistically in terms of degrees of unacceptability:[24]

Definition 4 (Validity). *A principle of the form $\Gamma \vdash \Delta$ is* valid *if and only if for any Popper function $\mathcal{P}$ and topic model $\mathcal{T}$,*

$$\sum_{\varphi \in \Gamma} \mathcal{U}(\varphi) \geqslant \mathcal{U}(\psi),$$

for all $\psi \in \Delta$. When $\Gamma = \varnothing$, we say $\vdash \Delta$ is valid *if and only if $\mathcal{U}(\psi) = 0$ for all $\psi \in \Delta$. $\Gamma \vdash \Delta$ is* invalid *otherwise.*

Our notion of validity depends on two relatively independent constraints: (i) probability and (ii) relevance or topicality, as per our two-component account of acceptability. Besides investigating valid closure principles, we want to check that the invalid ones fail for the right reason. So we consider *probabilistic validity* and *topical validity* separately, and highlight the distinct sources of invalidity. However, our focus keeps being the notion of validity given in Definition 4. We use the notions of probabilistic and topical validity in order to point out the subtle reasons for invalidity.

We say that $\Gamma \vdash \Delta$ is *probabilistically valid* (*p*-valid) iff for any Popper function $\mathcal{P}$ and singleton topic model $\mathcal{T}$, $\sum_{\varphi \in \Gamma} \mathcal{U}(\varphi) \geqslant \mathcal{U}(\psi)$, for all $\psi \in \Delta$. When $\Gamma = \varnothing$, we say $\vdash \Delta$ is *p-valid* if and only if $\mathcal{U}(\psi) = 0$ for all $\psi \in \Delta$; and $\Gamma \vdash \Delta$ is *p-invalid* otherwise. We say $\Gamma \vdash \Delta$ is *topically valid* (*t*-valid) iff for any topic model $\mathcal{T} = \langle T, \oplus, t, f \rangle$, if every conditional in Γ is an on-topic conditional wrt $\mathcal{T}$ then every conditional in Δ is also an on-topic conditional wrt $\mathcal{T}$; and $\Gamma \vdash \Delta$ is *t-invalid* otherwise.

Our *p*-validity works similarly to Adams' *p*-validity – except that we define it in terms of Popper functions instead of unconditional probability functions – and it bypasses the topicality constraint. *t*-validity ignores probabilistic constraints and checks whether a closure principle satisfies the required relevance condition. The following lemma relates validity *simpliciter* to *p*- and *t*-validity and vice versa:

Lemma 2. *If $\Gamma \vdash \Delta$ is valid then it is p-valid but not necessarily t-valid. If $\Gamma \vdash \Delta$ is both p- and t-valid, then it is valid.*

Proof. For the first part, it is easy to see that validity implies *p*-validity by definition: the latter is a special case of the former obtained by restricting validity to the class of singleton topic models. As an example of an inference which is valid but *t*-invalid, consider $p \wedge \neg p \vdash p \rightarrow q$. To show its validity, let $\mathcal{P}$ be a Popper function and $\mathcal{T}$ be a topic model on $\mathcal{L}_{PL}$. By Definitions 1 and 3, we have $\mathcal{A}(p \wedge \neg p) = \mathcal{P}(p \wedge \neg p | \top) = 0$,

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[24]Adams calls his analogous notion *uncertainty*; the terminology is a bit misleading (for it's actually the certainty of negation or falsity), but has gained currency.

thus, $\mathcal{U}(p \wedge \neg p) = 1$. As $\mathcal{U}(p \to q) \in [0, 1]$ by the definition of $\mathcal{U}$, we obtain that $\mathcal{U}(p \wedge \neg p) \geqslant \mathcal{U}(p \to q)$. To show its t-invalidity, consider the topic model $\langle \{a, b\}, \oplus, f, t \rangle$ such that $\oplus$ is idempotent and $a \oplus b = a$, thus, $b \sqsubset a$. Moreover, f is a constant function and $t_p = b$ and $t_q = a$. Therefore, $b = t_p = f(t_p)$ but $a = t_q \not\sqsubseteq f(t_p) = b$ (see Figure 1).
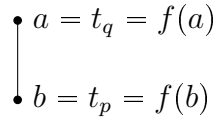
$$
\begin{array}{l}
\bullet\; a = t_q = f(a) \\
\Big| \\
\bullet\; b = t_p = f(b)
\end{array}
$$

Figure 1: Topic model $\langle \{a, b\}, \oplus, t, f \rangle$

For the second part, suppose that $\Gamma \vdash \Delta$ is both p- and t-valid. Let $\mathcal{P}$ a Popper function and $\mathcal{T}$ a topic model. Since $\Gamma \vdash \Delta$ is t-valid, we have two cases:

Case 1: Every conditional in $\Gamma \cup \Delta$ is an on-topic conditional wrt $\mathcal{T}$.
Then, it is easy to see that validity and p-validity coincide, thus, $\Gamma \vdash \Delta$ is valid.

Case 2: There is a conditional in Γ that is not an on-topic conditional wrt $\mathcal{T}$.
Wlog, suppose that $\varphi \in \Gamma$ is not an on-topic conditional wrt $\mathcal{T}$. This means that $\mathcal{U}_{\mathcal{P}, \mathcal{T}}(\varphi) = 1$ (by the definition of $\mathcal{U}_{\mathcal{P}, \mathcal{T}}$). Recall that $\mathcal{U}_{\mathcal{P}, \mathcal{T}}(\chi) \in [0, 1]$ for all $\chi \in \mathcal{L}$. Therefore, we conclude that $\sum_{\varphi \in \Gamma} \mathcal{U}_{\mathcal{P}, \mathcal{T}}(\varphi) \geqslant \mathcal{U}_{\mathcal{P}, \mathcal{T}}(\chi)$ for all $\chi \in \Delta$. $\qquad\square$

We now focus on the closure principles given in Table 1 (we label them, again, following Douven (2016), who sticks to popular names from the literature). The following is our main technical result:

Theorem 3.

1. *REF, ANT, CM, CC, CSO, CT, CMon, OR, and Modus Ponens are both p- and t-valid. Therefore, they all are valid.*

2. *MOD, RCE, RCEA, RCEC, RCK, RCM, and And-to-If are p-valid but t-invalid.*

3. *Trans and SA are p-invalid but t-valid.*

4. *Or-to-if, Contraposition, and SDA are both p-invalid and t-invalid.*

5. *MOD, RCE, RCEA, RCEC, RCK, RCM, And-to-If, Trans, SA, Or-to-If, Contraposition, and SDA are invalid.*

Proof. See Appendix A. $\qquad\square$

We comment on some notable validities and invalidities. As for the former, REF (Reflexivity) and ANT appear fairly obvious. CC, CT and CMon have already been discussed above.[25] Modus Ponens, we have argued, is desirable. The other validities hold

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[25]Countenancing the paradoxical Lottery cases, one might object to the validity of CC (see footnote 14). However, Lottery scenarios seem to rely on a more qualitative interpretation based on a (non-graded) notion of acceptability with respect to a threshold θ. And the most natural qualitative version of our proposal invalidates CC for $\theta \in (0.5, 1)$. Take a simple indicative $\varphi \to \psi$ to be (plainly) acceptable iff (i) $\mathcal{P}(\psi | \varphi) \geqslant \theta$ and (ii) $t_\psi \sqsubseteq f(t_\varphi)$; define the corresponding notion of validity as Douven (2016) does (see also p. 7): then CC becomes invalid for all threshold values $\theta \in (0.5, 1)$.

(REF)	$\vdash \varphi \to \varphi$
(ANT)	$\varphi \to \psi \vdash \varphi \to (\varphi \wedge \psi)$
(CM)	$\varphi \to (\psi \wedge \chi) \vdash \varphi \to \psi, \varphi \to \chi$
(CC)	$\varphi \to \psi, \varphi \to \chi \vdash \varphi \to (\psi \wedge \chi)$
(CSO)	$\varphi \to \psi, \psi \to \varphi, \varphi \to \chi \vdash \psi \to \chi$
(CT)	$\varphi \to \psi, (\varphi \wedge \psi) \to \chi \vdash \varphi \to \chi$
(CMon)	$\varphi \to \psi, \varphi \to \chi \vdash (\varphi \wedge \psi) \to \chi$
(OR)	$\varphi \to \psi, \chi \to \psi \vdash (\varphi \vee \chi) \to \psi$
(M. Ponens)	$\varphi, \varphi \to \psi \vdash \psi$
(Trans)	$\varphi \to \psi, \psi \to \chi \vdash \varphi \to \chi$
(SA)	$\varphi \to \psi \vdash (\varphi \wedge \chi) \to \psi$
(MOD)	$\neg\varphi \to \varphi \vdash \psi \to \varphi$
(RCE)	If $\varphi \vdash_{PL} \psi$, then $\vdash \varphi \to \psi$
(RCEA)	If $\vdash_{PL} \varphi \equiv \psi$, then $\varphi \to \chi \dashv\vdash \psi \to \chi$
(RCEC)	If $\vdash_{PL} \varphi \equiv \psi$, then $\chi \to \varphi \dashv\vdash \chi \to \psi$
(RCK)	If $\vdash_{PL} (\varphi_1 \wedge \cdots \wedge \varphi_n) \supset \psi$, then $\chi \to \varphi_1, \ldots, \chi \to \varphi_n \vdash$ $\chi \to \psi$
(RCM)	$\vdash_{PL} \varphi \supset \psi$, then $\chi \to \varphi \vdash \chi \to \psi$
(And-to-If)	$\varphi \wedge \psi \vdash \varphi \to \psi$
(Or-to-If)	$\varphi \vee \psi \vdash \neg\varphi \to \psi$
(Contr.)	$\varphi \to \neg\psi \vdash \psi \to \neg\varphi$
(SDA)	$(\varphi \vee \psi) \to \chi \vdash \varphi \to \chi, \psi \to \chi$

Table 1: Closure principles of interest

in most conditional logics and theories of non-monotonic entailment (see Nute, 1984, for a classic survey).

The invalidities in group 2 are all related to the hyperintensional acceptability conditions of conditionals: they are *p*-valid, but fail to be valid due to topicality. Look for instance at RCEA and RCEC: that φ and ψ are classically-necessarily equivalent doesn't guarantee their replacement in the antecedent or consequent of a conditional to preserve acceptability. Taking '5 is prime' and 'One cannot square the circle' as necessarily equivalent (qua true in all possible worlds), our sample conditionals above, (9) ('If all even numbers are prime and 5 is even, then 5 is prime') and (10) ('If all even numbers are prime and 5 is even, then one cannot square the circle') are not both acceptable: only the former's consequent is on-topic with respect to the antecedent. Or, look at RCE: that φ classically entails ψ doesn't make the corresponding conditional acceptable. 'Obama is tall' entails 'Either it is raining in Melbourne or not' in classical logic, but we don't assert 'If Obama is tall, then either it is raining in Melbourne or not', as the latter is patently off-topic.

One validity in group 1, namely CSO, stands out for comparison with such invalidities, for it limits the hyperintensional anarchy of indicatives. Even if replacement of necessary equivalents fails to preserve acceptability, CSO guarantees that replacement of conditional equivalents does: when both $\varphi \to \psi$ and the converse $\psi \to \varphi$ are among the premises (φ

	valid	p-valid	t-valid
REF	✓	✓	✓
ANT	✓	✓	✓
CM	✓	✓	✓
CC	✓	✓	✓
CSO	✓	✓	✓
CT	✓	✓	✓
CMon	✓	✓	✓
OR	✓	✓	✓
Modus Ponens	✓	✓	✓
MOD	X	✓	X
RCE	X	✓	X
RCEA	X	✓	X
RCEC	X	✓	X
RCK	X	✓	X
RCM	X	✓	X
And-to-If	X	✓	X
Trans	X	X	✓
SA	X	X	✓
Or-to-If	X	X	X
Contraposition	X	X	X
SDA	X	X	X

Table 2: Validities (✓) and invalidities (X): summary of the results in Theorem 3.

and ψ are taken as 'conditional equivalents' in this sense), the inference from these and $\varphi \to \chi$ to the conditional obtained by replacing φ with ψ in the latter, namely $\psi \to \chi$, is valid.

Groups 3 and 4 include inferences generally agreed to be invalid for any *ceteris paribus* conditional in the indicative and even in the subjunctive-counterfactual mood: SA (Strengthening the Antecedent), Contraposition, Transitivity, SDA (Simplification of Disjunctive Antecedents), Or-to-If, fail both in the Adams (1998) probabilistic semantics for indicatives and in the possible worlds semantics for indicatives and/or counterfactuals by Stalnaker (1968) and Lewis (1973).

Finally, And-to-If fails in the most natural way: the inference from $\varphi \land \psi$ to $\varphi \to \psi$ fails to be acceptability-preserving due to the topicality constraint: the latter may be an off-topic conditional like our (5) above ('If raccoons have no wings, then they cannot breath under water') although the former is a true and acceptable conjunction like our (4) ('Raccoons have no wings and they cannot breath under water'). Although the conjuncts plausibly overlap in topic (they are both about *raccoons*), which makes the conjunction coherently assertable in discourse, the topic of 'Raccoons cannot breath under water' is not fully included in that of the background assumptions contextually triggered by 'Raccoons have no wings'.

Such (in)validities make for a conditional logic that is not only theoretically desirable,

but also empirically plausible. As noted e.g. in (Evans and Over, 2004, pp. 44-5), the vast majority of experimental results concerning how people reason with conditionals only involve four simple inferences: Modus Ponens, Modus Tollens, and the usual fallacies of Affirming the Consequent and Denying the Antecedent. There are few studies investigating other conditional inferences (we mentioned (Pfeifer and Kleiter, 2010) above as one notable exception). However, an initial and tentative assessment of the psychological plausibility of our logic is possible, thanks to a sophisticated experiment reported in (Douven, 2016, Ch. 5).

Acknowledging that many inferences considered in the literature on conditional logics, including various among those in our table, are of a kind that people would rarely make in everyday reasoning, Douven went on to test them experimentally in a more roundabout way. Here's a summary of what he did (the detailed presentation is in pp. 140ff of Douven's book).

Most inferences involving simple conditionals in conditional logic feature at most three propositions (expressed by sentences) φ, ψ, and χ. So Douven asked over 1,000 subjects to rate the probabilities of conjunctions of the form $\pm\varphi \wedge \pm\psi \wedge \pm\chi$ (called *atoms*), with '$\pm\$$' indicating that sentence $\$$ may occur negated or unnegated, and φ, ψ, χ taken from news websites. For each triple of sentences, there are eight mutually exclusive and jointly exhaustive combinations (exactly one of them has to be true), composing an *atom matrix*. Subjects were instructed that the truth of any element of the matrix would exclude that of all the others, and that one of the elements had to be true, so that the probabilities assigned to the atoms had to add up to 100%. Douven then computed which conditional-involving inferences with φ, ψ, χ end up acceptability-preserving. He checked acceptability-preservation for two thresholds, $\theta = 0.5$ and $\theta = 0.9$. The results summarized in the table of (Douven, 2016, p. 144) show that the validities of our logic tested in the experiment correspond to highly popular inferences: CC has percentages of 100/100 endorsement (for 0.5 and 0.9 respectively); CSO has 75/100; CT has 87/94; CMon has 86/100; Modus Ponens has 91/78. Vice versa, some invalidities have low endorsement rates: Or-to-If has 28/9; SDA has 44/56.

An open problem is that the three inferences of Contraposition (70/78), SA (76/97) and Transitivity (78/100) are highly endorsed. However, these are invalid, as we mentioned, in *any* conditional logic for non-monotonic and *ceteris paribus* conditionals. Their invalidity, furthermore, is not due specifically to the distinctive element of our semantics for the indicative, namely our topicality constraint: they are invalidated purely probabilistically in semantics à la Adams, and they fail also in similarity-based possible worlds semantics à la Stalnaker-Lewis, due to conditionals being 'variably strict' (Lewis, 1973, p. 13) in this approach. Unsurprisingly, therefore, authors endorsing some variant of any of these treatments of conditionals have come up with explanations for the popularity of such invalid inferences. In particular, the fact that the three of them are more popular with higher 0.9 threshold than with lower 0.5, may corroborate the story proposed in Adams (1998); Bennett (2003): such inferences fail for non-perfectly-certain propositions, so it is plausible that their endorsement grows as we lift the threshold towards certainty, i.e., probability 1. They tend to be endorsed to the extent that they are mistaken for their limited counterparts, like Cautious Monotonicity (CMon) and Cautious Transitivity (CT).

19

We have focused on closure principles that are more commonly discussed in the context of conditional logics and non-monotonic reasoning, and empirically tested by Douven (2016). A more exhaustive list can be found, e.g., in (Douven, 2016, p. 129) and (Crupi and Iacona, 2019, p. 6). How to extend our analysis for those additional principles should be obvious. We can adopt the components of Crupi and Iacona (2019)'s framework concerning the operators necessity ($\Box$), possibility ($\Diamond$), and negation ($\sim$), and evaluate the principles involving them with respect to our topic-sensitive semantics.

6 Conclusion

We have presented a general semantics giving acceptability conditions for simple indicatives. The semantics preserves good features of probabilistic approaches to conditionals while fixing their troubles with relevance: we accept a conditional to the extent that its consequent is likely conditional on the antecedent, provided there's a relevant connection between antecedent and consequent. We have proposed that relevance be understood as topic-sensitivity, adopting a theory of topics that captures common features of various recently burgeoning subject matter semantics.

We have, then, presented the logic of on-topic conditionals; and we have argued that the closure principles (in)validated by the logic are both theoretically plausible, and in line with empirical results on how people reason with conditionals. In particular, our logic of on-topic indicatives fares better than EST with Douven (2016)'s experimental results as it validates the highly endorsed principles – CC, CSO, CT, CMon, and Modus Ponens – that fail with respect to EST. Our proposal also does a better job than AT in tracking relevance between the antecedent and consequent of an indicative conditional. Invalidities MOD, RCE, RCEA, RCEC, RCK, RCM, and And-to-If are all related to the hyperintensional acceptability conditions of conditionals: they are valid in Adams (1998)'s probabilistic semantics for indicatives, but fail to be valid in our account due to our topicality constraint.

We haven't conducted new experiments on reasoning with conditionals, but presented a logic that aligns nicely with the existing experimental results. However, we mention the following as a possible direction of further work (thanks to an anonymous referee): one could operationalize the notion of topicality for conditionals, and then carry out experiments specifically designed to test the original component of our logic, topicality, separately from its probabilistic component.

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A Proof of Theorem 3

The following lemma is well known and will be useful in proving Theorem 3.

Lemma 4. *For any Popper function $\mathcal{P} : \mathcal{L}_{PL} \times \mathcal{L}_{PL} \to [0,1]$ and $\varphi, \psi, \chi \in \mathcal{L}_{PL}$, we have*

1. *if φ is normal, then $\mathcal{P}(\neg\psi|\varphi) = 1 - \mathcal{P}(\psi|\varphi)$;*

2. *$\models_{PL} \varphi \equiv \psi$, then $\mathcal{P}(\varphi|\chi) = \mathcal{P}(\psi|\chi)$;*

3. *if $\varphi \models_{PL} \psi$, $\mathcal{P}(\psi|\chi) \geqslant \mathcal{P}(\varphi|\chi)$; and*

4. *$\mathcal{P}(\varphi \wedge \psi|\chi) = \mathcal{P}(\varphi|\chi) + \mathcal{P}(\psi|\chi) - \mathcal{P}(\varphi \vee \psi|\chi)$.*

Proof. Let $\mathcal{P}$ be a Popper function and $\varphi, \psi, \chi \in \mathcal{L}_{PL}$.

1. Suppose that φ is normal. We know that $\varphi \models_{PL} \psi \vee \neg\psi$ for all $\psi \in \mathcal{L}_{PL}$. Then, by Definition 1.3, $\mathcal{P}(\psi \vee \neg\psi|\varphi) = 1$. Moreover, ψ and $\neg\psi$ are obviously logically inconsistent. Then, by Definition 1.4, we have that $\mathcal{P}(\neg\psi|\varphi) = 1 - \mathcal{P}(\psi|\varphi)$.

2. We have two cases:

    Case 1: χ is abnormal
    Then, the result follows by the definition of abnormal sentences. In particular we have $\mathcal{P}(\varphi|\chi) = \mathcal{P}(\psi|\chi) = 1$.

    Case 2: χ is normal
    Suppose $\models_{PL} \varphi \equiv \psi$. Then, $\chi \models_{PL} \neg\varphi \vee \psi$ and $\chi \models_{PL} \varphi \vee \neg\psi$. Thus, by Definition 1.3, we have that $\mathcal{P}(\neg\varphi \vee \psi|\chi) = 1$ and $\mathcal{P}(\varphi \vee \neg\psi|\chi) = 1$. Moreover, we know that $\models_{PL} \neg(\neg\varphi \wedge \psi)$ and $\models_{PL} \neg(\varphi \wedge \neg\psi)$. Thus, by Definition 1.4, we obtain that

    $$\mathcal{P}(\neg\varphi \vee \psi|\chi) = \mathcal{P}(\neg\varphi|\chi) + \mathcal{P}(\psi|\chi) \text{ and}$$

    $$\mathcal{P}(\varphi \vee \neg\psi|\chi) = \mathcal{P}(\varphi|\chi) + \mathcal{P}(\neg\psi|\chi).$$

    As $\mathcal{P}(\neg\varphi \vee \psi|\chi) = \mathcal{P}(\varphi \vee \neg\psi|\chi) = 1$, we conclude that $\mathcal{P}(\varphi|\chi) = \mathcal{P}(\psi|\chi)$.

3. We have two cases:

    Case 1: χ is abnormal
    Follows by the definition of abnormal sentences.

    Case 2: χ is normal
    Suppose that $\varphi \models_{PL} \psi$. By classical propositional logic, we have that $\models_{PL} \neg((\neg\varphi \wedge \psi) \wedge \varphi)$. Then, by Definition 1.4, we obtain that $\mathcal{P}((\neg\varphi \wedge \psi) \vee \varphi|\chi) = \mathcal{P}(\neg\varphi \wedge \psi|\chi) + \mathcal{P}(\varphi|\chi)$. As $\varphi \models_{PL} \psi$, we have that $\models_{PL} ((\neg\varphi \wedge \psi) \vee \varphi) \equiv \psi$, thus, by Lemma 4.2, we obtain $\mathcal{P}((\neg\varphi \wedge \psi) \vee \varphi|\chi) = \mathcal{P}(\psi|\chi)$, i.e., $\mathcal{P}(\neg\varphi \wedge \psi|\chi) + \mathcal{P}(\varphi|\chi) = \mathcal{P}(\psi|\chi)$. Then, since $\mathcal{P}(\neg\varphi \wedge \psi|\chi) \in [0,1]$, we obtain that $\mathcal{P}(\psi|\chi) \geqslant \mathcal{P}(\varphi|\chi)$.

4. We have two cases:

    Case 1: χ is abnormal
    Then, the result follows by the definition of abnormal sentences.

    Case 2: χ is normal
    By Definition 1.4 and Lemma 4.2, we have

(a) $\mathcal{P}(\varphi|\chi) = \mathcal{P}(\varphi \wedge \psi|\chi) + \mathcal{P}(\varphi \wedge \neg\psi|\chi)$, and

(b) $\mathcal{P}(\psi|\chi) = \mathcal{P}(\varphi \wedge \psi|\chi) + \mathcal{P}(\neg\varphi \wedge \psi|\chi)$.

Thus,

$$\begin{aligned}
\mathcal{P}(\varphi|\chi) + \mathcal{P}(\psi|\chi) &= \mathcal{P}(\varphi \wedge \psi|\chi) + (\mathcal{P}(\varphi \wedge \psi|\chi) + \mathcal{P}(\varphi \wedge \neg\psi|\chi) + \mathcal{P}(\neg\varphi \wedge \psi|\chi)) \\
&= \mathcal{P}(\varphi \wedge \psi|\chi) + \mathcal{P}(\varphi \vee \psi|\chi) \qquad \text{(Defn. 1.4 and Lemma 4.2)}
\end{aligned}$$

I.e., $\mathcal{P}(\varphi \wedge \psi|\chi) = \mathcal{P}(\varphi|\chi) + \mathcal{P}(\psi|\chi) - \mathcal{P}(\varphi \vee \psi|\chi)$.

$\square$

A.1   Proof of Theorem 3.1

Let $\mathcal{P}$ be a Popper function and $\mathcal{T} = \langle T, \oplus, t, f \rangle$ be a topic model. We first show that REF, ANT, CM, CC, CSO, CT, CMon, OR, and Modus Ponens are both p- and t-valid. Therefore, by Lemma 2, they are also valid.

Given a premise-conclusion rule $\Gamma \vdash \Delta$, we only consider the cases where the antecedents of the conditionals in Δ are normal with respect to the given Popper function since otherwise the degrees of unacceptability of the conclusions equal to 0. And, due to the structure of the rules in Theorem 3.1, we cannot have that the antecedents of all the premises are abnormal but the antecedent of the conclusion is not.

REF: $\vdash \varphi \to \varphi$
t-valid: By Definition 2.4a, we have that $t_\varphi \sqsubseteq f(t_\varphi)$.
p-valid: By Definition 1.3, we have $\mathcal{P}(\varphi|\varphi) = 1$. Moreover, we also have that $t_\varphi \sqsubseteq f(t_\varphi)$. Therefore, $\mathcal{U}(\varphi \to \varphi) = 1 - \mathcal{P}(\varphi|\varphi) = 0$.

ANT: $\varphi \to \psi \vdash \varphi \to (\varphi \wedge \psi)$
t-valid: Suppose that $t_\psi \sqsubseteq f(t_\varphi)$. Then, since $t_\varphi \sqsubseteq f(t_\varphi)$ (Definition 2.4a), we obtain that $t_{\varphi \wedge \psi} = t_\varphi \oplus t_\psi \sqsubseteq f(t_\varphi)$.
p-valid: Now suppose that T is a singleton. Therefore, $\mathcal{U}(\varphi \to \psi) = 1 - \mathcal{P}(\psi|\varphi)$ and $\mathcal{U}(\varphi \to (\varphi \wedge \psi)) = 1 - \mathcal{P}(\varphi \wedge \psi|\varphi)$. Observe that $\mathcal{P}(\varphi \wedge \psi|\varphi) = \mathcal{P}(\varphi|\varphi) + \mathcal{P}(\psi|\varphi) - \mathcal{P}(\varphi \vee \psi|\varphi)$ (by Lemma 4.4), where $\mathcal{P}(\varphi|\varphi) = 1$ and $\mathcal{P}(\varphi \vee \psi|\varphi) = 1$ (by Definition 1.3). Therefore, $\mathcal{U}(\varphi \to \psi) = \mathcal{U}(\varphi \to (\varphi \wedge \psi))$.

CM: $\varphi \to (\psi \wedge \chi) \vdash \varphi \to \psi, \varphi \to \chi$
t-valid: Suppose that $t_{\psi \wedge \chi} \sqsubseteq f(t_\varphi)$. Then, since $t_\psi, t_\chi \sqsubseteq t_\psi \oplus t_\chi = t_{\psi \wedge \chi} \sqsubseteq f(t_\varphi)$ and $\sqsubseteq$ is transitive, we conclude that $t_\psi, t_\chi \sqsubseteq f(t_\varphi)$.
p-valid: Now suppose that T is a singleton. Therefore, $\mathcal{U}(\varphi \to (\psi \wedge \chi)) = 1 - \mathcal{P}(\psi \wedge \chi|\varphi)$, $\mathcal{U}(\varphi \to \psi) = 1 - \mathcal{P}(\psi|\varphi)$, and $\mathcal{U}(\varphi \to \chi) = 1 - \mathcal{P}(\chi|\varphi)$. Since $\psi \wedge \chi \vDash_{PL} \psi$ and $\psi \wedge \chi \vDash_{PL} \chi$, by Lemma 4.3, we know that $\mathcal{P}(\psi \wedge \chi|\varphi) \leqslant \mathcal{P}(\psi|\varphi)$ and $\mathcal{P}(\psi \wedge \chi|\varphi) \leqslant \mathcal{P}(\chi|\varphi)$. Therefore, $\mathcal{U}(\varphi \to (\psi \wedge \chi)) \geqslant \mathcal{U}(\varphi \to \psi)$ and $\mathcal{U}(\varphi \to (\psi \wedge \chi)) \geqslant \mathcal{U}(\varphi \to \chi)$.

CC: $\varphi \to \psi, \varphi \to \chi \vdash \varphi \to (\psi \wedge \chi)$

t-valid: Suppose that $t_\psi \sqsubseteq f(t_\varphi)$ and $t_\chi \sqsubseteq f(t_\varphi)$. Then, by the properties of $\oplus$, we obtain that $t_{\psi \wedge \chi} \sqsubseteq t_\psi \oplus t_\chi \sqsubseteq f(t_\varphi)$.

p-valid: Now suppose that T is a singleton. We then have

$$
\begin{aligned}
\mathcal{U}(\varphi \to \psi) + \mathcal{U}(\varphi \to \chi) &= 1 - \mathcal{P}(\psi|\varphi) + 1 - \mathcal{P}(\chi|\varphi) \\
&\geqslant 1 - \mathcal{P}(\psi|\varphi) - \mathcal{P}(\chi|\varphi) + \mathcal{P}(\psi \vee \chi|\varphi) \qquad (\mathcal{P}(\psi \vee \chi|\varphi) \leqslant 1) \\
&= 1 - \mathcal{P}(\psi \wedge \chi|\varphi) \qquad\qquad\qquad\qquad\quad (\text{Lemma 4.4}) \\
&= \mathcal{U}(\varphi \to (\psi \wedge \chi)) \qquad\qquad\qquad\quad (\text{by the defn. of } \mathcal{U})
\end{aligned}
$$

CSO: $\varphi \to \psi, \psi \to \varphi, \varphi \to \chi \vdash \psi \to \chi$

t-valid: Suppose that $t_\varphi \sqsubseteq f(t_\psi)$, $t_\psi \sqsubseteq f(t_\varphi)$, and $t_\chi \sqsubseteq f(t_\varphi)$. Then, by the former two, Lemma 1, and Definition 2.4b, we have $f(t_\psi) = f(t_\varphi)$. Therefore, we obtain that $t_\chi \sqsubseteq f(t_\psi)$.

p-valid: Now suppose that T is a singleton. Then, $\mathcal{U}(\varphi \to \psi) = 1 - \mathcal{P}(\psi|\varphi)$, $\mathcal{U}(\psi \to \varphi) = 1 - \mathcal{P}(\varphi|\psi)$, $\mathcal{U}(\varphi \to \chi) = 1 - \mathcal{P}(\chi|\varphi)$, and $\mathcal{U}(\psi \to \chi) = 1 - \mathcal{P}(\chi|\psi)$. To simplify notation, we represent the conditional probabilities on $\psi \vee \chi \vee \varphi$ by using the diagram in Figure 2 (similar to Adams (1998)'s diagrams). Each region of the diagram is identified with the set theoretic counterpart of Boolean combinations of ψ, χ, and φ, and each letter represents the probability of its region conditional on $\psi \vee \chi \vee \varphi$. For example, $\mathcal{P}(\varphi \wedge \neg\psi \wedge \neg\chi|\psi \vee \chi \vee \varphi) = x$, $\mathcal{P}(\varphi \wedge \psi \wedge \neg\chi|\psi \vee \chi \vee \varphi) = y$, and $\mathcal{P}(\varphi \wedge \psi \wedge \chi|\psi \vee \chi \vee \varphi) = k$ etc.
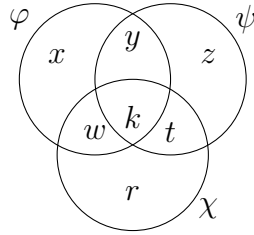

Figure 2: Diagram for the Popper function $\mathcal{P}(\cdot|\psi \vee \chi \vee \varphi)$

By Definition 1.2, 1.4 and 1.5, observe that

1. $\mathcal{P}(\psi|\varphi) = \mathcal{P}(\psi|\varphi \wedge (\psi \vee \chi \vee \varphi)) = \frac{\mathcal{P}(\psi \wedge \varphi|\psi \vee \chi \vee \varphi)}{\mathcal{P}(\varphi|\psi \vee \chi \vee \varphi)} = \frac{y+k}{x+y+w+k}$,

2. $\mathcal{P}(\varphi|\psi) = \mathcal{P}(\varphi|\psi \wedge (\psi \vee \chi \vee \varphi)) = \frac{\mathcal{P}(\psi \wedge \varphi|\psi \vee \chi \vee \varphi)}{\mathcal{P}(\psi|\psi \vee \chi \vee \varphi)} = \frac{y+k}{y+z+k+t}$,

3. $\mathcal{P}(\chi|\varphi) = \mathcal{P}(\chi|\varphi \wedge (\psi \vee \chi \vee \varphi)) = \frac{\mathcal{P}(\chi \wedge \varphi|\psi \vee \chi \vee \varphi)}{\mathcal{P}(\varphi|\psi \vee \chi \vee \varphi)} = \frac{w+k}{x+y+w+k}$, and

4. $\mathcal{P}(\chi|\psi) = \mathcal{P}(\chi|\psi \wedge (\psi \vee \chi \vee \varphi)) = \frac{\mathcal{P}(\chi \wedge \psi|\psi \vee \chi \vee \varphi)}{\mathcal{P}(\psi|\psi \vee \chi \vee \varphi)} = \frac{t+k}{y+z+k+t}$.

Note also that $1 = \mathcal{P}(\psi \vee \chi \vee \varphi|\psi \vee \chi \vee \varphi) = x + y + z + w + k + t + r$. Thus, we need to show that

$$
\frac{w+x}{x+y+w+k} + \frac{z+t}{y+z+k+t} + \frac{x+y}{x+y+w+k} \geqslant \frac{y+z}{y+z+k+t}.
$$

This can easily be established by tedious but simple algebra.

CT: $\varphi \to \psi, (\varphi \wedge \psi) \to \chi \vdash \varphi \to \chi$

t-valid: Suppose that $t_\psi \sqsubseteq f(t_\varphi)$ and $t_\chi \sqsubseteq f(t_{\varphi \wedge \psi})$. The former, by Lemma 1 and Definition 2.4b, implies that $f(t_\psi) \sqsubseteq f(t_\varphi)$. Then, by Definition 2.4c, we obtain that $f(t_{\varphi \wedge \psi}) = f(t_\varphi \oplus t_\psi) = f(t_\varphi) \oplus f(t_\psi) = f(t_\varphi)$. Hence, since $t_\chi \sqsubseteq f(t_{\varphi \wedge \psi})$, we have $t_\chi \sqsubseteq f(t_\varphi)$.

p-valid: Now suppose that T is a singleton. Therefore, we have that $\mathcal{U}(\varphi \to \psi) = 1 - \mathcal{P}(\psi|\varphi)$, $\mathcal{U}((\varphi \wedge \psi) \to \chi) = 1 - \mathcal{P}(\chi|\varphi \wedge \psi)$, and $\mathcal{U}(\varphi \to \chi) = 1 - \mathcal{P}(\chi|\varphi)$. Observe that, by Definition 1.4 and Lemma 4.2, we have $\mathcal{P}(\psi|\varphi) = \mathcal{P}(\psi \wedge \chi|\varphi) + \mathcal{P}(\psi \wedge \neg\chi|\varphi)$, $\mathcal{P}(\chi|\varphi) = \mathcal{P}(\psi \wedge \chi|\varphi) + \mathcal{P}(\neg\psi \wedge \chi|\varphi)$, and by Definition 1.5, $\mathcal{P}(\chi|\psi \wedge \varphi) = \frac{\mathcal{P}(\psi \wedge \chi|\varphi)}{\mathcal{P}(\psi|\varphi)}$. To ease the notation, let $\mathcal{P}(\psi \wedge \chi|\varphi) = x$, $\mathcal{P}(\psi \wedge \neg\chi|\varphi) = y$, $\mathcal{P}(\neg\psi \wedge \chi|\varphi) = z$, and $\mathcal{P}(\neg\psi \wedge \neg\chi|\varphi) = w$. Obviously, $x + y + z + w = \mathcal{P}(\top|\varphi)$ (by Definition 1.4) $= 1$ (by Definition 1.3). Then, $\mathcal{U}(\varphi \to \psi) = z + w$, $\mathcal{U}(\varphi \to \chi) = y + w$ and $\mathcal{U}((\varphi \wedge \psi) \to \chi) = \frac{y}{x+y}$. Since $\frac{y}{x+y} \geqslant y$, we conclude that $z + w + \frac{y}{x+y} \geqslant y + w$, that is, $\mathcal{U}(\varphi \to \psi) + \mathcal{U}((\varphi \wedge \psi) \to \chi) \geqslant \mathcal{U}(\varphi \to \chi)$.

CMon: $\varphi \to \psi, \varphi \to \chi \vdash (\varphi \wedge \psi) \to \chi$

t-valid: Suppose that $t_\psi \sqsubseteq f(t_\varphi)$ and $t_\chi \sqsubseteq f(t_\varphi)$. The latter, by Definition 2.4, implies that $t_\chi \sqsubseteq f(t_{\varphi \wedge \psi})$.

p-valid: Now suppose that T is a singleton. Then, $\mathcal{U}(\varphi \to \psi) = 1 - \mathcal{P}(\psi|\varphi)$, $\mathcal{U}(\varphi \to \chi) = 1 - \mathcal{P}(\chi|\varphi)$, and $\mathcal{U}((\varphi \wedge \psi) \to \chi) = 1 - \mathcal{P}(\chi|\varphi \wedge \psi)$. Observe that

$$
\begin{aligned}
\mathcal{P}(\chi|\varphi \wedge \psi) &= \frac{\mathcal{P}(\psi \wedge \chi|\varphi)}{\mathcal{P}(\psi|\varphi)} &&\text{(Definition 1.5)} \\
&\geqslant \mathcal{P}(\psi \wedge \chi|\varphi) &&\text{(since } \mathcal{P}(\psi|\varphi) \in [0,1]) \\
&= \mathcal{P}(\psi|\varphi) + \mathcal{P}(\chi|\varphi) - \mathcal{P}(\psi \vee \chi|\varphi) &&\text{(Lemma 4.4)} \\
&\geqslant \mathcal{P}(\psi|\varphi) + \mathcal{P}(\chi|\varphi) - 1 &&(\mathcal{P}(\psi \vee \chi|\varphi) \leqslant 1)
\end{aligned}
$$

Therefore, $\mathcal{U}(\varphi \to \psi) + \mathcal{U}(\varphi \to \chi) = 2 - \mathcal{P}(\psi|\varphi) - \mathcal{P}(\chi|\varphi) \geqslant 1 - \mathcal{P}(\chi|\varphi \wedge \psi) = \mathcal{U}((\varphi \wedge \psi) \to \chi)$.

OR: $\varphi \to \psi, \chi \to \psi \vdash (\varphi \vee \chi) \to \psi$

t-valid: Suppose that $t_\psi \sqsubseteq f(t_\varphi)$ and $t_\psi \sqsubseteq f(t_\chi)$. The latter implies that $t_\psi \sqsubseteq f(t_{\varphi \vee \chi})$ (by Lemma 1 and the fact that $t_\chi \sqsubseteq t_{\varphi \vee \chi}$).

p-valid: Now suppose that T is a singleton. Then, $\mathcal{U}(\varphi \to \psi) = 1 - \mathcal{P}(\psi|\varphi)$, $\mathcal{U}(\chi \to \psi) = 1 - \mathcal{P}(\psi|\chi)$, and $\mathcal{U}((\varphi \vee \chi) \to \psi) = 1 - \mathcal{P}(\psi|\varphi \vee \chi)$. By Definition 1.2, 1.4 and 1.5, observe that

1. $\mathcal{P}(\psi|\varphi) = \mathcal{P}(\psi|\varphi \wedge (\varphi \vee \chi)) = \frac{\mathcal{P}(\varphi \wedge \psi|\varphi \vee \chi)}{\mathcal{P}(\varphi|\varphi \vee \chi)} = \frac{\mathcal{P}(\varphi \wedge \psi|\varphi \vee \chi)}{\mathcal{P}(\varphi \wedge \psi|\varphi \vee \chi) + \mathcal{P}(\varphi \wedge \neg\psi|\varphi \vee \chi)}$,

2. $\mathcal{P}(\psi|\chi) = \mathcal{P}(\psi|\chi \wedge (\varphi \vee \chi)) = \frac{\mathcal{P}(\chi \wedge \psi|\varphi \vee \chi)}{\mathcal{P}(\chi|\varphi \vee \chi)} = \frac{\mathcal{P}(\chi \wedge \psi|\varphi \vee \chi)}{\mathcal{P}(\chi \wedge \psi|\varphi \vee \chi) + \mathcal{P}(\chi \wedge \neg\psi|\varphi \vee \chi)}$, and

3. $\mathcal{P}(\psi|\varphi \vee \chi) = \mathcal{P}(\psi \wedge (\chi \wedge \varphi)|\varphi \vee \chi) + \mathcal{P}(\psi \wedge (\neg\chi \wedge \varphi)|\varphi \vee \chi) + \mathcal{P}(\psi \wedge (\chi \wedge \neg\varphi)|\varphi \vee \chi)$.

To simplify the proof, we represent the conditional probabilities on $\varphi \vee \chi$ by using the diagram in Figure 3, similar to the one in the proof for CSO.
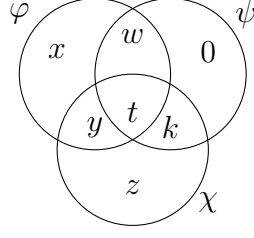
Then, by Definition 1.4, we have

Figure 3: Diagram for the Popper function $\mathcal{P}(\cdot | \varphi \vee \chi)$

4. $\mathcal{P}(\varphi \wedge \psi | \varphi \vee \chi) = \underbrace{\mathcal{P}((\varphi \wedge \psi) \wedge \chi | \varphi \vee \chi)}_{t} + \underbrace{\mathcal{P}((\varphi \wedge \psi) \wedge \neg \chi | \varphi \vee \chi)}_{w},$

5. $\mathcal{P}(\varphi \wedge \neg \psi | \varphi \vee \chi) = \underbrace{\mathcal{P}((\varphi \wedge \neg \psi) \wedge \chi | \varphi \vee \chi)}_{y} + \underbrace{\mathcal{P}((\varphi \wedge \neg \psi) \wedge \neg \chi | \varphi \vee \chi)}_{x},$

6. $\mathcal{P}(\chi \wedge \psi | \varphi \vee \chi) = \underbrace{\mathcal{P}((\chi \wedge \psi) \wedge \varphi | \varphi \vee \chi)}_{t} + \underbrace{\mathcal{P}((\chi \wedge \psi) \wedge \neg \varphi | \varphi \vee \chi)}_{k},$

7. $\mathcal{P}(\chi \wedge \neg \psi | \varphi \vee \chi) = \underbrace{\mathcal{P}((\chi \wedge \neg \psi) \wedge \varphi | \varphi \vee \chi)}_{y} + \underbrace{\mathcal{P}((\chi \wedge \neg \psi) \wedge \neg \varphi | \varphi \vee \chi)}_{z},$

Again by Definition 1.3 and 1.4, we have that $1 = \mathcal{P}(\varphi \vee \chi | \varphi \vee \chi) = x + y + z + t + k + w$. Then, simple calculations show that $\mathcal{U}(\varphi \to \psi) = \frac{x+y}{x+y+w+t}$, $\mathcal{U}(\chi \to \psi) = \frac{z+y}{t+k+z+y}$, and $\mathcal{U}((\varphi \vee \chi) \to \psi) = x + y + z$. Then, since $\frac{x+y}{x+y+w+t} \geqslant x + y$ and $\frac{z+y}{t+k+z+y} \geqslant z$, we obtain that $\frac{x+y}{x+y+w+t} + \frac{z+y}{t+k+z+y} \geqslant x + y + z$. That is, $\mathcal{U}(\varphi \to \psi) + \mathcal{U}((\chi \to \psi) \geqslant \mathcal{U}((\varphi \vee \chi) \to \psi)$.

Modus Ponens: $\varphi, \varphi \to \psi \vdash \psi$
t-valid: Modus Ponens is vacuously t-valid since the conclusion is not a conditional.
p-valid: Now suppose that T is a singleton.
Case 1: $\mathcal{P}(\varphi | \top) = 0$:
Then, $\mathcal{U}(\varphi | \top) = 1$. Therefore, $\mathcal{U}(\varphi) + \mathcal{U}(\varphi \to \psi) \geqslant 1 \geqslant \mathcal{U}(\psi)$.
Case 2: $\mathcal{P}(\varphi | \top) \neq 0$:
Then, $\mathcal{P}(\psi | \varphi) = \frac{\mathcal{P}(\varphi \wedge \psi | \top)}{\mathcal{P}(\varphi | \top)} = \frac{\mathcal{P}(\varphi \wedge \psi | \top)}{\mathcal{P}(\varphi \wedge \psi | \top) + \mathcal{P}(\varphi \wedge \neg \psi | \top)}$. Moreover, by Definition 1.4, we have

1. $1 = \mathcal{P}(\top | \top) = \underbrace{\mathcal{P}(\varphi \wedge \psi | \top)}_{x} + \underbrace{\mathcal{P}(\neg \varphi \wedge \psi | \top)}_{y} + \underbrace{\mathcal{P}(\varphi \wedge \neg \psi | \top)}_{z} + \underbrace{\mathcal{P}(\neg \varphi \wedge \neg \psi | \top)}_{w},$

2. $\mathcal{P}(\varphi | \top) = \mathcal{P}(\varphi \wedge \psi | \top) + \mathcal{P}(\varphi \wedge \neg \psi | \top) = x + z$, and

3. $\mathcal{P}(\psi | \top) = \mathcal{P}(\varphi \wedge \psi | \top) + \mathcal{P}(\neg \varphi \wedge \psi | \top) = x + y$.

Therefore, $\mathcal{P}(\psi | \varphi) = \frac{x}{x+z}$. Then, by simple calculations, we obtain that $\mathcal{U}(\varphi) + \mathcal{U}(\varphi \to \psi) = y + w + \frac{z}{x+z} \geqslant z + w = \mathcal{U}(\psi)$ (since $\frac{z}{x+z} \geqslant z$).

A.2 Proof of Theorem 3.2

MOD: $\neg\varphi \to \varphi \vdash \psi \to \varphi$

p-validity: Let $\mathcal{P}$ be a Popper function and $\mathcal{T} = \langle T, \oplus, t, f \rangle$ be a singleton topic model on $\mathcal{L}_{PL}$. Since $\mathcal{T}$ is a singleton topic model, $\psi \to \varphi$ is an on-topic conditional wrt $\mathcal{T}$. We then have two cases:

Case 1: $\neg\varphi$ is normal

Then, $\mathcal{P}(\varphi|\neg\varphi) = 0$. Therefore, $\mathcal{U}(\neg\varphi \to \varphi) = 1 - \mathcal{P}(\varphi|\neg\varphi) = 1$. Since $\mathcal{U}(\psi \to \varphi) \in [0, 1]$, we obtain the result.

Case 2: $\neg\varphi$ is abnormal

Then, $\mathcal{P}(\varphi|\neg\varphi) = 1$, thus, $\mathcal{U}(\neg\varphi \to \varphi) = 0$. Since $\mathcal{T}$ is a singleton model, we have $\mathcal{U}(\psi \to \varphi) = 1 - \mathcal{P}(\varphi|\psi)$. If ψ is abnormal, then $\mathcal{P}(\varphi|\psi) = 1$, thus, $\mathcal{U}(\psi \to \varphi) = 0$. If ψ is normal, then $\mathcal{U}(\psi \to \varphi) = \mathcal{P}(\neg\varphi|\psi)$ (by Lemma 4.1). Since $\neg\varphi$ is abnormal, we have that $\mathcal{P}(\neg\varphi|\psi) = 0$, i.e., $\mathcal{U}(\psi \to \varphi) = 0$.

t-invalidity: Consider the instance $\neg p \to p \vdash q \to p$ and the topic model $\langle \{a, b\}, \oplus, t, f \rangle$ such that $\oplus$ is idempotent and $a \oplus b = a$, thus, $b \sqsubset a$. Moreover, f is a constant function and $t_q = b$ and $t_p = a$. Therefore, $a = t_p = f(t_{\neg p})$ but $a = t_p \not\sqsubseteq f(t_q) = b$ (see Figure 4).
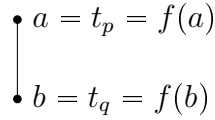
$$\bullet\ a = t_p = f(a)$$
$$\bullet\ b = t_q = f(b)$$

Figure 4: Topic model $\langle \{a, b\}, \oplus, t, f \rangle$

RCE: If $\varphi \vdash_{PL} \psi$, then $\vdash \varphi \to \psi$

p-validity: Follows immediately from Definition 1.3

t-invalidity: Consider the counterexample given in Figure 4 and take $\varphi := q$ and $\psi : p \vee \neg p$.

RCEA: If $\vdash_{PL} \varphi \equiv \psi$, then $\varphi \to \chi \dashv\vdash \psi \to \chi$

p-validity: Follows immediately from Definition 1.2.

t-invalidity: Consider the counterexample given in Figure 4 and take $\varphi := p \vee \neg p$, $\psi := q \vee \neg q$, $\chi := r \vee \neg r$ such that $t_r = a$. We then have that $\models_{PL} \varphi \equiv \psi$, $(p \vee \neg p) \to (r \vee \neg r)$ is an on-topic conditional wrt $\mathcal{T}$ but $(q \vee \neg q) \to (r \vee \neg r)$ is not.

RCEC: If $\vdash_{PL} \varphi \equiv \psi$, then $\chi \to \varphi \dashv\vdash \chi \to \psi$

p-validity: Follows immediately from Lemma 4.2.

t-invalidity: Consider the counterexample above but set $t_p = t_r = b$ and $t_q = a$.

RCK: If $\vdash_{PL} (\varphi_1 \wedge \cdots \wedge \varphi_n) \supset \psi$, then $\chi \to \varphi_1, \ldots, \chi \to \varphi_n \vdash \chi \to \psi$

p-validity: Suppose that $\mathcal{T}$ is a singleton topic model and $\vdash_{PL} (\varphi_1 \wedge \cdots \wedge \varphi_n) \supset \psi$. We want to show that $\mathcal{U}(\chi \to \varphi_1) + \cdots + \mathcal{U}(\chi \to \varphi_n) \geqslant \mathcal{U}(\chi \to \psi)$, i.e., that $1 - \mathcal{P}(\varphi_1|\chi) + \cdots + 1 - \mathcal{P}(\varphi_n|\chi) \geqslant 1 - \mathcal{P}(\psi|\chi)$, i.e., that $\mathcal{P}(\psi|\chi) \geqslant \mathcal{P}(\varphi_1|\chi) + \ldots \mathcal{P}(\varphi_n|\chi) - n + 1$. It is easy to see that

$$\mathcal{P}(\psi|\chi) \geqslant \mathcal{P}(\varphi_1 \wedge \cdots \wedge \varphi_n|\chi) \qquad \text{(by the assumption and Lemma 4.3)}$$
$$\geqslant \mathcal{P}(\varphi_1|\chi) + \ldots \mathcal{P}(\varphi_n|\chi) - n + 1 \quad \text{(by repeated applications of Lemma 4.4)}$$

t-invalidity: Same as the proof of RCEC.

RCM: $\vdash_{PL} \varphi \supset \psi$, then $\chi \to \varphi \vdash \chi \to \psi$
p-validity: Follows immediately by Lemma 4.3.

t-invalidity: Same as the proof of RCEC.

And-to-If: $\varphi \wedge \psi \vdash \varphi \to \psi$
p-validity: Let $\mathcal{P}$ be a Popper function and $\mathcal{T}$ be a singleton topic model. Then, $\mathcal{U}(\varphi \wedge \psi) = 1 - \mathcal{P}(\varphi \wedge \psi | \top)$ and $\mathcal{U}(\varphi \to \psi) = 1 - \mathcal{P}(\psi | \varphi)$. If $\mathcal{P}(\varphi | \top) = 0$, then $\mathcal{P}(\varphi \wedge \psi | \top) = 0$ (by Lemma 4.3). Therefore, $\mathcal{U}(\varphi \wedge \psi) = 1 \geqslant \mathcal{U}(\varphi \to \psi)$. If $\mathcal{P}(\varphi | \top) \neq 0$, then $\mathcal{P}(\psi | \varphi) = \frac{\mathcal{P}(\varphi \wedge \psi | \top)}{\mathcal{P}(\varphi | \top)}$. Thus, since $\mathcal{P}(\varphi | \top) \in [0, 1]$, we obtain that $\mathcal{P}(\psi | \varphi) \geqslant \mathcal{P}(\varphi \wedge \psi | \top)$. Thefore, $\mathcal{U}(\varphi \wedge \psi) \geqslant \mathcal{U}(\varphi \to \psi)$.

t-invalidity: See the counterexample given in Figure 4 and take $\varphi := q$ and $\psi := p$.

A.3   Proof of Theorem 3.3

We prove Theorems 3.3 and 3.4 by showing that every inference $\Gamma \vdash \Delta$ that is invalid with respect to AT and Adams' notion of probabilistic validity is also *p*-invalid with respect to our proposal. Let us first recall Adams' framework and fix notation (Adams, 1998). The reader who is familiar with Adams' probabilistic conditional logic and the connection between Popper and unconditional probability functions should feel free to skip until Corollary 6.

Let $p : \mathcal{L} \to [0, 1]$ be a(n) (unconditional) probability function defined on the language $\mathcal{L}$ such that for all $\varphi, \psi \in \mathcal{L}_{PL}$, $p(\varphi)$ and $p(\psi)$ satisfy the Kolmogorov axioms and

$$ p(\varphi \to \psi) = \begin{cases} \frac{p(\psi \wedge \varphi)}{p(\varphi)}, & \text{if } p(\varphi) \neq 0 \\ 1 & \text{otherwise.} \end{cases} $$

The corresponding uncertainty function $u_p : \mathcal{L} \to [0, 1]$ is defined as $u_p(\varphi) = 1 - p(\varphi)$ for all $\varphi \in \mathcal{L}$ (see Adams, 1998, Chapters 6 & 7). In fact, every unconditional probability measure p on $\mathcal{L}$ leads to a Popper function $\mathcal{P}_p : \mathcal{L}_{PL} \times \mathcal{L}_{PL} \to [0, 1]$ defined as $\mathcal{P}_p(\psi | \varphi) = p(\varphi \to \psi)$, for all $\varphi, \psi \in \mathcal{L}_{PL}$. The following lemma proves this result, which will be useful in proving Theorems 3.3 and 3.4.

Lemma 5. *Given an unconditional probability function* $p : \mathcal{L} \to [0, 1]$*, the conditional probability measure* $\mathcal{P}_p : \mathcal{L}_{PL} \times \mathcal{L}_{PL} \to [0, 1]$ *defined as* $\mathcal{P}_p(\psi | \varphi) = p(\varphi \to \psi)$ *is a Popper function.*

Proof. We need to show that $\mathcal{P}_p$ satisfies the conditions given in Definition 1.

1. By the definition of p, we know that $p(\top) = 1$ and $p(\bot) = 0$. Therefore, $\mathcal{P}_p(\bot | \top) = \frac{p(\bot \wedge \top)}{p(\top)} = \frac{0}{1} = 0 \neq 1$.

Let $\varphi, \psi, \chi \in \mathcal{L}_{PL}$.

2. Suppose that $\vDash_{PL} \psi \equiv \chi$. Then, by classical propositional logic, we know that $\vDash_{PL} (\psi \wedge \varphi) \equiv (\chi \wedge \varphi)$. By the properties of p, we also have that $p(\psi) = p(\chi)$ and $p(\psi \wedge \varphi) = p(\chi \wedge \varphi)$. Hence,

$$ \mathcal{P}_p(\varphi | \psi) = \begin{cases} \frac{p(\psi \wedge \varphi)}{p(\psi)}, & \text{if } p(\psi) \neq 0 \\ 1 & \text{otherwise} \end{cases} = \mathcal{P}_p(\varphi | \chi) = \begin{cases} \frac{p(\chi \wedge \varphi)}{p(\chi)}, & \text{if } p(\chi) \neq 0 \\ 1 & \text{otherwise.} \end{cases} $$

3. Suppose that $\varphi \models_{PL} \psi$. Then, by classical propositional logic, we know that $\models_{PL}$ $(\varphi \wedge \psi) \equiv \varphi$. Therefore, by the properties of p, we have that $p(\varphi \wedge \psi) = p(\varphi)$. Hence, in both cases where $p(\varphi) \neq 0$ and $p(\varphi) = 0$, we have $\mathcal{P}_p(\psi|\varphi) = 1$.

4. Suppose that $\varphi \models_{PL} \neg(\psi \wedge \chi)$. We have two cases:
   Case 1: $p(\varphi) = 0$
   Then, by the definition of $\mathcal{P}_p$, we know that $\mathcal{P}(\eta|\varphi) = 1$ for all $\eta \in \mathcal{L}_{PL}$.

   Case 2: $p(\varphi) \neq 0$
   Then,

$$
\begin{aligned}
\mathcal{P}_p(\psi \vee \chi | \varphi) &= \frac{p((\psi \vee \chi) \wedge \varphi)}{p(\varphi)} &&\text{(by the defn. of } \mathcal{P}_p) \\
&= \frac{p((\psi \wedge \varphi) \vee (\chi \wedge \varphi))}{p(\varphi)} \\
&\qquad\qquad (\text{since } \models_{PL} ((\psi \vee \chi) \wedge \varphi) \equiv ((\psi \wedge \varphi) \vee (\chi \wedge \varphi))) \\
&= \frac{p(\psi \wedge \varphi) + p(\chi \wedge \varphi)}{p(\varphi)} &&(\text{since } \models_{PL} \neg((\psi \wedge \varphi) \wedge (\chi \wedge \varphi))) \\
&= \frac{p(\psi \wedge \varphi)}{p(\varphi)} + \frac{p(\chi \wedge \varphi)}{p(\varphi)} \\
&= \mathcal{P}_p(\psi|\varphi) + \mathcal{P}_p(\chi|\varphi) &&\text{(by the defn. of } \mathcal{P}_p)
\end{aligned}
$$

5. We have two cases:
   Case 1: $p(\varphi) = 0$
   Then, by the properties of p, $p(\psi \wedge \varphi) = 0$. Therefore, $1 = \mathcal{P}_p(\psi \wedge \chi|\varphi) = \mathcal{P}_p(\psi|\varphi)\mathcal{P}_p(\chi|\psi \wedge \varphi)$.

   Case 2: $p(\varphi) \neq 0$
   Then, $\mathcal{P}_p(\psi \wedge \chi|\varphi) = \frac{p(\psi \wedge \chi \wedge \varphi)}{p(\varphi)}$ and $\mathcal{P}_p(\psi|\varphi) = \frac{p(\psi \wedge \varphi)}{p(\varphi)}$. Then, if $p(\psi \wedge \varphi) = 0$, we have $0 = \mathcal{P}_p(\psi \wedge \chi|\varphi) = \mathcal{P}_p(\psi|\varphi)\mathcal{P}_p(\chi|\psi \wedge \varphi)$. If $p(\psi \wedge \varphi) \neq 0$, we obtain that $\mathcal{P}_p(\chi|\psi \wedge \varphi) = \frac{p(\psi \wedge \chi \wedge \varphi)}{p(\psi \wedge \varphi)}$. Therefore,

$$
\begin{aligned}
\mathcal{P}_p(\psi|\varphi)\mathcal{P}_p(\chi|\psi \wedge \varphi) &= \frac{p(\psi \wedge \varphi)}{p(\varphi)} \cdot \frac{p(\psi \wedge \chi \wedge \varphi)}{p(\psi \wedge \varphi)} \\
&= \frac{p(\psi \wedge \chi \wedge \varphi)}{p(\varphi)} \\
&= \mathcal{P}_p(\psi \wedge \chi|\varphi).
\end{aligned}
$$

$\square$

Corollary 6. *Given an unconditional probability function* $p : \mathcal{L} \to [0, 1]$, *a topic model* $\mathcal{T} = \langle T, \oplus, t, f \rangle$ *on* $\mathcal{L}_{PL}$ *with* T *a singleton, and* $\varphi \in \mathcal{L}$, *we have* $\mathcal{U}_{\mathcal{P}_p, \mathcal{T}}(\varphi) = u_p(\varphi)$, *where* $\mathcal{U}_{\mathcal{P}_p, \mathcal{T}}$ *is the degree of unacceptability as given in Definition 3.*

Proof. Observe that, since T is singleton, we have $t_\psi = f(t_\varphi)$ for all $\varphi, \psi \in \mathcal{L}_{PL}$. Therefore,

$$
\mathcal{U}_{\mathcal{P}_p, \mathcal{T}}(\varphi) = \begin{cases} 1 - \mathcal{P}_p(\chi|\psi), & \text{if } \varphi := \psi \to \chi, \\ 1 - \mathcal{P}_p(\varphi|\top) & \text{if } \varphi \in \mathcal{L}_{PL}. \end{cases}
$$

32

Then, by the definition of $\mathcal{P}_{\mathrm{p}}$, we obtain that

$$
\mathcal{U}_{\mathcal{P}_{\mathrm{p}},\mathcal{T}}(\varphi) = \begin{cases} 1 - \mathrm{p}(\psi \to \chi), & \text{if } \varphi := \psi \to \chi, \\ 1 - \mathrm{p}(\top \to \varphi) & \text{if } \varphi \in \mathcal{L}_{PL}. \end{cases}
$$

It is easy to see that $\mathrm{p}(\top \to \varphi) = \mathrm{p}(\varphi)$. Therefore, by the definition of u_{p}, we conclude that $\mathcal{U}_{\mathcal{P}_{\mathrm{p}},\mathcal{T}}(\varphi) = \mathrm{u}_{\mathrm{p}}(\varphi)$. $\qquad\square$

For the sake of the following argument, we call Adams' probabilistic (in)validity defined in (Adams, 1998, p. 151) *A-(in)validity*.

Corollary 7. *Any A-invalid principle $\Gamma \vdash \Delta$ is also p-invalid.*

Proof. Let $\Gamma \vdash \Delta$ be an *A*-invalid inference. This means that there is an unconditional probability measure $\mathrm{p} : \mathcal{L} \to [0,1]$ and a corresponding uncertainty function $\mathrm{u}_{\mathrm{p}} : \mathcal{L} \to [0,1]$ such that $\sum_{\varphi \in \Gamma} \mathrm{u}_{\mathrm{p}}(\varphi) < \mathrm{u}_{\mathrm{p}}(\psi)$, for some $\psi \in \Delta$. Then, by Corollary 6, we obtain that $\sum_{\varphi \in \Gamma} \mathcal{U}_{\mathcal{P}_{\mathrm{p}},\mathcal{T}}(\varphi) < \mathcal{U}_{\mathcal{P}_{\mathrm{p}},\mathcal{T}}(\psi)$, where $\mathcal{U}_{\mathcal{P}_{\mathrm{p}},\mathcal{T}}$ is the degree of unacceptability wrt $\mathcal{P}_{\mathrm{p}}$ and some $\mathcal{T} = \langle T, \oplus, t, f \rangle$ with a singleton T. $\qquad\square$

Proof of Theorem 3.3:
p-invalidity of Trans and SA follow from Corollary 7 since they are also *A*-invalid, as shown in (Adams, 1998, p. 125-126).

t-validity:
Trans: $\varphi \to \psi, \psi \to \chi \vdash \varphi \to \chi$
Suppose that $t_\psi \sqsubseteq f(t_\varphi)$ and $t_\chi \sqsubseteq f(t_\psi)$. The former, by Lemma 1 and Definition 2.4b, implies that $f(t_\psi) \sqsubseteq f(t_\varphi)$. Since $\sqsubseteq$ is transitive, by the latter, we obtain that $t_\chi \sqsubseteq f(t_\varphi)$.
SA: $\varphi \to \psi \vdash (\varphi \wedge \chi) \to \psi$
Suppose that $t_\psi \sqsubseteq f(t_\varphi)$. Then, since $t_\varphi \sqsubseteq t_{\varphi \wedge \chi}$, we obtain that $f(t_\varphi) \sqsubseteq f(t_{\varphi \wedge \chi})$ (by Lemma 1). Since $\sqsubseteq$ is transitive, we conclude that $t_\psi \sqsubseteq f(t_{\varphi \wedge \chi})$.

A.4   Proof of Theorem 3.4

p-invalidity of Or-to-If, Contraposition, and SDA follow from Corollary 7 since they are also *A*-invalid. See (Adams, 1998, p. 121) for the *A*-invalidity of Or-to-if and Contraposition and (Adams, 1998, p. 331, counterexample-e) for the *A*-invalidity of SDA.

t-invalidity:
For Or-to-If and Contraposition, consider the topic model given in Figure 4. This model *t*-invalidates Or-to-If since $q \vee p$ is not a conditional and $\neg q \to p$ is not an on-topic conditional wrt $\mathcal{T}$: $a = t_p \not\sqsubseteq f(t_{\neg q}) = b$. And, it *t*-invalidates Contraposition since $p \to \neg q$ is an on-topic conditional wrt $\mathcal{T}$ (since $b = t_{\neg q} \sqsubseteq f(t_p) = a$) but $q \to \neg p$ is not (since $a = t_{\neg p} \not\sqsubseteq f(t_q) = b$). For SDA, consider the topic model $\mathcal{T}' = \langle \{a, b, c\}, \oplus', f', t' \rangle$ where $\oplus'$ is as depicted in Figure 5, f' is a constant function, and $t'_p = b$, $t'_q = c$, and $t'_r = a$. It is then easy to see that $(p \vee q) \to r$ is an on-topic conditional wrt $\mathcal{T}$, however, neither $p \to r$ nor $q \to r$ is.

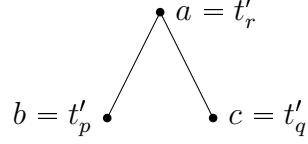Figure 5: Counterexample for the t-invalidity of SDA

A.5   Proof of Theorem 3.5

That Trans, SA, Or-to-If, Contraposition, and SDA are invalid follows from Lemma 2, Theorems 3.3, and 3.4. For RCE, RCEA, RCEC, RCK, and RCM, the counter-topic models given in the proof of Theorem 3.2 together with any arbitrary Popper function constitute counterexamples for their validity, respectively, since in each case the degree of unacceptability of the elements in Γ is 0 and the degree of unacceptability of the elements in Δ is 1 (since the conditionals in Δ are off-topic conditionals with respect to the corresponding topic models). For And-to-If, consider the instance $q \wedge p \vdash q \rightarrow p$. This is invalidated by the topic model given in Figure 4 together with any Popper function $\mathcal{P}$ such that $\mathcal{P}(p \wedge q | \top) \neq 0$. Finally, MOD is invalid only when $\neg \varphi$ is abnormal. Consider the instance $\neg(p \vee \neg p) \rightarrow (p \vee \neg p) \vdash q \rightarrow (p \vee \neg p)$. Observe that, for any Popper function $\mathcal{P}$, we have (by Definition 1.3), $\mathcal{P}(p \vee \neg p | \neg(p \vee \neg p)) = 1$. Consider also the topic model given in Figure 4. Then, we have that $\mathcal{U}(\neg(p \vee \neg p) \rightarrow (p \vee \neg p)) = 1 - \mathcal{P}(p \vee \neg p | \neg(p \vee \neg p)) = 0$. However, as $t_{p \vee \neg p} \not\sqsubseteq f(t_q)$, we have $\mathcal{U}(q \rightarrow (p \vee \neg p)) = 1$. Therefore, $\mathcal{U}(\neg(p \vee \neg p) \rightarrow (p \vee \neg p)) < \mathcal{U}_{\mathcal{P}, \mathcal{T}}(q \rightarrow (p \vee \neg p))$.