Near Closeness and Conditionals

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This paper presents a new system of conditional logic **B2**, which is strictly intermediate in strength between the existing systems **B1** and **B3** from John Burgess (1981) and David Lewis (1973a). After presenting and motivating the new system, we will show that it is characterized by a natural class of frames. These frames correspond to the idea that conditionals are about which worlds are *nearly* closest, rather than which worlds are closest. Along the way, we will also give new characterization results for **B1** and **B3**, along with two other new systems **B1.1** and **B1.2**.

1 Systems

$\mathcal{L}$ is the language of propositional logic extended with the two-place sentential operator $\Box\!\!\rightarrow$. We will generally think of this operator as expressing the subjunctive conditional, so read $A \Box\!\!\rightarrow B$ as saying that had it been that A, it would have been that B. Most of what follows, though, applies equally well to the indicative reading.

Besides this first operator, we will also have a second defined operator $\Diamond\!\!\rightarrow$ that is its dual.

$$A \Diamond\!\!\rightarrow B \equiv \neg(A \Box\!\!\rightarrow \neg B) \tag{1}$$

Because we are reading $\Box\!\!\rightarrow$ as expressing the subjunctive conditional, we will read $A \Diamond\!\!\rightarrow B$ as saying that had it been that A, it might have been that B.[1]

What we are going to call **B1** is a Hilbert-style system from John Burgess (1981).[2]

1. This is not meant to beg any questions about the duality of would and might counterfactuals. Those who deny duality can either (a) read $A \Diamond\!\!\rightarrow B$ as saying that it is false that had it been that A, it would not have been that B or (b) read $A \Box\!\!\rightarrow B$ as saying that it is false that had it been that A, it might not have been that B. If duality is denied, I prefer the second option.

2. Note that the rules and axioms given below are in fact slightly different than the ones given by Burgess.

That system extends classical logic[3] with two inference rules and several axioms. The inference rules are:

SUB $\vdash (A \mathrel{\Box\!\!\rightarrow} B) \supset (A^* \mathrel{\Box\!\!\rightarrow} B)$ when $A \dashv\vdash A^*$
WEA $\vdash (A \mathrel{\Box\!\!\rightarrow} B) \supset (A \mathrel{\Box\!\!\rightarrow} B^*)$ when $B \vdash B^*$

The first rule is Substitution. This lets us replace the antecedent of a counterfactual with anything that is logically equivalent. The second is Weakening, which lets us replace the consequent of a counterfactual with anything that it logically implies. The axioms are then:

ID $A \mathrel{\Box\!\!\rightarrow} A$
IM $(A \mathrel{\Box\!\!\rightarrow} B \wedge C) \supset (A \wedge B \mathrel{\Box\!\!\rightarrow} C)$
CO $(A \mathrel{\Box\!\!\rightarrow} B) \wedge (A \mathrel{\Box\!\!\rightarrow} C) \supset (A \mathrel{\Box\!\!\rightarrow} B \wedge C)$
DI $(A \mathrel{\Box\!\!\rightarrow} C) \wedge (B \mathrel{\Box\!\!\rightarrow} C) \supset (A \vee B \mathrel{\Box\!\!\rightarrow} C)$

The first is Identity, which says that every sentence counterfactually entails itself. The second is Import, which lets us import a conjunct from the consequent of a counterfactual into the antecedent. The last two axioms are Conjunction and Disjunction, which have a kind of symmetry. Conjunction lets us conjoin the *consequents* of counterfactuals that share an *antecedent*. Disjunction lets us disjoin the *antecedents* of counterfactuals that share a *consequent*.

We can build other systems by extending **B1** with various axioms. Our main interest will be in the following:

DB $(A \vee B \mathrel{\Box\!\!\rightarrow} C) \supset (A \mathrel{\Box\!\!\rightarrow} C) \vee (B \mathrel{\Box\!\!\rightarrow} C)$
DM $(A \vee B \mathrel{\Box\!\!\rightarrow} \neg A) \wedge (B \vee C \mathrel{\Box\!\!\rightarrow} \neg B) \supset (A \vee D \mathrel{\Box\!\!\rightarrow} \neg A) \vee (D \vee C \mathrel{\Box\!\!\rightarrow} \neg D)$
SM $(A \mathrel{\Box\!\!\rightarrow} C) \wedge (A \mathrel{\Diamond\!\!\rightarrow} B) \supset (A \wedge B \mathrel{\Box\!\!\rightarrow} C)$
CEM $(A \mathrel{\Box\!\!\rightarrow} B) \vee (A \mathrel{\Box\!\!\rightarrow} \neg B)$
ST $(A \mathrel{\Box\!\!\rightarrow} C) \supset (A \wedge B \mathrel{\Box\!\!\rightarrow} C)$

We will call the first axiom Distribution. Distribution says that when you have a counterfactual with a disjunctive antecedent, the counterfactual operator can be distributed over that disjunction. The second is Diamond, the contents of which will be explained shortly.[4] The third is Strengthen Might, which lets us strengthen antecedents using might counterfactuals.[5] The fourth is the Counterfactual Law of Excluded Middle, which is a sort of counterfactual analogue of the Law of Excluded

3. See Hartry Field's (2016) for a non-classical variation of **B1**.

4. Distribution and Diamond have not, to my knowledge, been discussed elsewhere in the literature. [BLINDED] reports that [BLINDED] has suggested Distribution as a plausible principle in conversation.

5. This axiom is also sometimes called Rational Monotonicity.

Middle. The fifth is Strengthening, which lets us strengthen antecedents using any sentence whatsoever. Adding these axioms to **B1** gives us the following taxonomy of systems:

System	**Additional Axioms**
B1	
B1.1	DB
B1.2	DM
B2	DB, DM
B3	SM
B4	CEM
B5	ST

These systems are generally numbered in order of strength, with two exceptions. **B1.1** and **B1.2** are strictly stronger than **B1**, but neither is strictly stronger than the other. Similarly, **B4** and **B5** are both strictly stronger than **B1**-**B3**, but neither is strictly stronger than the other.

Many of these systems have prominent defenders in the literature. Proponents of the strict conditional analysis, like Kai von Fintel (2001) and Tony Gillies (2007), accept **B5**, along with Strong Centering and Weak Centering.[6]

SC $(A \wedge B) \supset (A \mathrel{\Box\!\!\rightarrow} B)$
WC $(A \mathrel{\Box\!\!\rightarrow} B) \supset (A \supset B)$

Robert Stalnaker (1970; 1975, 1968, 1980) accepts a system that he calls **C2**, which is the result of adding both centering axioms to **B4**. David Lewis (1973b, 1973a) accepts a system that he calls **VC**, which results from adding both centering axioms to **B3**.[7] John Pollock (1975, 1976b, 1976a) rejects Strengthen Might, and so rejects not only **B3**, but any system extending it. His preferred system is what he calls **SS**.[8] That system

6. I will sometimes talk about *accepting* a system of counterfactual logic. For present purposes, accepting a system means accepting that there is a proof in the system from $A_1, \ldots, A_n$ to B iff the inference from $A_1, \ldots, A_n$ to B is *correct*. Rejecting a system means rejecting the same schematic biconditional. What it means for an inference to be correct will depend on your substantive views in the philosophy of logic. You might think, for example, that an inference is correct when it necessarily preserves truth. Finally, note that given this way of thinking about acceptance and rejection, if you accept a system, then you are committed to rejecting any system that is either strictly stronger or strictly weaker.

7. Lewis refers to the system that we are calling **B3** as **V**. This system is notable, from his perspective, because it is sound and complete with respect to the class of all sphere models. **B3** is thus the *weakest* system that can be modeled using systems of spheres.

8. Kratzer (1981) suggests modeling counterfactuals with what she calls premise semantics. Her model theory is, on the face of it, very different from the modal theory used by Pollock. But as Lewis (1981)

is the result of adding both Strong Centering and Weak Centering to **B1**.

Setting aside centering axioms, then, we can think of the proponents of the strict conditional analysis as accepting **B5**, Stalnaker as accepting **B4**, Lewis as accepting **B3**, and Pollock as accepting **B1**.

My own view is that **B2** is the right logic for counterfactuals.[9] My reasons for thinking this are, in broad outline, as follows.

First, like many others, I reject Strengthening for broadly the same reasons as Lewis (1973a). Suppose that had Margaux gone to the party, she would have had a good time. From this, it simply does not follow that had Margaux gone to the party *and learned that an asteroid was about to destroy the earth*, she would have had a good time.[10] Since I reject Strengthening, this rules out **B5**.

Second, like Lewis, I deny Counterfactual Excluded Middle.[11] It is neither true that had I flipped a coin one minute ago, it would have landed heads, nor is it true that had I flipped a coin one minute ago, it would have landed tails. This rules out **B4**.

Third, I reject Strengthen Might, and so side with Pollock against Lewis, though I do so for different reasons. Pollock proposes various counterexamples to Strengthen Might, all of which I find unconvincing.[12] Instead, I reject Strengthen Might because this strikes me as the most natural way to resolve the paradox of counterfactual tolerance. This paradox was introduced in my (2021) and will be briefly sketched in §2. Since I reject Strengthen Might, this rules out **B3**.

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points out, the two approaches are in fact equivalent. Every Kratzer model determines a unique Pollock model, and visa-versa. Thus, Kratzer is also committed to accepting **SS**.

9. Again, setting aside centering axioms. I am inclined to accept Counterfactual Modus Ponens, which is equivalent to Weak Centering, given a classical background logic. I am undecided about Strong Centering, though am somewhat inclined to deny it.

10. von Fintel (2001) and Gillies (2007) argue that such apparent counterexamples can be explained away using context shifts. But even if they could, I maintain that the paradox of counterfactual tolerance (to be sketched in §2) gives us a counterexample to Strengthen Might, and so gives us a different style of counterexample to Strengthening. But counterexamples to Strengthen Might cannot be explained away using the same sorts of context shifting mechanisms. See Boylan and Schultheis (forthcoming) for more on this. Perhaps other context shifting mechanisms can do the job but, if so, defenders of the strict conditional analysis will need to say what they are.

11. This is a bit of a simplification. My considered view is that counterfactuals often have both a Lewis reading and a Stalnaker reading. On the Lewis reading, CEM fails; on the Stalnaker reading, CEM holds. In the present context, I am interested in the Lewis reading, so reject CEM.

12. See for example pp. 473-474 of Pollock (1976a). Loewer (1979) has a nice discussion of why Pollock-style counterexamples to Strengthen Might are not convincing. Boylan and Schultheis (forthcoming) have recently suggested another, much cleaner, Pollock-style counterexample. I ultimately reject Pollock-style counterexamples to Strengthen Might because I can find no uniform reading that generates all the necessary judgments. This, I suspect, is because such counterfactuals run afoul of another principle I like called Strengthen Easy. For discussion of that principle, see my (2021).

Fourth, while I side with Pollock on the question of Strengthen Might, we part ways when it comes to Distribution. From my perspective, Distribution is obviously correct. For example, suppose that:

Had it either rained or snowed, Naomi would have been pleased. (2)

From this it would seem to follow that the *disjunction* of the following two claims is true:

Had it rained, Naomi would have been pleased. (3)

Had it snowed, Naomi would have been pleased. (4)

The problem for Pollock is that he accepts **B1** and so is committed to rejecting Distribution.[13] As a result, he is committed to the view that the truth of (2) is entirely consistent with the falsity of both (3) and (4). This strikes me as absurd, so I reject **B1**.[14,15]

It may be worth pointing out that Distribution has a certain passing resemblance to a controversial principle called the Simplification of Disjunctive Antecedents (or Simplification for short).

SDA $(A \vee B \mathrel{\Box\!\!\rightarrow} C) \supset (A \mathrel{\Box\!\!\rightarrow} C) \wedge (B \mathrel{\Box\!\!\rightarrow} C)$

Simplification is the converse of Disjunction. The difference between Simplification and Distribution is that where Simplification has a *conjunction* in the consequent, Distribution only has a *disjunction*. This makes all the difference in the world. Suppose for example that (2) is true. In that case, Simplification tells us that *both* (3) and (4) are true. Distribution, on the other hand, only says that *at least one* of them is true. As further illustration of the difference, we might also note that while Distribution is a theorem of Lewis's **B3**, Simplification is not.

This leaves us to decide between **B1.1** and **B2**. The difference between the two is that **B2** validates Diamond, but **B1.1** does not. So what does Diamond say? Suppose that Alice had been given the choice between one dollar, two dollars, three dollars, and one beer. Plausibly:

Had Alice chosen either one dollar or two dollars, she would not have chosen one dollar. (5)

13. This follows from Theorem 6.21. For a countermodel, see Proposition 7.14.

14. For further discussion and motivation of Distribution, see my (2021).

15. Given our observation in Footnote 8, premise semantics invalidates Distribution, and so Kratzer is also committed to rejecting Distribution.

Had Alice chosen either two dollars or three dollars, she would not have chosen two dollars. (6)

Diamond then tells us that the disjunction of the following two claims is true:

Had Alice chosen either one dollar or one beer, she would not have chosen one dollar. (7)

Had Alice chosen either one beer or three dollars, she would not have chosen one beer. (8)

The reasoning here is complex and I grant that the correctness of Diamond is not completely obvious. Nevertheless, I think that the axiom is in fact correct.[16] Should it turn out to fail, I would be reasonably happy retreating from **B2** to **B1.1**.

From a theoretical perspective, there are two reason to like Diamond. The first is that while Strengthen Might may not be correct, it is still compelling. We would thus like to preserve as much of the content as we can without generating paradox. DM is strictly weaker than SM and does not generate paradox. So this gives us some reason to accept DM.

The second reason has to do with models. There is a common thought that the truth of counterfactuals can be modeled by saying which possible worlds are closest. In fact, I think this common view is mistaken. We will say more about this in §4, but my view is that counterfactual should instead be modeled by saying which worlds are *nearly* closest. But in that case, it is natural to think that our models will validate Diamond. Thus, the fact that **B2** has an especially natural model theory is some reason to think that Diamond is correct.

Here is the plan for the rest of this paper. In §2, we will sketch the paradox of counterfactual tolerance as a way of motivating our interest in systems weaker tha **B3**. We will then describe Burgess accessibility models in §3. §4 introduces the idea of using *as close* and *nearly as close* relations to think about counterfactual accessibility. §5 catalogues various useful properties of these relations. In §6, we will use those properties to give several new characterization results. The bulk of the technical material is then in the last two sections. §7 shows how to prove both soundness and inverse soundness. §8 shows how to prove completeness and decidability using a new canonical models

16. You might think that the inference fails because you determinately prefer three dollars to two dollars and determinately prefer two dollars to one dollar, but neither determinately prefer one beer to one dollar nor determinately prefer three dollars to one beer. Thus, (5) and (6) are determinately true, but neither (7) nor (8) is determinately true. But while this situation can arise, this is not a counterexample to Diamond. For Diamond to fail, we would need the *disjunction* of (7) and (8) to be indeterminate, but I can see no reason to think that it is. At best, we only have a case in which both disjuncts are indeterminate.

construction.

2 The Paradox of Counterfactual Tolerance

Planck lengths are incredibly small. You would quite literally need a billion trillion of them just to span the diameter of a proton. Now consider Barack Obama who is, it turns out, exactly h Planck lengths tall. It would then seem that:

Tolerance: For all positive integers $n > h$, had Obama been at least n Planck lengths, he might have been at least $n + 1$ Planck lengths.

Boundedness: There are positive integers $k > j > h$ such that had Obama been at least j Planck lengths, he would not have been at least k Planck lengths.

Heights: For all positive integers n, had Obama been at least $n + 1$ Planck lengths, he would have been at least n Planck lengths.

Tolerance says that had Obama been at least seven feet, he might have been at least one Planck length taller, and likewise for other heights. Boundedness will be true if, for example, had Obama been at least seven feet, he would not have been at least a thousand feet. Heights says that had Obama been at least seven feet and one Planck length, he would have been at least seven feet, and likewise for other heights.

Given these three attractive claims, we can prove a flat contradiction using any system extending **B3**. First, we observe that Limited Transitivity is valid in not only **B3**, but any system extending **B1**.[17]

LT $\quad (A \mathrel{\Box\!\!\rightarrow} B) \wedge (A \wedge B \mathrel{\Box\!\!\rightarrow} C) \supset (A \mathrel{\Box\!\!\rightarrow} C)$

Besides being derivable, the principle is also compelling in its own right, and so often taken as basic, even in systems that are not extensions of **B1**.[18]

This gives us everything we need to sketch the paradox. Suppose that Obama is in fact exactly h Planck lengths tall. For all n, let p_n express the claim that Obama is at least n Planck lengths tall. We then reason as follows:

17. This is proved in Lemma 8.7.
18. See for example Kit Fine's (2012).

1.	$p_j \boxright \neg p_k$	boundedness
2.	$p_j \diamondright p_{j+1}$	tolerance
3.	$p_{j+1} \boxright p_j$	heights
4.	$p_j \wedge p_{j+1} \boxright \neg p_k$	1,2, SM
5.	$p_{j+1} \wedge p_j \boxright \neg p_k$	4, SUB
6.	$p_{j+1} \boxright \neg p_k$	3, 5, LT

This argument is paradoxical because the reasoning used in lines two through six can be iterated. In particular, after $k - j - 1$ iterations, we get:

$$p_{k-1} \boxright \neg p_k \tag{9}$$

But tolerance tells us that

$$p_{k-1} \diamondright p_k \tag{10}$$

which is equivalent to

$$\neg(p_{k-1} \boxright \neg p_k) \tag{11}$$

and so we have a flat contradiction. Thus, something has to go. Either we have to reject one of the premises or we have to reject one of the rules or axioms of **B3**.

There are many strategies for responding to the paradox, several of which I consider at length in my (2021).[19] Rather than repeating that discussion here, I will simply note that denying Strengthen Might is one natural solution and, in fact, my own preferred solution. We thus have good reason to be interested in systems strictly weaker than **B3**.

3 Burgess Models

A natural thought is that counterfactuals should be modeled in terms of the relative closeness of possible worlds. Maybe $A \boxright B$ is true if and only if all of the closest A worlds are B worlds. Similarly, $A \diamondright B$ is true if and only some closest A world is also a B world.[20]

There are different ways of converting this intuitive idea into a formal model theory. Lewis (1973a) uses systems of spheres. Stalnaker (1968) uses selection functions. Here,

19. For example, one response would be to deny that would and might counterfactuals are duals. But in fact, this does not solve the problem, since there are other ways of staging the paradox that do not depend on duality. I say more about this in the referenced paper.

20. Following the usual convention, an A world is just a world at which A is true.

we are going to use Burgess models, which are most general. These were introduced by John Burgess in his (1981).

Definition 3.1: A **frame** $\mathcal{F} = \langle W, f, \leq \rangle$ consists of a non-empty set of worlds W, a function f assigning every world x a **local domain** $W_x \subseteq W$, and a function $\preceq$ assigning every world x an **accessibility relation** $\preceq_x \subseteq W_x \times W_x$.

The worlds in the local domain W_x of x are the worlds that are **counterfactually possible** relative to x. For each such world, there is a corresponding two-place accessibility relation $\preceq_x$ on that local domain. When $b \preceq_x a$, we say that b is **accessible** from a relative to x. We can then define other useful relations:

$b \prec_x a$	iff	$b \preceq_x a$ and not $a \preceq_x b$
$b \approx_x a$	iff	$b \preceq_x a$ and $a \preceq_x b$
$b \sim_x a$	iff	$b \preceq_x a$ or $a \preceq_x b$

The first is the relation of b being **strictly accessible** from a relative to x. The second is the relation of b and a being **coaccessible** relative to x. The third is the relation of b and a being **connected** relative to x.

These various relations can all be thought of as relative closeness relations. We will say more about this in §4 but, at a first pass, $b \preceq a$ says that b is at least as close as a to x. $b \prec a$ says that b is strictly closer than a to x. $b \approx_x a$ says that b and a are equally close to x. $b \sim_x a$ says that the distance of b from x is commensurable with the distance of a from x. Finally, when $a \in W_x$, this means that the distance of a from x is defined.

Definition 3.2: A **Burgess model** $\mathcal{M} = \langle \mathcal{F}, V \rangle$ consists of a frame $\mathcal{F}$ and a valuation function V assigning every atomic sentence p of $\mathcal{L}$ a denotation $V(p) \subseteq W$. A sentence A is **true at a world** x in a model $\mathcal{M}$ when $\mathcal{M}, x \vDash A$, which is defined recursively:

$\mathcal{M}, x \vDash p$	iff	$x \in V(p)$
$\mathcal{M}, x \vDash \neg A$	iff	$x \nvDash A$
$\mathcal{M}, x \vDash A \vee B$	iff	$x \vDash A$ or $x \vDash B$
$\mathcal{M}, x \vDash A \wedge B$	iff	$x \vDash A$ and $x \vDash B$
$\mathcal{M}, x \vDash A \supset B$	iff	$x \nvDash A$ or $x \vDash B$
$\mathcal{M}, x \vDash A \mathrel{\Box\!\!\rightarrow} B$	iff	for every $a \vDash A$ such that $a \in N_x$, there is a $b \vDash A$ such that $b \preceq_x a$ and, for all c such that $c \preceq_x b$, if $c \vDash A$ then $c \vDash B$
$\mathcal{M}, x \vDash A \mathrel{\Diamond\!\!\rightarrow} B$	iff	there is an $a \vDash A$ such that $a \in N_x$ and, for all $b \vDash A$ such that $b \preceq_x a$, there is a c such that $c \preceq_x b$ with $c \vDash A$ and $c \vDash B$

Once we have our models, soundness and completeness are defined in the usual way. That is, a system S is **sound** in a frame $\mathcal{F}$ when every theorem of S is true at every world in every model based on $\mathcal{F}$. A system S is sound in a class of frames Γ when S is sound in every frame in Γ. A set of sentences Δ is S-consistent when no sentence of the form $A \wedge \neg A$ can be derived in S from any subset of Δ. A system S is **complete** in a class of frames Γ when every S-consistent set of sentences is true at some world in some model based on some frame in Γ. Finally, we will say that a class of frames Γ **generates** a system S when S is sound and complete in Γ.

The class of all frames whatsoever generates a system that we will call **B0**. This system is of some technical interest, but is too weak to be a good match for our ordinary counterfactual practice. For example, **B0** fails to license Identity. But surely, if there are any logical truths involving counterfactuals, one of them is that had it been that A, it would have been that A.

Because the logic of **B0** is so weak, we need to add restrictions on the accessibility relation to generate more realistic systems. Some of the conditions most commonly used in the literature are listed below.

Property	**Definition**
Reflexive	$a \preceq a$
Pairwise Connected	$(a \preceq b) \vee (b \preceq a)$
Anti-Symmetric	$(b \preceq a) \supset \neg(a \preceq b)$
Symmetric	$(b \preceq a) \supset (a \preceq b)$
Fully Transitive	$(c \preceq b) \wedge (b \preceq a) \supset (c \preceq a)$
Preorder	reflexive, fully transitive
Total Preorder	pairwise connected, fully transitive
Total Order	pairwise connected, fully transitive, anti-symmetric
Universal	pairwise connected, symmetric

The general consensus has been that at a minimum, we should require accessibility relations to be reflexive and transitive. That is, the consensus has been that we should require accessibility relations to form a *preorder*. The most common systems are then generated by restricting to various classes of preorders.[21]

Definition 3.3: A sequences $\langle a_1, a_2, a_3, \dots \rangle$ forms an **infinite chain** when $a_{n+1} \preceq a_n$ for all n.

Definition 3.4: An infinite chain $\langle a_1, a_2, a_3, \dots \rangle$ is **trivial** when $a_n = a_m$ for all n

21. Here, and elsewhere, we will sometimes run together the distinction between frames and accessibility relations. We will say, for example, that **B1** is generated by the class of all *preorders*, when what we really mean is that **B1** is generated by the class of all *frames* in which every world is assigned an accessibility relation that forms a preorder.

and m.

Definition 3.5: A relation is **well-founded (wf)** when it does not contain any non-trivial infinite chains.

Theorem 3.6 (Burgess)**:** ***B1*** *is generated by the class of all preorders.*

Theorem 3.7 (Lewis)**:** ***B3*** *is generated by the class of all total preorders.*

Theorem 3.8 (Stalnaker)**:** ***B4*** *is generated by the class of all well-founded total orders.*

Theorem 3.9 (Kripke)**:** ***B5*** *is generated by the class of all universal relations.*

The problem is that while using preorders to model counterfactuals is natural and useful, it can also be misleading. For while certain systems may be generated by certain classes of preorders, they are not *characterized* by them. There is thus a clear sense in which the systems and the classes of frames are not an exact match.

Definition 3.10: A system S is **inverse sound** in a class of frames Γ when S is not sound in any frame not in Γ.

Definition 3.11: A system S is **exactly sound** in a class of frames Γ when S is both sound and inverse sound in Γ.

Definition 3.12: A system S **characterizes** a class of frames Γ relative to class of frames Δ when S is exactly sound in $\Delta \cap \Gamma$ and S is complete in $\Delta \cap \Gamma$.

Definition 3.13: A system S **fully characterizes** a class of frames Γ when S characterizes Γ relative to the class of all frames.

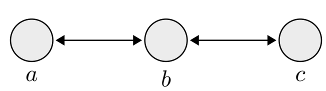
Characterization is stronger than soundness and completeness. To show that a system is characterized by a class, we need to show that it is not just sound and complete, but *exactly* sound and complete.[22] This lets us make several observations.

Proposition 3.14: ***B1*** *is not fully characterized by the class of all preorders.*

Proposition 3.15: ***B3*** *is not fully characterized by the class of all total preorders.*

Proposition 3.16: ***B5*** *is not fully characterized by the class of all universal relations.*

Proof. These three propositions can be demonstrated by considering the following frame. Accessibility relations are represented by arrows.



Here, c and b are coaccessible and b and a are coaccessible, but c and a are not connected,

22. We will generally treat characterization as a symmetric relation. So, when a system characterizes a class of frames, we will also say that the class of frames characterizes the system.

with these accessibility relations all holding relative to some fixed x. As a result, this frame is not fully transitive. Still, **B1**, **B3**, and **B5** are all sound in this frame by Theorem 7.20.[23] Thus, none of these systems is characterized by any fully transitive class of frames. Preorders are fully transitive. So none of these systems is fully characterized by any class of preorders. ■

Theorem 3.17: ***B4*** *is fully characterized by the class of well-founded total orders.*

Proof. Soundness and completeness are by Theorem 3.8. The inverse of soundness is straightforward, so left to the reader. ■

There is nothing wrong with using a class of models that merely generates a system. We have soundness and completeness after all! Still, proving characterization results can be illuminating.

When a class of frames merely generates a system, that class has what you might think of as unnecessary structure. This unnecessary structure may be useful and is generally harmless, but is also not needed to support the theorems of the system being modelled. When a class of frames *characterizes* a class of frames, on the other hand, that class has all *and only* the structure needed to support the theorems. The system and the class of frames exactly match. We thus have good reason to want characterization results, and not just soundness and completeness results, whenever we can get them.

To my knowledge, there are no existing full characterization results for **B1**, **B3**, and **B5** in the literature. In what follows, then, one of our goals will be to fill in this gap. This is in addition to our main focus, which is giving characterization results for the new systems **B1.1**, **B1.2**, and **B2**.

4 Nearly as Close

We are going to introduce several new frame conditions in the coming sections, many of which are non-transitive. To get a feel for how these conditions work, and how they might be motivated, it will be helpful to distinguish between two kinds of distance relations.

Imagine that you are standing in the middle of a grassy field and are surrounded by several brightly colored balls arranged at various distances. These balls stand in **as close**

23. The frame here is only partially specified, but I trust that the reader can fill in the details. The stated results hold, for example, if no other worlds are counterfactually possible relative to any other worlds. As we go along, I will give only partial descriptions of frames and countermodels when filling in the rest of the details would be straightforward.

relations. It might be, for example, that the blue ball is as close as the red ball.[24] If the red ball is also as close as the blue ball, then the two balls are equally close. If not, then the blue ball is strictly closer than the red ball.

Besides standing in *as close* relations, the balls in the grassy field also stand in *nearly as close* relations. We will say that b is **nearly as close** as a to x when b is no more than t farther away than a from x. The value of t is what we will call the **tolerance margin**.

So for example, suppose there is a red ball that is three feet away and a blue ball that is four feet away. Suppose also that the tolerance margin is one foot. In that case, the blue ball is not as close as the red ball, but it is still *nearly* as close, since its additional distance is within the tolerance margin.

In ordinary contexts, when we say that something is nearly as close, this often implies that it is not as close. For example, if you tell a friend that the coffee shop is nearly as close as the bakery, this will often imply that the coffee shop is not as close as the bakery. Whether this implication is semantic or pragmatic is an interesting question. For our purposes, we are going to stipulate that there is no such implication. If b is nearly as close as a, it could be that b is as close as a. It could even be that b is strictly closer.

We can now define other useful relations. We will say that b is **much closer** than a when b is nearly as close as a and a is not nearly as close as b. The distance of b and the distance of a is **roughly equal** when b is nearly as close as a and a is nearly as close as b.

So for example, suppose again that we have a one foot tolerance margin. In that case, the blue ball is much closer than the red ball if and only if it is more than one foot closer. The distance of the two balls is roughly equal if and only if neither is more than one foot closer than the other.

The main observation now is that while *as close* relations are transitive, *nearly as close* relations are not. Suppose that a is three feet away, b is four feet away, and c is five feet away. In that case, using a one foot tolerance margin, b is nearly as close as a and c is nearly as close as b, but c is not nearly as close as a. In fact, a is much closer.

Once we have both kinds of relations on the table, we can think about counterfactual accessibility using either *as close* or *nearly as close* relations. The reading in terms of *as close* relations was given early. In terms of *nearly as close* relations, $b \preceq_x a$ says that b is nearly as close as a to x. $b \prec_x a$ says that b is much closer than a to x. $b \approx_x a$ says that the distance of b from x and the distance of a from x is roughly equal. $b \sim_x a$ says that the distance of b from x and the distance of a from x is commensurable.

24. When we say one ball is as close as another in English, this claim is often ambiguous. Sometimes, we mean that the first as *at least* as close as the second. Sometimes we mean that the first is *exactly* as close as the second. For our purposes, we will stipulate that we always have the first reading in mind.

5 Properties of Closeness Relations

There is by now a long tradition of thinking about counterfactuals in terms of *as close* relations. Say that an A world is among the closest A worlds when there are no other A worlds that are strictly closer. The usual thought is that, given the limit assumption, $A \mathrel{\Box\!\!\to} B$ is true iff all of the closest A worlds are B worlds. Similarly, $A \mathrel{\Diamond\!\!\to} B$ is true iff some of the closest A worlds are B worlds.

The problem is that *as close* relations are pairwise connected and fully transitive. Thus, if we model counterfactuals using *as close* relations, we will generate Lewis's **B3**.[25] But as we saw in §2, we can use **B3** to prove a contradiction from Tolerance, Boundedness, and Heights. Thus, those of us who accept Tolerance, Boundedness, and Heights are committed to denying **B3**, and so committed to denying that counterfactuals can be correctly modeled using *as close* relations.

If we are not going to model counterfactuals using *as close* relations, then how should we model them? What I think is that we should replace *as close* relations with *nearly as close* relations.[26]

Say that an A world is among the nearly closest A worlds when there are no A worlds that are much closer. Given the limit assumption, $A \mathrel{\Box\!\!\to} B$ is then true iff all of the nearly closest A worlds are B worlds. $A \mathrel{\Diamond\!\!\to} B$ is true iff some of the nearly closest A worlds are B worlds.

Since *nearly as close* relations are not fully transitive, modeling counterfactuals with *nearly as close* relations means denying that counterfactual accessibility is fully transitive. We thus cannot require counterfactual accessibility relations to be preorders. Instead, I think we should require them to be *semiorders*. If we do, the system that will be generated is **B2**.[27]

In the rest of this section, we are going to catalogue several properties of *nearly as close* and *as close* relations. These can be broadly divided into three kinds: transitivity properties, connectedness properties, and directedness properties. Once we have a suitable catalogue of properties, we will use said properties to give our characterization results in §6.

5.1 Transitivity Properties

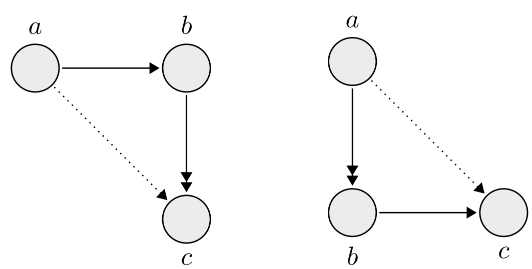
A **transitivity property** is a non-trivial property that is entailed by full transitivity. Thus, full transitivity is itself one example of a transitivity property, but not the only

25. This is one of the upshots of Theorem 6.20.
26. See my (2021) for a more detailed discussion.
27. This is one of the implications of Theorem 6.20.

one. There are many others.

Property	Definition
Left Transitive	$(c \prec b) \wedge (b \preceq a) \supset (c \preceq a)$
Right Transitive	$(c \preceq b) \wedge (b \prec a) \supset (c \preceq a)$
Strong Left Transitive	$(c \prec b) \wedge (b \preceq a) \supset (c \prec a)$
Strong Right Transitive	$(c \preceq b) \wedge (b \prec a) \supset (c \prec a)$
Double Transitive	$(c \prec b) \wedge (b \prec a) \supset (c \prec a)$
Zigzag Transitive	$(d \prec c) \wedge (c \preceq b) \wedge (b \prec a) \supset (d \prec a)$
Double Left Transitive	$(d \prec c) \wedge (c \prec b) \wedge (b \preceq a) \supset (d \prec a)$
Double Right Transitive	$(d \preceq c) \wedge (c \prec b) \wedge (b \prec a) \supset (d \prec a)$
Weak Transitive	left transitive, right transitive
Strong Transitive	strong left transitive, strong right transitive
Semitransitive	zigzag transitive, double left transitive, double right transitive

The first pair of transitivity properties are left transitivity and right transitivity. These are illustrated below:



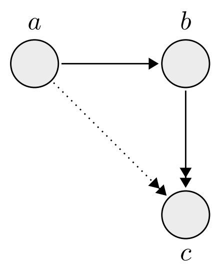
Left Transitive Right Transitive

In this diagram, and the ones to follow, solid arrows represent accessibility relations that appear in the antecedent. Dotted arrows represents accessibility relations that appear in the consequent. Arrows with a single tip represent accessibility relations that may or may not be strict. Arrows with a double tip represent accessibility relations that are strict.

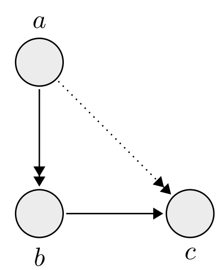
In terms of *nearly as close* relations, left transitivity says that if c is much closer than b and b is nearly as close as a, then c is nearly as close as a. Right transitivity says that if c is nearly as close as b and b is much closer that a, then c is nearly as close as a. A relation is weakly transitive when it is both left transitive and weak right transitive.

The second pair of properties are strong left transitivity and strong right transitivity.

These are properties of *as close* relations, but not *nearly as close* relations.



Strong Left Transitive



Strong Right Transitive

Thinking in terms of *as close* relations, strong left transitivity says that if c is strictly closer than b and b is as close as a, then c is strictly closer than a. Strong right transitivity says that if c is as close as b and b is strictly closer than a, then c is strictly closer than a. When a relation is both strong left transitive and strong right transitive, we will say that it is strongly transitive.

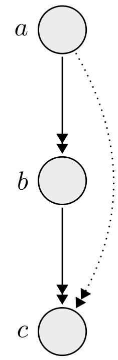
Strong left transitivity fails for *nearly as close* relations. To see this, suppose that c is three feet away, a is four feet away, and b is five feet away. Using a one-foot tolerance margin, b is nearly as close as a and c is much closer than b, but c is not much closer than a. Thus, strong left transitivity fails. A similar countermodel shows that strong right transitivity also fails.

Observation 5.1: Every strongly left transitive relation is left transitive.[28]

Observation 5.2: Every strongly right transitive relation is right transitive.

The fifth property on our list is double transitivity. This property is called double transitivity because it takes two strict accessibility relation as input, then gives back a third strict accessibility relation as output.

28. Here and elsewhere, I will call especially straightforward propositions *observations*. Proofs of observations are left to the reader.



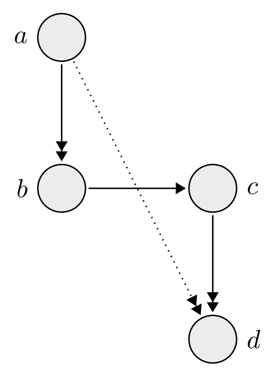
Double Transitive

In terms of *nearly as close* relations, double transitivity says that if c is much closer than b and b is much closer than a, then c is much closer than a.

Observation 5.3: If a relation is either right transitive or left transitive, then it is also double transitive.

Observation 5.4: If a relation is pairwise connected and double transitive, then it is also weak transitive.

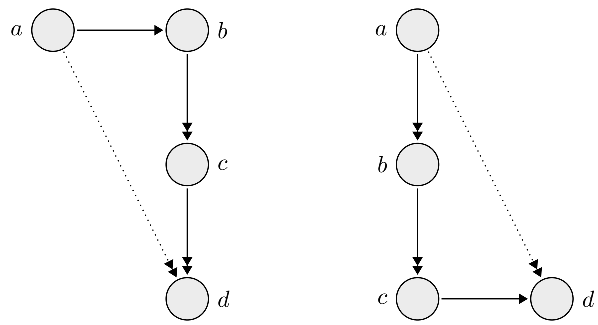
The sixth property is zigzag transitivity. Unlike the other properties considered so far, zigzag transitivity takes three accessibility relations as input, then gives us a strict accessibility relation as output.



Zigzag Transitive

In terms of *nearly as close* relations, zigzag transitivity says that if d is much closer than c and c is nearly as close as b and b is much closer than a, then d is much closer than a.

The last two basic transitivity properties are double left transitivity and double right transitivity. These are like zigzag transitivity, but permute the patterns of strict and nonstrict accessibility relations in the antecedent.



Double Left Transitive Double Right Transitive

Thinking in terms of *nearly as close* relations, double left transitivity says that if d is much closer than c and c is much closer than b and b is nearly as close as a, then d is much closer than a. Double right transitivity says that if d is nearly as close as c and c is much closer than b and b is much closer than a, then d is much closer than a. When a relation is zigzag transitive, double left transitive, and double right transitive, we will say that it is semitransitive.

Observation 5.5: If a relation is reflexive and semitransitive, then it is also double transitive.

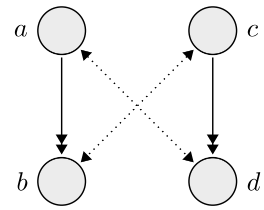
Observation 5.6: If a relation is pairwise connected and semitransitive, then it is also weakly transitive.

5.2 Connectedness Properties

A property is a **connectedness property** when it is a non-trivial property entailed by pairwise connectedness. Pairwise connectedness, then, is one example of a connectedness property, but not the only one. There are many others as well.

Property	**Definition**
Bowtie Connected	$(d \prec c) \wedge (b \prec a) \supset (d \sim a) \vee (b \sim c)$
Diamond Connected	$(c \prec b) \wedge (b \prec a) \supset (d \sim c) \vee (d \sim a)$
Triangle Connected	$(b \prec a) \supset (c \sim b) \vee (c \sim a)$

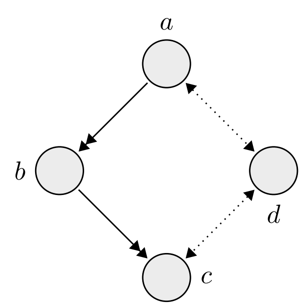
Bowtie connectedness says that given any pair of strict accessibility relations, one of the crossing pairs is connected. As such, it might also be called the property of *cross* *connectedness*.



Bowtie Connected

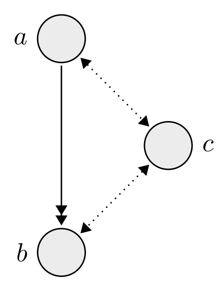
In terms of *nearly as close* relations, bowtie connectedness says that if d is much closer than c and b is much closer than a, then either d and a are commensurable or b and c are commensurable.

The second property is diamond connectedness. It says that if b is much closer than a and c is much closer than b, then either d and a are commensurable or d and c are commensurable.



Diamond Connected

The last connectedness property is triangle connectedness. Triangle connectedness says that if b is much closer than a , then either c and a are commensurable or b and c are commensurable.



Triangle Connected

Observation 5.7: If a relation is diamond connected, then it is double right transitive iff it is double left transitive.

Observation 5.8: If a relation is triangle connected, then it is also bowtie connected.

Observation 5.9: If a relation is triangle connected, then it is strongly left transitive iff it is strongly right transitive.

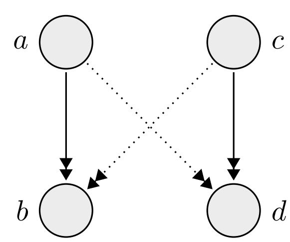
5.3 Directedness Properties

What we are going to call **directedness properties** are what you might think of as hybrid properties. They generally result from pairing a connectedness property with an appropriate transitivity property. Alternatively, you might also think of directedness properties as connectedness properties of the *strict* accessibility relations.

Directedness properties will be important, for our purposes, because our model theory is often sensitive to directedness properties, even when it is not sensitive to the underlying connectedness and transitivity properties, taken individually.

Property	**Definition**
Bowtie Directed	$(d \prec c) \wedge (b \prec a) \supset (d \prec a) \vee (b \prec c)$
Diamond Directed	$(c \prec b) \wedge (b \prec a) \supset (c \prec d) \vee (d \prec a)$
Triangle Directed	$(b \prec a) \supset (b \prec c) \vee (c \prec a)$
Pairwise Directed	$(b \prec a) \vee (a \prec b)$

Our first directedness property is bowtie directedness. It says that if b is much closer than a and d is much closer than c, then either b is much closer than c or d is much closer than a.



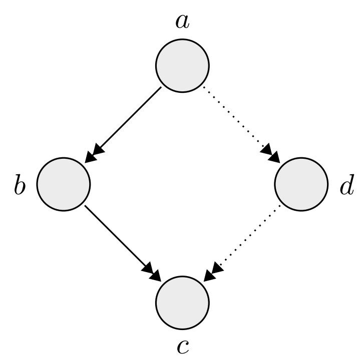
Bowtie Directed

When a relation is pairwise connected and bowtie directed, it forms an **interval order**. Such relations are called interval orders because they have the same structure as the intervals on the real line. In particular, let S and T be any such intervals. We then let $S \preceq T$ whenever there is some $x \in S$ and $y \in T$ such that $x \preceq y$. The resulting relation is both pairwise connected and bowtie directed, and so forms an interval order.

Observation 5.10: A relation is bowtie directed iff it is bowtie connected and zigzag transitive.

The second property is diamond directedness. It says that if b is much closer than

a and c is much closer than b, then either d is much closer than a or c is much closer than d.



Diamond Directed

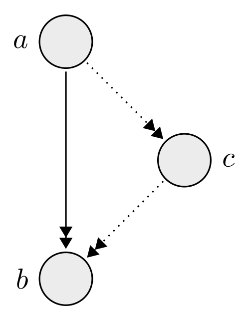
Observation 5.11: A relation is diamond directed iff it is diamond connected, double left transitive, and double right transitive.

A relation forms a **semiorder** when it is pairwise connected, zigzag transitive, and diamond directed. Like preorders, semiorders are strictly weaker than total orders, but they weaken full transitivity rather than pairwise connectedness. They are generally attributed to Luce (1956), who originally introduced them to model intransitive preferences.[29]

Observation 5.12: A relation is a semiorder iff it is pairwise connected and semitransitive.

Observation 5.13: Every semiorder is weakly transitive.

The third relation is triangle directedness, which is also sometimes called *almost connectedness*. This is a property of *as close* relations, but not *nearly as close* relations.



29. While semiorders are generally attributed to Luce, they were in fact first introduced almost forty years earlier by Wiener (1914), a math prodigy who studied under Bertrand Russell and completed his Ph.D. from Harvard at the age of eighteen. See Fishburn and Monjardet (1992) for more on this.

Triangle Directed

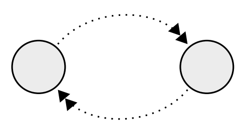
In terms of *as close* relations, triangle directedness says that if b is strictly closer than a, then either c is strictly closer than a or b is strictly closer than c.

Observation 5.14: A relation is triangle directed iff it is triangle connected and strongly transitive.

Observation 5.15: If a relation is triangle directed, then it is also diamond directed and bowtie directed.

While *as close* relations are triangle directed, *nearly as close* relation are not. Suppose that b is three feet away, c is four feet away, and a is five feet away. Suppose also that the tolerance margin is one foot. In that case, b is much closer than a, but c is not much closer than a, nor is b much closer than c.

The last directedness property is pairwise directedness. In terms of *as close* relations, this says that for any pair of worlds b and a, either b is strictly closer than a or a is strictly closer than b.

a  b

Pairwise Directed

Unlike our other directedness properties, pairwise directedness is not the result of conjoining a connectedness property with a transitivity property. Instead, it is the result of conjoining pairwise connectedness with anti-symmetry.

Observation 5.16: A relation is pairwise directed iff it is pairwise connected and anti-symmetric.

Observation 5.17: A pairwise directed relation is a total order iff it is reflexive and left transitive.

6 Characterization Results

In the last section, we introduced several properties of *as close* and *nearly as close* relations. In this section, we are going to use those properties to give characterization results. This will be done in two steps. First, we will give characterization results for finite frames. These are important because, among other things, they entail corresponding decidability results. We will then deal with certain difficulties raised by infinite frames, which will let us give full characterization results.

6.1 Finite Frames

§8 will show how to prove completeness using a new canonical models construction. That procedure if different from, but in many way complementary to, the step-by-step procedure given by Burgess (1981).

As you will see in §8, the new canonical modals procedure builds pairwise connected models for any set of sentences that are consistent in **B1.1**. The Burgess step-by-step procedure, on the other hand, builds *fully transitive* models for any set of sentences that are consistent in **B1**. As a result, it is helpful to be able to switch back and forth between the two. To build models that are pairwise connected, but not fully transitive, we can use the canonical models procedure. To build models that are fully transitive, but not pairwise connected, we can use the step-by-step procedure.

We are going to start by describing what we can show using the canonical models procedure. The results for **B3**-**B5** are already known. The results for **B1.1** and **B2** are new.

Definition 6.1: A frame is **finite** when the domain of worlds is finite. Otherwise, a frame is **infinite**.

Theorem 6.2: *When restricting to the class of finite frames that are pairwise connected, each system on the left is characterized by the class listed in the center. The class or pairwise connected relations meeting this condition is also known by the condition listed on the right.*

System	*Added Condition*	*AKA*
B1.1	*zigzag transitive*	*interval order*
B2	*semitransitive*	*semiorder*
B3	*fully transitive*	*total partial order*
B4	*fully transitive, anti-symmetric*	*total order*
B5	*fully transitive, symmetric*	*universal relation*

Proof. Exact soundness is by Theorem 7.20, since the conditions listed here are stronger than the conditions listed there. Completeness is by Theorem 8.15. ■

Corollary 6.3: ***B1.1**-**B5** are decidable.*

Proof. Immediate by Theorem 6.2. ■

Suppose we start with the assumption that accessibility relations are always pairwise connected. The above theorem then tells us which systems we can generate by imposing additional transitivity requirements. If we add zigzag transitivity, we get **B1.1**. If we add semitransitivity, we get **B2**. If we add full transitivity, we get **B3**.

Theorem 6.2 also demonstrates the usefulness of our new canonical models procedure. If we could only build models that were fully transitive, then we could not prove the first two lines of the table. This is because there are consistent sets of sentences in both **B1.1** and **B2** whose only models are not total preorders.

If we assume pairwise connectedness for counterfactual accessibility, there is an especially natural analogy between Strengthen Might and the S4 axiom from modal logic.[30]. After all, if we adopt pairwise connectedness for counterfactual accessibility as a background assumption, each of those axioms characterizes the class of fully transitive frames. Thus, while the two axioms have very different syntax, they turn out to be similar from a model-theoretic perspective.

On reflection, this is perhaps not surprising, given that our motivation for denying Strengthen Might was the paradox of counterfactual tolerance, and tolerance arguments of various kinds are often deployed against S4.[31]

Theorem 6.4: *When restricting to the class of finite frames that are preorders, each system on the left is characterized by the class listed in the center. The class of preorders listed in the center is also known by the condition listed on the right.*

System	*Added Conditions*	*AKA*
B1		*preorder*
B1.1	*bowtie connected*	
B1.2	*diamond connected*	
B2	*bowtie connected, diamond connected*	
B3	*triangle connected*	
B4	*pairwise connected, anti-symmetric*	*total order*
B5	*symmetric*	*universal relation*

Proof. Exact soundness is by Theorem 7.20, since the conditions listed here are stronger than the conditions listed there. Completeness can be shown using the Burgess step-by-step procedure. ■

Corollary 6.5: ***B1-B5*** *are decidable.*

Proof. Immediate by Theorem 6.4. ■

Suppose that we start with the assumption that accessibility relations are always fully transitive. The above table then tells us which systems we can generate by adding

30. The S4 axiom says that $\Box(A) \supset \Box\Box(A)$.
31. See for example Chandler (1976), Salmon (1979), and Williamson (1992).

various connectedness requirements. The results for **B.1**-**B3** are new. The other results are already known, though they are usually framed as soundness and completeness results, rather than relativized characterization results.

Theorem 6.4 demonstrates the usefulness of having the Burgess step-by-step procedure in addition to our canonical models procedure. If we only had the canonical models construction, we would only be able to build pairwise connected models. But in that case, we would not be able to prove most of the lines in the above table.

Theorem 6.2 and Theorem 6.4 tell us which systems can be characterized, given that we have as a background assumption that the domain of worlds is finite and that we either assume that accessibility relations are always pairwise connected, or that we assume that they are always fully transitive. What happens, though, if we relax those assumptions? What if we only assume that the domain is finite? In that case, **B1**-**B5** are characterized using the frame conditions listed in Theorem 6.7.

Definition 6.6: A relation is a **left order** when it is reflexive and left transitive.

Theorem 6.7 (Characterization for Finite Frames)**:** *When restricting to the class of finite frames, each system on the left is characterized by the class of left orders in the center. The class of left orders meeting that condition is also known by the condition listed on the right.*

System	***Added Condition***	***AKA***
B1	*zigzag transitive*	
B1.1	*bowtie directed*	
B1.2	*zigzag transitive, diamond directed*	
B2	*bowtie directed, diamond directed*	
B3	*triangle directed*	
B4	*pairwise directed*	*total order*
B5	*symmetric*	

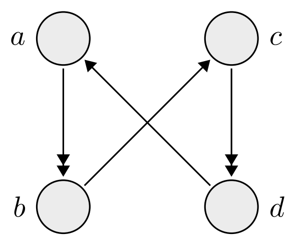
Proof. Exact soundness is by Theorem 7.20, since the conditions listed here are stronger than the conditions listed there. Completeness for **B1.1** and **B3**-**B5** is by Theorem 8.15, since the conditions listed here are weaker than the conditions listed there. Completeness for **B1** and **B1.2** is by Theorem 6.4. ■

Most of the results in Theorem 6.7 are new, including those for the new systems **B1.1** - **B2**, as well as those for the old systems **B1**, **B3**, and **B5**. Given Observation 5.17, the result for **B4** is just Theorem 3.17 in another guise, and so this line of the table is already known.

6.2 Infinite Frames

We have shown that **B1**-**B5** are characterized by various frame conditions, so long as we restrict our attention to finite frames. While this is a good start, we would also like to allow for infinite frames. This requires additional work.

Definition 6.8: The worlds a, b, c and d form a **bowtie cycle** when $d \prec c$ and $b \prec a$ and $a \preceq d$ and $c \preceq b$.



Thinking in terms of *nearly as close* relations, four worlds form a bowtie cycle when d is much closer than c and b is much closer than a but, nevertheless, a is nearly as close as d and c is nearly as close as b.

We can rule out bowtie cycles by accepting zigzag transitivity. The problem is that once we allow for infinite models, there can still be infinite sequences of worlds that *look* just like bowtie cycles, at least from the perspective of the model theory. These sequences are what we are going to call bowtie sequences. Thus, once we allow for infinite frames, **B1** is no longer sound with respect to the class of all left orders that are zigzag transitive.[32]

Definition 6.9: A **bowtie sequence** $\langle a_1, b_1, c_1, d_1, a_2, b_2 \ldots \rangle$ is an infinite chain such that $a_n \not\preceq b_m$ and $c_n \not\preceq d_m$ for all n and m.

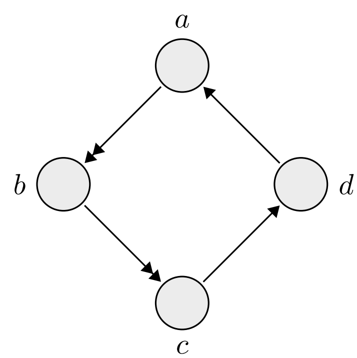
Definition 6.10: A relation is **bowtie well-founded (bwf)** when it does not have any bowtie sequences.

There is a similar challenge when it comes to characterizing **B2**. In particular, if we allow for infinite frames, **B2** is no longer sound with respect to the class of all diamond

32. This follows from Proposition 7.13 given that there are left orders that are zigzag transitive, but not bowtie well-founded.

orders.[33]

Definition 6.11: The worlds a, b, c and d form a **diamond cycle** when $c \prec b$ and $b \prec a$ and $d \preceq c$ and $a \preceq c$.



Four worlds form a diamond cycle when b is much closer than a and c is much closer than b but, nevertheless, d is nearly as close as c and a is nearly as close as d.

Diamond directedness rules out diamond cycles, which is needed to ensure that **B2** is sound. The problem is that if we allow for infinite frames, there can be certain infinite sequences that *look* just like diamond cycles, at least so far as the model theory is concerned. We will call these sequences diamond sequences.

Definition 6.12: A **diamond sequence** $\langle a_1, b_1, c_1, d_1, a_2, b_2 \dots \rangle$ is an infinite chain such that $a_n \npreceq_x b_m$ and $b_n \npreceq_x c_m$ for all n and m.

Definition 6.13: A relation is **diamond well-founded (dwf)** when it does not have any diamond sequences.

Observation 6.14: If relation is fully transitive, then it is bowtie well-founded and diamond well-founded.

Observation 6.15: If relation is pairwise connected, then it is zigzag transitive iff it is bowtie well-founded.

Observation 6.16: If relation is pairwise connected, then it is diamond transitive iff it is diamond well-founded.

There is a familiar, and related, problem when it comes to characterizing **B4**. For while **B4** is sound in the class of all total orders, so long as we have only finite frames,

33. This follows from Proposition 7.16 given that there are left orders that are diamond directed, but not diamond well-founded.

it is not sound in the class of all total orders, once we allow for infinite frames. The problem is that infinite sequences of strict accessibility relations *look* just like symmetric accessibility relations from the perspective of the model theory.

Definition 6.17: A **loop sequence** $\langle a_1, a_2, a_3 \dots \rangle$ is an infinite chain such that $a_n \not\leq_x a_m$ for all $n < m$.

Definition 6.18: A relation satisfies the **limit assumption (lma)** when it does not have any loop sequences.

Observation 6.19: If a relation is well-founded, then it satisfies the limit assumption, is bowtie well-founded iff it is zigzag transitive, and is diamond well-founded iff it is diamond directed.

We are now in a position to give results analogous to Theorems 6.2-6.7 while allowing for infinite frames.

Theorem 6.20: *When restricting to the class of frames that are pairwise connected, each system on the left is characterized by the class listed in the center. This class is also known by the condition listed on the right.*

System	***Added Condition***	***AKA***
B1.1	*zigzag transitive*	*interval order*
B2	*semitransitive*	*semiorder*
B3	*fully transitive*	*total partial order*
B4	*fully transitive, anti-symmetric, lma*	*wf total order*
B5	*fully transitive, symmetric*	*universal relation*

Proof. Exact soundness is by Theorem 7.20, since the conditions listed here are stronger than the conditions listed there. Completeness is by Theorem 8.15, since the conditions listed here are weaker than the conditions listed there. ■

Theorem 6.20 is almost exactly like Theorem 6.2, the only difference being that the limit assumption is needed to characterize **B4**. The reason that the two theorems are so similar is Observation 6.15 and Observation 6.16. These tell us that so long as we have pairwise connectedness as a background assumption, zigzag transitivity is equivalent to bowtie well-foundedness and diamond directedness is equivalent to diamond well-foundedness. There is thus no need to appeal to these as additional constraints when characterizing **B1** and **B2**.

Theorem 6.21: *When restricting to the class of preorders, each system on the left is* *characterized by the class listed in the center. This class is also known by the condition listed on the right.*

System	*Added Condition*	*AKA*
B1		*preorder*
B1.1	*bowtie connected*	
B1.2	*diamond connected*	
B2	*bowtie connected, diamond connected*	
B3	*triangle connected*	
B4	*pairwise connected, anti-symmetric, lma*	*wf total order*
B5	*symmetric*	*universal relation*

Proof. Exact soundness is by Theorem 7.20, since the conditions listed here are stronger than the conditions listed there. Completeness is by Theorem 6.4, since the conditions here are strictly weaker than the conditions listed there. ■

Again, Theorem 6.21 is almost exactly the same as Theorem 6.4, with the only difference being that we need the limit assumption to characterize **B4**. In this case, the similarity is explained by Observation 6.14, which tells us that transitivity on its own entails bowtie well-foundedness and diamond well-foundedness.

Theorem 6.22 (Full Characterization)**:** *Each of the systems on the left is fully characterized by the class of left orders listed in the center. That class is also known by the condition listed on the right.*

System	*Added Condition*	*AKA*
B1	*zigzag transitive, bwf*	
B1.1	*bowtie directed, bwf*	
B1.2	*zigzag transitive, diamond directed, bwf, dwf*	
B2	*bowtie directed, diamond directed, bwf, dwf*	
B3	*triangle directed*	
B4	*pairwise directed, lma*	*wf total order*
B5	*symmetric*	

Proof. Exact soundness is by Theorem 7.20. Completeness is by Theorem 6.7, since the conditions here are weaker than the conditions listed there. ■

All of the results listed in Theorem 6.22 are new. As you can see, once we are no longer assuming either full transitivity or pairwise connectedness, bowtie well-foundedness and diamond well-foundedness are needed to fully characterize **B1**-**B2**. We also still need the limit assumption to characterize **B4**.

One implication of this is that most of our systems are characterized by second-order frame conditions, with **B3** and **B5** being striking exceptions. But while that may be, there is also an important difference between **B1**-**B2** and **B4**. Given Theorems 6.20 and 6.21, **B1**-**B2** can be *generated* using first-order frame conditions, even though they cannot be characterized using first-order frame conditions. In contrast, **B4** can neither be generated nor characterized using first-order frame conditions. There is thus a clear sense in which **B4** is more deeply second-order than **B1**-**B2**.

7 Exact Soundness

Soundness proofs are often straightforward, and so left to the reader. In the present case, though, the proofs can be somewhat less obvious than usual. As such, I have included them.

Exact soundness proofs have two steps, and so this section is divided into two parts. The first half proves soundness. The second half proves inverse soundness.

7.1 Soundness

Say that a world c is an **example** of $A \;\Box\!\!\rightarrow\; B$ when $c \vDash A \wedge B$. A world b is a **witness** of $A \;\Box\!\!\rightarrow\; B$ relative to x when $b \vDash A$ and every c such that $c \preceq_x b$ and $c \vDash A$ is an example. A world a **confirms** $A \;\Box\!\!\rightarrow\; B$ relative to x when $a \vDash A$ and there is some b that is a witness relative to x such that $b \preceq_x a$.

Similarly, a world c is an **example** of $A \;\Diamond\!\!\rightarrow\; B$ when $c \vDash A \wedge B$. A world b is a **supporter** of $A \;\Diamond\!\!\rightarrow\; B$ relative to x when $b \vDash A$ and there is some c such that $c \preceq b$ that is an example. A world a **confirms** $A \;\Diamond\!\!\rightarrow\; B$ relative to x when $a \vDash A$ and every b such that $b \preceq_x a$ and $b \vDash A$ is a supporter.

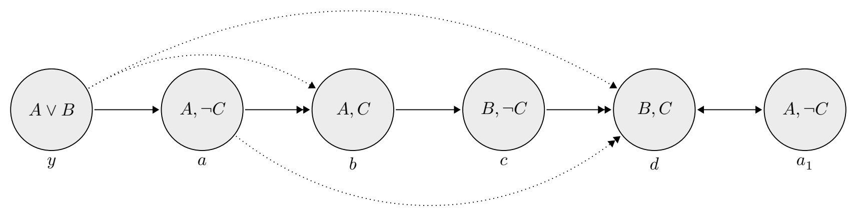
Putting these definitions together, $A \;\Box\!\!\rightarrow\; B$ is true at a world x iff every counterfactually possible A world relative to x confirms it. Similarly, $A \;\Diamond\!\!\rightarrow\; B$ is true at a world x iff some counterfactually possible A world relative to x confirms it.

Observation 7.1: SUB, WEA, and IM are valid in all frames.

Observation 7.2: If a frame is reflexive, ID is valid.

Proposition 7.3: *If a frame is reflexive, left transitive, bowtie directed, and bowtie well-founded, then DI is valid.*

Proof. Suppose for reductio that $x \vDash (A \;\Box\!\!\rightarrow\; C) \wedge (B \;\Box\!\!\rightarrow\; C)$ and $x \vDash (A \vee B) \;\Diamond\!\!\rightarrow\; \neg C$. There is thus some y that confirms $(A \vee B) \;\Diamond\!\!\rightarrow\; \neg C$ and, given reflexivity, there is an example a such that $a \preceq y$. Either $a \vDash A$ or $a \vDash B$. The basic reasoning is the same either way, so suppose that $a \vDash A$. We know that a confirms $A \;\Box\!\!\rightarrow C$, since $x \vDash A \;\Box\!\!\rightarrow C$, so there is a witness b such that $b \prec_x a$, with $b \preceq_x y$ by left transitivity. This gives us the first three worlds below.[34]



Since $b \preceq_x y$, b is a supporter of $A \vee B \;\Box\!\!\rightarrow \neg C$, so there is an example c such that $c \preceq_x b$. Moreover, since b is a witness of $A \;\Box\!\!\rightarrow C$, this means that $c \vDash B \wedge \neg C$. This world confirms $B \;\Box\!\!\rightarrow C$, since $x \vDash B \;\Box\!\!\rightarrow C$, so there is also a witness d such that $d \prec_x c$. It thus follows that $d \prec_x a$ by zigzag transitivity and $d \preceq_x y$ by left transitivity.

This last observation means that we will need yet another example a_1 of $A \;\Diamond\!\!\rightarrow B$, with this example being such that $a_1 \preceq_x d$. The problem now is that we cannot stop with a_1. We in fact need to add a full copy $\langle a_1, b_1, c_1, d_1 \rangle$ of the worlds $\langle a, b, c, d \rangle$ on the right of the diagram. Once we do, we will be in a similar position, so will need to add yet another copy $\langle a_2, b_2, c_2, d_2 \rangle$ to the right of that diagram, and so on and so forth for all n. The result is a bowtie sequence, which is contrary to the assumption that the relation is bowtie well-founded. ■

Proposition 7.4: *If a frame is reflexive, left transitive, bowtie directed, and bowtie well-founded, then CO is valid.*

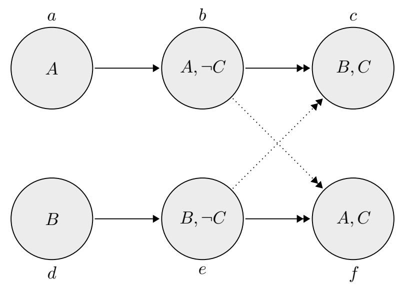
Proof. By reasoning similar to that used to prove Proposition 7.3. ■

Proposition 7.5: *If a frame is reflexive, left transitive, and bowtie directed, then DB is valid.*

Proof. Suppose that $x \vDash (A \vee B \;\Diamond\!\!\rightarrow C)$ and $x \vDash A \;\Diamond\!\!\rightarrow \neg C$ and $x \vDash B \;\Diamond\!\!\rightarrow \neg C$. We thus have a world a that confirms $A \;\Diamond\!\!\rightarrow \neg C$ with an example b such that $b \preceq_x a$. Since b confirms $A \vee B \;\Box\!\!\rightarrow C$, there is also a witness c such that $c \prec b$, with $c \preceq_x a$ by weak left transitivity. If it were the case that $c \vDash A$, there would need to be an example y of $A \;\Diamond\!\!\rightarrow \neg C$ such that $y \preceq_x c$. But this would contradict the fact that c witnesses $A \vee B \;\Box\!\!\rightarrow C$. So $c \nvDash A$. This gives us the worlds on the top row in the diagram below.

34. Reflexive arrows will generally be left implicit to simplify the diagrams.

Analogous reasoning gives us the worlds on the bottom, given that there must be a d that confirms $B \diamondto \neg C$.

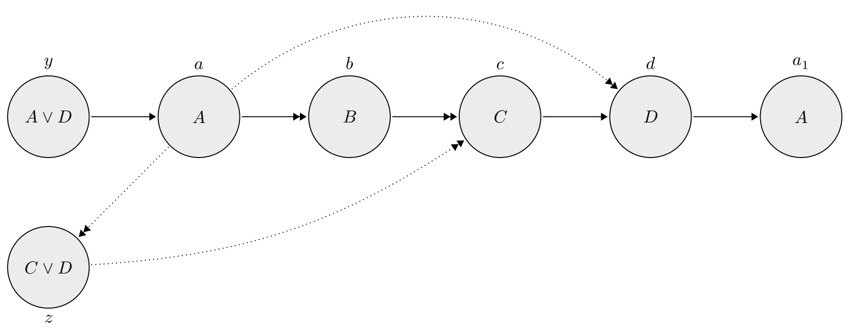


There are now two possibilities given bowtie directedness. These are that $f \prec_x b$ or $c \prec_x e$. Each of these cases leads to contradiction.

First, suppose that $f \prec_x b$. This gives us $f \preceq_x a$ by weak left transitivity, so there must be an example y of $A \diamondto \neg C$ such that $y \preceq_x f$. But this contradicts the fact that f witnesses $A \vee B \boxto C$. If we instead suppose that $c \prec_x e$, we get a contradiction for similar reasons. ■

Proposition 7.6: *If a frame is reflexive, diamond directed, and diamond well-founded, then DM is valid.*

Proof. Suppose that $x \vDash (A \vee B \boxto \neg A) \wedge (B \vee C \boxto \neg B)$ and $x \vDash C \vee D \diamondto C$ and $x \vDash A \vee D \diamondto A$. There is thus a y that confirms $A \vee D \diamondto A$ and so a witness a such that $a \preceq_x y$. Since a confirms $A \vee B \boxto \neg A$, there is a witness such that $b \prec_x a$. And since b confirms $B \vee C \boxto \neg B$, there is a witness c such that $c \prec_x b$.



Since $x \vDash C \vee D \diamondto D$, there is a z that confirms it. By diamond directedness, either $c \prec_x z$ or $a \prec_x z$.

First, suppose that $c \prec_x z$. From this it follows that there is an example d of $C \vee D \diamondto D$ such that $d \preceq_x c$. By diamond connectedness $d \prec_x a$ and so $d \preceq_x y$

by left transitivity. This means that we need a further example a_1 of $A \vee D \mathrel{\Diamond\!\!\rightarrow} A$ such that $a_1 \preceq_x d$.

The problem is that we are now back where we started. We will thus need to add not only a_1, but a full copy $\langle a_1, b_1, c_1, d_1 \rangle$ of $\langle a, b, c, d \rangle$ on the right. Once we do, we will need to add yet another copy, and so on and so forth for all n. The result is a diamond sequence, which violates diamond well-foundedness.

Now suppose instead that $a \prec_x z$. This results in $z \preceq_x y$ by left transitivity. In that case, we will need to construct a similar infinite sequence of worlds, either on the top line or the bottom line. Since the reasoning in that case is similar, this half of the proof is left to the reader. Since we once again have a violation of diamond well-foundedness, the result follows. ■

Observation 7.7: If a frame is reflexive and triangle connected, then SM is valid.

Observation 7.8: If a frame is a total order and well-founded, then CEM is valid.

Observation 7.9: If a frame is symmetric, then ST is valid.

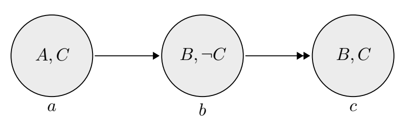
7.2 Inverse Soundness

Now that we have shown soundness, we turn to showing the inverse. This is done by providing several countermodels.

Observation 7.10: If a frame is not reflexive, then ID is not valid.

Proposition 7.11: *If a frame is reflexive, but not left transitive, then DI is not valid.*

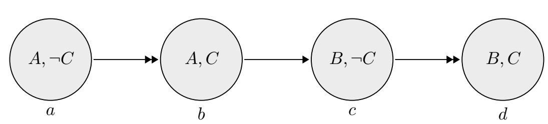
Proof. The following is a countermodel.



■

Proposition 7.12: *If a frame is not zigzag transitive, then DI is not valid.*

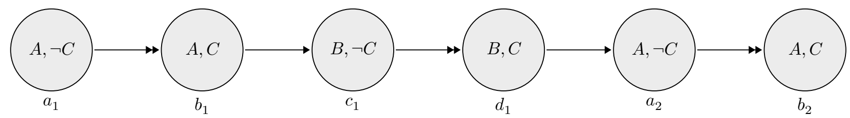
Proof. The following is a countermodel.



■

Proposition 7.13: *If a frame is reflexive, but not bowtie well-founded, then DI is not* *valid.*

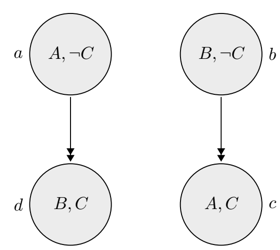
Proof. Suppose that we have a bowtie sequence extending to the right. The following assignment of propositions is then a counterexample.



The countermodel is just like the one used for Proposition 7.12, except there is a copy $\langle a_n, b_n, c_n, d_n \rangle$ of $\langle a, b, c, d \rangle$ for each n pasted on the right. The additional arrows required by ■

Proposition 7.14: *If a frame is reflexive, but not bowtie directed, then DB is not valid.*

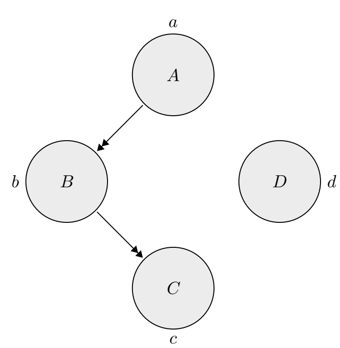
Proof. The following is a countermodel.



■

Proposition 7.15: *If a frame is reflexive, but not diamond directed, then DM is not valid.*

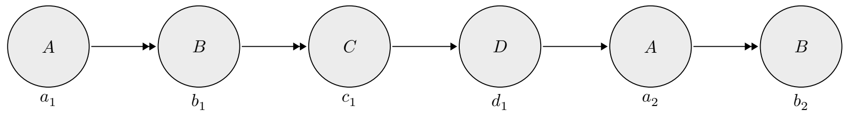
Proof. The following is a countermodel.



■

Proposition 7.16: *If a frame is reflexive, but not diamond well-founded, then DM is* *not valid.*

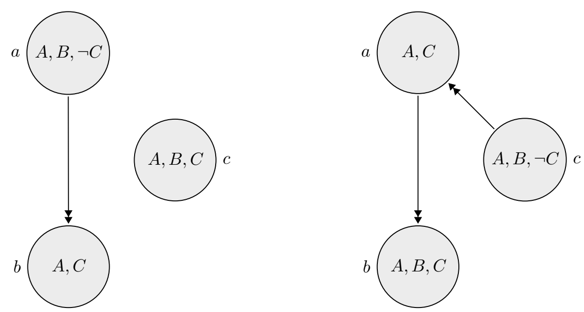
Proof. The following is a countermodel, with the diamond sequence extending infinitely to the right.



■

Proposition 7.17: *If a frame is reflexive, but not triangle directed, then SM is not valid.*

Proof. Consider any frame that is reflexive but not triangle connected. There are thus worlds a and b and c such that $b \prec a$ but neither $c \prec_x a$ nor $c \prec b$. There are then two cases to consider. If not $a \prec_x c$, we can use the countermodel on the left. If $a \prec_x c$, on the other hand, we can use the countermodel on the right instead.



■

Observation 7.18: If a frame is not well-founded, then CEM is not valid.

Observation 7.19: If a frame is not symmetric, then ST is not valid.

Theorem 7.20 (Exact Soundness)**:** *Each system on the left is characterized by the class of left orders meeting the condition listed in the center, which is equivalent to the condition* *listed on the right.*

System	Frames	AKA
B1	zigzag transitive, bwf	
B1.1	bowtie directed, bwf	
B1.2	diamond directed, dwf, bwf	
B2	bowtie directed, diamond directed, dwf, bwf	
B3	triangle directed	
B4	pairwise directed, lma	wf total order
B5	symmetric order	

Proof. By the preceding. ∎

8 Completeness

The most familiar canonical constructions, like those from Henkin (1949), build a single canonical model that is a model for every consistent sentence. What we are going to do instead is assign each sentence a type and then, for each type, build a canonical model that is a model for every consistent sentence of that type. Since we have the compactness theorem, this means that we can also build a model for every consistent *set* of sentences.

In what follows, we are going to show that every set of sentences that is consistent in **B1.1** has not only a model, but a finite model. The basic procedure can be easily extended to any system extending **B1.1**. This includes my preferred system **B2**, along with **B3**-**B5**.

Definition 8.1: Fix an enumeration $p_1, p_2, \ldots$ of the atomic sentence of $\mathcal{L}$. The **atomic type** of a sentence A is the smallest n such that $p_1, \ldots, p_n$ includes all the atomic sentences in A.

Definition 8.2: The **modal depth** of a sentence A is given by $f(A)$, where this is defined recursively with:

$f(A) = 0$ when A is an atom.
$f(\neg A) = f(A)$
$f(A \wedge B) = f(A \vee B) = f(A \supset B) = max(f(A), f(B))$
$f(A \mathrel{\Box\!\!\rightarrow} B) = f(A \mathrel{\Diamond\!\!\rightarrow} B) = max(f(A), f(B)) + 1$

Definition 8.3: The **type** of a sentence is $t = \langle n, m \rangle$, where n is the atomic type and

m is the modal depth.

Definition 8.4: The **states** of type $\langle n, m\rangle$ are the members of $Y^{n,m}$, where this set is defined recursively with:

$X^{n,0}$	$=$	the set of atomic sentence of type $\langle n, 0\rangle$
$Y^{n,m}$	$=$	the set of consistent conjunctions $A_1 \wedge \ldots \wedge A_n$ with A_i being either B_i or $\neg B_i$ for the enumerated $B_i \in X^{n,m}$
$X^{n,m+1}$	$=$	the union of the $X^{n,m}$ and all sentences of the form $A \vee B \mathrel{\Box\!\!\rightarrow} \neg B$ for $A, B \in Y^{n,m}$

Note that in the above construction, we always fix an enumeration of the relative atoms in $X^{n,m}$ in order to form the states in $Y^{n,m}$. This is important because it ensures that numerically distinct states are always logically inconsistent.

We now have almost everything needed to build our canonical models. As a final bit of preamble, we are going to fix a function mapping each state x to a maximal consistent set x^* such that $x \in x^*$. We then institute the following shorthand, where a and b are also states.

$$
\begin{array}{lll}
a \trianglelefteq_x b & \text{iff} & x^* \vdash a \vee b \mathrel{\Diamond\!\!\rightarrow} a \\
a \triangleleft_x b & \text{iff} & a \trianglelefteq_x b \text{ and } a \not\trianglerighteq_x b
\end{array}
$$

When the x is arbitrary or clear from context, we will drop the corresponding subscript, and so just write $a \trianglelefteq b$ and $a \triangleleft b$.

Definition 8.5: For every type t, the corresponding canonical model $\mathcal{M}^t$ is constructed as follows:

$$
\begin{array}{lll}
W & = & \{x \mid x \text{ is a state of type } t\} \\
N_x & = & \{y \mid y \trianglelefteq_x z \text{ for some } z\} \\
a \preceq_x b & \text{iff} & a \trianglelefteq_x b \\
V(p) & = & \{x \in W \mid p \in x^*\}
\end{array}
$$

Proposition 8.6 (Deduction Theorem)**:** $A \vdash B$ *iff* $\vdash A \supset B$.

Proof. The proof is the same as in the purely propositional case, and so left to the reader. ■

Lemma 8.7: *The following schemas are all valid in any system extending **B1**:*

LT	$(A \mathrel{\Box\!\!\rightarrow} B) \wedge (A \wedge B \mathrel{\Box\!\!\rightarrow} C) \supset (A \mathrel{\Box\!\!\rightarrow} C)$
A1	$A \mathrel{\Box\!\!\rightarrow} B$ *when* $\vdash A \supset B$
A2	$(A \mathrel{\Box\!\!\rightarrow} C) \supset (A \vee B \mathrel{\Box\!\!\rightarrow} C \vee B)$
A3	$(A \vee B \vee C \mathrel{\Box\!\!\rightarrow} \neg B \wedge \neg C) \supset (A \vee B \mathrel{\Box\!\!\rightarrow} \neg B)$
A4	$(A \vee B \mathrel{\Box\!\!\rightarrow} \neg B) \wedge (A \vee C \mathrel{\Box\!\!\rightarrow} \neg C) \supset (A \vee B \vee C \mathrel{\Box\!\!\rightarrow} \neg B \wedge \neg C)$ *when* $B \vdash \neg C$
A5	$(B \mathrel{\Box\!\!\rightarrow} C) \supset (A \vee B \mathrel{\Box\!\!\rightarrow} C)$ *when* $A \vdash C$
A6	$(A \mathrel{\Box\!\!\rightarrow} C) \wedge (B \mathrel{\Box\!\!\rightarrow} D) \supset (A \vee B \mathrel{\Box\!\!\rightarrow} C \vee D)$

Proof. To show A1, let $A \vdash B$. Then:

1.	$A \mathrel{\Box\!\!\rightarrow} A$	ID
2.	$A \mathrel{\Box\!\!\rightarrow} B$	1, WEA

To show A2:

1.	$A \mathrel{\Box\!\!\rightarrow} B$	
2.	$A \mathrel{\Box\!\!\rightarrow} B \vee C$	1, WEA
3.	$C \mathrel{\Box\!\!\rightarrow} B \vee C$	A1
4.	$A \vee C \mathrel{\Box\!\!\rightarrow} B \vee C$	2, 3, DI

To show LT:

1.	$A \mathrel{\Box\!\!\rightarrow} B$	
2.	$A \wedge B \mathrel{\Box\!\!\rightarrow} C$	
3.	$(A \wedge B) \vee (A \wedge \neg B) \mathrel{\Box\!\!\rightarrow} C \vee (A \wedge \neg B)$	2, A2
4.	$A \mathrel{\Box\!\!\rightarrow} C \vee (A \wedge \neg B)$	3, SUB
5.	$A \mathrel{\Box\!\!\rightarrow} B \wedge (C \vee (A \wedge \neg B))$	1,4, CO
6.	$A \mathrel{\Box\!\!\rightarrow} C$	5, WEA

To show A4, let $B \vdash \neg C$. Then:

1.	$A \vee B \mathrel{\Box\!\!\rightarrow} \neg B$	
2.	$A \vee C \mathrel{\Box\!\!\rightarrow} \neg C$	
3.	$A \vee B \vee C \mathrel{\Box\!\!\rightarrow} \neg B \vee C$	1, A2
4.	$A \vee C \vee B \mathrel{\Box\!\!\rightarrow} \neg C \vee B$	2, A2
5.	$A \vee B \vee C \mathrel{\Box\!\!\rightarrow} \neg C \vee B$	4, SUB
6.	$A \vee B \vee C \mathrel{\Box\!\!\rightarrow} \neg C$	5, WEA
7.	$A \vee B \vee C \mathrel{\Box\!\!\rightarrow} (\neg B \vee C) \wedge \neg C$	3, 6, CO
8.	$A \vee B \vee C \mathrel{\Box\!\!\rightarrow} \neg B \wedge \neg C$	7, SUB

To show A5, let $A \vdash C$. Then:

1. $B \mathrel{\Box\!\!\rightarrow} C$
2. $A \mathrel{\Box\!\!\rightarrow} C$ — A1
3. $A \lor B \mathrel{\Box\!\!\rightarrow} C$ — 2, 1, DI

To show A6:

1. $A \mathrel{\Box\!\!\rightarrow} C$
2. $B \mathrel{\Box\!\!\rightarrow} D$
3. $A \mathrel{\Box\!\!\rightarrow} C \lor D$ — 1, WEA
4. $B \mathrel{\Box\!\!\rightarrow} C \lor D$ — 2, WEA
5. $A \lor B \mathrel{\Box\!\!\rightarrow} C \lor D$ — 3,4, DI

■

Proposition 8.8: $\trianglelefteq$ *is pairwise connected.*

Proof. We need to show that if $a \trianglelefteq c$ for some c and $b \trianglelefteq d$ for some d, then either $a \trianglelefteq b$ or $b \trianglelefteq a$.

1. $(a \lor b \mathrel{\Box\!\!\rightarrow} \neg a) \land (a \lor b \mathrel{\Box\!\!\rightarrow} \neg b)$
2. $a \lor b \mathrel{\Box\!\!\rightarrow} \neg a \land \neg b$ — 1, CO
3. $(a \mathrel{\Box\!\!\rightarrow} \neg a \land \neg b) \lor (b \mathrel{\Box\!\!\rightarrow} \neg a \land \neg b)$ — 2, DB
4. $(a \mathrel{\Box\!\!\rightarrow} \neg a) \lor (b \mathrel{\Box\!\!\rightarrow} \neg b)$ — 3, CL, PL
5. $c \mathrel{\Box\!\!\rightarrow} \neg a$ — A1
6. $d \mathrel{\Box\!\!\rightarrow} \neg b$ — A1
7. $(a \lor c \mathrel{\Box\!\!\rightarrow} \neg a) \lor (b \lor d \mathrel{\Box\!\!\rightarrow} \neg b)$ — 6, DI, PL

Pairwise connectedness follows by contraposition. ■

Proposition 8.9: $\trianglelefteq$ *is zigzag transitive.*

Proof. We need to show that if $a \trianglelefteq e$ for some e and $b \triangleleft a$ and $c \trianglelefteq b$ and $d \triangleleft c$, then $d \triangleleft a$.

1. $a \lor b \mathrel{\Box\!\!\rightarrow} \neg a$
2. $\neg(b \lor c \mathrel{\Box\!\!\rightarrow} \neg c)$
3. $c \lor d \mathrel{\Box\!\!\rightarrow} \neg c$
4. $a \lor b \lor c \lor d \mathrel{\Box\!\!\rightarrow} \neg a \lor \neg c$ — 1, 3 A6
5. $a \lor b \lor c \lor d \mathrel{\Box\!\!\rightarrow} a \lor b \lor c \lor d$ — ID
6. $(a \lor d) \lor (b \lor c) \mathrel{\Box\!\!\rightarrow} b \lor d$ — 4, 5, CO, SUB
7. $(a \lor d \mathrel{\Box\!\!\rightarrow} b \lor d) \lor (b \lor c \mathrel{\Box\!\!\rightarrow} b \lor d)$ — 6, DB
8. $(a \lor d \mathrel{\Box\!\!\rightarrow} \neg a) \lor (b \lor c \mathrel{\Box\!\!\rightarrow} \neg c)$ — 7, WEA
9. $a \lor d \mathrel{\Box\!\!\rightarrow} \neg a$ — 2, 8, PL

■

Proposition 8.10: *Every sentence A of type $\langle n, m\rangle$ is logically equivalent to a sentence*

B of type $\langle i, j\rangle$ whenever $i \geq n$ and $j \geq j$.

Proof. Given any A of type $\langle n, m\rangle$, we can find a logically equivalent B_1 of type $\langle n + 1, m\rangle$ by using $A \wedge (A \vee p_{n+1})$, and a logically equivalent B_2 of type $\langle n, m+1\rangle$ by using $A \wedge (A \vee (A \mathrel{\Box\!\!\rightarrow} A))$. ■

Lemma 8.11: *Let A and B be sentences of type t. Then $A \mathrel{\Box\!\!\rightarrow} B$ is logically equivalent to $\bigvee_i \bigwedge_j (p_i \vee q_j \mathrel{\Box\!\!\rightarrow} \neg q_j)$, where the p_i and q_j are the states of type t such that $p_i \vdash A \wedge B$ and $q_j \vdash A \wedge \neg B$ respectively.*

Proof. The proof is by induction. For the base case, suppose A and B are both of type $\langle m, 0\rangle$ and consider the sentence $A \mathrel{\Box\!\!\rightarrow} B$. This is equivalent to

$$(A \wedge B) \vee (A \wedge \neg B) \mathrel{\Box\!\!\rightarrow} \neg(A \wedge \neg B) \tag{12}$$

by substitution and weakening. Every sentence of type $\langle m, 0\rangle$ is equivalent to a sentence of type $\langle m, 0\rangle$ in disjunctive normal form in which every conjunction is maximal with respect to sentences of type $\langle m, 0\rangle$. As such, (12) is equivalent to

$$\bigvee_i (p_i) \vee \bigvee_j (q_j) \mathrel{\Box\!\!\rightarrow} \bigwedge_j \neg(q_j) \tag{13}$$

where the p_i and q_j are as described. The only thing left to show is that this is equivalent to the target sentence:

$$\bigvee_i \bigwedge_j (p_i \vee q_j \mathrel{\Box\!\!\rightarrow} \neg q_j) \tag{14}$$

The proof from (13) to (14) uses repeated applications of DB and A3. The other direction uses disjunctive syllogism and repeated applications of A4 and A5. This gives us the base case. The induction step is essentially the same, with the exception that we use the induction hypothesis when showing that (12) is equivalent to (13). So the full result follows. ■

Observation 8.12: If A is a state of type t and x is a state of type t, then $x \vdash A$ iff $x^* \vdash A$.

Lemma 8.13: *Let A and B be sentences of type t. Then the following are equivalent:*

$$x \vdash A \mathrel{\Box\!\!\rightarrow} B \tag{15}$$

For every state $a \vdash A$ of type t such that $a \trianglelefteq g$ for some state g of type t, there is a state $b \vdash A$ of type t such that $b \trianglelefteq a$ and, for every state $c \vdash A$ of type t, if $c \trianglelefteq b$, then $c \vdash B$. (16)

Proof. Suppose (15). By Lemma 8.11 and the fact that x^* is maximal, there is some $d \vdash A \wedge B$ such that $d \triangleleft e$ for all $e \vdash A \wedge \neg B$. Now consider any a of the type described. If $d \trianglelefteq a$, then d is the needed b, so suppose otherwise. In that case, $a \triangleleft d$ (because $\trianglelefteq$ is pairwise connected), and so $a \triangleleft e$ (because $\triangleleft$ is transitive). But then a is itself is the requisite b (since $\trianglelefteq$ is reflexive), and so (16).

For the other direction, suppose (16) and consider any $a \vdash A$. What we are going to show is that there is always a $b \vdash A$ such that $x^* \vdash a \vee b \;\Box\!\!\rightarrow B$. But in that case, we can use DI to disjoin the antecedents of all such counterfactuals, with the result being a disjunctive antecedent that is logically equivalent to A. So $x^* \vdash A \;\Box\!\!\rightarrow B$ by substitution, and therefore (15) by Observation 8.12.

Suppose then that $a \vdash A$ and that there is no d such that $a \trianglelefteq d$. In that case, we have $x^* \vdash a \vee a \;\Box\!\!\rightarrow \neg a$ and so $x^* \vdash a \vee a \;\Box\!\!\rightarrow B$ by (B8), and so a itself can be the requisite b. Now suppose instead that $a \vdash A$ and that there is some d such that $a \trianglelefteq d$. It thus follows that there is some b as descried in (16) and, furthermore, $b \vDash B$ because $\trianglelefteq$ is reflexive. There are then two cases. If $a \trianglelefteq b$, then $a \vdash B$, and so $x^* \vdash a \vee b \;\Box\!\!\rightarrow B$ by ID and WEA. On the other hand, if $a \not\trianglelefteq b$, then $x^* \vdash a \vee b \;\Box\!\!\rightarrow b$ and so $x^* \vdash a \vee b \;\Box\!\!\rightarrow B$. The upshot is that for any $a \vDash A$, there is a $b \vdash A$ such that $x^* \vdash a \vee b \;\Box\!\!\rightarrow B$, as claimed. ■

Lemma 8.14 (Truth Lemma)**:** *Let A be a sentence of type* t *and $\mathcal{M}^t$ the canonical model of that same type. Then for all $x \in W^{m,n}$:*

$\mathcal{M}^t, x \vDash A$ *iff* $x \vdash A$

Proof. The proposition holds for atomic sentences by construction and, whenever it holds for a set of sentences, it also holds for the truth functional compounds of those sentences. The proof thus reduces to the case in which A has the form $B \;\Box\!\!\rightarrow C$. That it holds in this case follows from Proposition 8.10 and Lemma 8.13 by induction and the construction of $\mathcal{M}^t$. ■

Theorem 8.15 (Completeness)**:** *Given a consistent set of sentences of any of the systems on the left, there is a finite pairwise connected model meeting the added conditions on the right.*

System	**Added Condition**	**AKA**
B1.1	zigzag transitive	
B2	semitransitive	semiorder
B3	fully transitive	total partial order
B4	fully transitive, anti-symmetric	total order
B5	fully transitive, symmetric	universal relation

Proof. By the preceding. ■

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